Linear Independence

Geometric Algorithms
Lecture 7

CAS CS 132

Practice Problem

Do these three vectors span all of $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

$\begin{bmatrix}0\\0\\1\end{bmatrix} \notin \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Answer

$$\mathbf{v}_1 = \begin{bmatrix}-4\\4\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}-3\\6\\-3\end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix}-5\\8\\-2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4 & -3 & -5 & b_1 \\ 4_{-4} & 6_{-3} & 8_{-5} & b_2 \\ 4 & -6 & -4 & b_3 \\ -4 & -3 & -5 & \end{array}\right] \sim \left[\begin{array}{ccc|c} -4 & -3 & 5 & b_1 \\ 0 & 3 & 3 & b_2+b_1 \\ 0 & -9 & -9 & b_3+b_1 \end{array}\right]$$

$\boxed{\text{NO}}$

$b_1 = 0$
$b_2 = 0$
$b_3 = 1$

$$\sim \left[\begin{array}{ccc|c} -4 & -3 & 5 & b_1 \\ 0 & 3 & 3 & b_2+b_1 \\ 0 & 0 & 0 & b_3+b_1+3(b_2+b_1) \end{array}\right]$$

$b_3 + b_1 + 3(b_2+b_1) =$
$4b_1 + 3b_2 + b_3 = 0$

Answer: No

`Consider the matrix`

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 2 & -3 & -2 \end{bmatrix}$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 4 & -6 & -4 \end{bmatrix}$$

$$R_3 \leftarrow 2R_3$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & -9 & -9 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$
$$R_3 \leftarrow R_3 + R_1$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 3R_2$$

Answer: No

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Third row has no pivot

Objectives

1. Recap on the notion of **full span**
2. Motivate the and define **linear independence**
3. See several perspectives on linear independence
4. ~~If there's time: see an application of linear systems to **network flows**~~

Keywords

linear independence

linear dependence

homogenous systems of linear equations

trivial and nontrivial solutions

Recap: Full Span

Recall: Span

Recall: Span

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

Recall: Span

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$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. the ***span*** of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is:

$$\mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

Spans (with Matrices)

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

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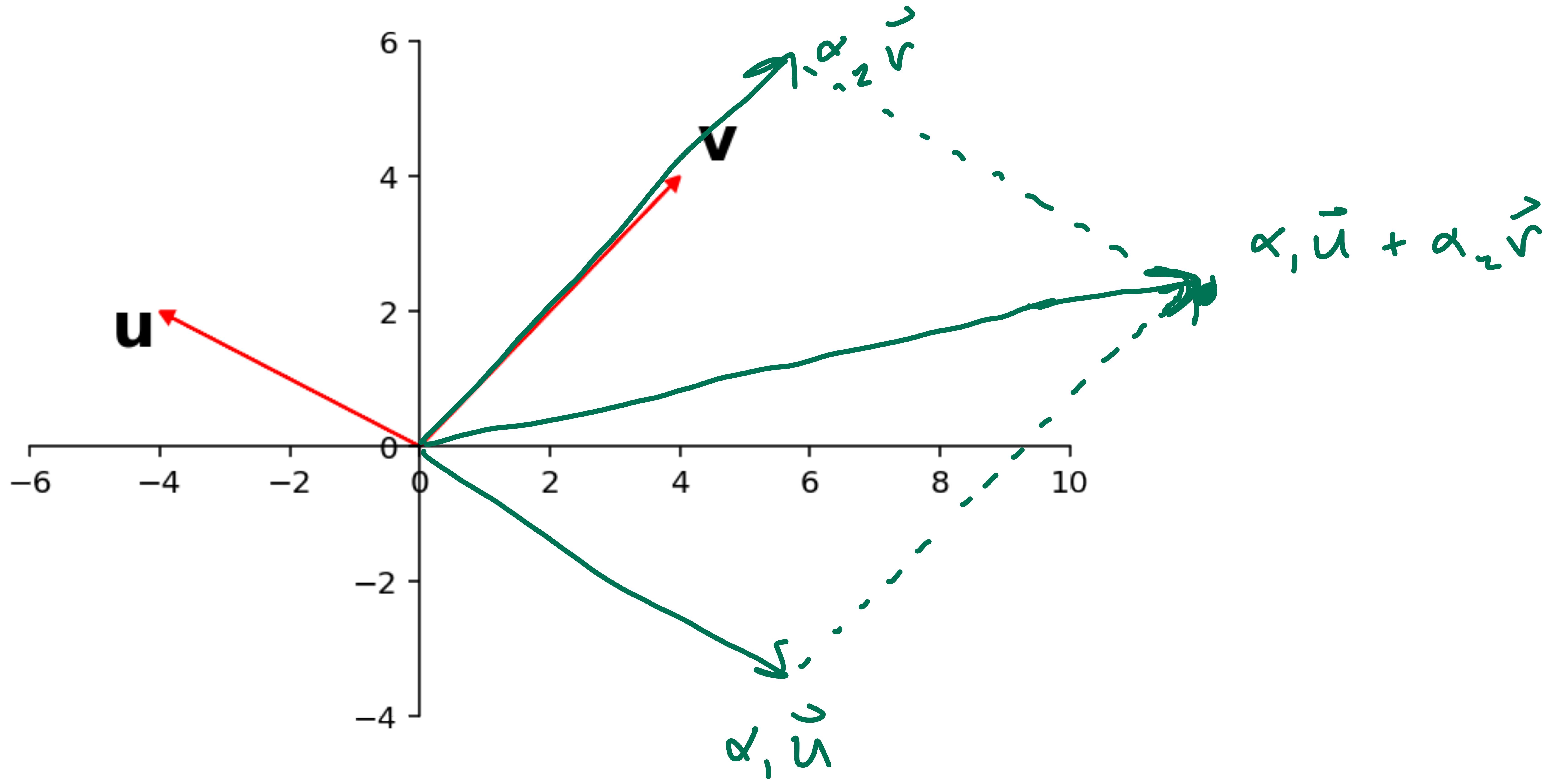
the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

Spanning all of $\mathbb{R}^2$

if two (or more) vectors in $\mathbb{R}^2$ span a plane, they must span all of $\mathbb{R}^2$. They "fill up" $\mathbb{R}^2$



What about $\mathbb{R}^n$?

When do a set of vectors span all of $\mathbb{R}^n$?
When do a set of vectors "fill up" $\mathbb{R}^n$?

A Thought Experiment

suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

`then we reduce it to echelon form`

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

```
then we reduce it to echelon form
```

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Yes. It doesn't have an inconsistent row

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix}$$

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$ [redacted]

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

what about this system?

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{array}\right]$$

0

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

`it depends...`

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

if it doesn't matter what the last column is,
then **every choice must be possible**

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

if it doesn't matter what the last column is, then **every choice must be possible**

every vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Spanning $\mathbb{R}^m$

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution
2. The columns of A span $\mathbb{R}^m$
3. A has a pivot position in every row

Spanning $\mathbb{R}^m$

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution

2. The columns of A span $\mathbb{R}^m$

3. A has a pivot position in every row

HOW TO: Spanning $\mathbb{R}^m$

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{a}_n]$ to echelon form and check if every row has a pivot

HOW TO: Spanning $\mathbb{R}^m$

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

!! We only need the echelon form !!

Example

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ *span all of* $\mathbb{R}^3$*?*

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 3 & 2023 \end{bmatrix}$$

$\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

`in this case the choice matters`

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

in this case the choice matters

we can't make the last column $[0 \ \ 0 \ \ 0 \ \ \blacksquare]$ for nonzero $\blacksquare$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

in this case the choice matters

we can't make the last column $\begin{bmatrix} 0 & 0 & 0 & \blacksquare \end{bmatrix}$ for nonzero $\blacksquare$

but we can make the last column <u>parameters</u> to find equations that must hold

Not spanning $\mathbb{R}^m$

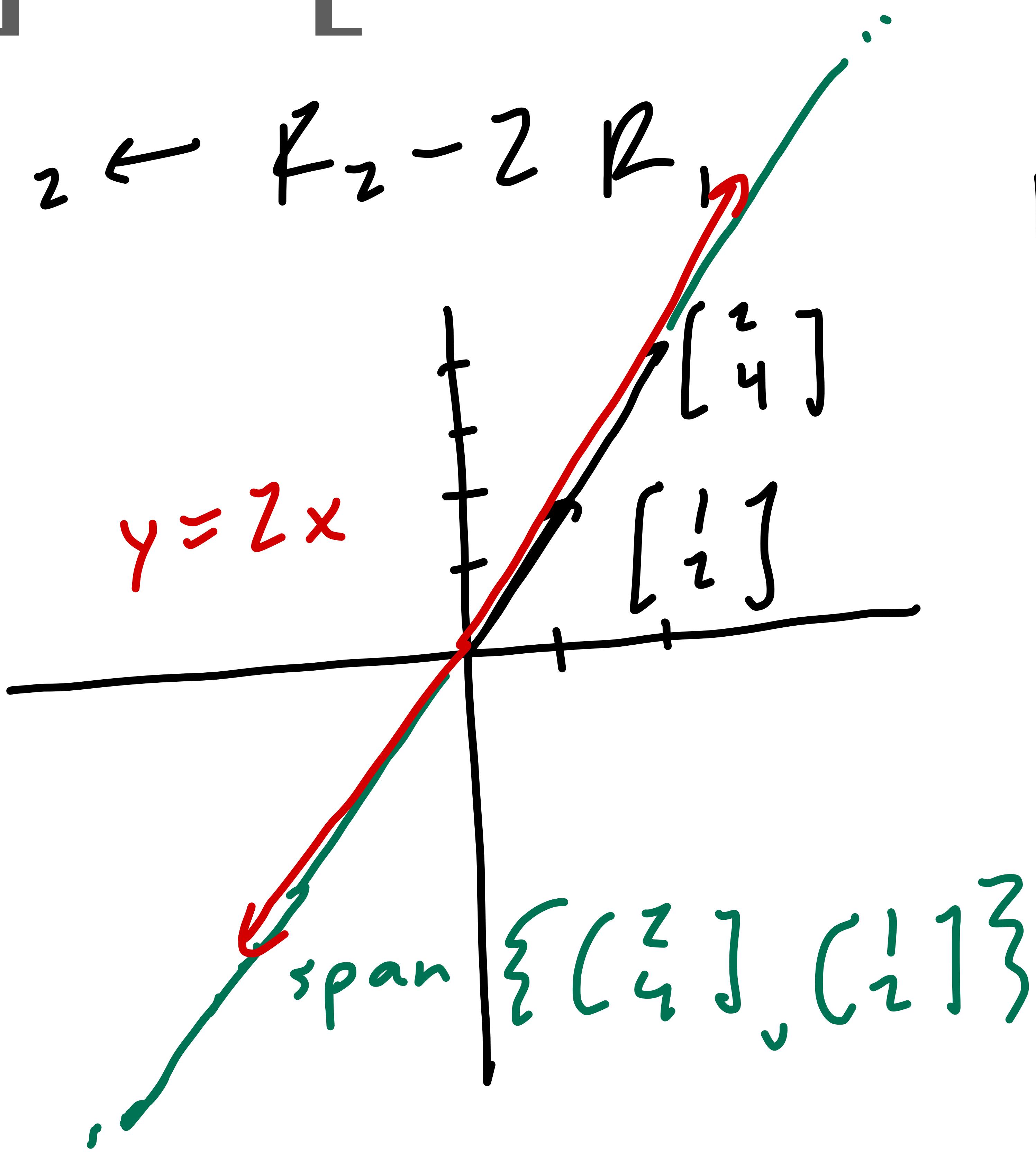
$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$\boxed{-2x_1 + x_2 = 0}$

$b_2 - 2b_1 = 0$

$-2x + y = 0$



Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

as long as $(-2)b_1 + b_2 = 0$, the system is consistent

Not spanning $\mathbb{R}^m$

$R_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$-2b_1$

as long as $(-2)b_1 + b_2 = 0$, the system is consistent

this gives use a <u>linear equation</u> which describes the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Question (Understanding Check)

True* or *False*, the echelon form of any matrix has at most one row of the form** $[0 \ \ 0 \ \ \ldots \ \ 0 \ \ \blacksquare]$ ***where $\blacksquare$ ***is nonzero.***

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entry not to the right

this is not in echelon form

Example

Give a linear equation for the span of the vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ *and* $\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$.

$$x_1\begin{bmatrix}1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\begin{matrix}0 & -1\\0 & -1\\1 & -1\end{matrix}$$

$$-2(1)+2+0=0 \checkmark$$

$$-2(-1)+(-1)+(-1)=0$$

$$\left[\begin{array}{ccc}1 & -1 & b_1\\ 2 & -1 & b_2\\ 0 & -1 & b_3\end{array}\right]\begin{matrix}0 & -1\\0 & -1\\1 & -1\end{matrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc}1 & -1 & b_1\\ 0 & 1 & b_2-2b_1\\ 0 & -1 & b_3\end{array}\right]\begin{matrix}0 & -1\\0 & 1\\1 & -1\end{matrix} \xrightarrow{R_3 \leftarrow R_3 + R_2}$$

$-2 \quad +2 \quad -2b_1$

$+0 \quad +1 \quad + b_2 - 2b_1$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \notin \operatorname{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}$$

$$\left[\begin{array}{cccc}1 & -1 & -1 & b_1\\ 0 & 1 & 1 & b_2-2b_1\\ 0 & 0 & 0 & b_3+b_2-2b_1\end{array}\right]\begin{matrix}0\\0\\1\end{matrix}$$

$$b_3 + b_2 - 2b_1 = 0$$

$$\boxed{-2x_1 + x_2 + x_3 = 0}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

back to linear independence...

Homogeneous Linear Systems

Recall: The Zero Vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big| \, n$$

the dimension is implicit in the notation

Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

a_{ij} : coeff of row i, col j

Trivial Solutions

Definition. For the matrix equation

$$A\mathbf{x} = \mathbf{0}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***.

Any other solution is called ***nontrivial***.

Trivial Solutions

Definition. For the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***.

Any other solution is called ***nontrivial***.

Trivial Solutions

Definition. For the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the ***trivial solution***.

Any other solution is called ***nontrivial***.

Questions about Homogeneous Systems

When does $A\mathbf{x} = \mathbf{0}$ have only the trivial solution?

When does $A\mathbf{x} = \mathbf{0}$ have nontrivial solutions?

What does it mean *geometrically* in each case?

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

What do we know about the covered column?

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

What do we know about the covered column?

It has to be all zeros.

Linear Independence

Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution).

Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution).

The columns of A are linearly independent if $A\mathrm{x} = 0$ has exactly one solution.

Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has a nontrivial solution.

Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has a nontrivial solution.

A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals $\mathbf{0}$.

Linear Dependence (Alternative)

Definition. A set of vectors is **linearly dependent** if it is <u>not</u> linearly independent.

Linear Dependence (Alternative)

Definition. A set of vectors is **linearly dependent** if it is not linearly independent.

$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

$\equiv$

$A\mathbf{x} = \mathbf{0}$ does not have only the trivial solution

Examples

$$\{\ \}$$

linearly independent

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_1 \\ -x_1 \end{bmatrix} \qquad \begin{matrix} 2x_1 = 0 \\ 3x_1 = 0 \\ -x_1 = 0 \end{matrix}$$

linearly independent

$x_1 = 0$

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -2$

$x_1 = \underline{1}$

linearly dep.

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

L.I.

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Another Interpretation of Linear Dependence

demo
(from ILA)

Three Vectors in $\mathbb{R}^3$

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It's possible for three vectors in $\mathbb{R}^3$ to span all of $\mathbb{R}^3$, but it's <u>not</u> guaranteed

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Three Vectors in $\mathbb{R}^3$

It's possible for three vectors in $\mathbb{R}^3$ to span all of $\mathbb{R}^3$, but it's <u>not</u> guaranteed

There may be vectors which lies in the plane spanned by two other vectors.

Or even two vectors which lie in the span of one of the others.

Fundamental Concern

How do we classify when a set of vectors does <u>not</u> span as much as it possibly can? When it is "smaller" than it could be?

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How do we classify when a set of vectors does <u>not</u> span as much as it possibly can? When it is "smaller" than it could be?

This is the role of linear dependence.

Linear Dependence (Another Alternative)

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Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is ***linearly dependent*** if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself).

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e.g.,

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

(the recap problem)

The Linear Combination Perspective

Suppose we have four vectors such that

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \qquad + 5\mathbf{v}_4$$

what do we know about the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

$$2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3 + 5\vec{v}_4 = \vec{0}$$

The Linear Combination Perspective

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \qquad + 5\mathbf{v}_4$$

implies

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

has a nontrivial solution:

$$(2,3,-1,5)$$

The Vector Equation Perspective

Suppose

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

has a nontrivial solution:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

where, say, $\alpha_2 \neq 0$

The Vector Equation Perspective

Suppose

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

has a nontrivial solution:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

where, say, $\alpha_2 \neq 0$

We can turn this into a linear combination.

The Vector Equation Perspective

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}$$

$$\alpha_2 \neq 0$$

$$\alpha_2 \vec{v}_2 = -\alpha_1 \vec{v}_1 - \alpha_3 \vec{v}_3 - \alpha_4 \vec{v}_4$$

$$\vec{v}_2 = \left(\frac{-\alpha_1}{\alpha_2}\right)\vec{v}_1 + \left(\frac{-\alpha_3}{\alpha_2}\right)\vec{v}_3 + \left(\frac{-\alpha_4}{\alpha_2}\right)\vec{v}_4$$

The Vector Equation Perspective

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = \mathbf{0}$$

The Vector Equation Perspective

$$\alpha_1 \mathbf{v}_1 + \qquad\qquad\qquad \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = -\alpha_2 \mathbf{v}_2$$

The Vector Equation Perspective

$$\frac{-\alpha_1}{\alpha_2}\mathbf{v}_1 + \qquad \frac{-\alpha_3}{\alpha_2}\mathbf{v}_3 + \frac{-\alpha_4}{\alpha_2}\mathbf{v}_4 = \qquad \mathbf{v}_2$$

We get one vector as a linear combination of the others.

The Vector Equation Perspective

This division only works because $\alpha_2 \neq 0$.

$$\frac{-\alpha_1}{\alpha_2}\mathbf{v}_1 + \qquad \frac{-\alpha_3}{\alpha_2}\mathbf{v}_3 + \frac{-\alpha_4}{\alpha_2}\mathbf{v}_4 = \qquad \mathbf{v}_2$$

We get one vector as a linear combination of the others.

In All

Theorem. A set of vectors is linearly dependent if and only if it is nonempty and *at least* one of its vectors can be written as a linear combination of the others.

P if and only if Q means
P implies Q and Q implies P

Linear Dependence Relation

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

A linear dependence relation *witnesses* the linear dependence.

How To: Linear Dependence Relation

How To: Linear Dependence Relation

Question. Write down a linear dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$.

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Solution. Find a nontrivial solution to the equation

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]\,\mathbf{x} = \mathbf{0}$$

How To: Linear Dependence Relation

Question. Write down a linear dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$.

Solution. Find a nontrivial solution to the equation

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]\,\mathbf{x} = \mathbf{0}$$

(there will be a free variable you can choose to be nonzero)

Example

Write down the linear dependence relation for the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Added $\mathbf{0}$ column

Where we left off

Answer

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2/3$$

Answer

$$\begin{bmatrix} -4 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

Answer

$$\begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1/(-4)$$

Answer

$$x_1 = -(0.5)x_3$$
$$x_2 = -x_3$$
$$x_3 \ \texttt{is free}$$

Answer

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

Answer

$$\begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} - 2 \begin{bmatrix} -5 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix}-4\\4\\2\end{bmatrix}+2\begin{bmatrix}-3\\6\\-3\end{bmatrix}-2\begin{bmatrix}-5\\8\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Note there are other solutions as well...

Simple Cases

The Empty Set

The Empty Set

$\{\}$ (a.k.a. $\varnothing$) is linearly independent

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We stretch the definition a bit: there is no nontrivial linear combination of the vectors equaling $\mathbf{0}$

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There are none at all...

The Empty Set

$\{\}$ (a.k.a. $\emptyset$) is linearly independent

We stretch the definition a bit: there is no nontrivial linear combination of the vectors equaling $\mathbf{0}$

There are none at all...

$\mathbf{0}$ is in every span, even the empty span.

One Vector

A single vector $\mathbf{v}$ is linearly independent if and only if it $\mathbf{v} \neq \mathbf{0}$.

Note that

$$x_1\mathbf{0} = \mathbf{0}$$

has many nontrivial solutions.

The Zero Vector and Linear Dependence

If a set of vectors V contains the $\mathbf{0}$, then it is linearly dependent.

The Zero Vector and Linear Dependence

If a set of vectors V contains the $\mathbf{0}$, then it is linearly dependent.

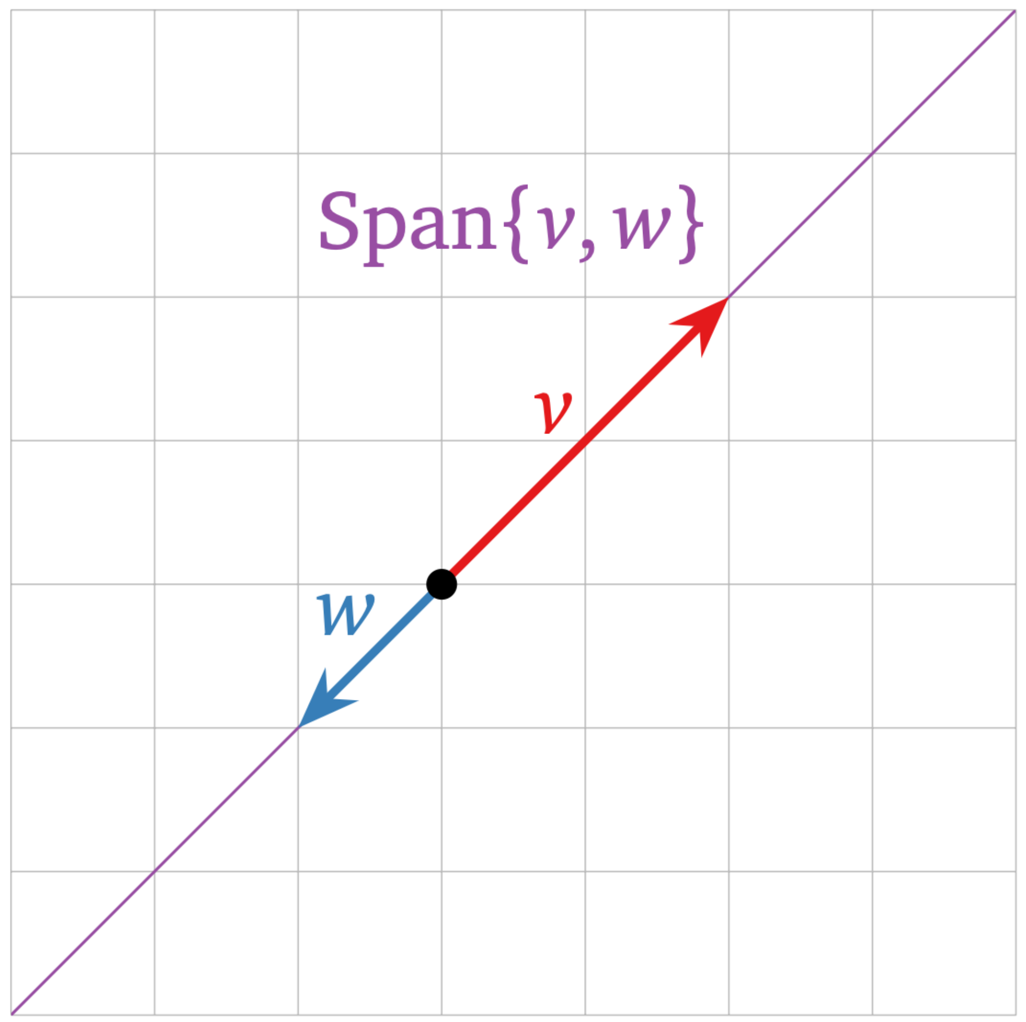
$$(1)\mathbf{0} + 0\mathbf{v}_2 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n = \mathbf{0}$$

There is a very simple nontrivial solution.

Two Vectors

Definition. Two vectors are ***colinear*** if they are scalar multiples of each other.

e.g., $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1.5\\1.5\\3\end{bmatrix}$ or $\begin{bmatrix}2\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$ $\longrightarrow$



Two vectors are linearly dependent if and only if they are colinear.

image source

Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane.

Three vectors are linearly dependent if an only if they are colinear or coplanar.

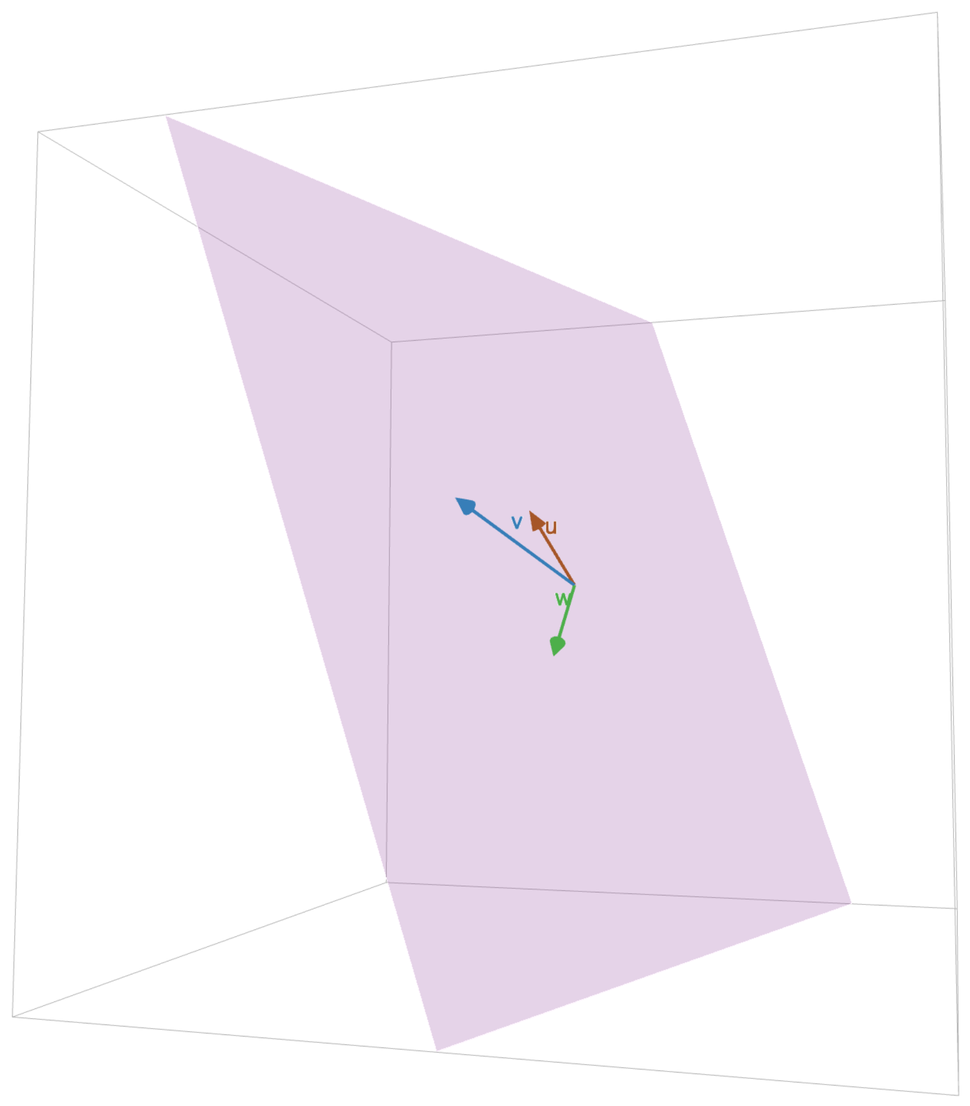


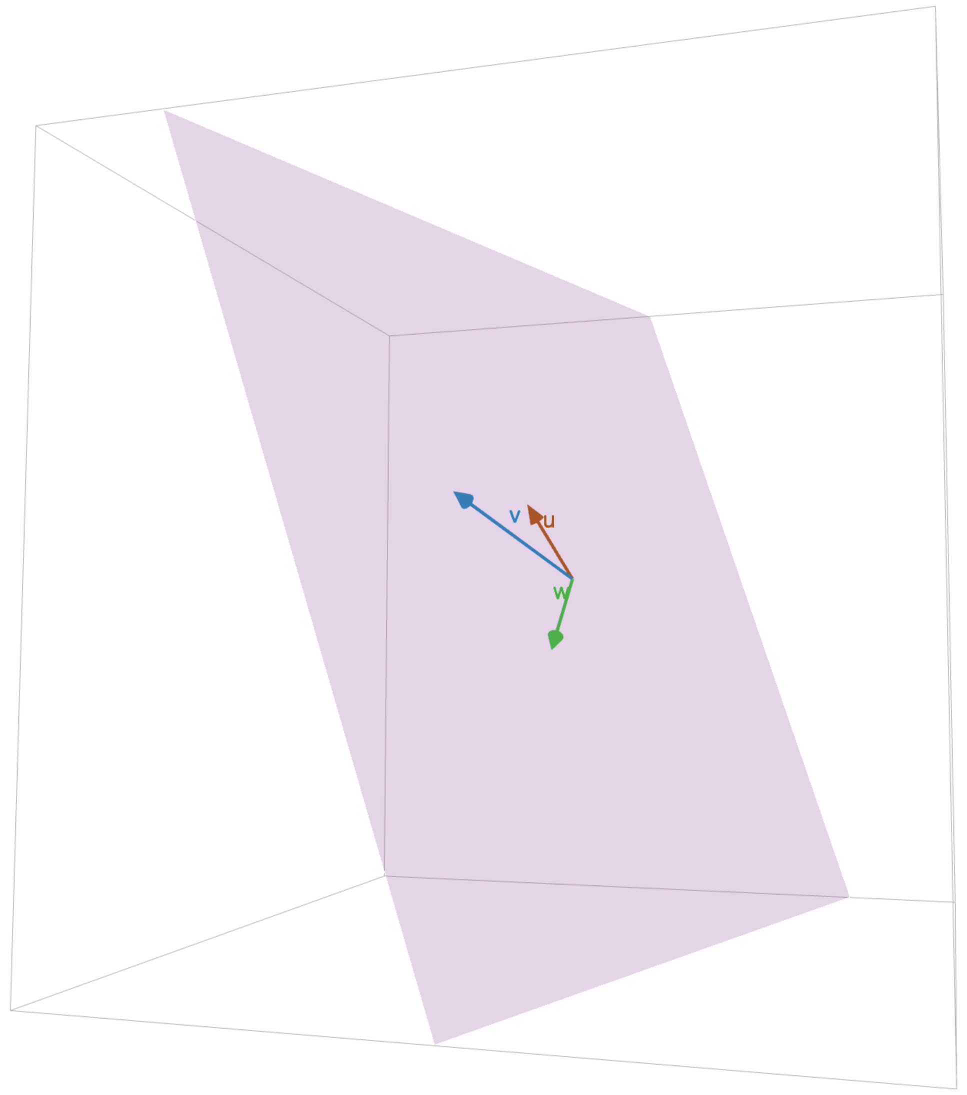
image source

Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane.

Three vectors are linearly dependent if an only if they are colinear or coplanar.

This can be reasoning can be extended to more vectors, but we run out of terminology



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Yet Another Interpretation

Increasing Span Criterion

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly *independent* then we cannot write one of it's vectors as a linear combination of the others.

But we get something stronger.

Increasing Span Criterion

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if for all $i \leq n$,

$$\mathbf{v}_i \notin \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

Increasing Span Criterion

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if for all $i \leq n$,

$$\mathbf{v}_i \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

As we add vectors, the span gets larger.

Increasing Span Criterion

```
So in this case, our span keeps getting "bigger"
```

Increasing Span Criterion

So in this case, our span keeps getting "bigger"

$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

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$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

$\text{span}\{\mathbf{v}_1\}$ is a line

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Increasing Span Criterion

So in this case, our span keeps getting "bigger"

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$\text{span}\{\mathbf{v}_1\}$ is a line

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a 3d-hyperplane

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a 4d-hyperlane

Increasing Span Criterion

So in this case, our span keeps getting "bigger"

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a 4d-hyperlane

...

Characterization of Linear Dependence

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only there is an $i \leq n$,

$$\mathbf{v}_i \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

Characterization of Linear Dependence

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$$\mathbf{v}_i \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

As we add vectors, we'll eventually find one in the span of the preceding ones.

Characterization of Linear Dependence

$\text{span}\{\}$ is a point $\{\mathbf{0}\}$

$\text{span}\{\mathbf{v}_1\}$ is a line

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is still a plane

...

Characterization of Linear Dependence

span$\{\}$ is a point $\{\mathbf{0}\}$

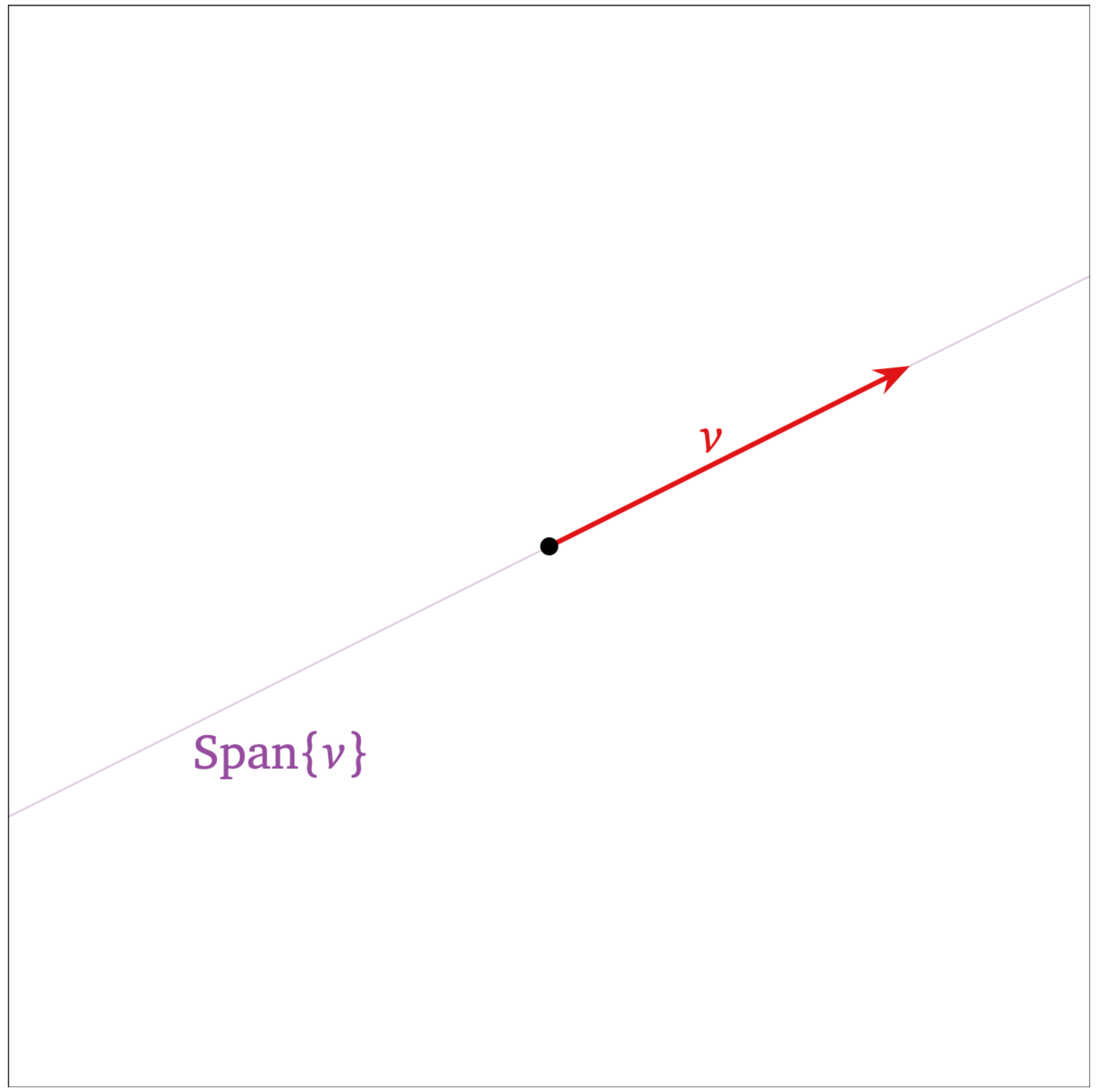
span$\{\mathbf{v}_1\}$ is a line

span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane

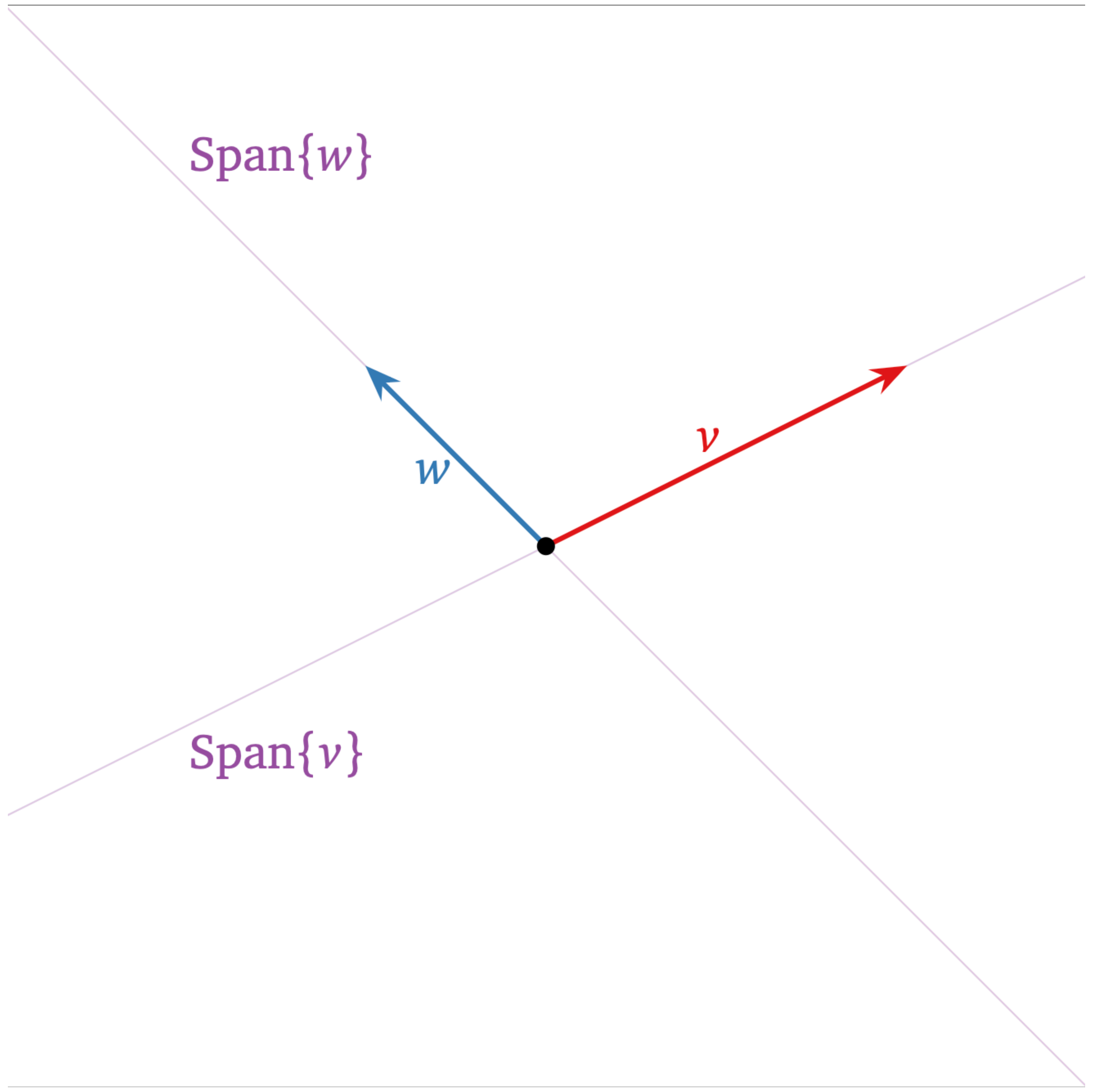
span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is still a plane

... (this is an example, it may take a lot more vectors before we find one in the span of the preceding vectors)

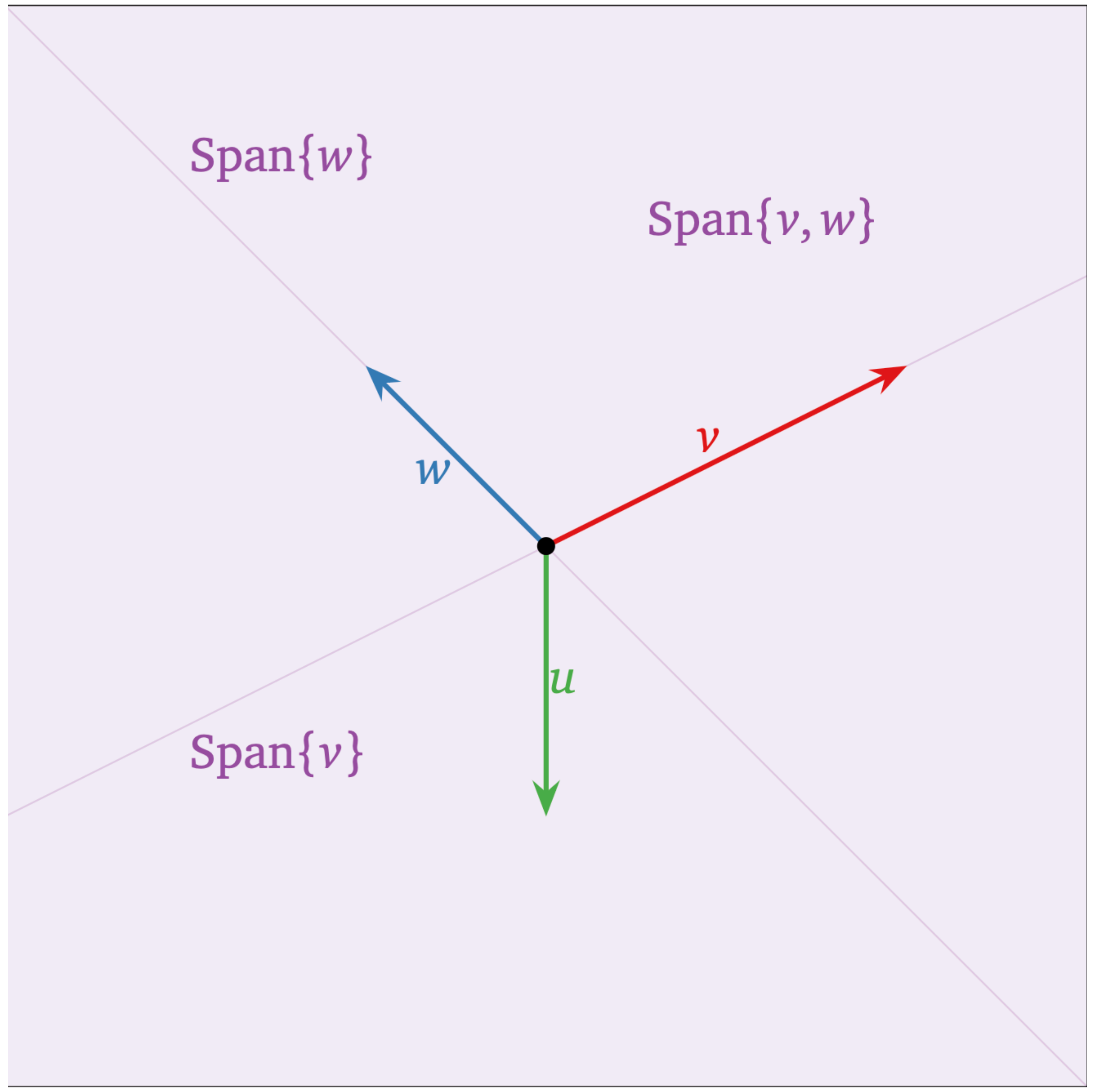
As a Picture



span of 1 vector
a line



span of 2 vector
a plane



span of 3 vector
still a plane

image source

Characterization of Linear Dependence

Corollary. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent, then for any vector $\mathbf{v}_{k+1}$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly dependent.

If we add a vector to a linearly dependent set, it remains linearly dependent

Question

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Answer: No

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Any three vectors can at most span a plane.

The first two are not colinear, so they span a plane ($\mathbb{R}^2$).

Linear Independence and Free Variables

Linear Dependence Relations (Again)

When finding a linear dependence relation, we came across a system which a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take x_3 to be free

Pivots and Linear Dependence

Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column.

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Remember that we choose our free variables to be the ones whose columns don't have pivots.

Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column.

Remember that we choose our free variables to be the ones whose columns don't have pivots.

Free variables allow for infinitely many (nontrivial) solution.

Recall: General Form Solutions

$$x_1 = -(0.5)x_3$$
$$x_2 = -x_3$$
$$x_3 \ \texttt{is free}$$

Recall: General Form Solutions

$$x_1 = -0.5$$
$$x_2 = -1$$
$$x_3 = 1$$

Recall: General Form Solutions

$$x_1 = 0.5$$
$$x_2 = 1$$
$$x_3 = -1$$

Recall: General Form Solutions

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = -2$$

Recall: General Form Solutions

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = -2$$

`The point: the solution is not unique.`

How To: Linear Independence

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$ has a unique solution.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{0}$ has a unique solution.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if the general form solution of $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0} \end{bmatrix}$ has any free variables.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_n]$ to echelon form and check if has a pivot position in every column.

Example: Recap Problem Again

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$ is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column without a pivot

Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row.

The columns of a matrix are linearly independent if there is a pivot in every column.

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}$$

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 rows.

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} * & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \end{bmatrix}$$

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

This matrix as at most 3 pivots, but 4 columns.

A Warning

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row.

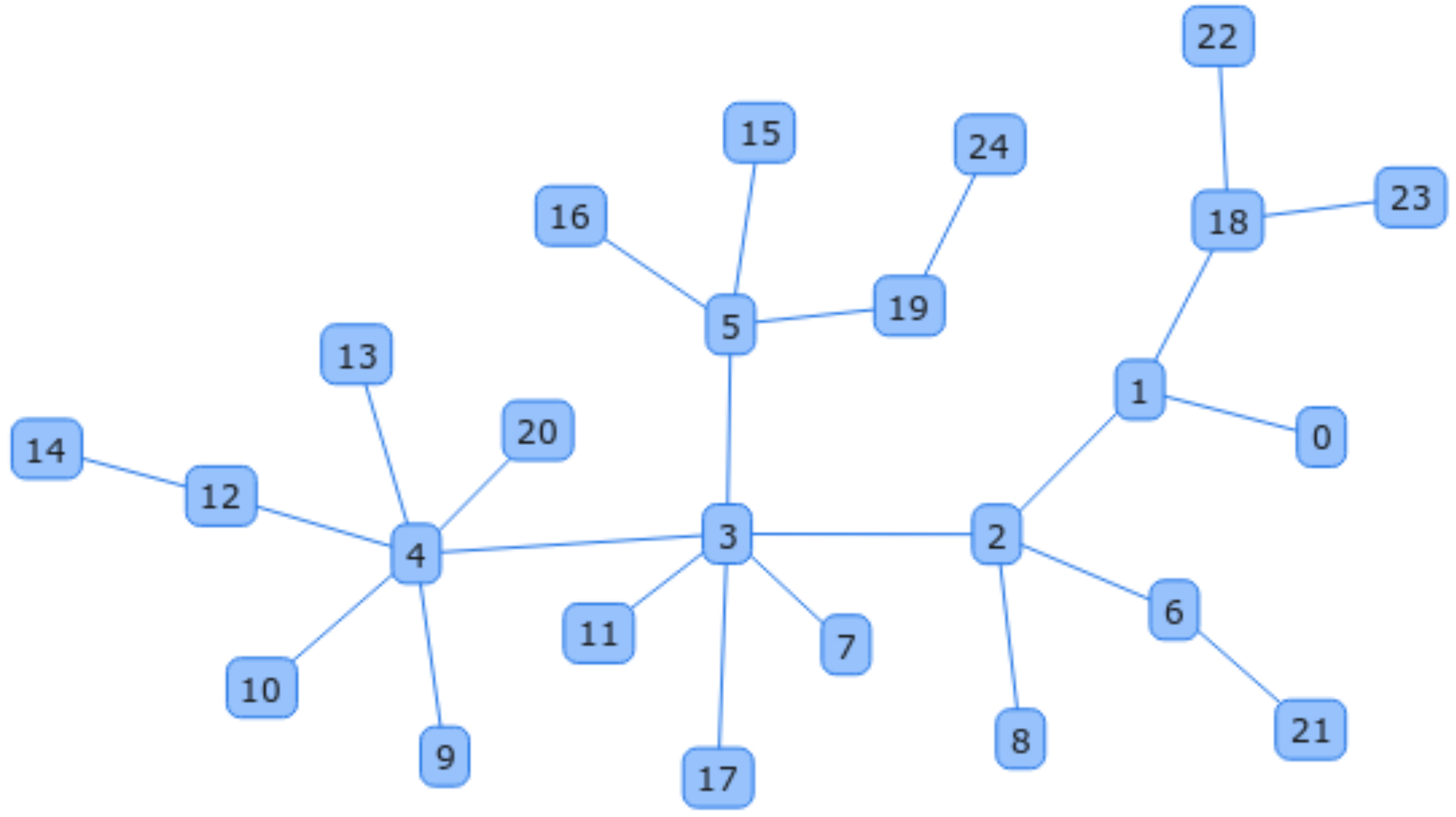
The columns of a matrix are linearly independent if there is a pivot in every column.

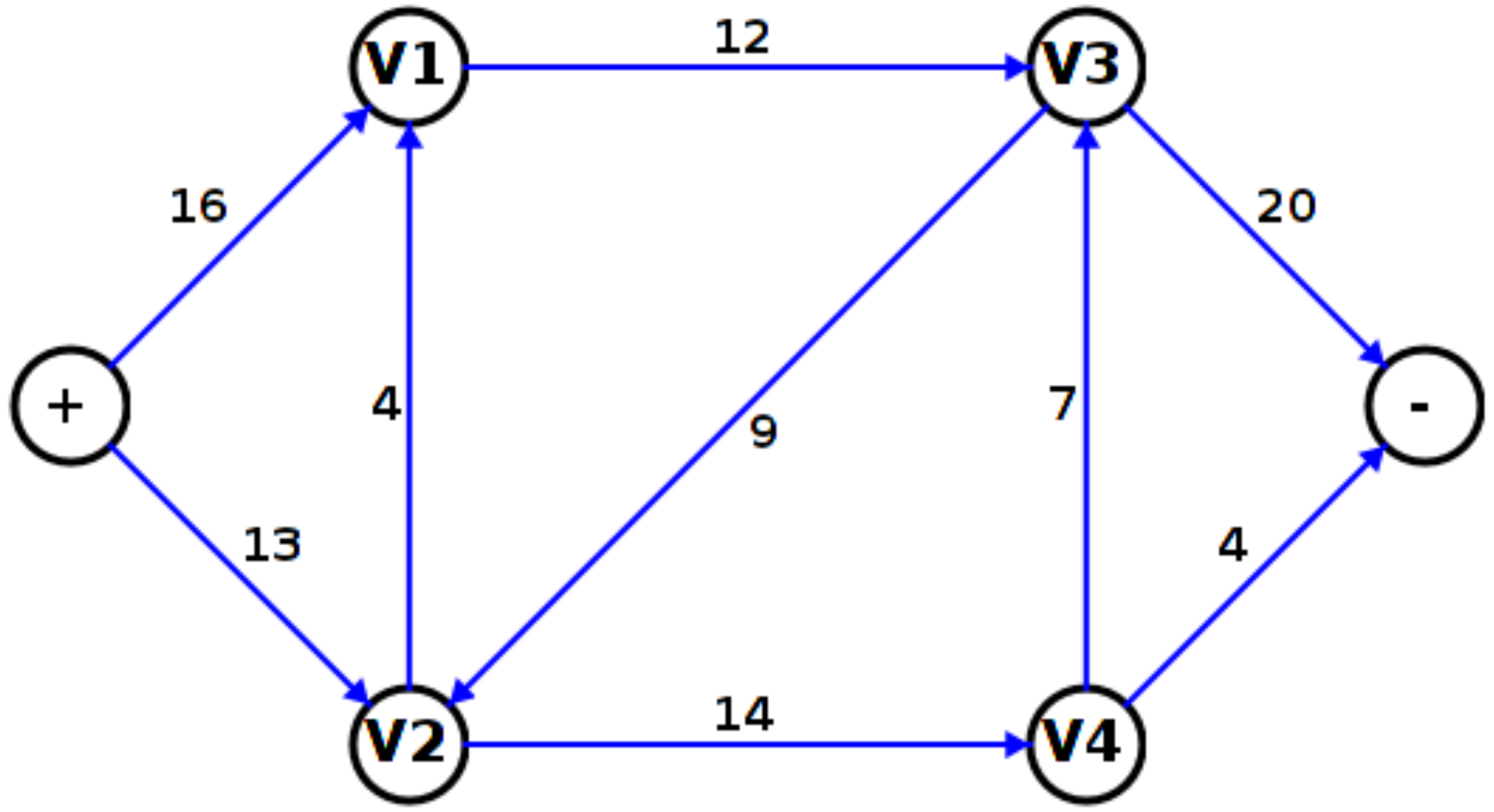
Don't confuse these!

Application: Networks and Flow

Graphs/Networks

A ***graph/network*** is a mathematical object representing collection of *nodes* and *edges* connecting them.





Graphs/Networks

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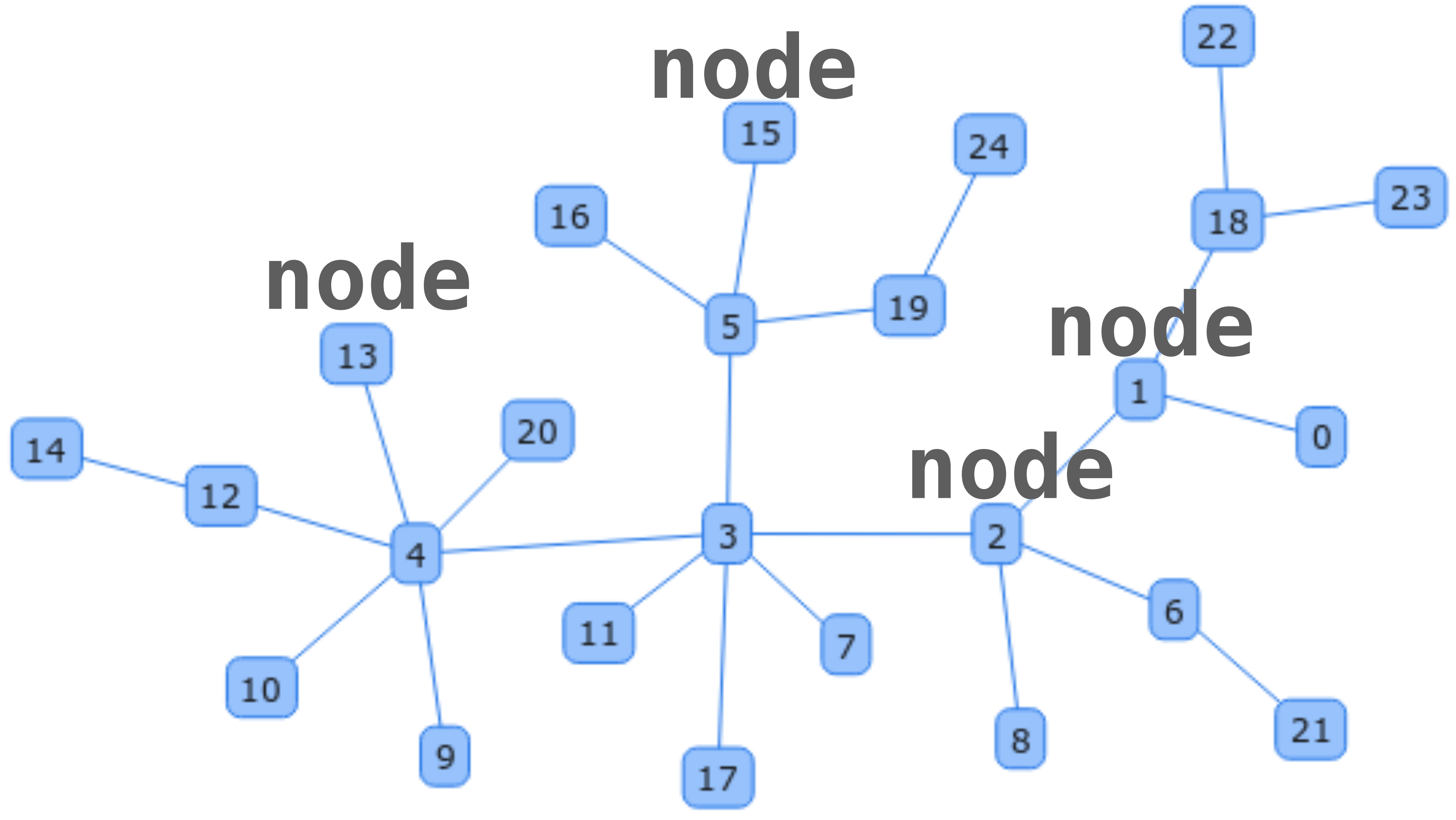

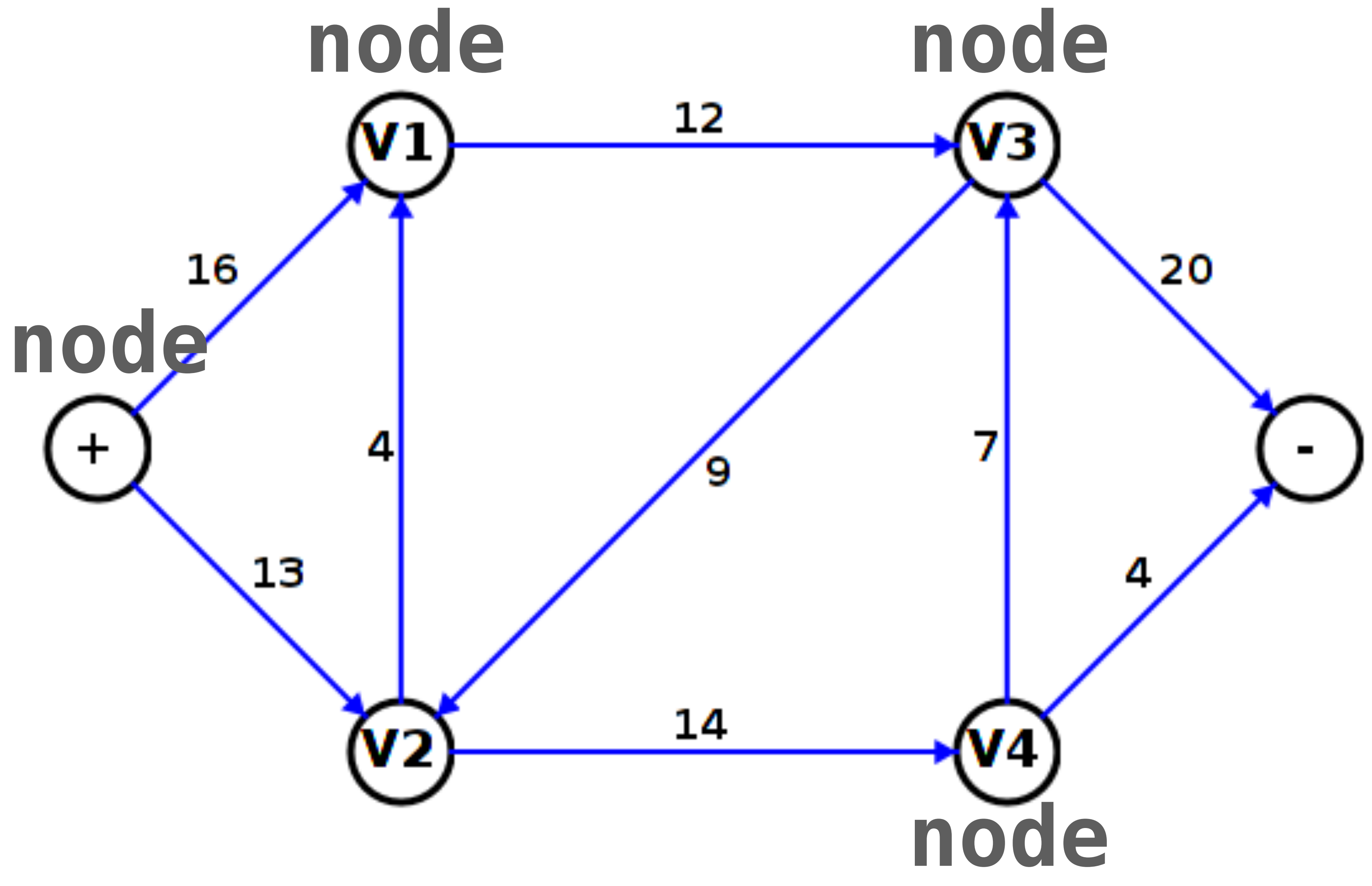

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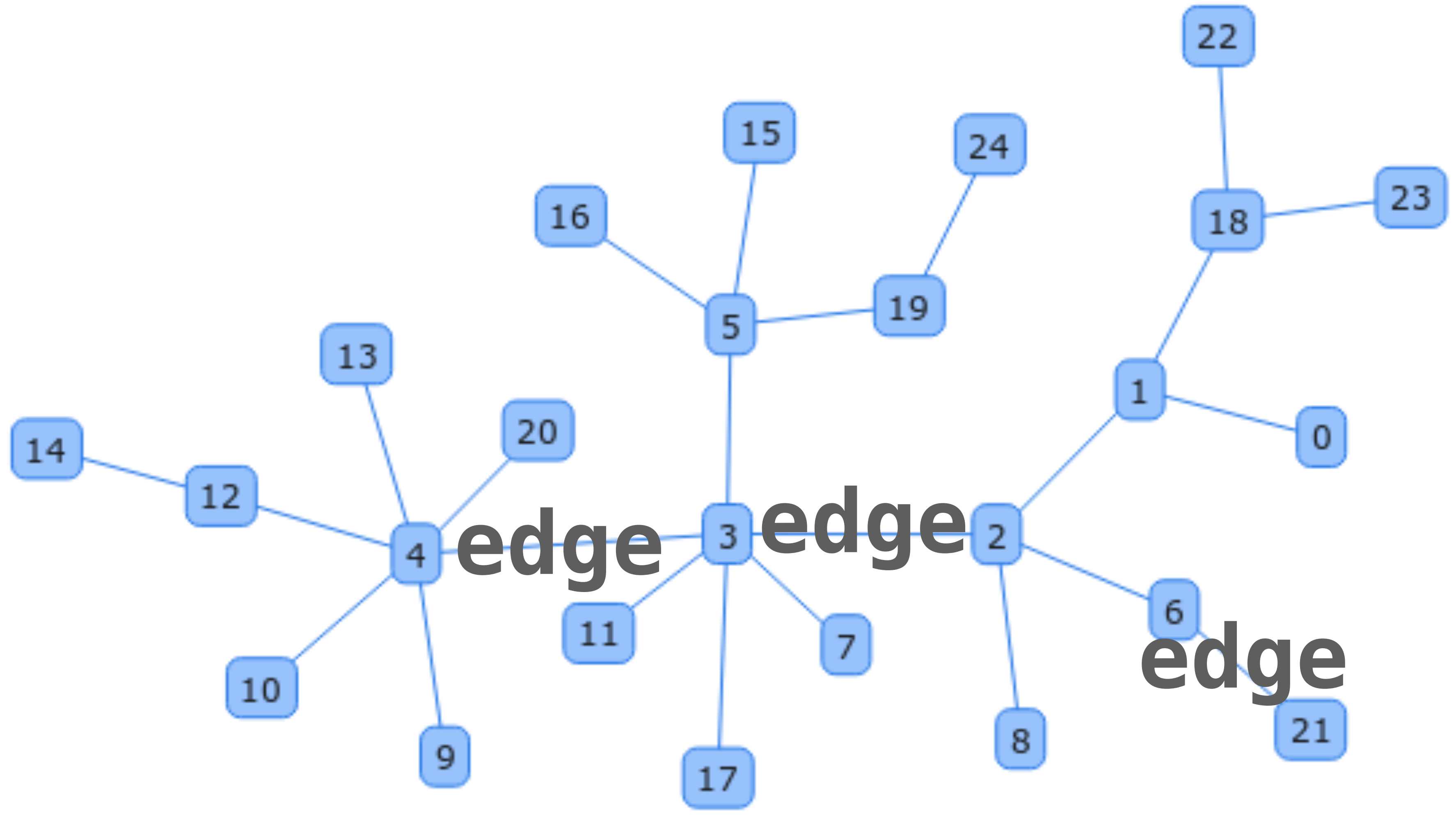
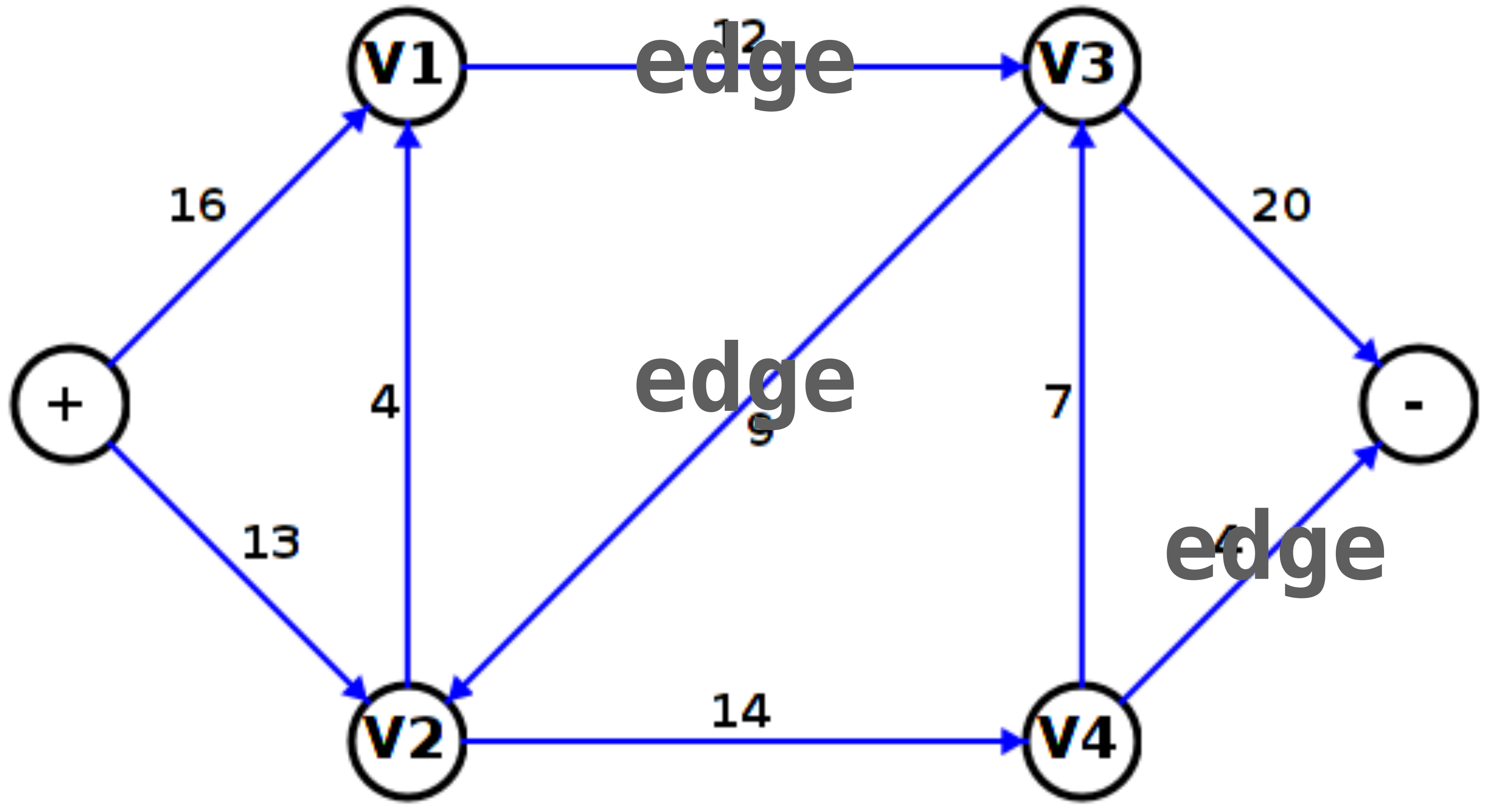
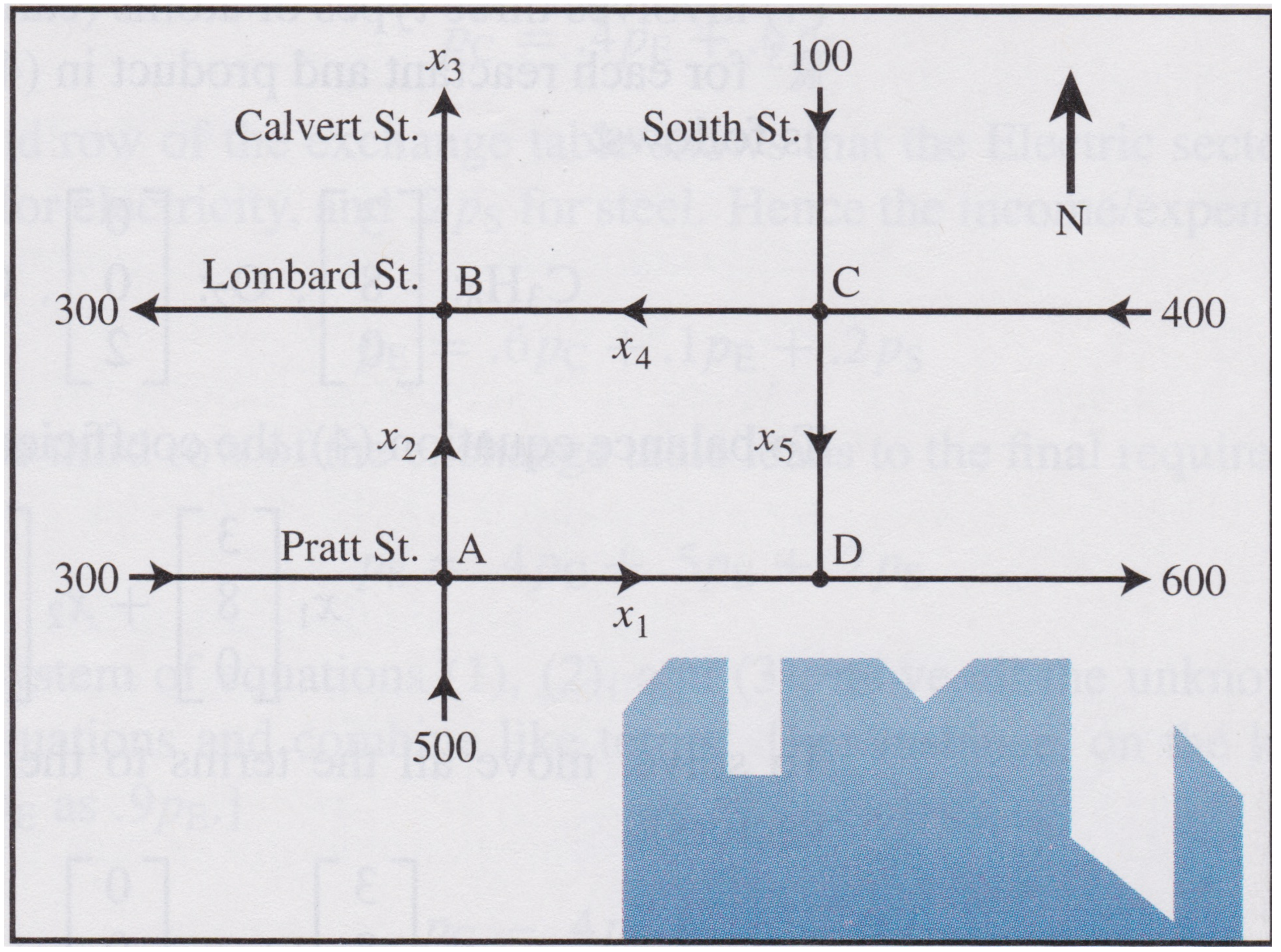




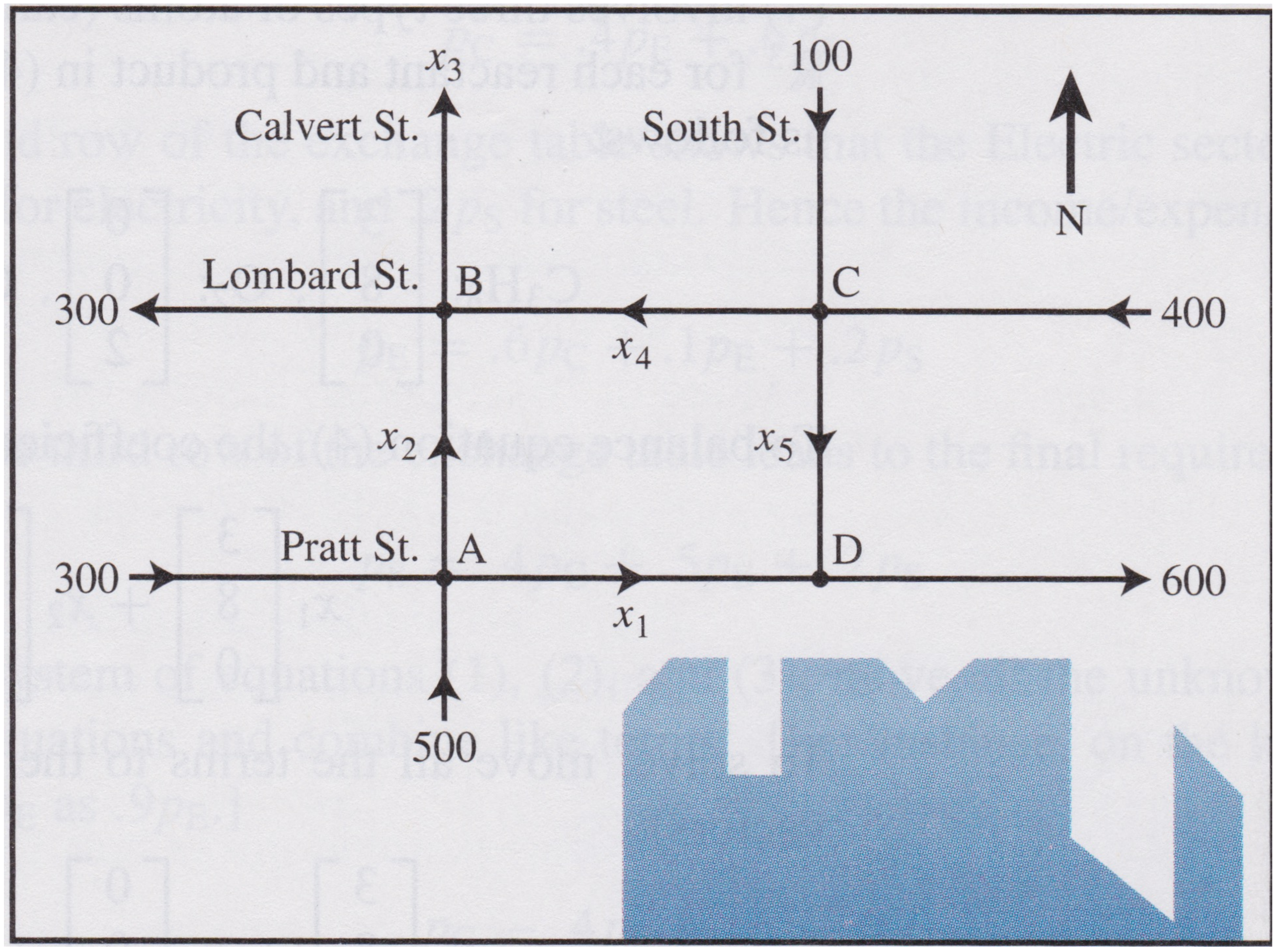
Directed Graphs

Today we focus on ***directed*** graphs, in which edges have a specified direction.



Think of these as one-way streets.

Flow



We are often interested in how much "stuff" we can push through the edges

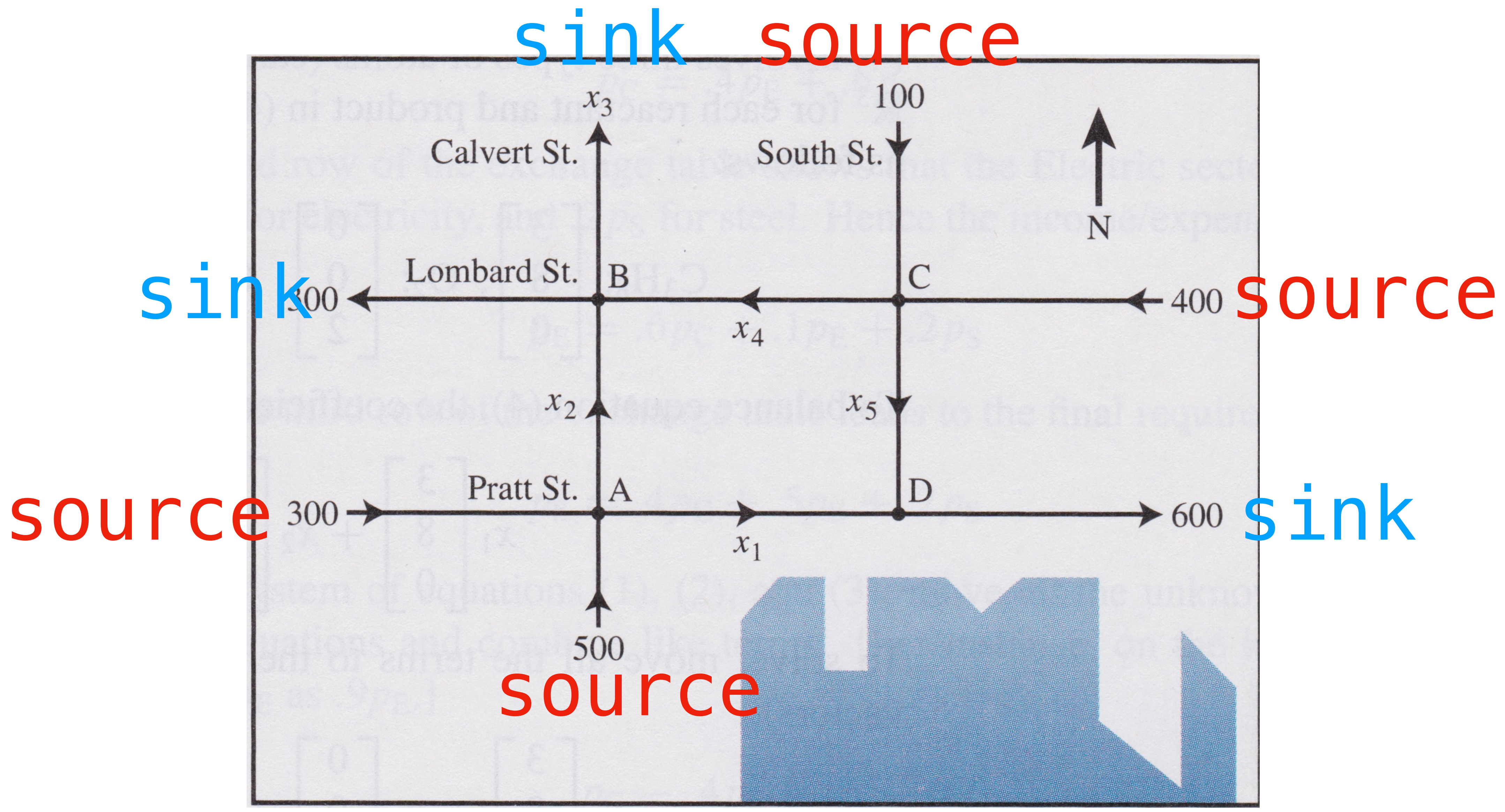
In the above example, the "stuff" is cars/hr.

I like to imagine water moving through a pipe, and splitting an joints in the pipe

Flow Network

A ***flow network*** is a directed graph with specified **source and sink nodes.**

Flow <u>comes out of</u> and <u>goes into</u> sources and sinks. They are assigned a flow value (or variable).



Flow

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conservation: flow into a node = flow out of a node

source/sink constraint: flow into a source/out of a sink is nonnegative.

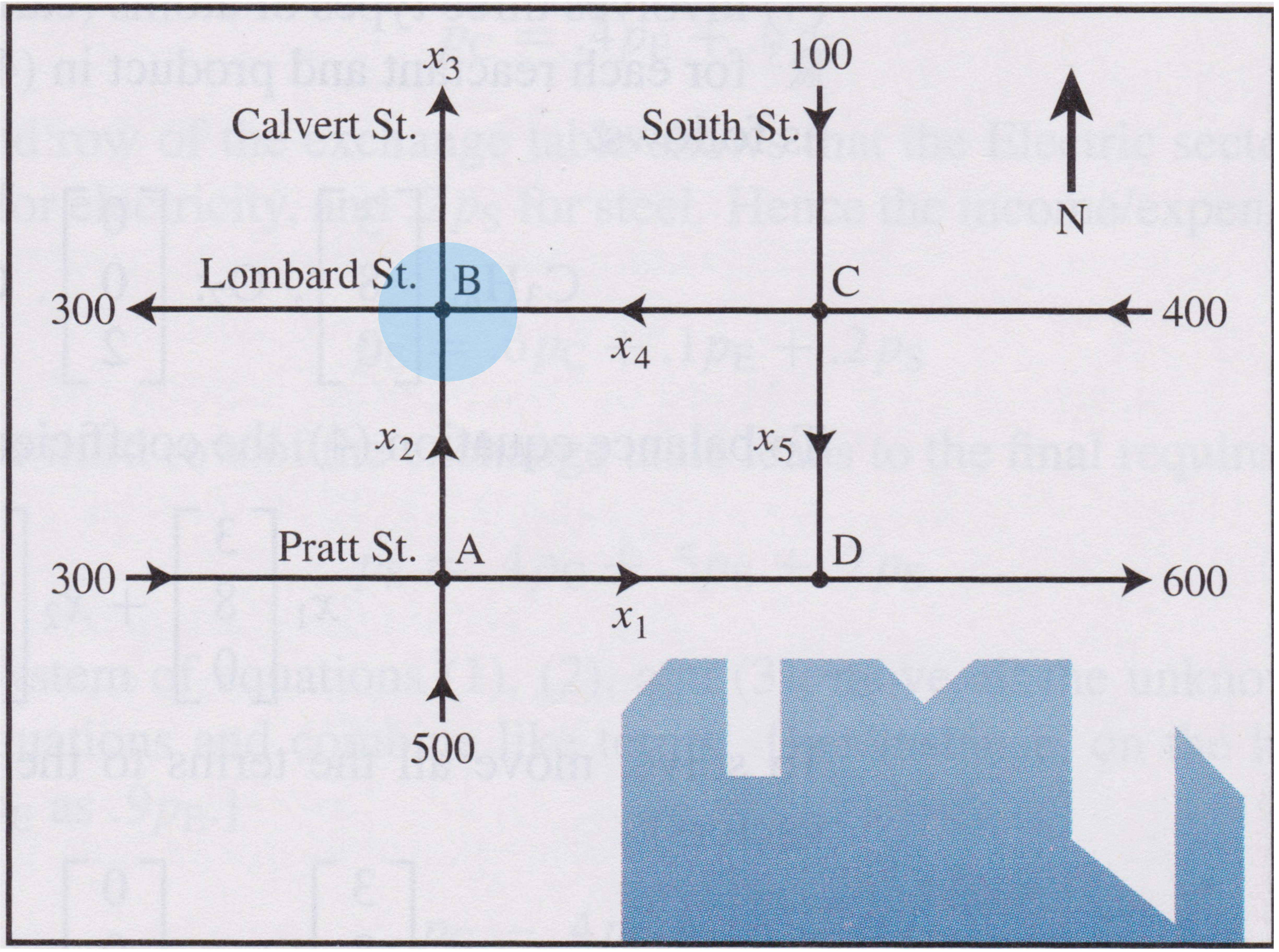
Flow Conservation

`Flow in = Flow out`

e.g.,

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$



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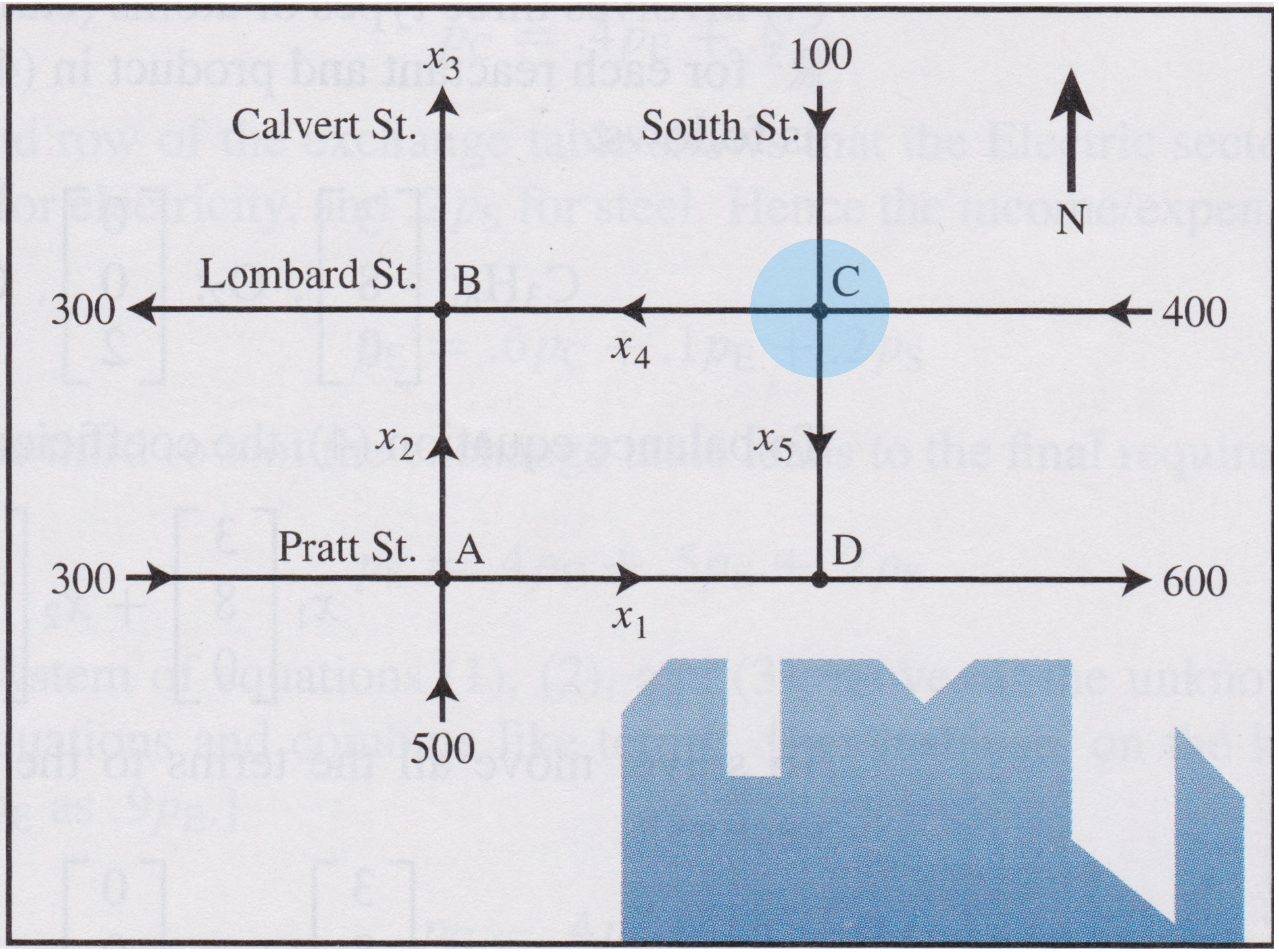

Flow Conservation

Flow in = Flow out

e.g.,

$$x_2 + x_4 = 300 + x_3$$

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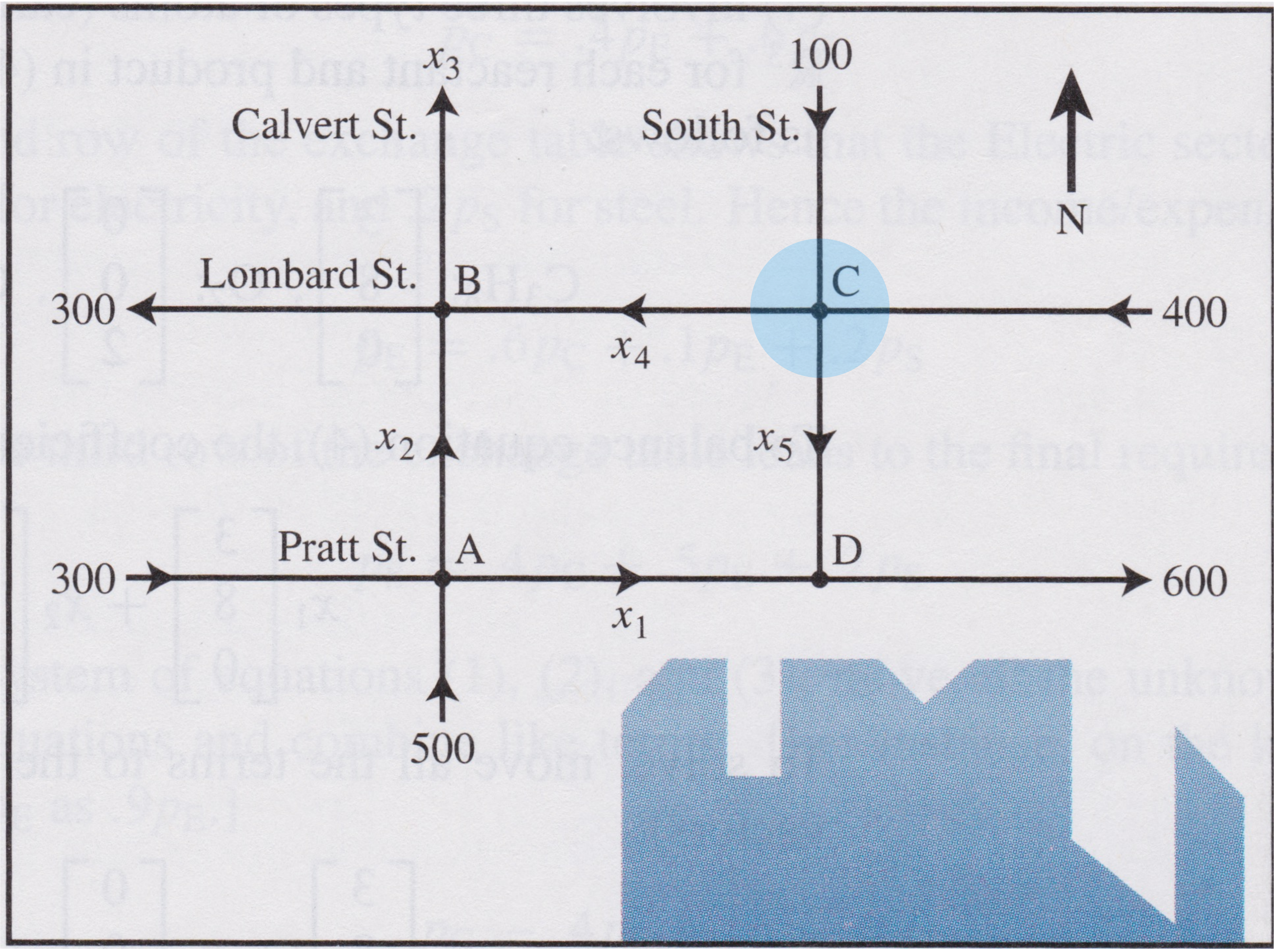
Every node determines a linear equation



How To: Network Flow

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Question. Find a general solution for the flow of a given graph.

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Solution. Write down the linear equations determined by <u>flow conservation</u> at non-source and non-sink nodes, and then solve.

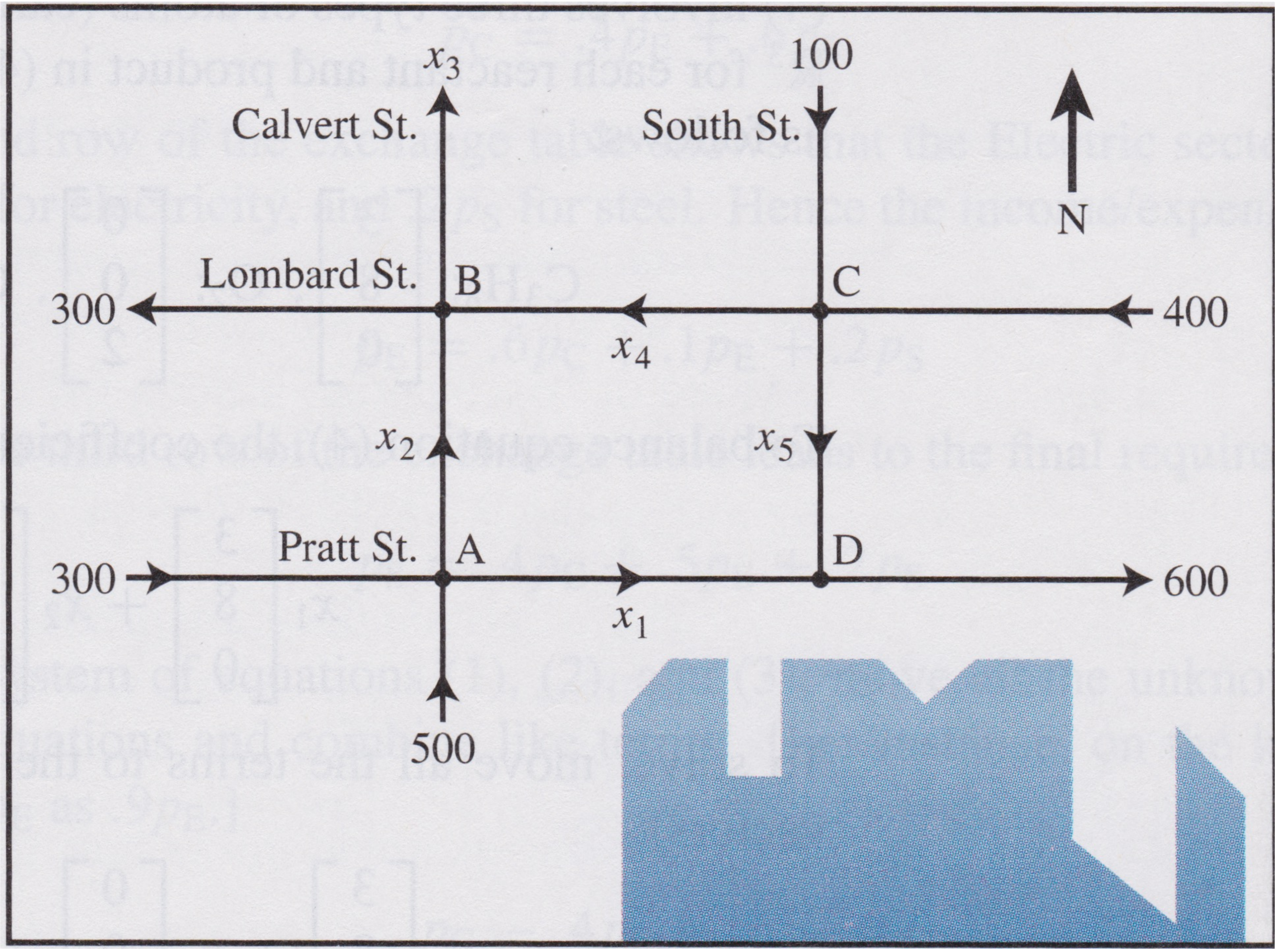
Example

(A) $500 + 300 = x_1 + x_2$

(B) $x_2 + x_4 = 300 + x_3$

(C) $100 + 400 = x_4 + x_5$

(D) $x_1 + x_5 = 600$



Example

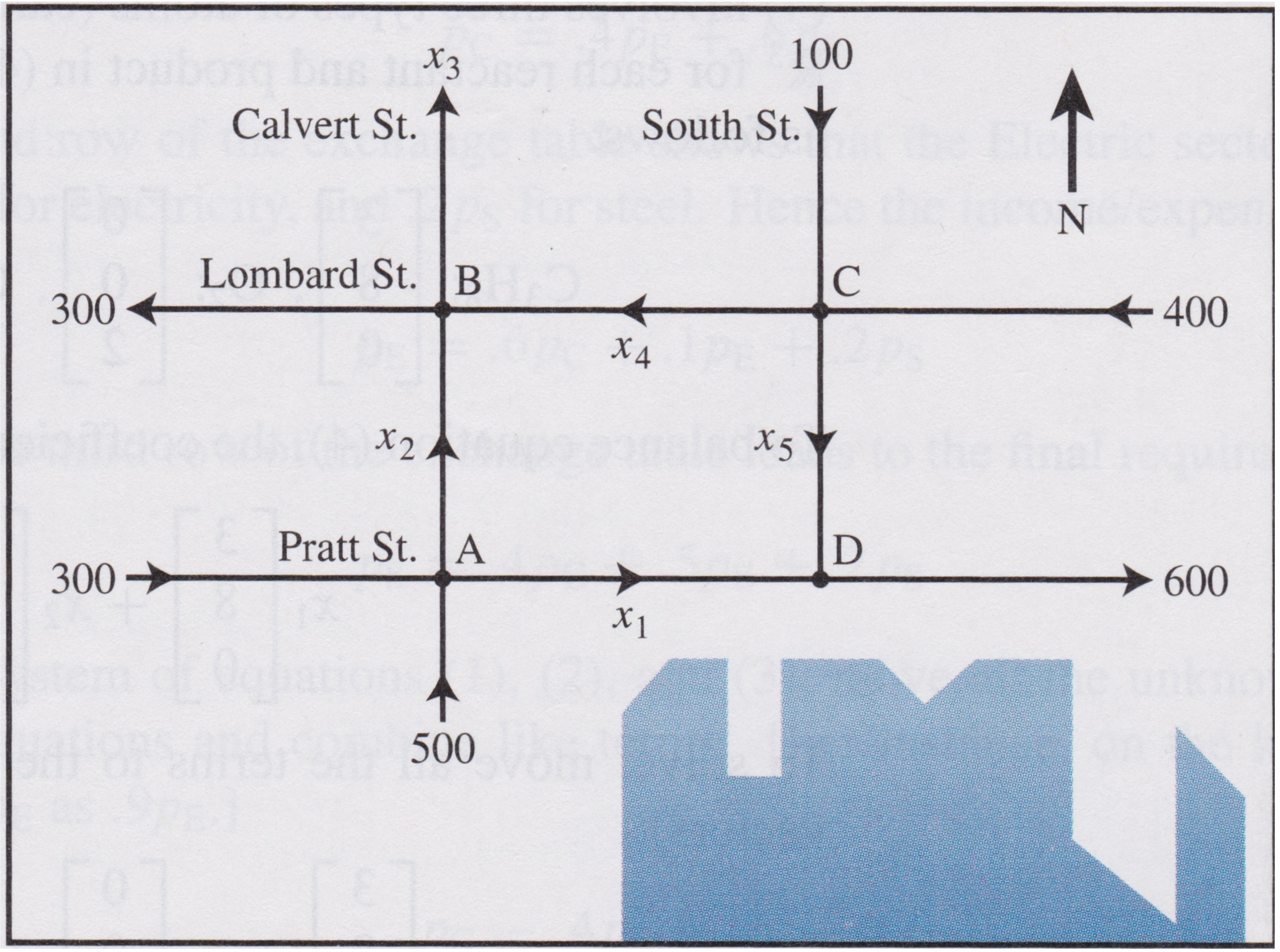
(A) $x_1 + x_2 = 800$

(B) $x_2 - x_3 + x_4 = 300$

(C) $x_4 + x_5 = 500$

(D) $x_1 + x_5 = 600$

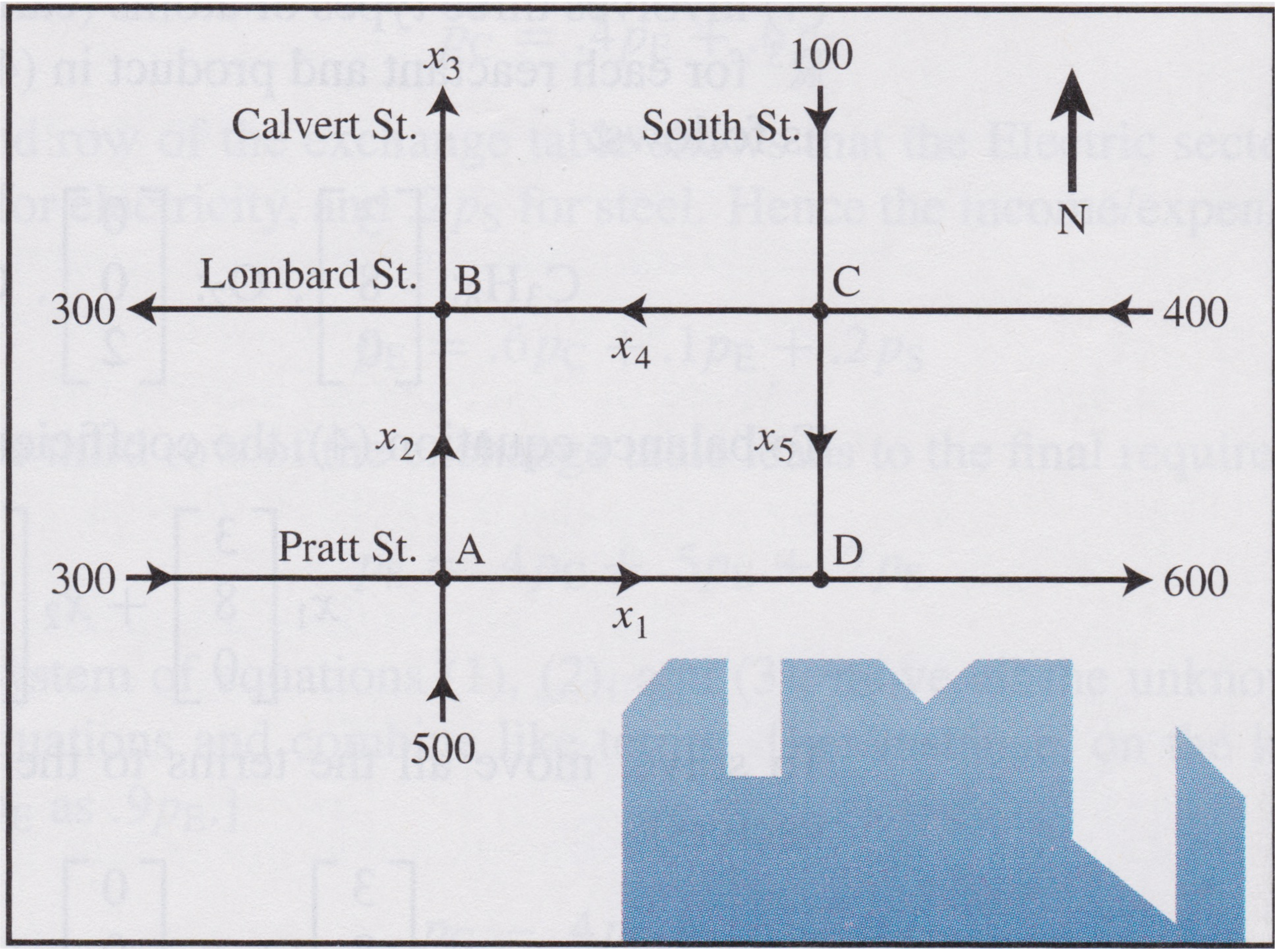
System of Linear Equations



Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{bmatrix}$$

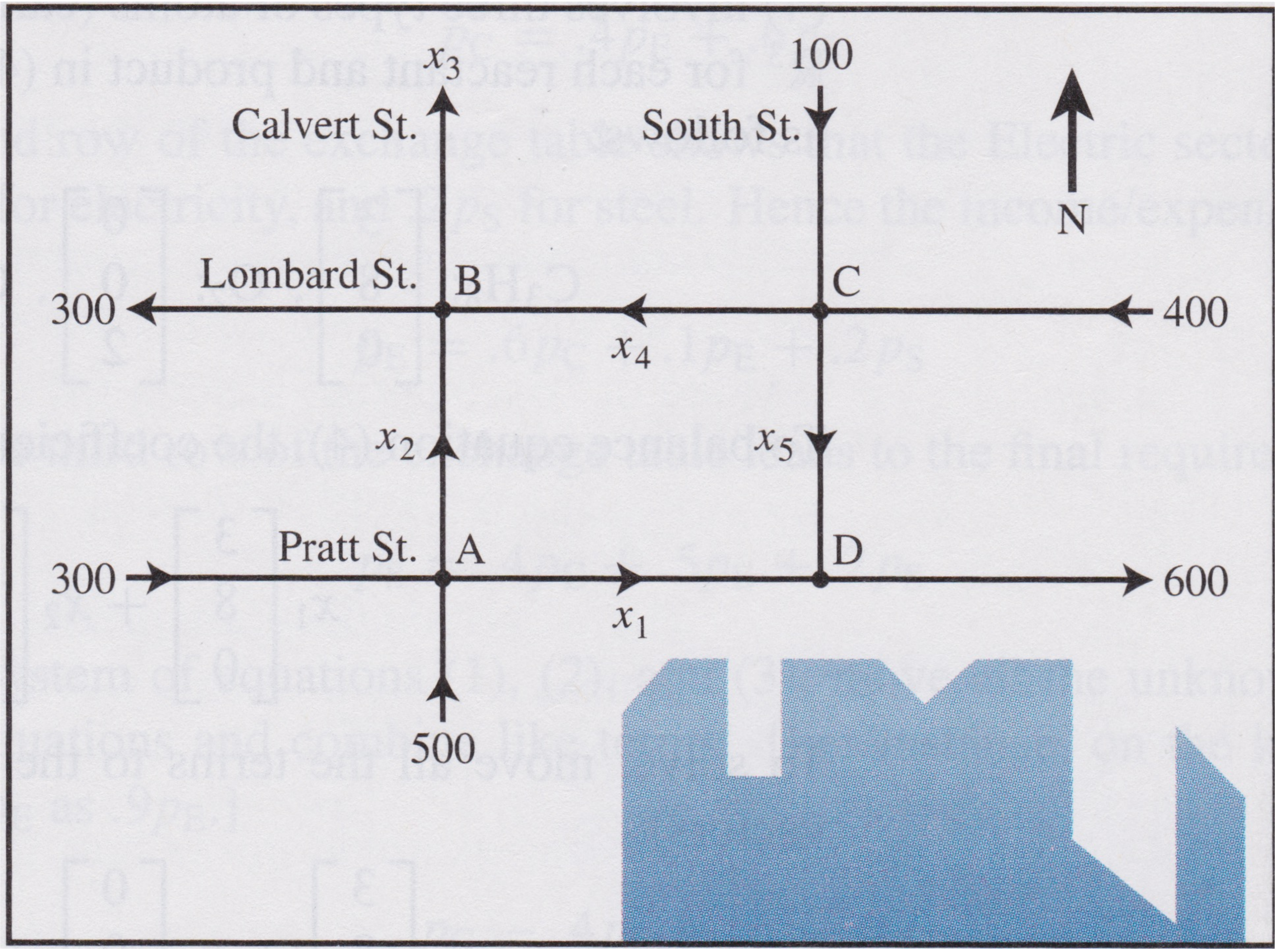
Augmented Matrix



Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{bmatrix}$$

Reduced Echelon Form

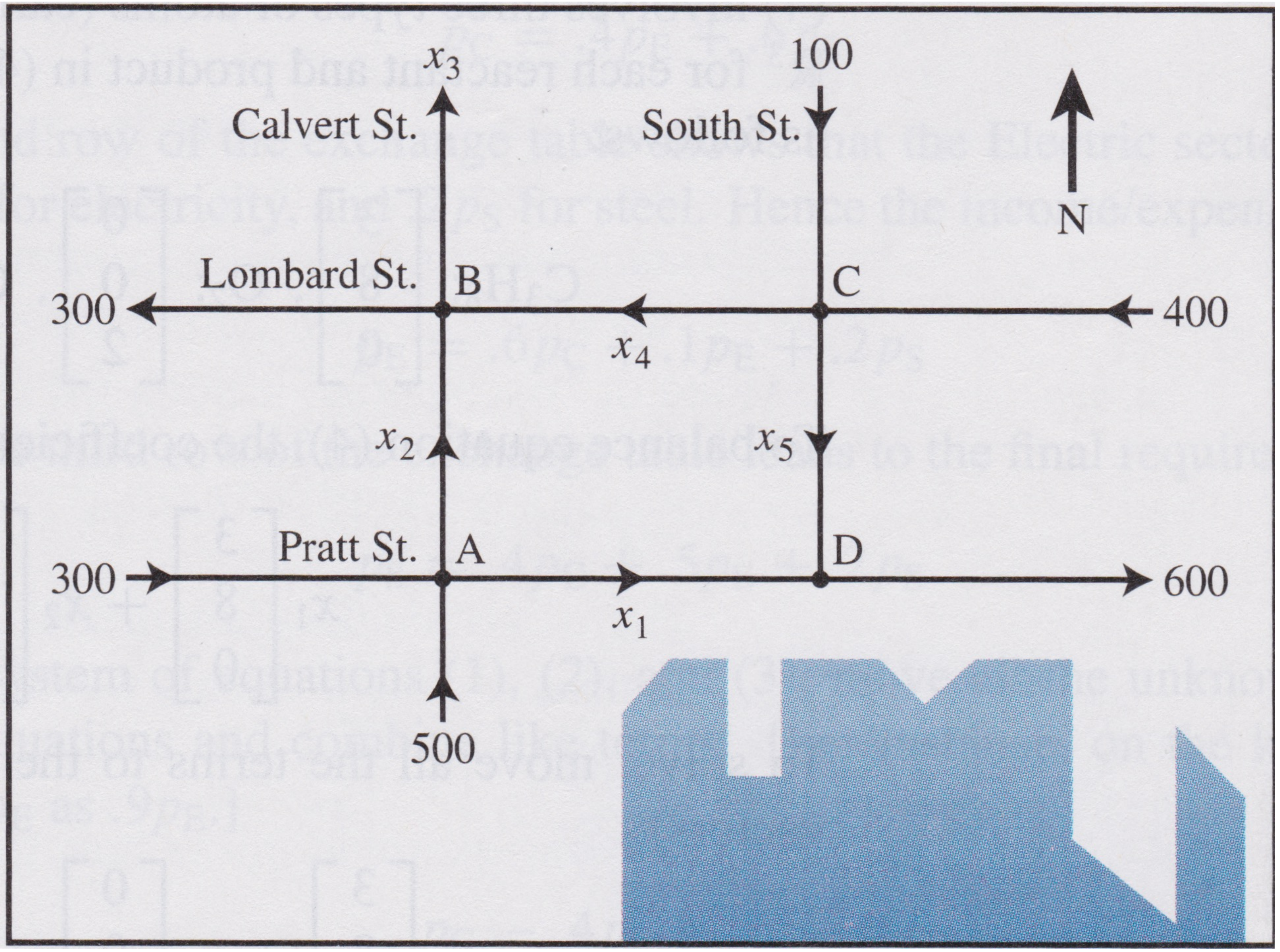


Note that global flow is conserved.

Example

$$x_1 = 600 - x_5$$
$$x_2 = 200 + x_5$$
$$x_3 = 400$$
$$x_4 = 500 - x_5$$
x_5 is free

General Solution



How To: Max Flow Value for an Edge

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Question. Find the maximum value of a flow variable in a given flow network.

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Solution. Remember that flow values must be positive. Look at the general form solution and see what makes this hold.

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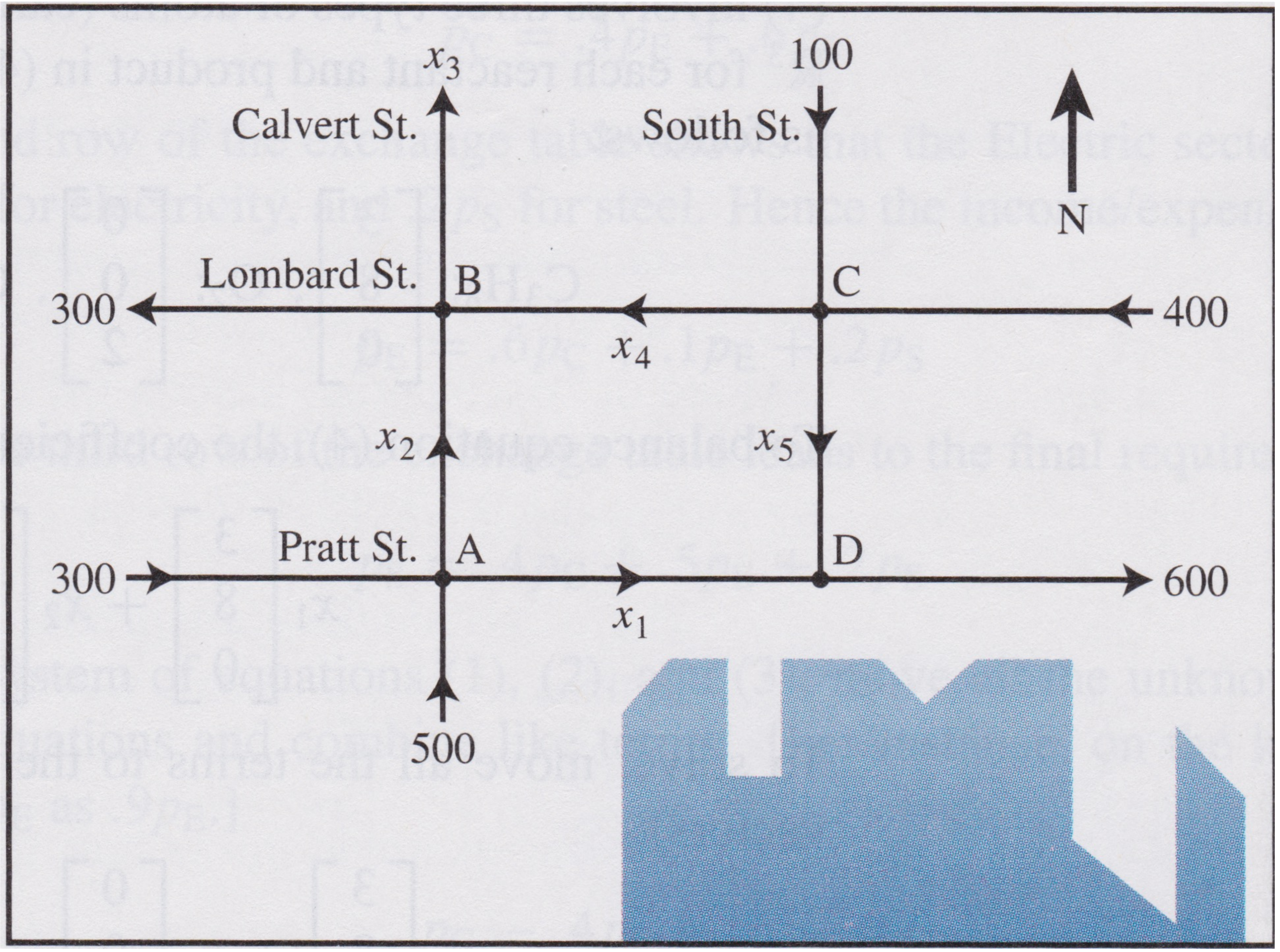
$$x_2 = 200 + x_5$$
$$x_3 = 400$$
$$x_4 = 500 - x_5$$
$$x_5 \text{ is free}$$

$x_4 \geq 0$ implies $x_5 \leq 500$
$x_1 \geq 0$ implies $x_5 \leq 600$



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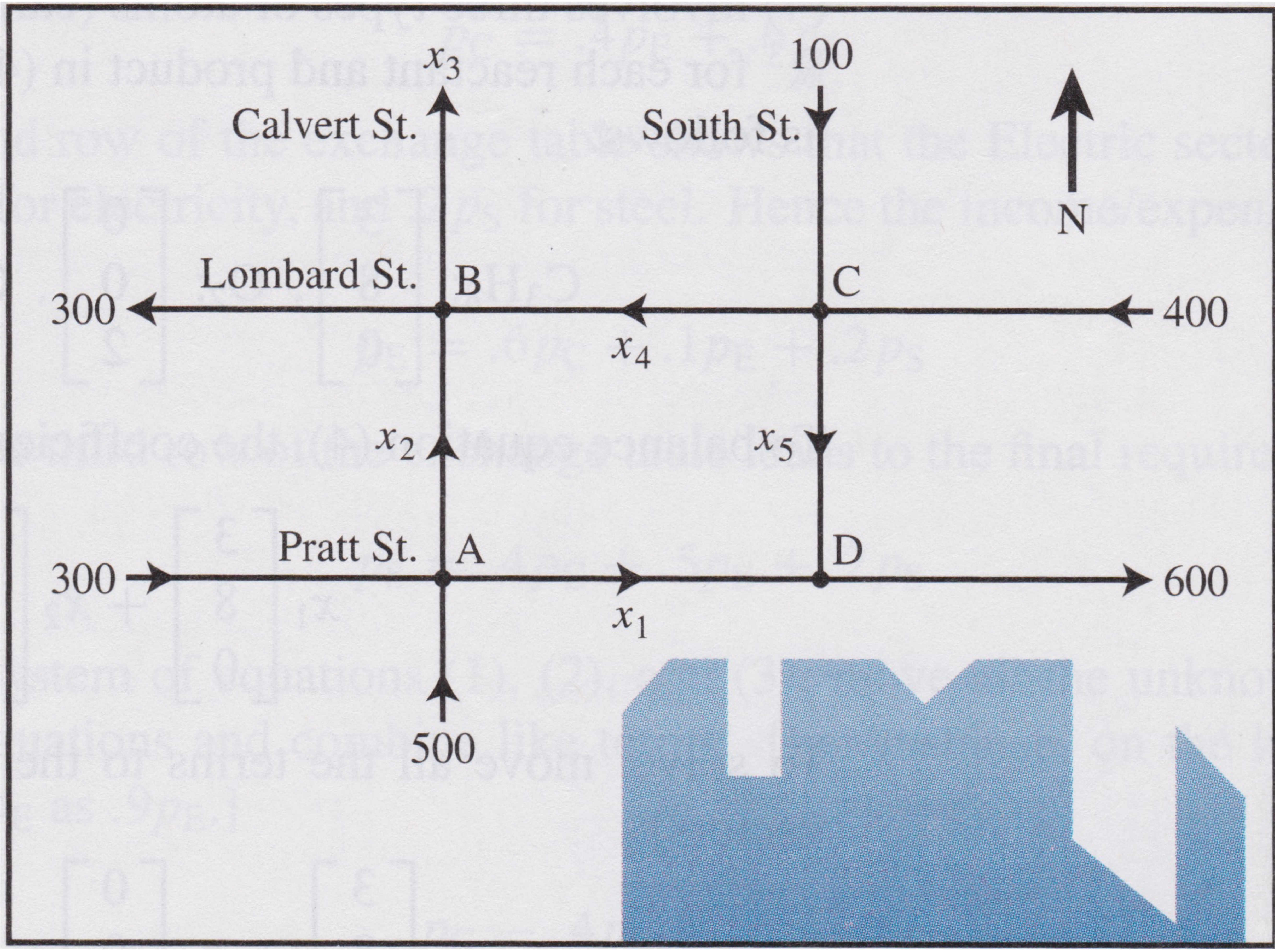

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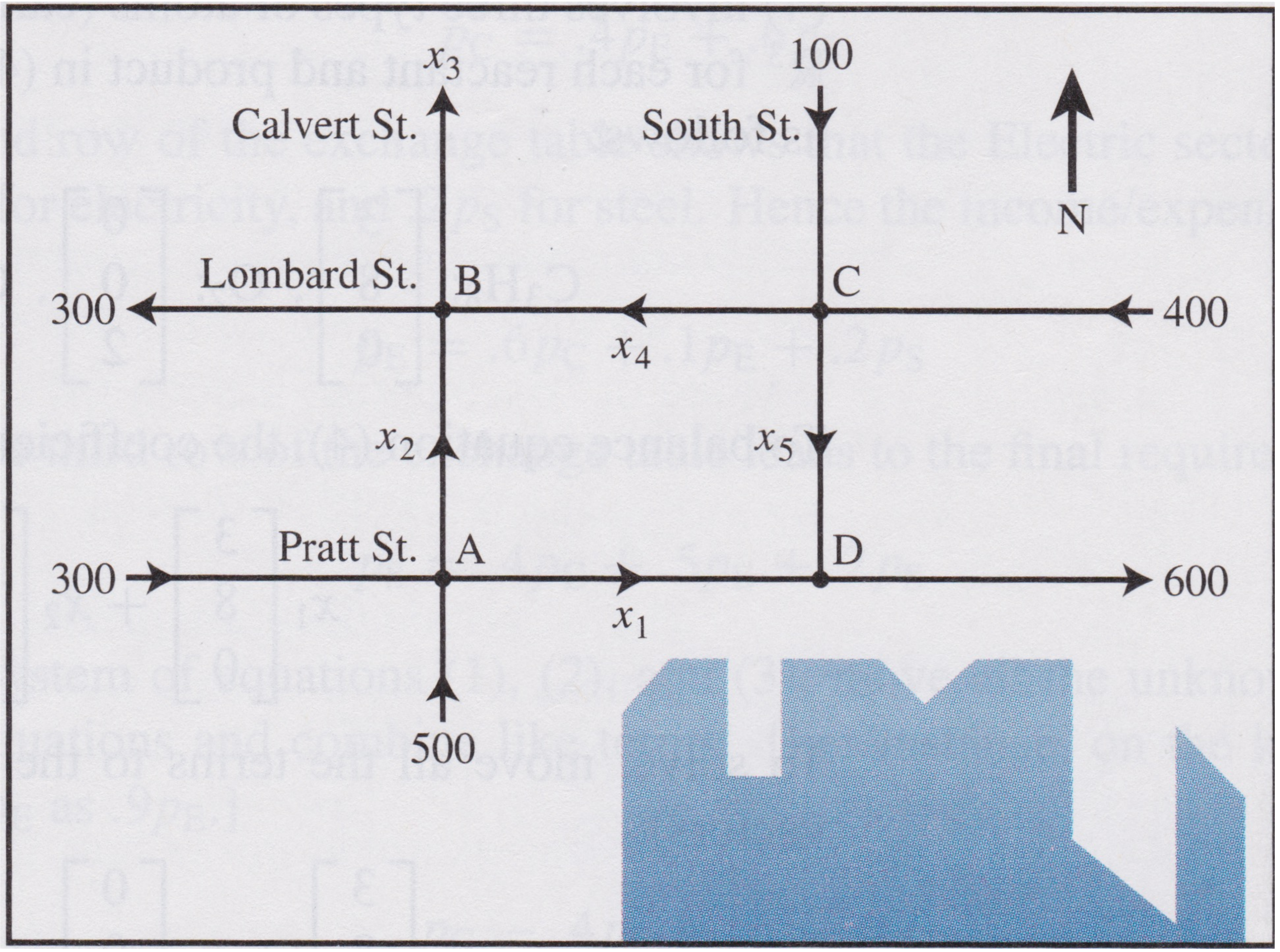

$x_4 \geq 0$ implies $x_5 \leq 500$

$x_1 \geq 0$ implies $x_5 \leq 600$

The maximum value of x_5 is 500

Summary

Linear independence helps us understand when a span is "as large as it can be."

We can reduce this seeing if a single homogeneous equation has a unique solution.

Network Flows define linear systems we can solve.