Linear Transformations

Geometric Algorithms
Lecture 8

CAS CS 132

Practice Problem

Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *in* $\mathbb{R}^3$ *such that*

- *every pair of vectors (i.e.,* $\{\mathbf{v}_1, \mathbf{v}_2\}$*,* $\{\mathbf{v}_1, \mathbf{v}_3\}$*,* $\{\mathbf{v}_2, \mathbf{v}_3\}$*)* ~~*are*~~ is *linearly independent*
- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly dependent*

Answer

$$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Objectives

1. Finish our discussion of Linear Independence
2. Introduce Matrix Transformations
3. Define Linear Transformations
4. Start looking at the Geometry of Linear Transformations

Keywords

Transformations

Domain, Codomain

Image, Range

Matrix Transformations

Linear Transformations

Additivity, Homogeneity

Dilation, Contraction, Shearing, Rotation

Recap

Recap: Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

Recall: Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution).

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has exactly one solution (the trivial solution).

The columns of A are linearly independent if $A\mathrm{x} = 0$ has exactly one solution.

Recall: Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

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has a nontrivial solution.

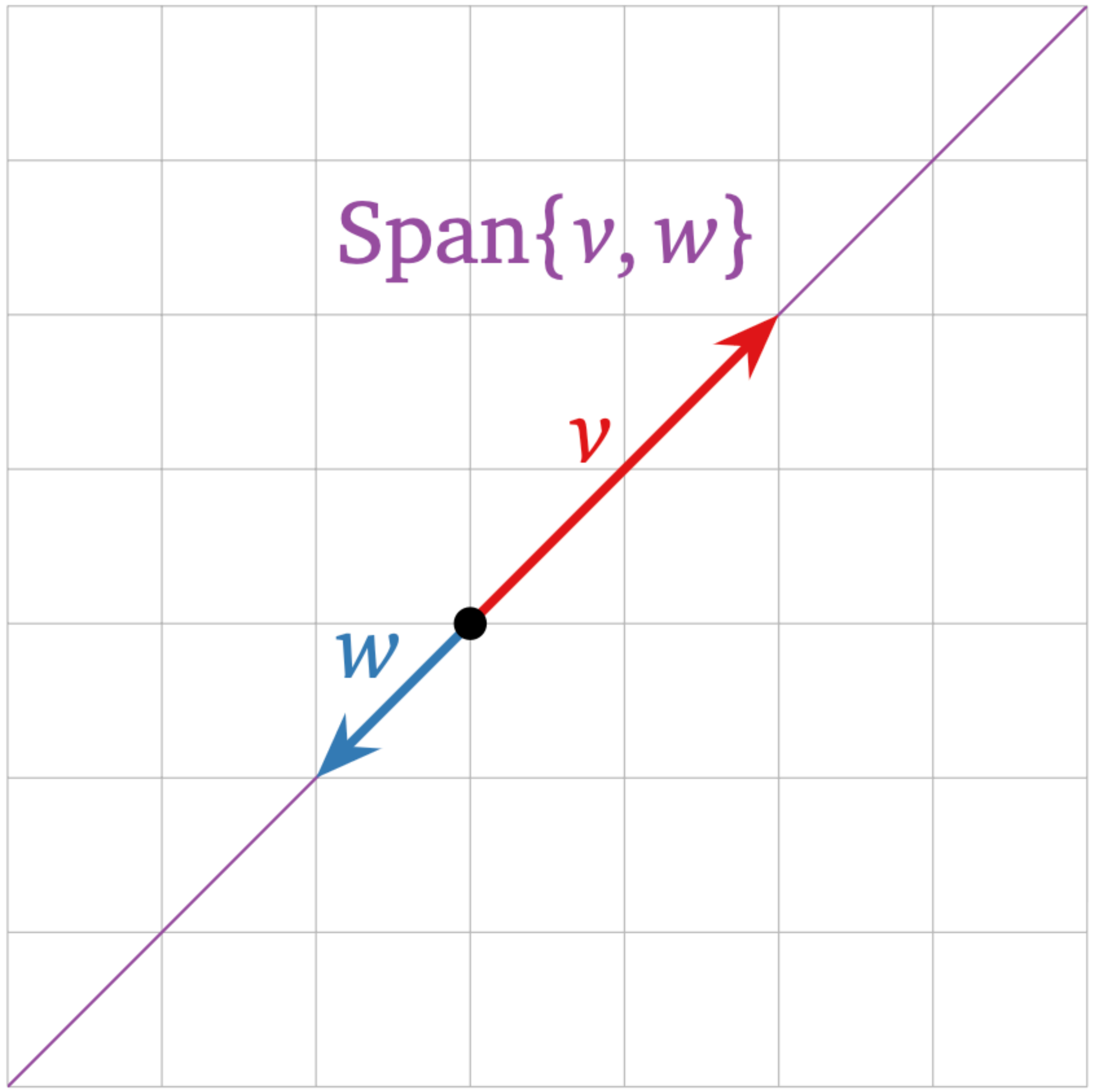
A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals $\mathbf{0}$.

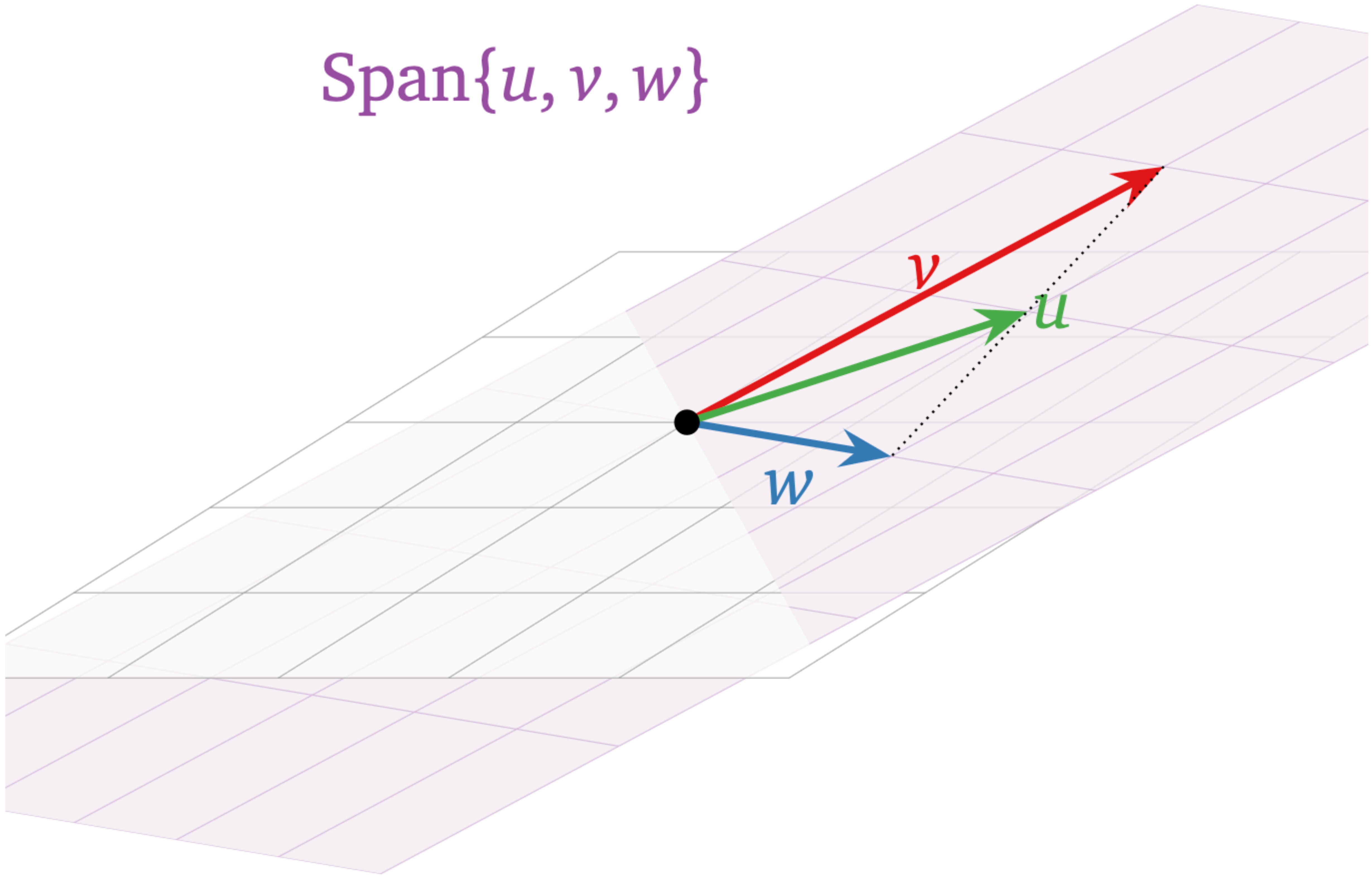
Recall: Linear Dependence

Recall: Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself).

Linear Dependence (Pictorally)





Recall: Linear Dependence Relation

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

A linear dependence relation *witnesses* the linear dependence.

Example

Write down the linear dependence relation for the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

$x_1 = -1$

$x_2 = -2$

$x_3 = 2$

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v} + \alpha_3 \vec{v} = \vec{0}$$

$$-\vec{v}_1 - 2\vec{v}_2 + 2\vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Answer

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 4 & 6 & 8 & 0 \\ 2 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

+0   +3   +3

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftarrow R_1 + 3R_2$

$$\begin{bmatrix} -4 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftarrow R_1 / -4$

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{2}x_3$

$x_2 = -x_3$

x_3 is free

$x_1 = -1$

$x_2 = -2$

$x_3 = 2$

Simple Cases

The Empty Set

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$\{\}$ (a.k.a. $\emptyset$) is linearly independent

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We stretch the definition a bit: there is no nontrivial linear combination of the vectors equaling $\mathbf{0}$. There are none at all...

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$\{\}$ (a.k.a. $\emptyset$) is linearly independent

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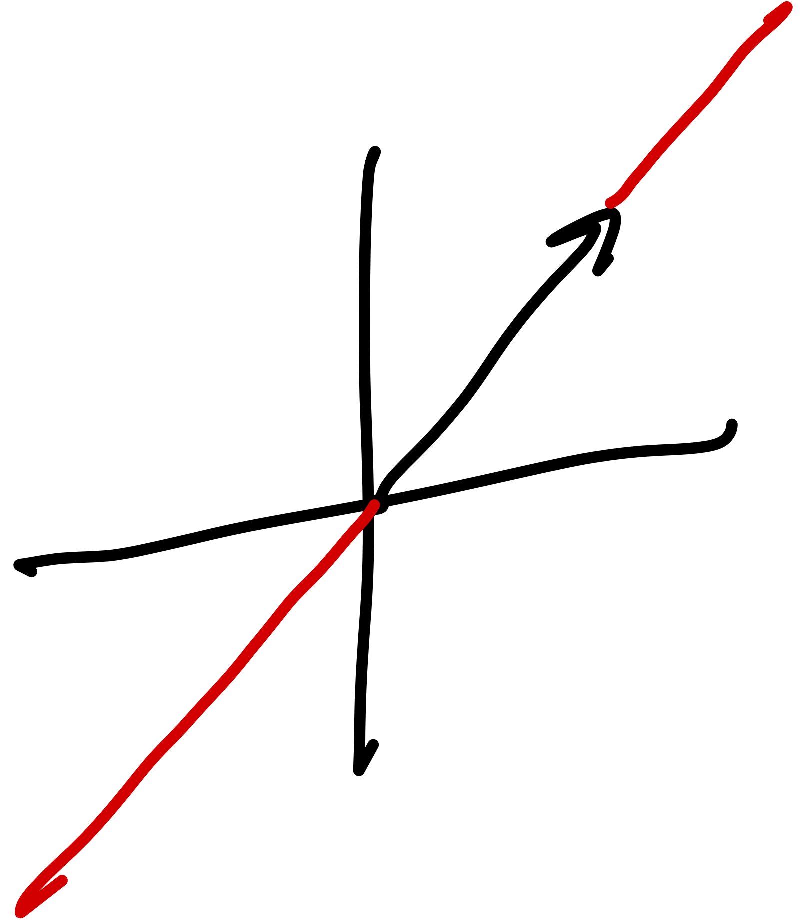
$\mathbf{0}$ is in every span, even the empty span.

One Vector



A single vector $\mathbf{v}$ is linearly independent if and only if it $\mathbf{v} \neq \mathbf{0}$.

Note that $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The Zero Vector and Linear Dependence

If a set of vectors V contains the $\mathbf{0}$, then it is linearly dependent.

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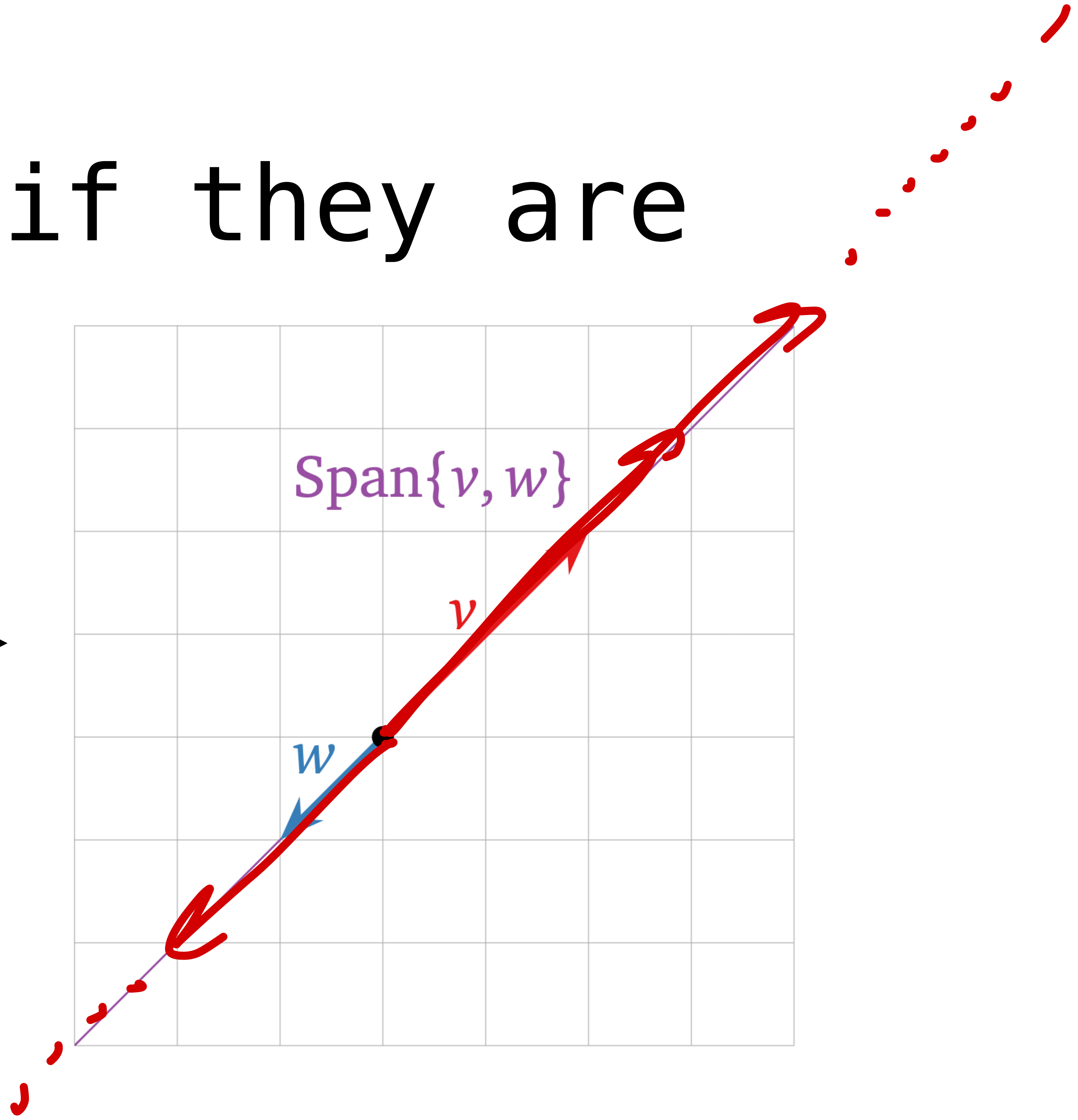
$$(1)\mathbf{0} + 0\mathbf{v}_2 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n = \mathbf{0}$$

There is a very simple nontrivial solution.

Two Vectors

Definition. Two vectors are ***colinear*** if they are scalar multiples of each other.

e.g., $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1.5\\1.5\\3\end{bmatrix}$ or $\begin{bmatrix}2\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$ →



Two vectors are linearly dependent if and only if they are colinear.

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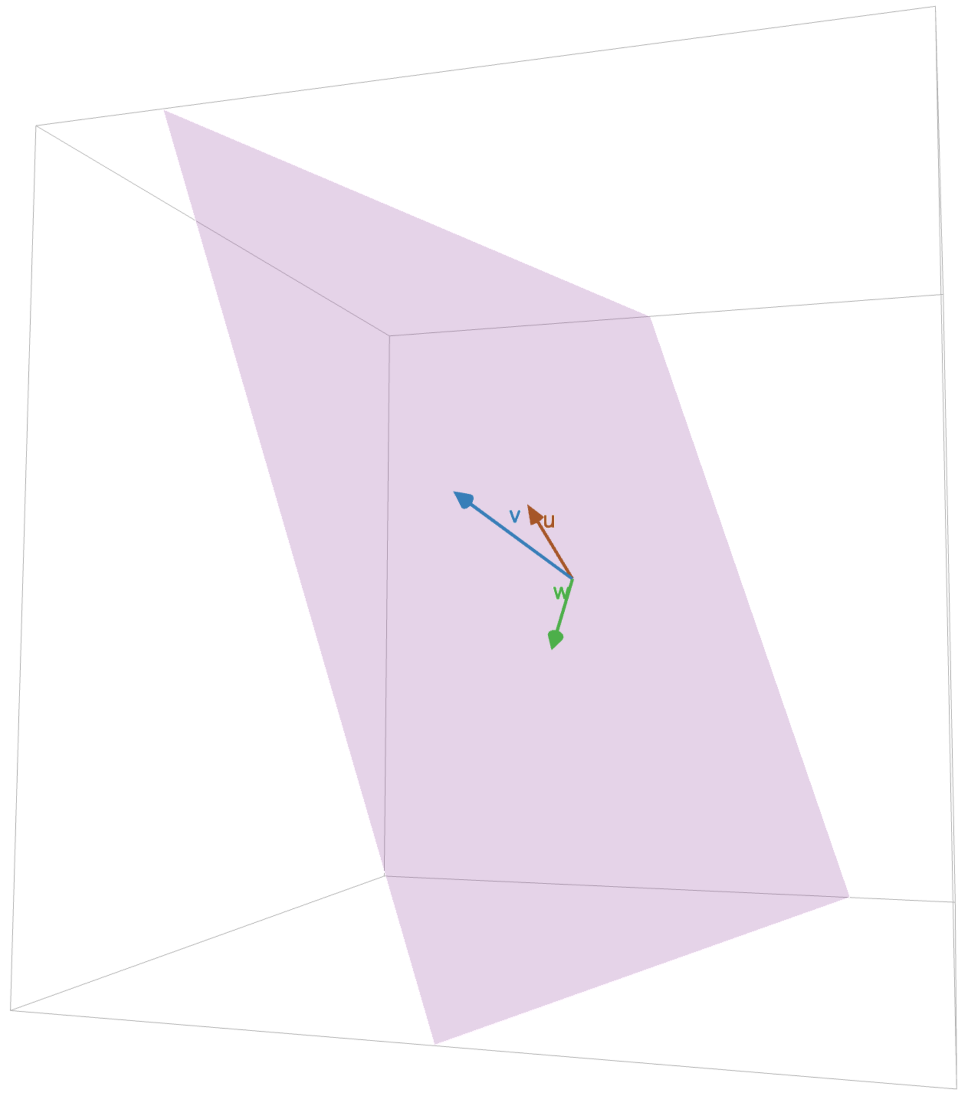
Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane.

Three vectors are linearly dependent if an only if they are colinear or coplanar.

or $\vec{0}$



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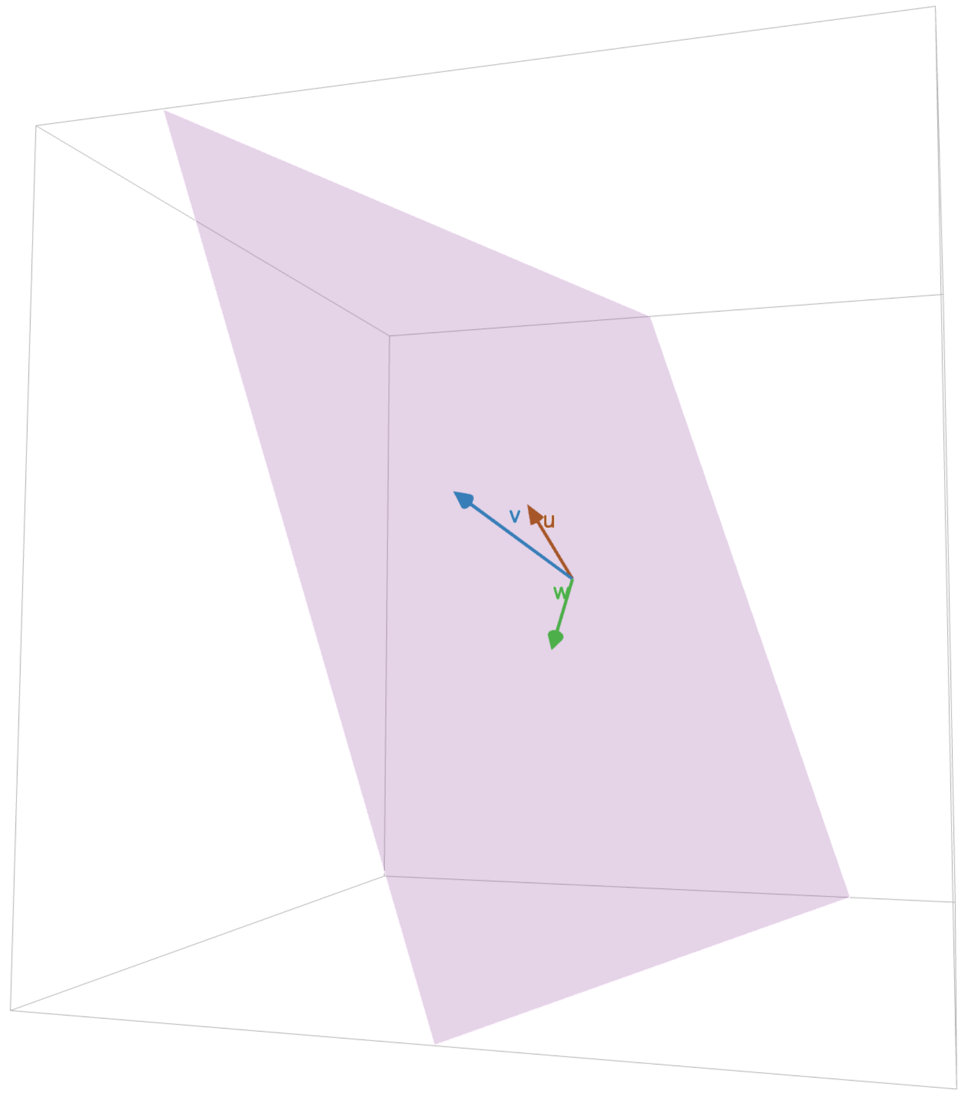
This can be reasoning can be extended to more vectors, but we run out of terminology



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Yet Another Interpretation

Increasing Span Criterion

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly *independent* then we cannot write one of it's vectors as a linear combination of the others.

But we get something stronger.

Increasing Span Criterion

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if for all $i \leq n$,

$$\mathbf{v}_i \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

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As we add vectors, the span gets larger.

Characterization of Linear Dependence

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only if there is an $i \leq n$,

$$\mathbf{v}_i \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

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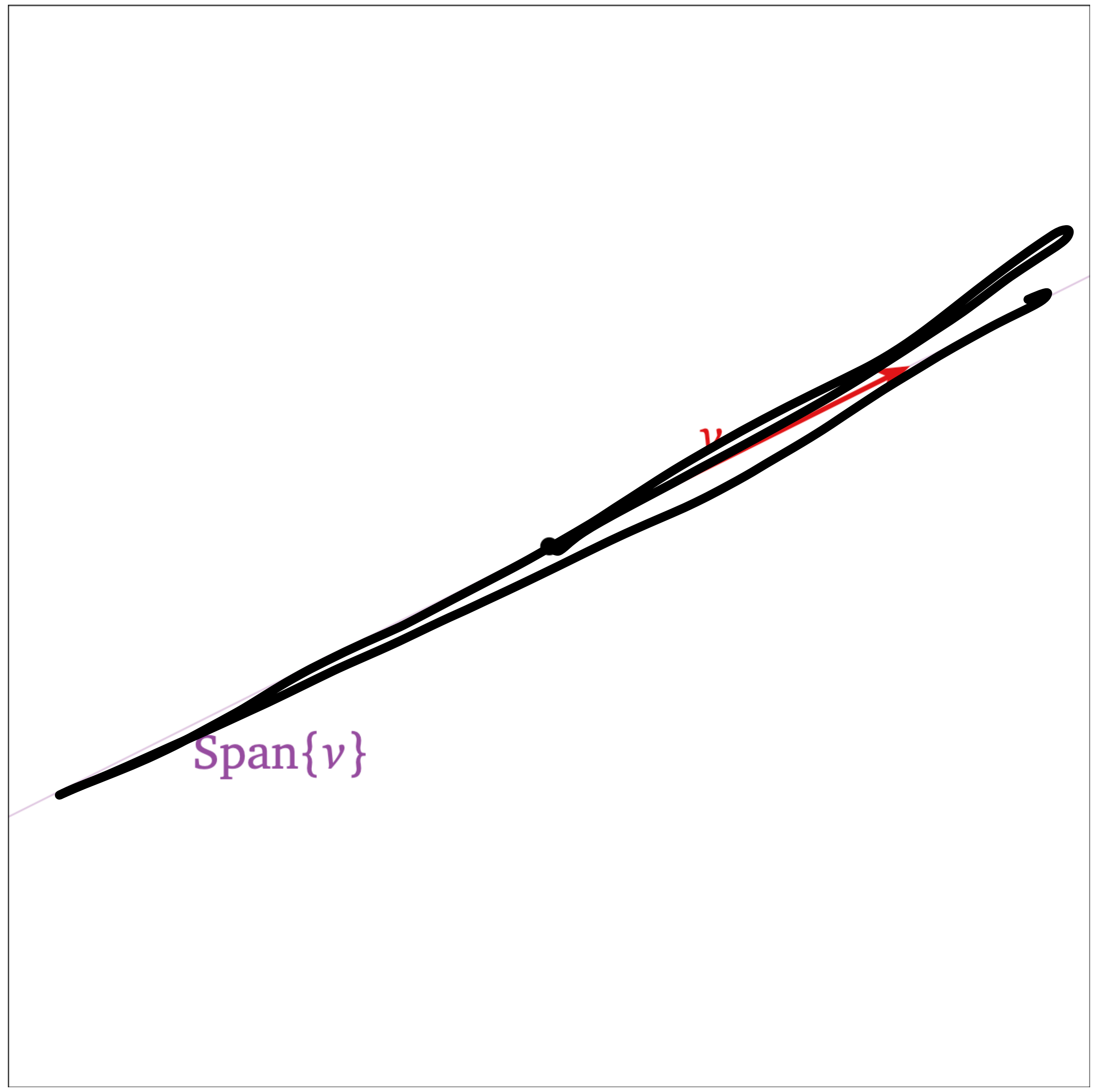
As we add vectors, we'll eventually find one in the span of the preceding ones.

As a Picture

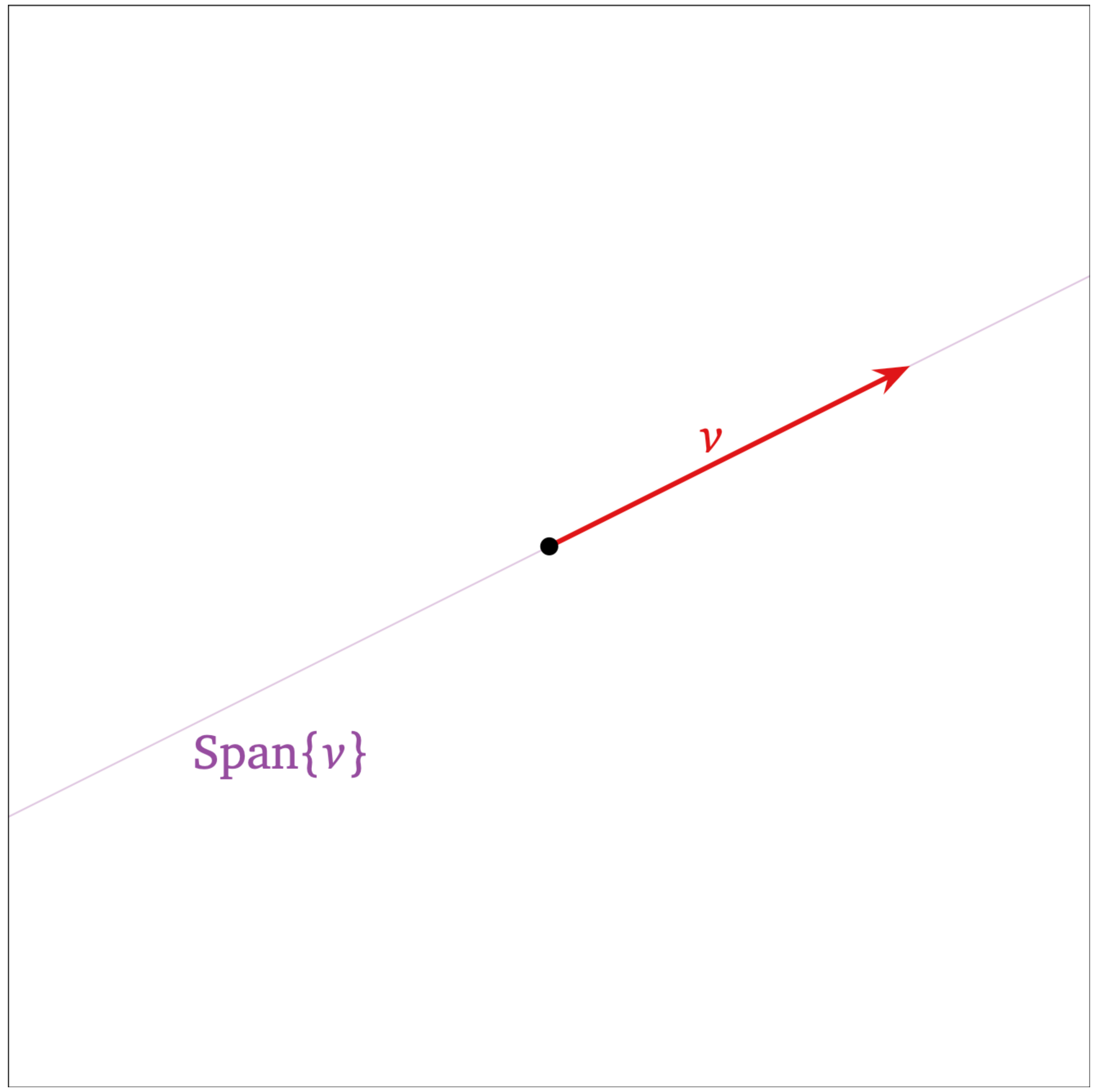


span of 1 vector
a line

image source

As a Picture

$\mathbb{R}^2$



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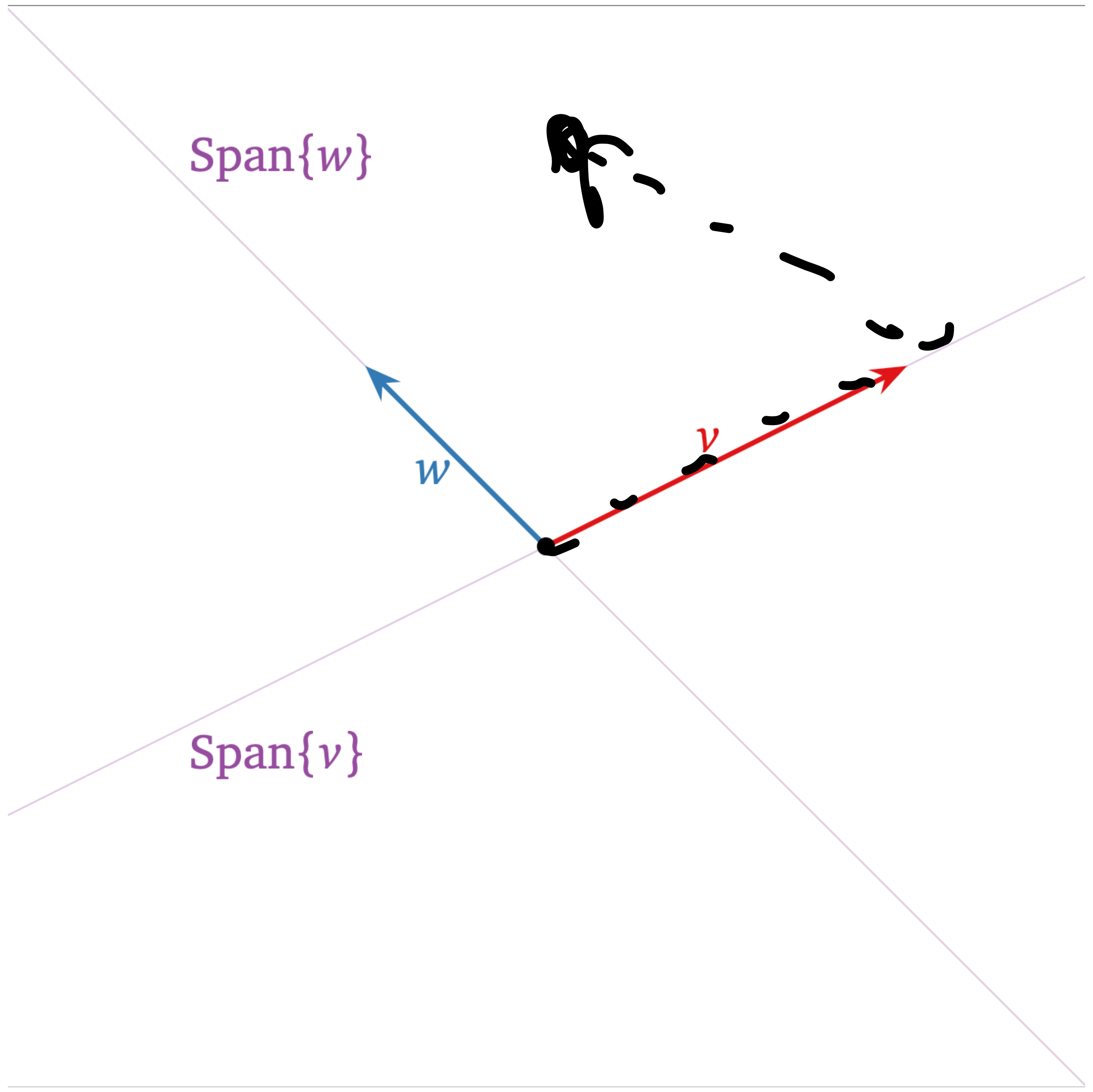
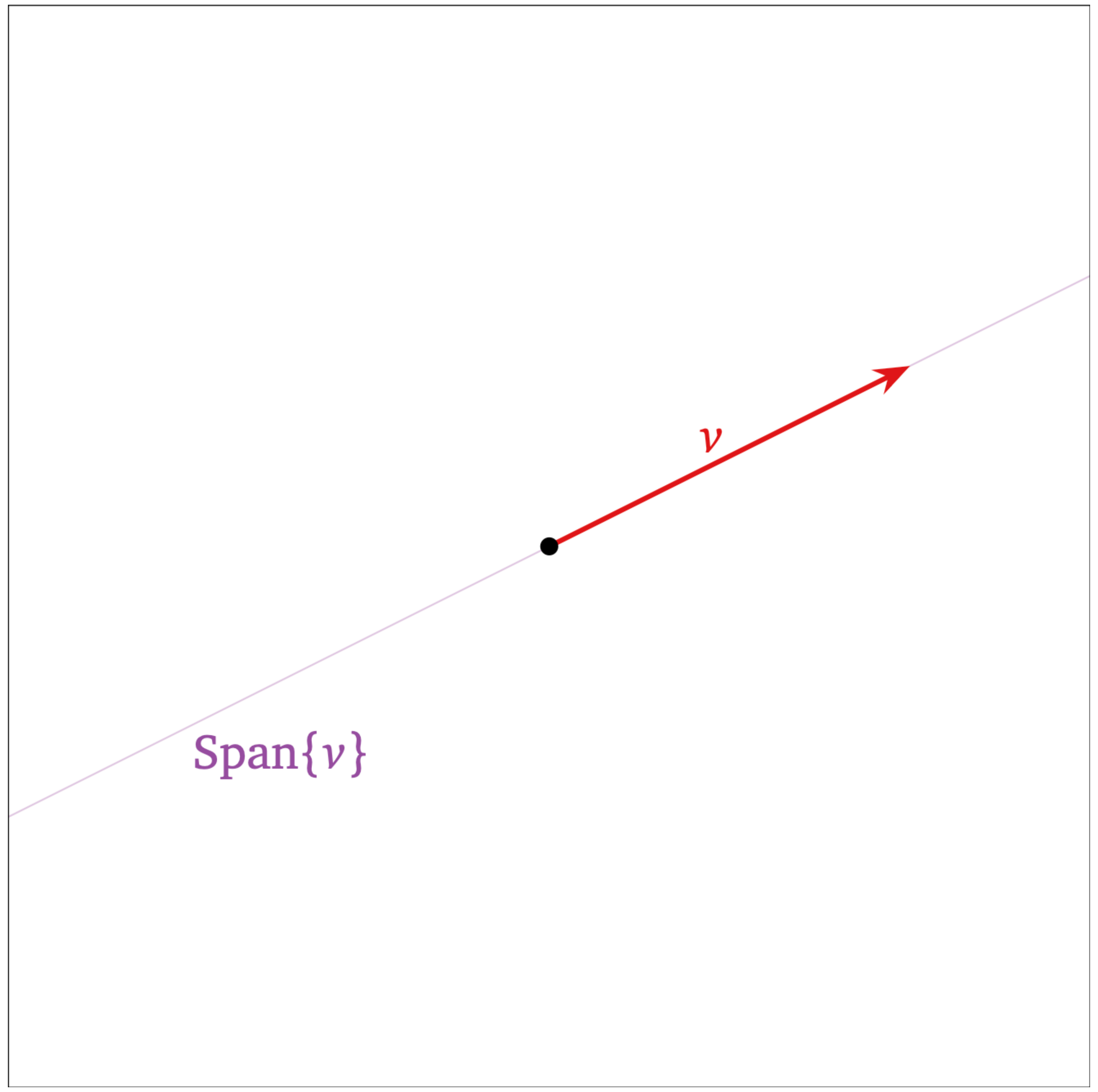


span of 2 vector
a plane

image source

As a Picture



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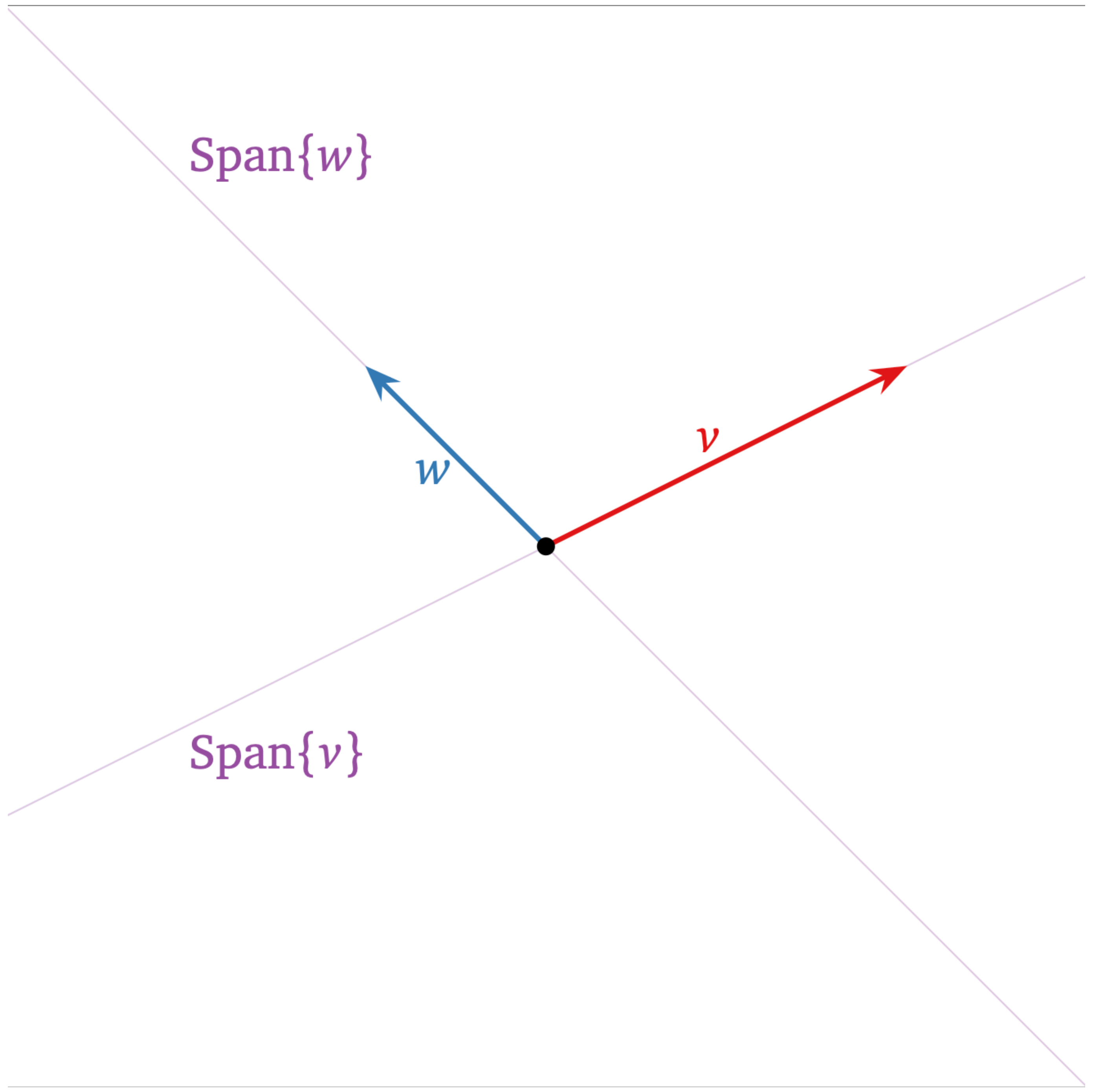

span of 2 vector
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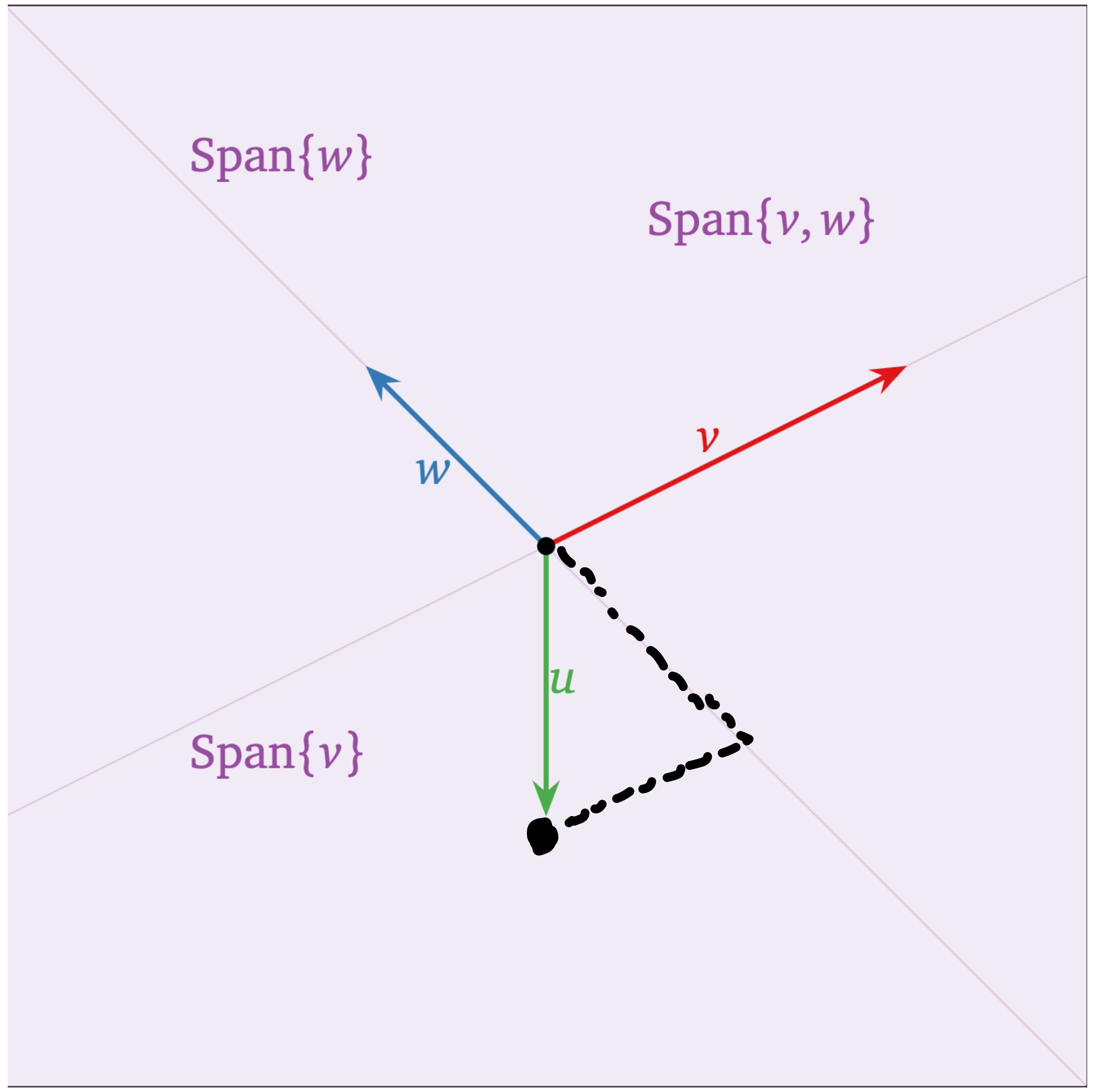

span of 3 vector
still a plane

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For linearly independent sets, our span keeps getting "bigger"

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a 4d-hyperlane

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...

Worth Noting...

Corollary. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent, then for any vector $\mathbf{v}_{k+1}$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly dependent.

If we add a vector to a linearly dependent set, it remains linearly dependent

Question

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Answer: No

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Any three vectors can at most span a plane.

The first two are not colinear, so they span a plane ($\mathbb{R}^2$).

Linear Independence and Free Variables

Linear Dependence Relations (Again)

When finding a linear dependence relation, we came across a system which a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take x_3 to be free

Pivots and Linear Dependence

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Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column.

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Remember that we choose our free variables to be the ones whose columns don't have pivots.

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Remember that we choose our free variables to be the ones whose columns don't have pivots.

Free variables allow for infinitely many (nontrivial) solution.

Recall: General Form Solutions

$$x_1 = -(0.5)x_3$$
$$x_2 = -x_3$$
$$x_3 \texttt{ is free}$$

Recall: General Form Solutions

$$x_1 = -0.5$$

$$x_2 = -1$$

$$x_3 = 1$$

Recall: General Form Solutions

$$x_1 = 0.5$$
$$x_2 = 1$$
$$x_3 = -1$$

Recall: General Form Solutions

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = -2$$

Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

`The point: the solution is not unique.`

If a homogenous linear system has a unique solution then it **must** be the trivial solution.

How To: Linear Independence

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Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

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Solution. Check if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$ has a unique solution.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{0}$ has a unique solution.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if the general form solution of $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0}\end{bmatrix}$ has any free variables.

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Reduce $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n]$ to echelon form and check if **has a pivot position in every column.**

Example

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column without a pivot

Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row.

The columns of a matrix are linearly independent if there is a pivot in every column.

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}$$

Tall Matrices

If $m > n$ then the columns cannot span $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 rows.

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

This matrix as at most 3 pivots, but 4 columns.

A Warning

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row.

The columns of a matrix are linearly independent if there is a pivot in every column.

Don't confuse these!

back to it...

Matrix Transformations

Recall: Spans (with Matrices)

Definition. The ***span*** of a set of vectors is the set of all possible linear combinations of them.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\, \mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

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The span of the columns of a matrix A is the set of of vectors resulting from multiplying A by any vector.

Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

Example (Algebraic)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\in \mathbb{R}^3} = \begin{bmatrix} 1 + 0 + 0 \\ 0 + 2 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\in \mathbb{R}^2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + 0 + 0 \\ 0 + 6 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

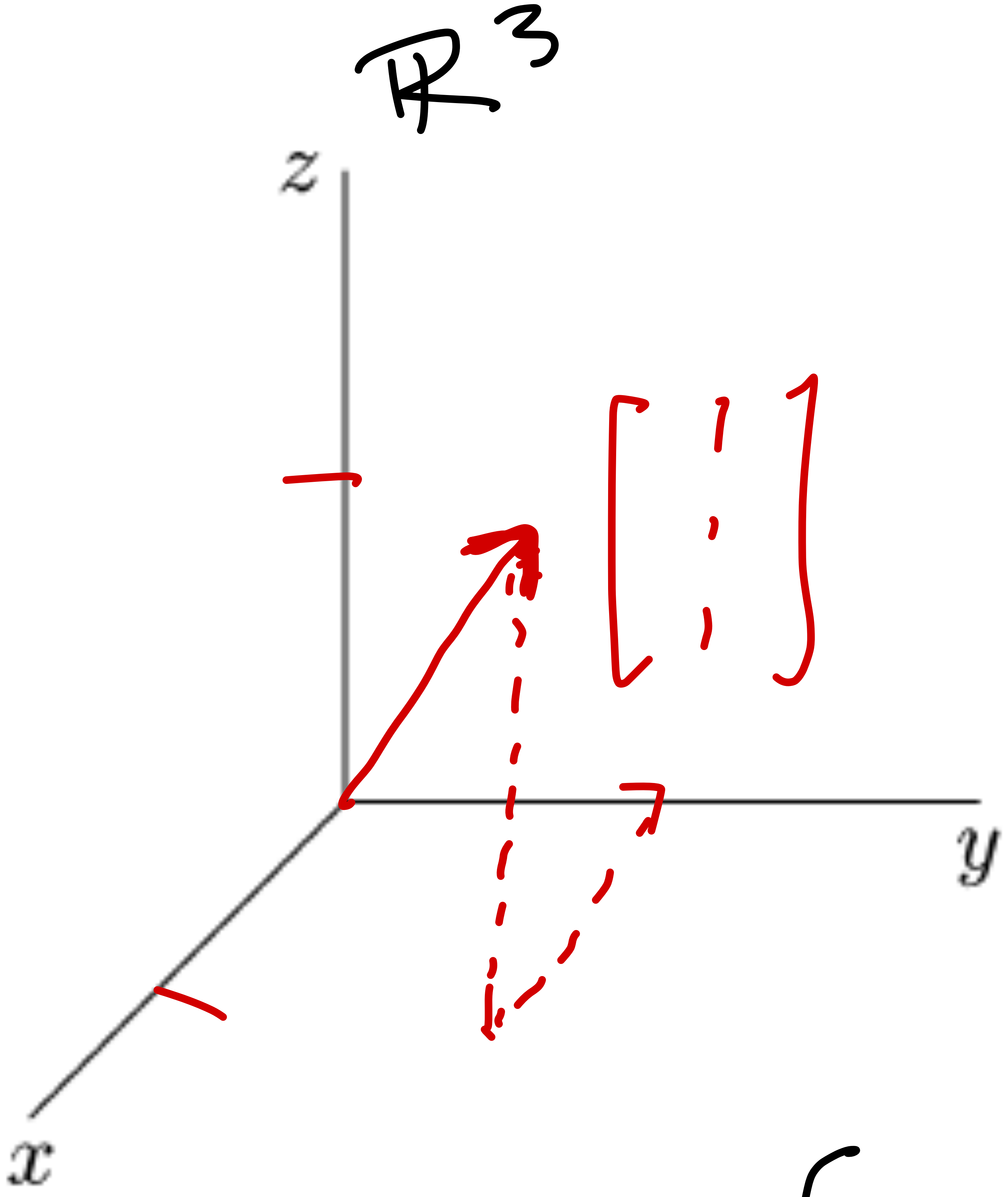
Example (Algebraic)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

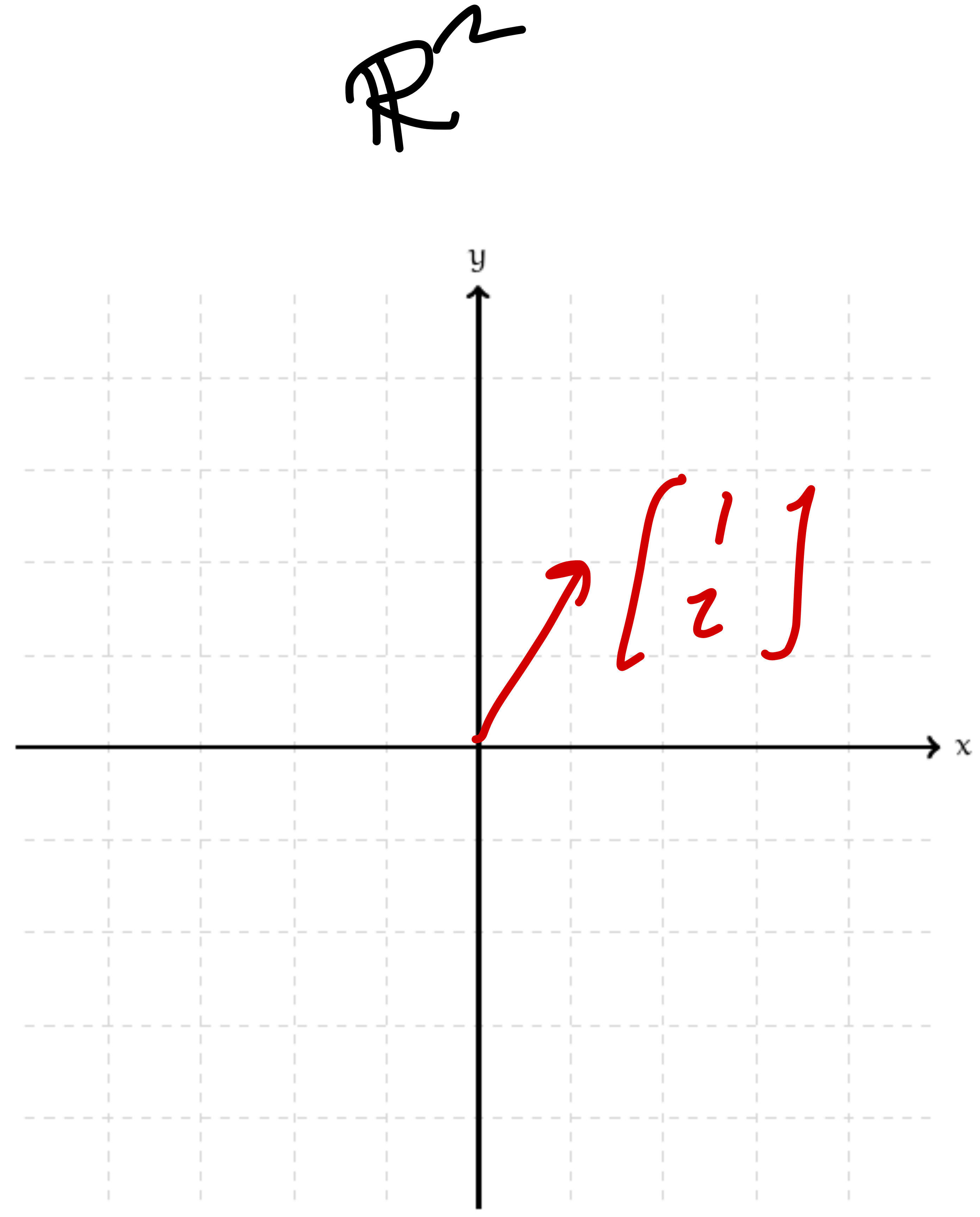
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 \\ 0 + 2x_2 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

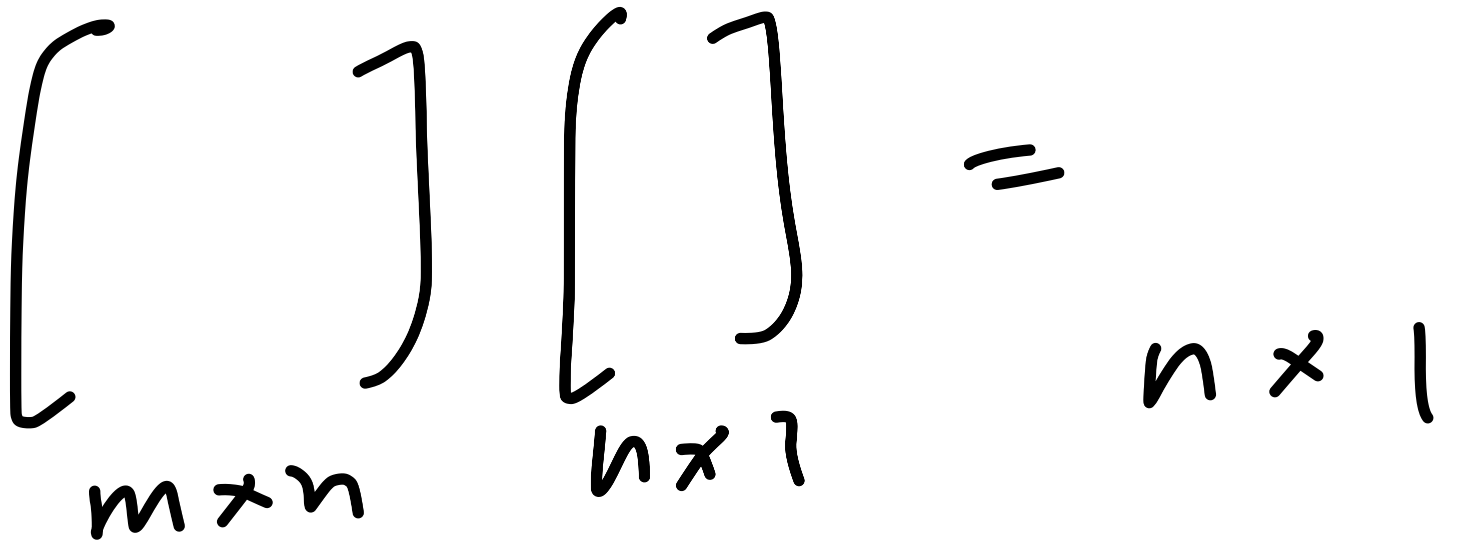
Example (Geometric)

$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\2x_2\end{bmatrix}$$



$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\end{bmatrix}\mathbf{x}$$





!!Important!!

The vector may be a different size after translation.

Recall: Matrix-Vector Multiplication and Dimension

matrix-vector multiplication only works if the number of *columns* of the matrix matches the dimension of the vector

$$\overbrace{m\left[\begin{matrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{matrix}\right]}^{n} \; n\left[\begin{matrix} * \\ \vdots \\ * \end{matrix}\right] = m\left[\begin{matrix} * \\ * \\ \vdots \\ * \\ * \end{matrix}\right]$$

$$(m \times n) \qquad \mathbb{R}^n \qquad \mathbb{R}^m$$

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

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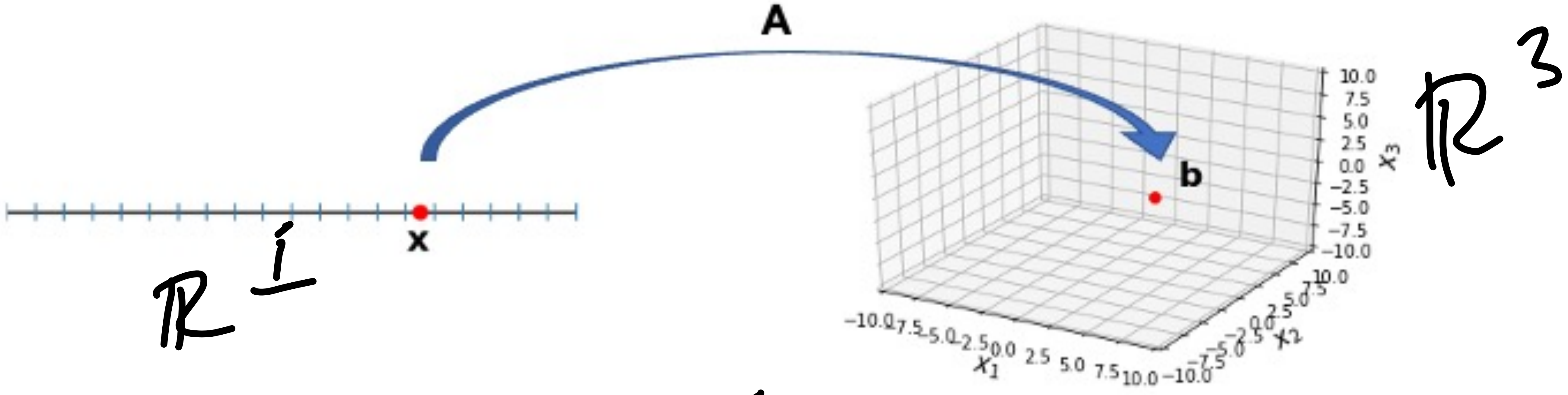
A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$?	$\equiv$	is there a vector which A transforms into $\mathbf{b}$?
Solve $A\mathbf{x} = \mathbf{b}$	$\equiv$	find a vector which A transforms into $\mathbf{b}$

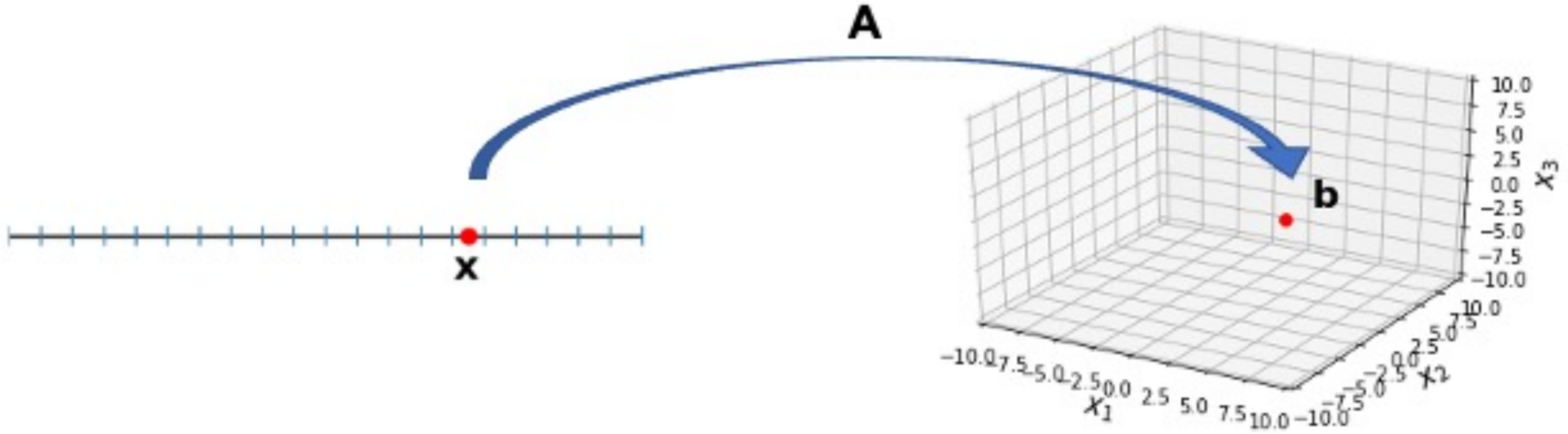
Question (Conceptual)

Suppose a matrix transforms a vector according to the following picture. What is the size of the matrix?



$$[m \times n][n \times 1] = [m \times 1]$$

Answer: 3×1



$$1 \mapsto \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

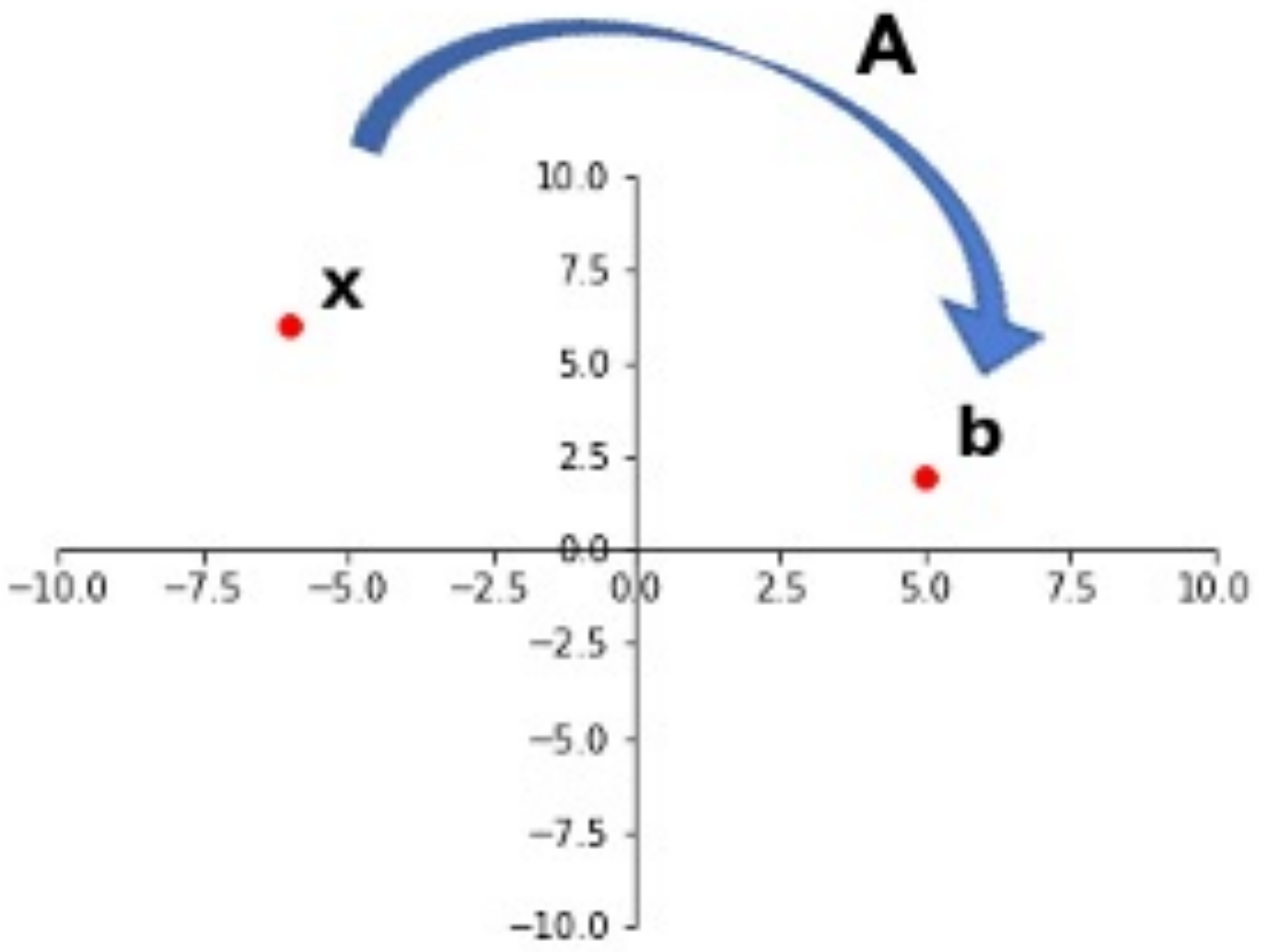
$$[x_1] \qquad x_1 \mapsto \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}$$

$$3 \mapsto \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbb{R}^n \to \mathbb{R}^n$$

Mapping between the same space can be viewed as a way of moving around points.



Transformations

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

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domain codomain

It's just a function, like in calculus.

Image and Range

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Definition. The ***range*** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under T.

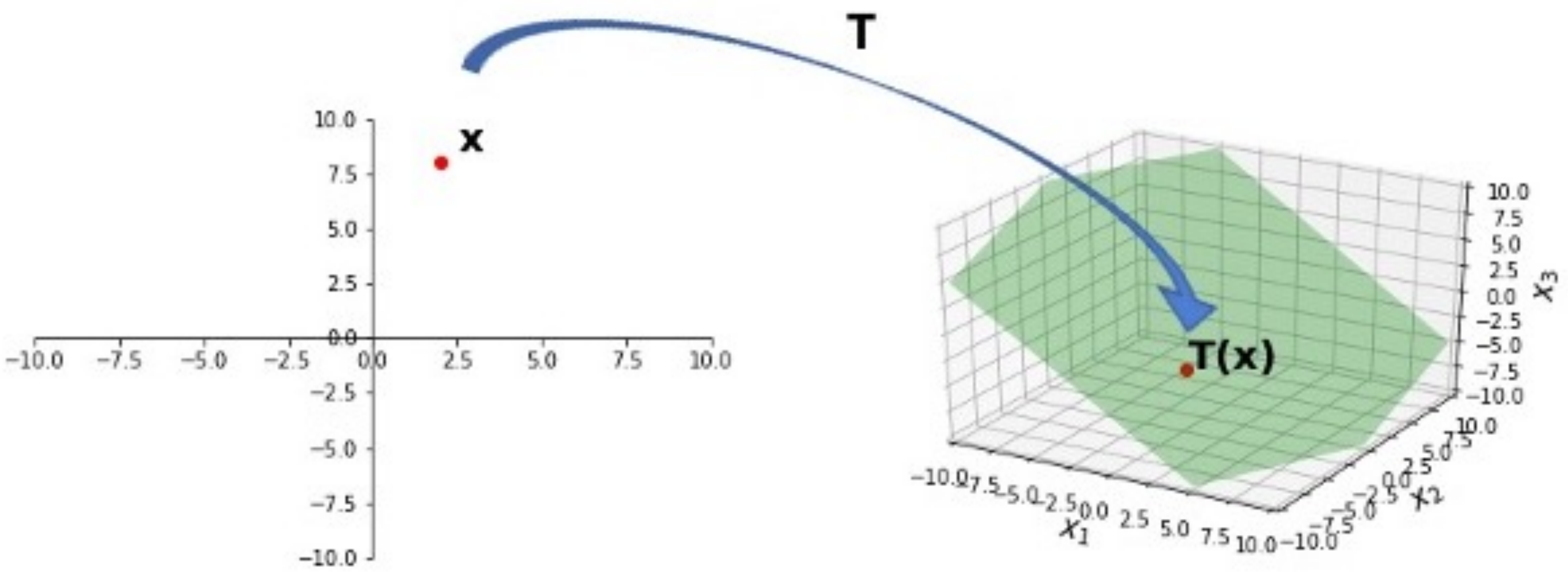
$$\text{ran}(T) = \{T(\mathbf{v}) : v \in \mathbb{R}^n\}$$

image of $\mathbf{v}$ under T $\equiv$ output of T applied to $\mathbf{v}$

range of T $\equiv$ all possible output of T

Codomain and Range

The codomain and range of a transformation may or may not be the same.



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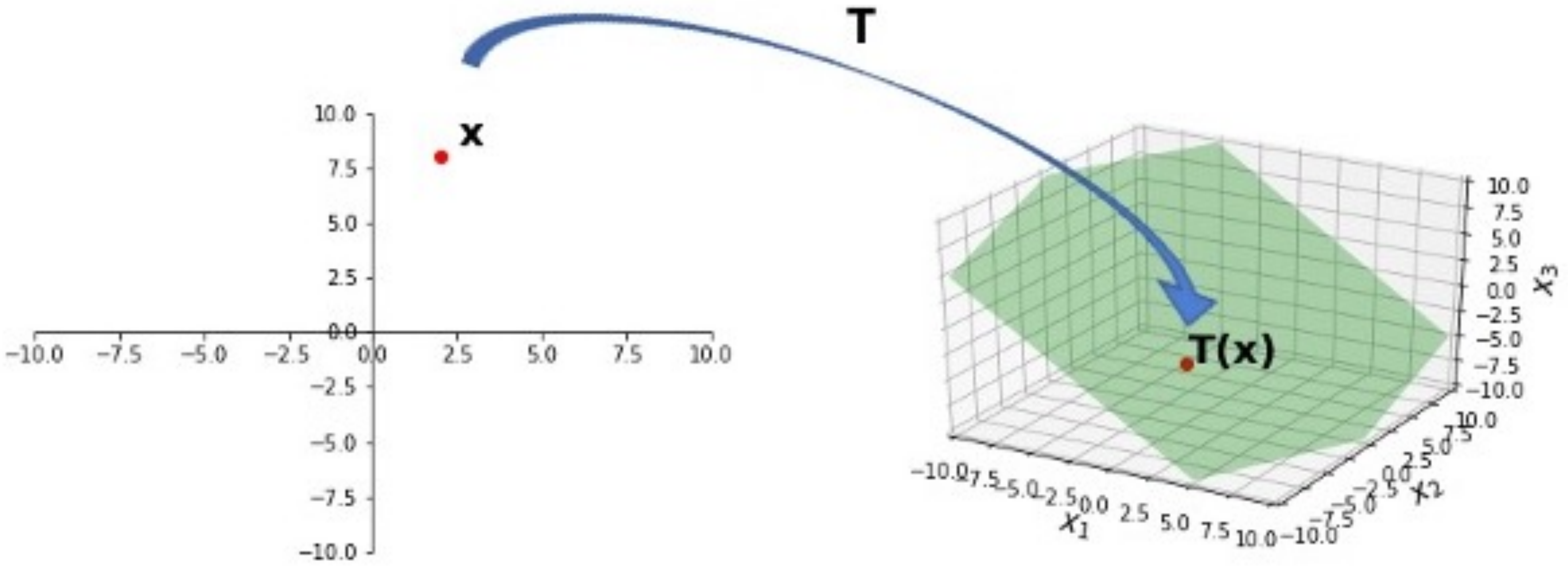

range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

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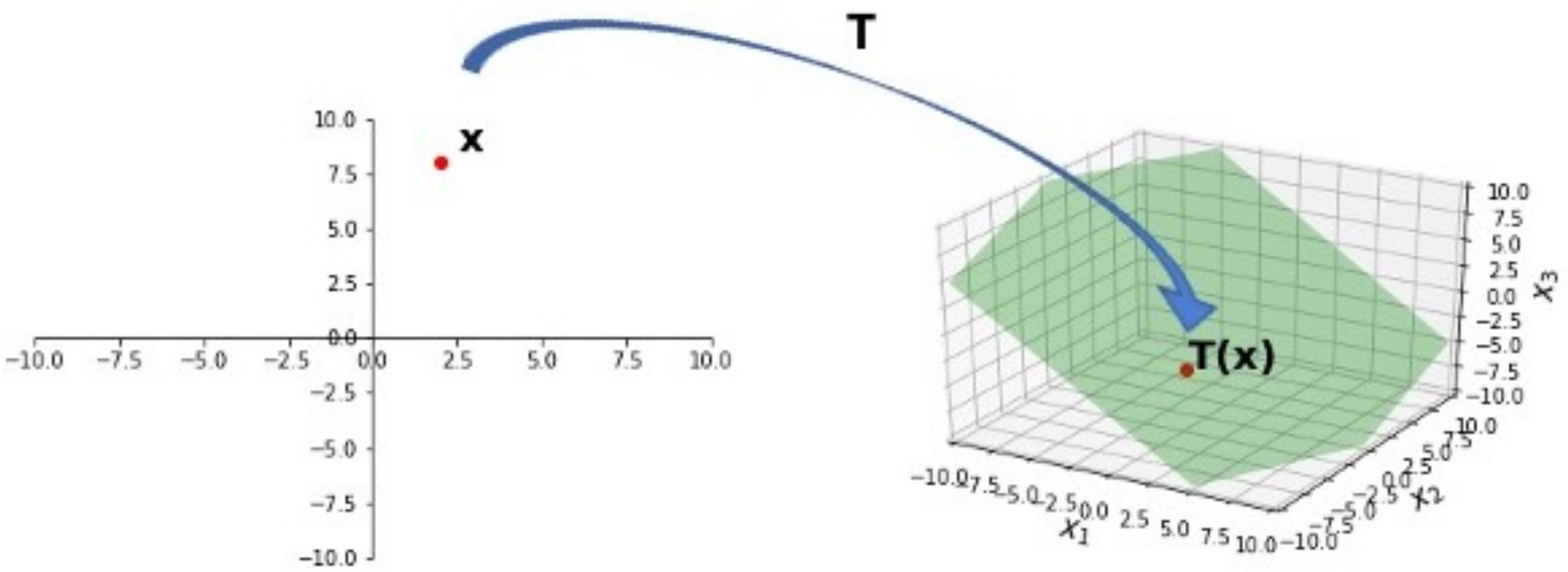

range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

The range is always contained in the codomain.

Matrix Transformations

Transformation of a Matrix

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The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

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$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

Range and Span

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The span of the columns of a matrix A is the set of all possible *images* under A.

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$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

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The span of the columns of a matrix A is the set of all possible *images* under A.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

$$= \{[a_1 \ \ldots \ a_n]\vec{v} : v \in \mathbb{R}^n\}$$

The transformation of a vector $\mathbf{v}$ under the matrix A always lies in the span of its columns.

Motivating Questions

What kind of functions can we define in this way?

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Linear Transformations

Recall: Algebraic Properties

Matrix–vector multiplication satisfies the following two properties:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

Example

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(2\begin{bmatrix}2\\3\end{bmatrix}\right)=2\left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\right)$$

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) =$$

Linear Transformations

Definition. A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Linear Transformations

Definition. A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties.

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Matrix transformations are linear transformations.

Example: Identity

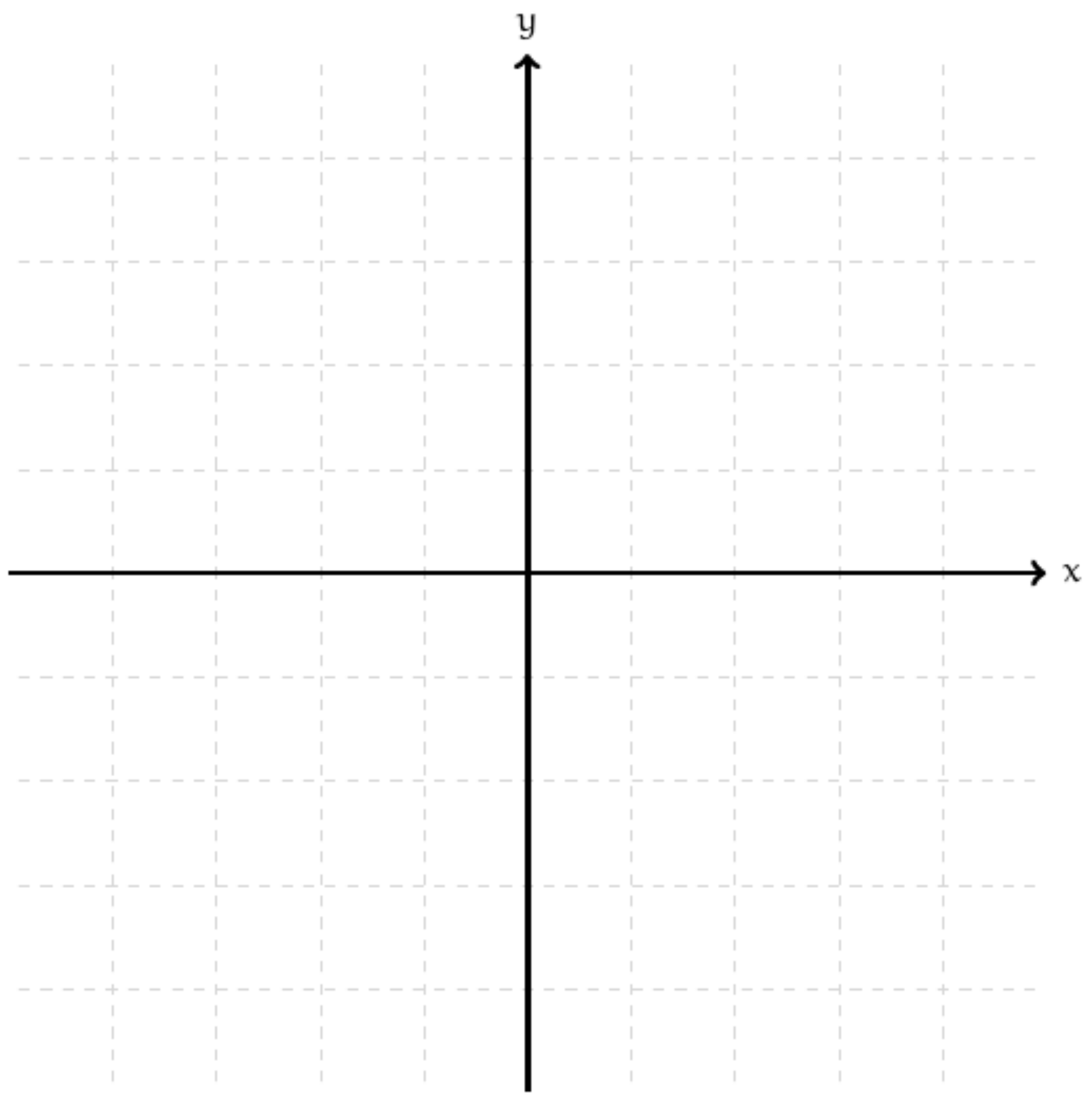
$$T(\mathbf{v}) = \mathbf{v}$$

Example: Zero

$$T(\mathbf{v}) = \mathbf{0}$$

Example: Rotation

We'll see this on Thursday, but we can reason about it geometrically for now.



Example: Indefinite Integrals

$$T(f) = \int f(x)dx$$

Disclaimer: Advanced Material

$$T(f+g) = \int (f+g)(x)dx = \int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx = T(f) + T(g)$$

$$T(cf) = \int (cf)(x)dx = \int cf(x)dx = c\int f(x)dx = cT(f)$$

the same goes for derivatives
(how are functions vectors???)

Example: Expectation

$$T(X) = \mathbb{E}[X]$$

Disclaimer:
Advanced
Material

This is exactly linearity of expectation.

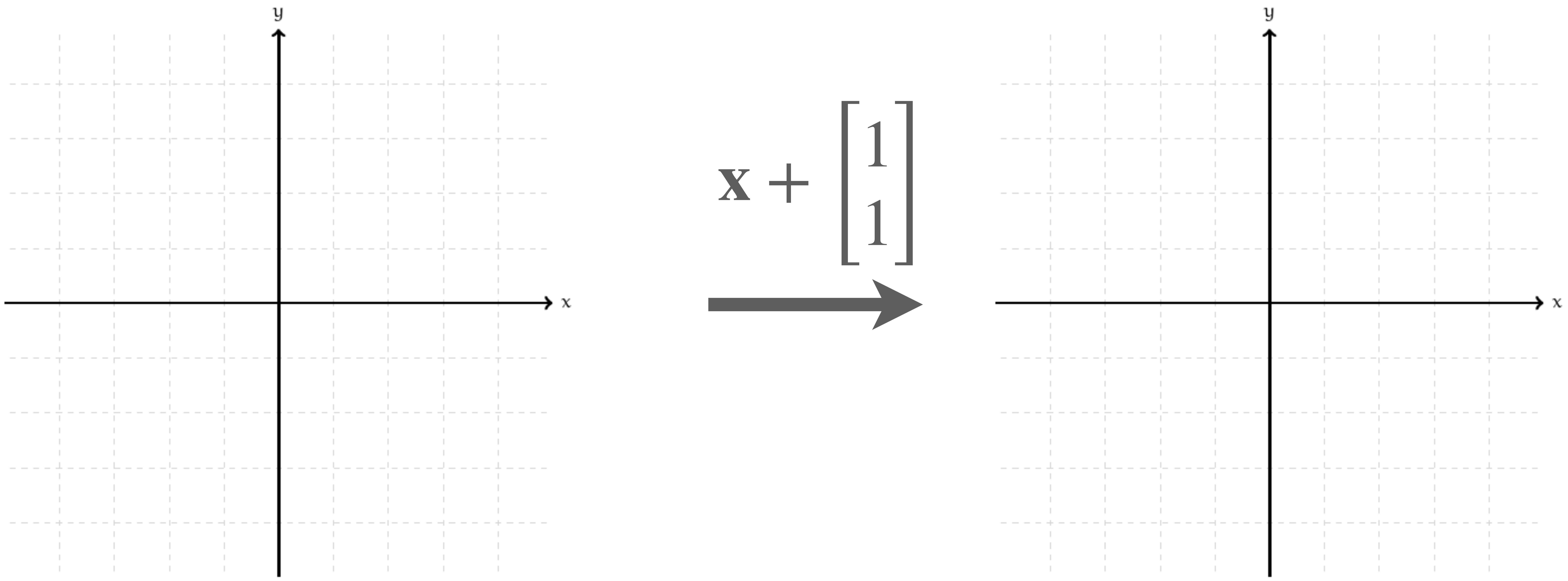
(how are random variables vectors???)

Non-Example: Squares

$$T(x) = x^2$$

Note that $T : \mathbb{R}^1 \to \mathbb{R}^1$

Non-Example: Translation



Example (Understanding Check)

$$T(\mathbf{v}) = 5\mathbf{v}$$

Example (Understanding Check)

$$T(x) = e^x$$

Properties of Linear Transformations

The Zero Vector

$$T(\mathbf{0}) = \;???$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The Zero Vector

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`The zero vector is` *`fixed`* `by linear transformations.`

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

Note: These may be different dimensions!

The zero vector is *fixed* by linear transformations.

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

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We can combine our linearity conditions:

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We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

$= T(a\mathbf{v}) + T(b\mathbf{u})$ (additivity)

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$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

$= T(a\mathbf{v}) + T(b\mathbf{u})$ (additivity)

$= aT(\mathbf{v}) + bT(\mathbf{u})$ (homogeneity for each term)

A Single Condition

Theorem. A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

A Single Condition

Theorem. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

It's often easiest to show this single condition.

Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_nT(\mathbf{v}_n)$$

We can generalize this condition to any linear combination.

Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

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Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

We can generalize this condition to any linear combination.

This is the most useful form.

Geometry of Matrix Transformations

Motivating Questions

What kind of functions can we define in this way?

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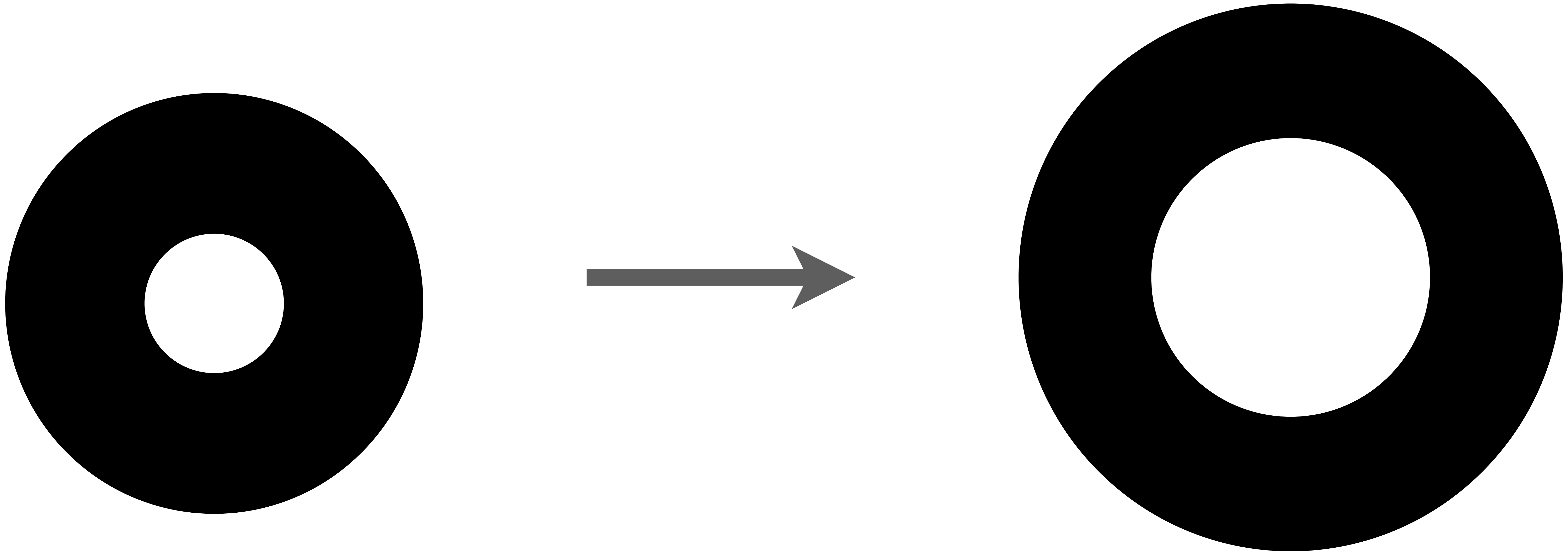
Motto

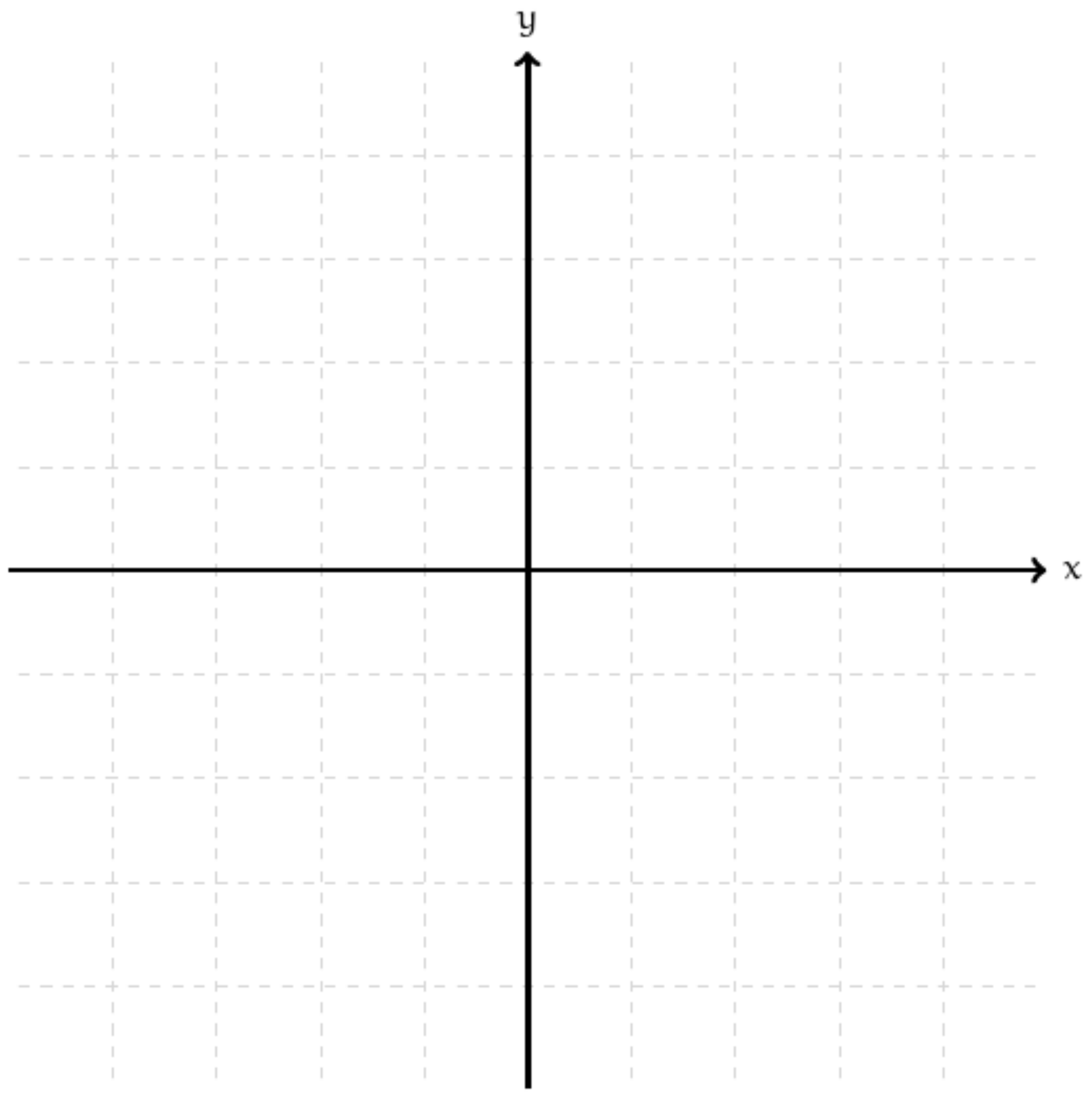
Matrix transformations change the "shape" of a set of set of vectors (points).

Example: Dilation

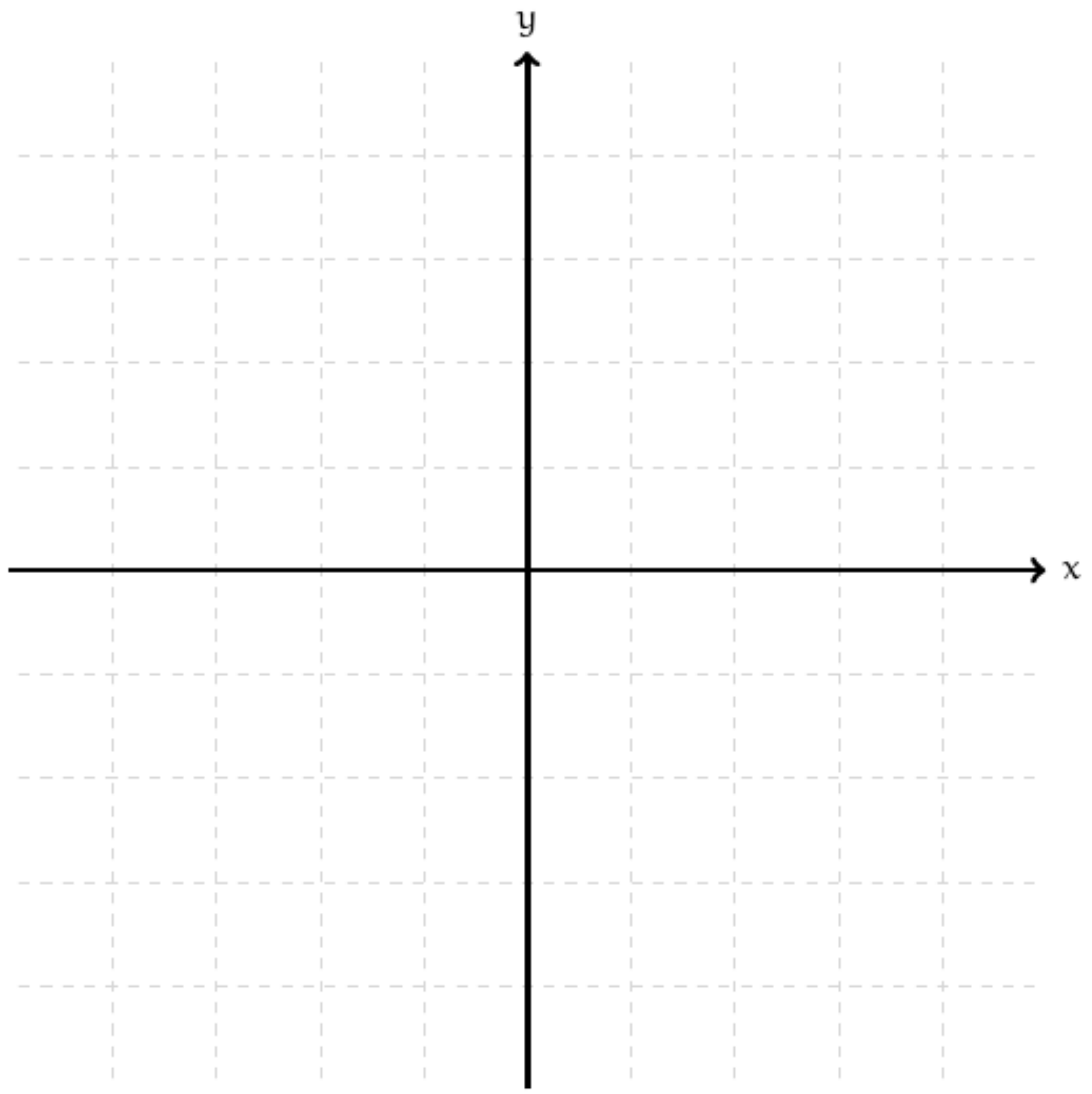


Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$

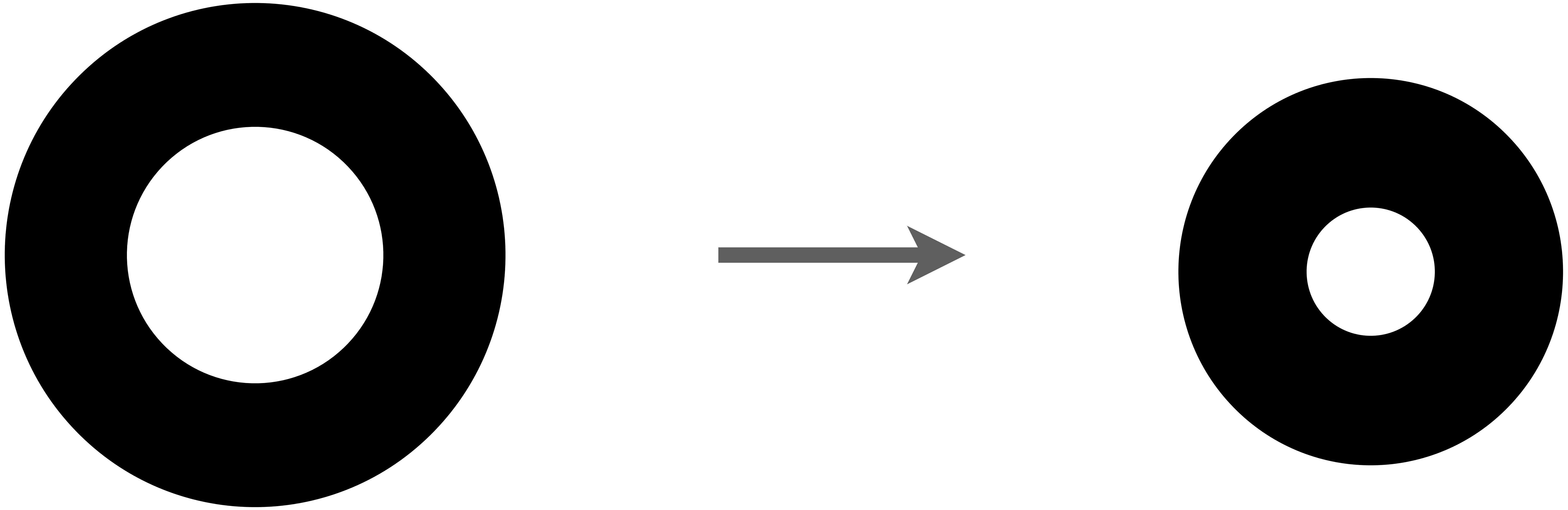


$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$



if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



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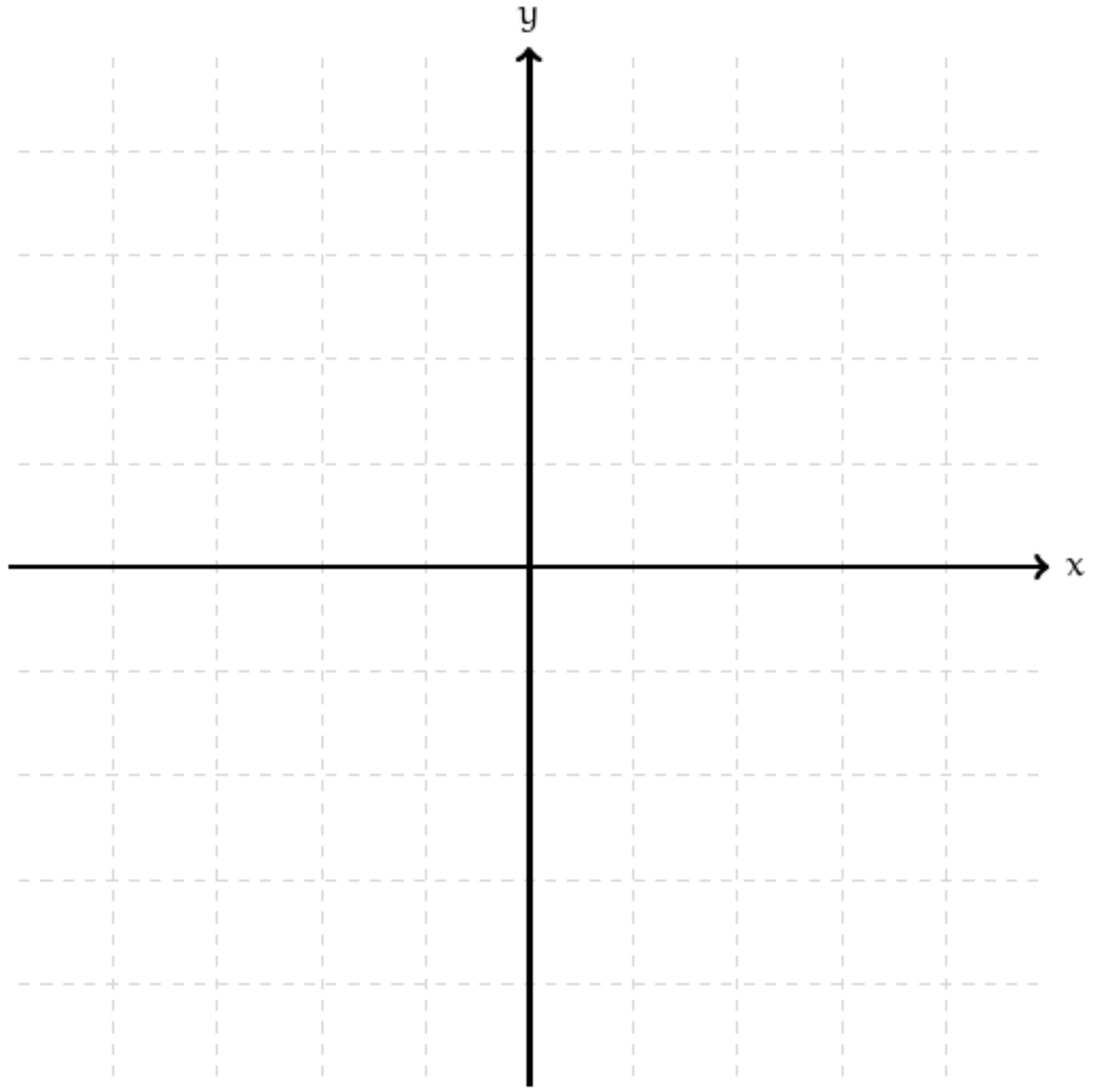
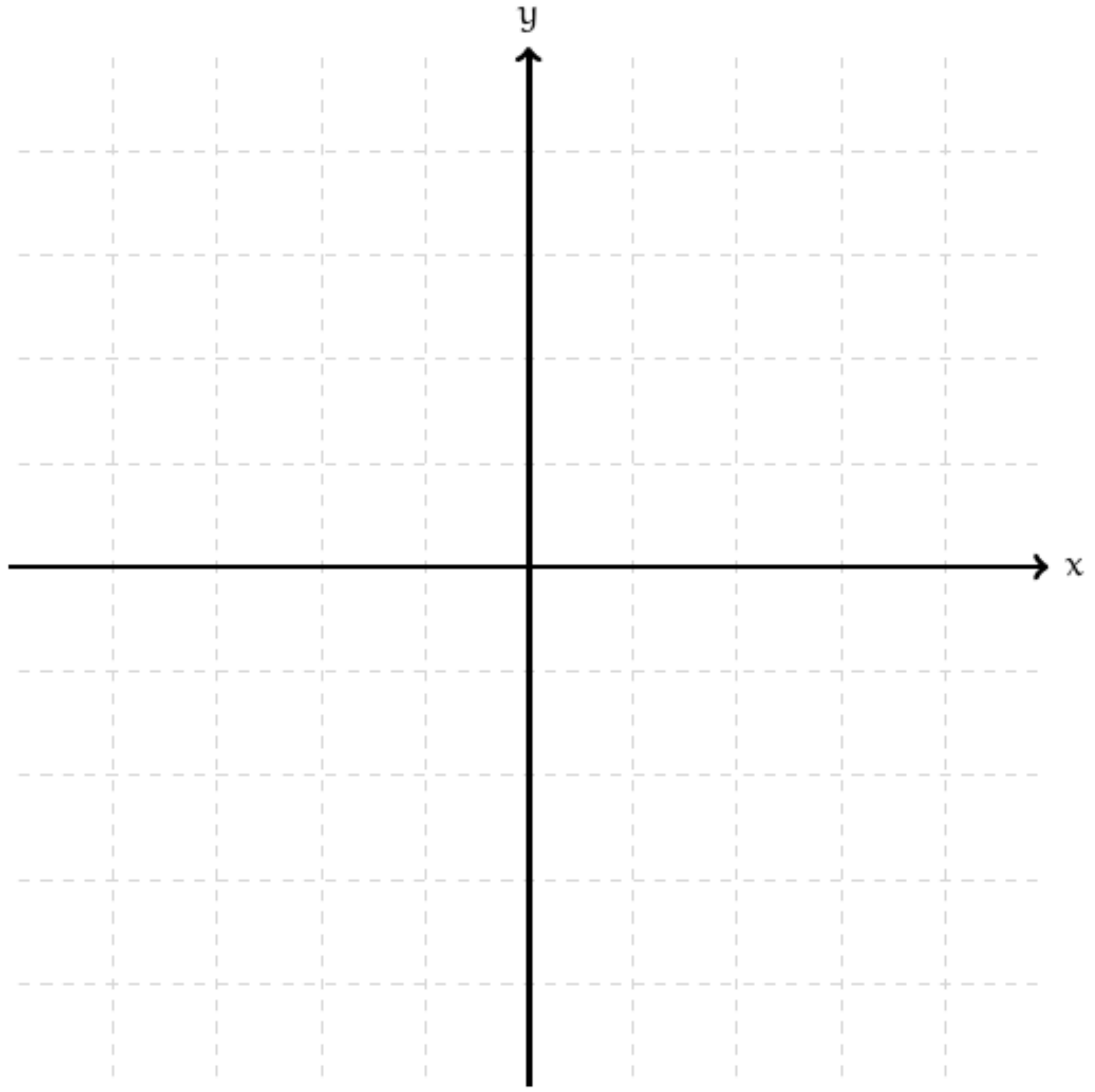


$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}$$

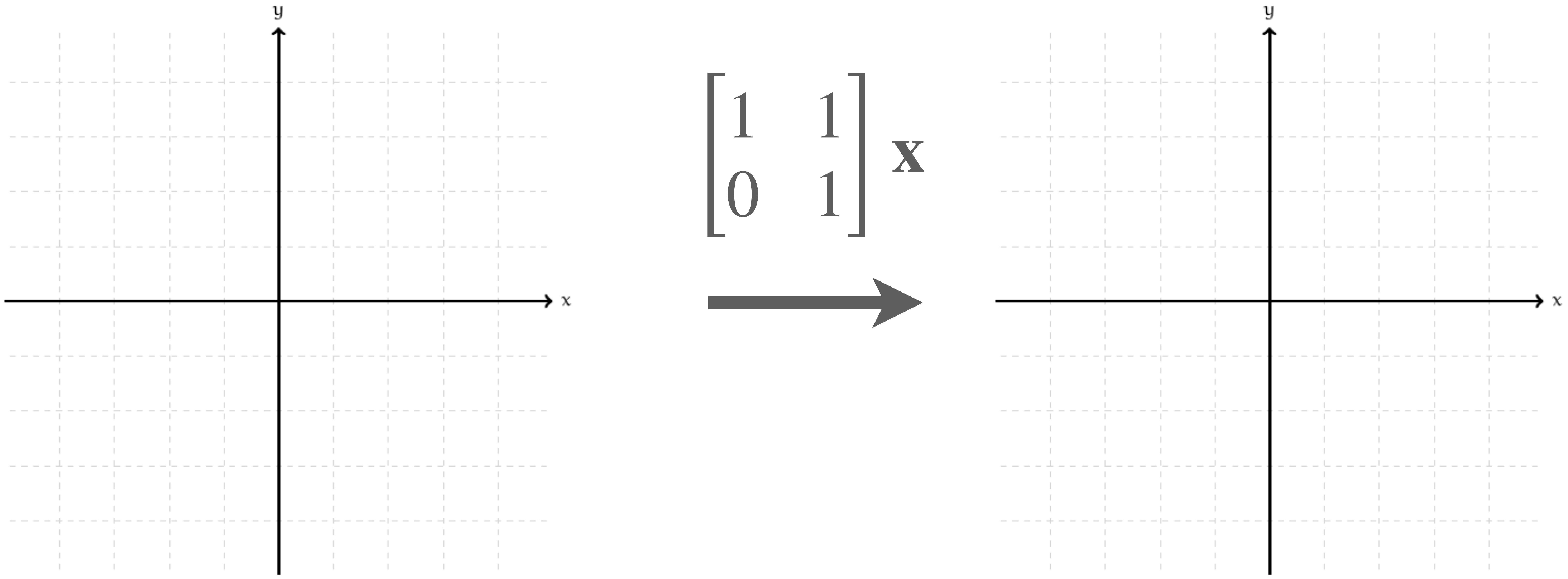


if $0 \le r \le 1$, then the transformation
pulls points towards the origin.

Example: Shearing

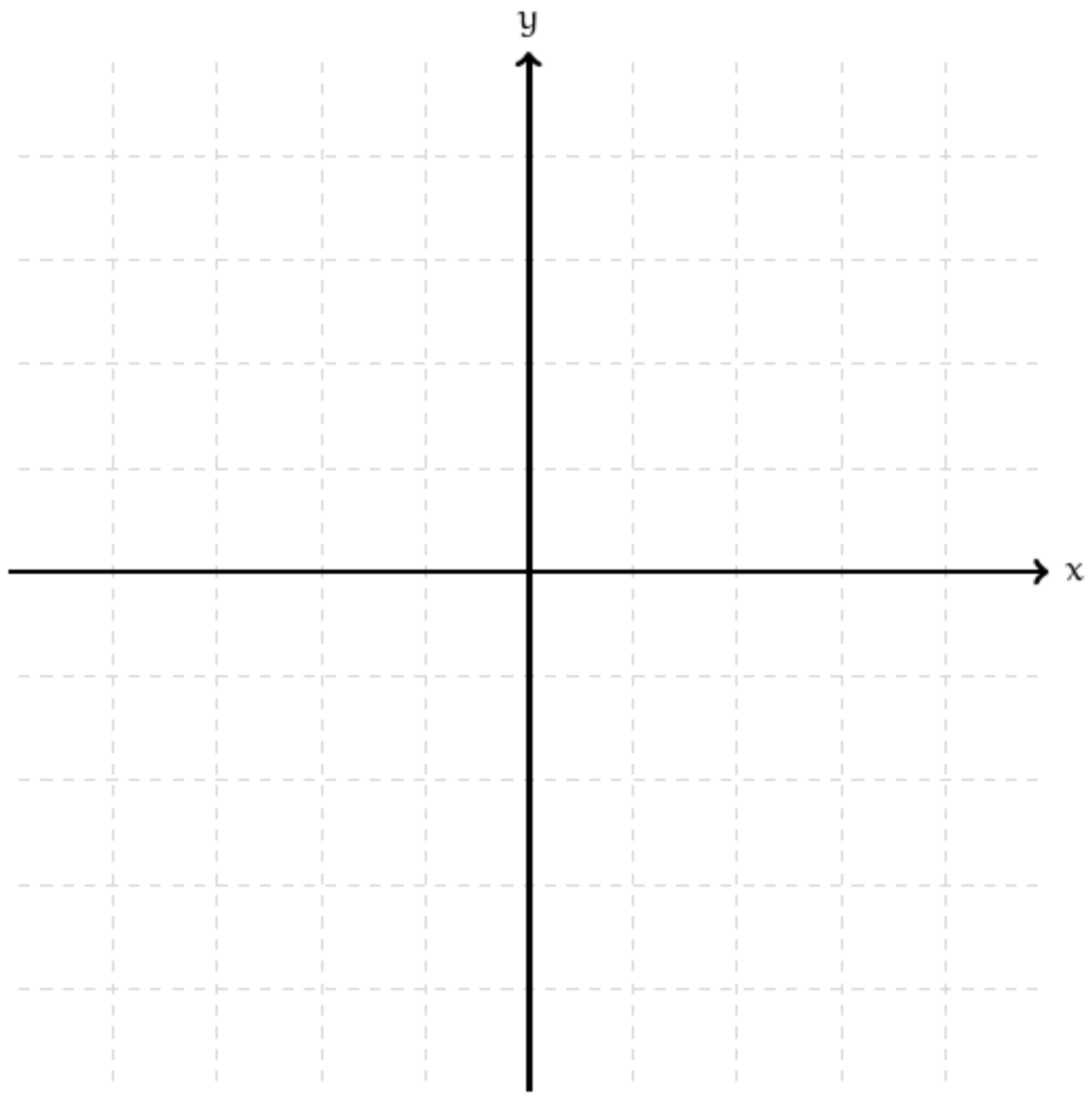
Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

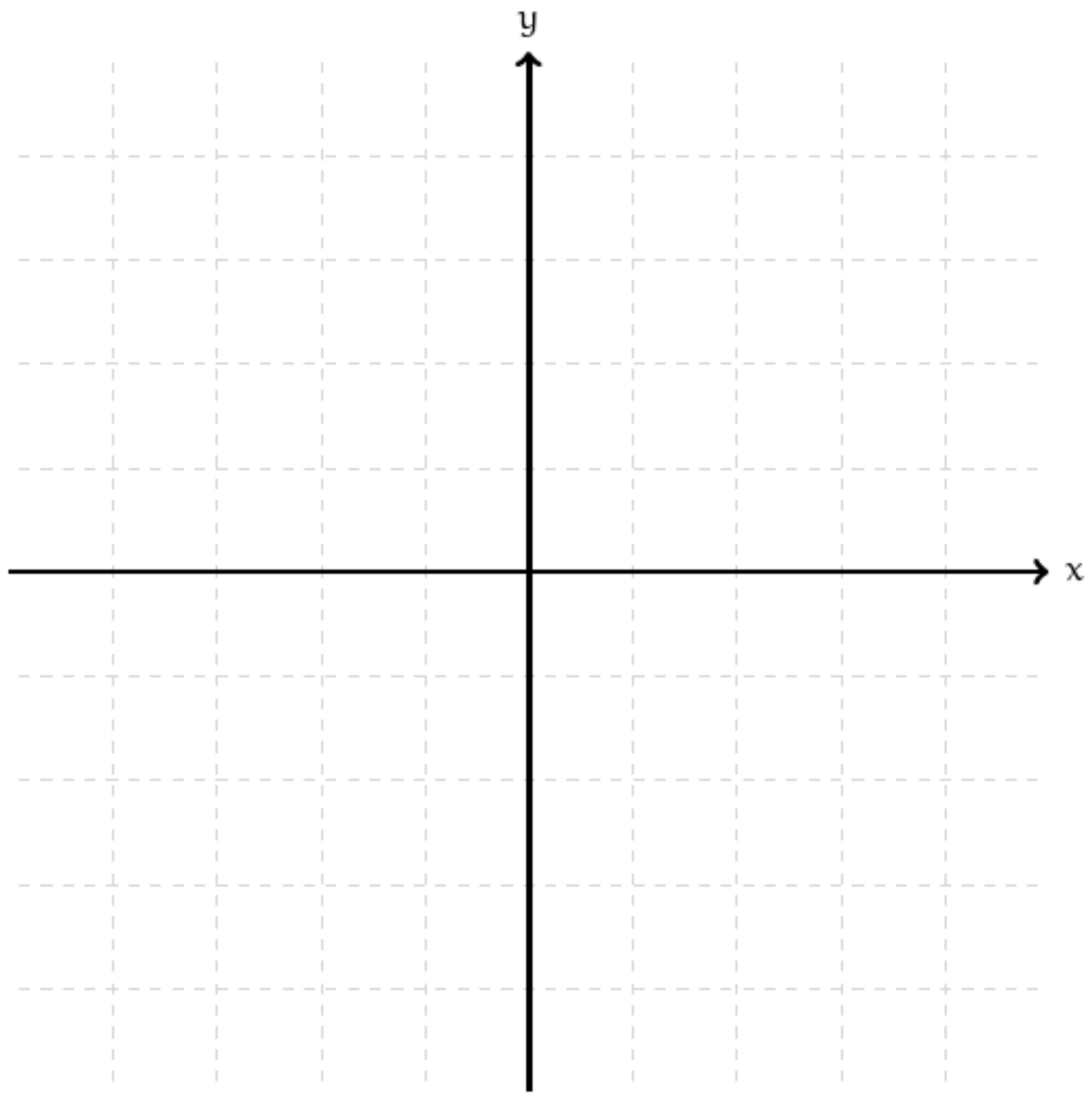


Imagine shearing like with rocks or metal.

Question

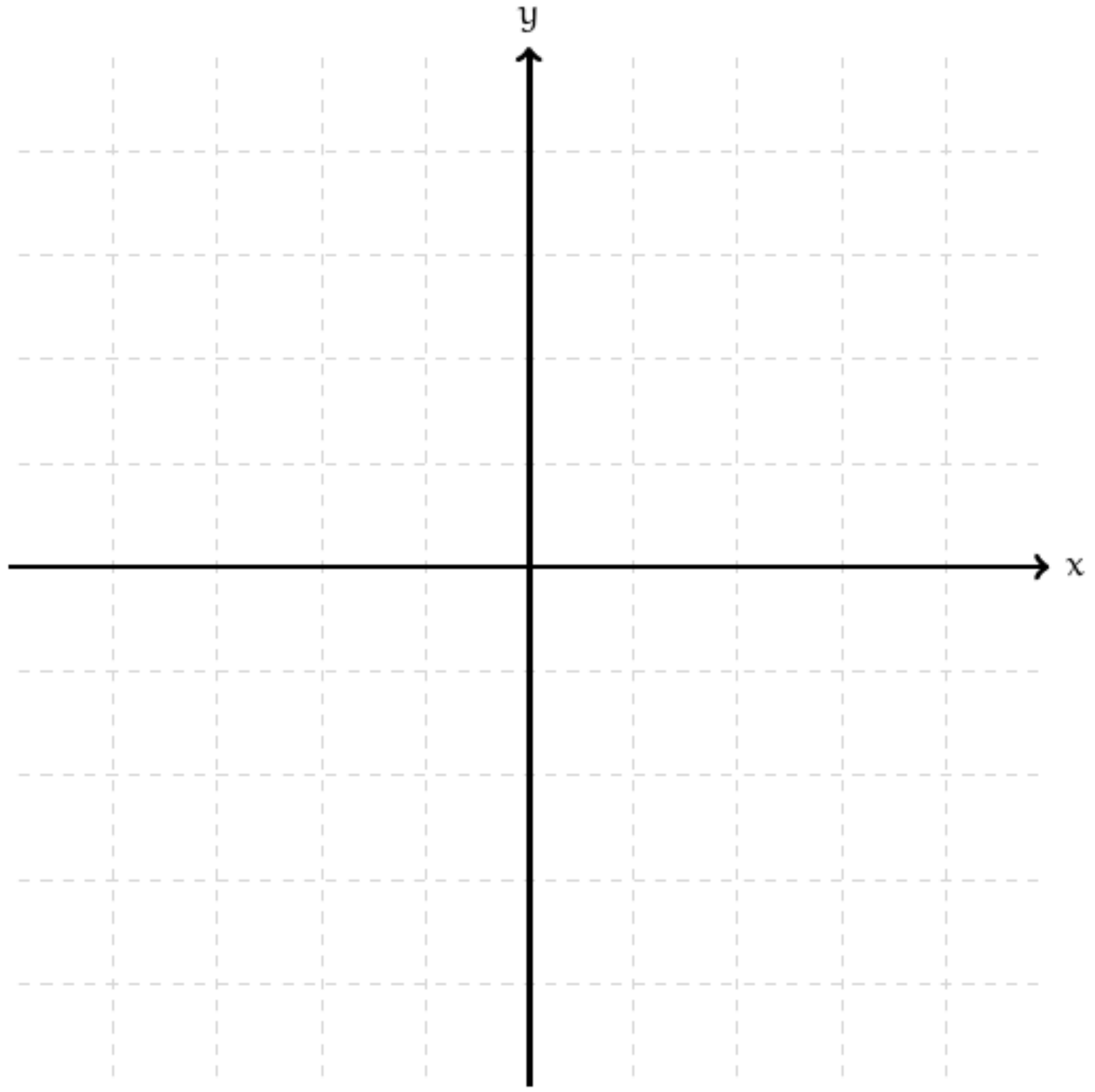


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



Draw how this matrix transforms points. What kind of transformation does it represent?

Answer: Reflection



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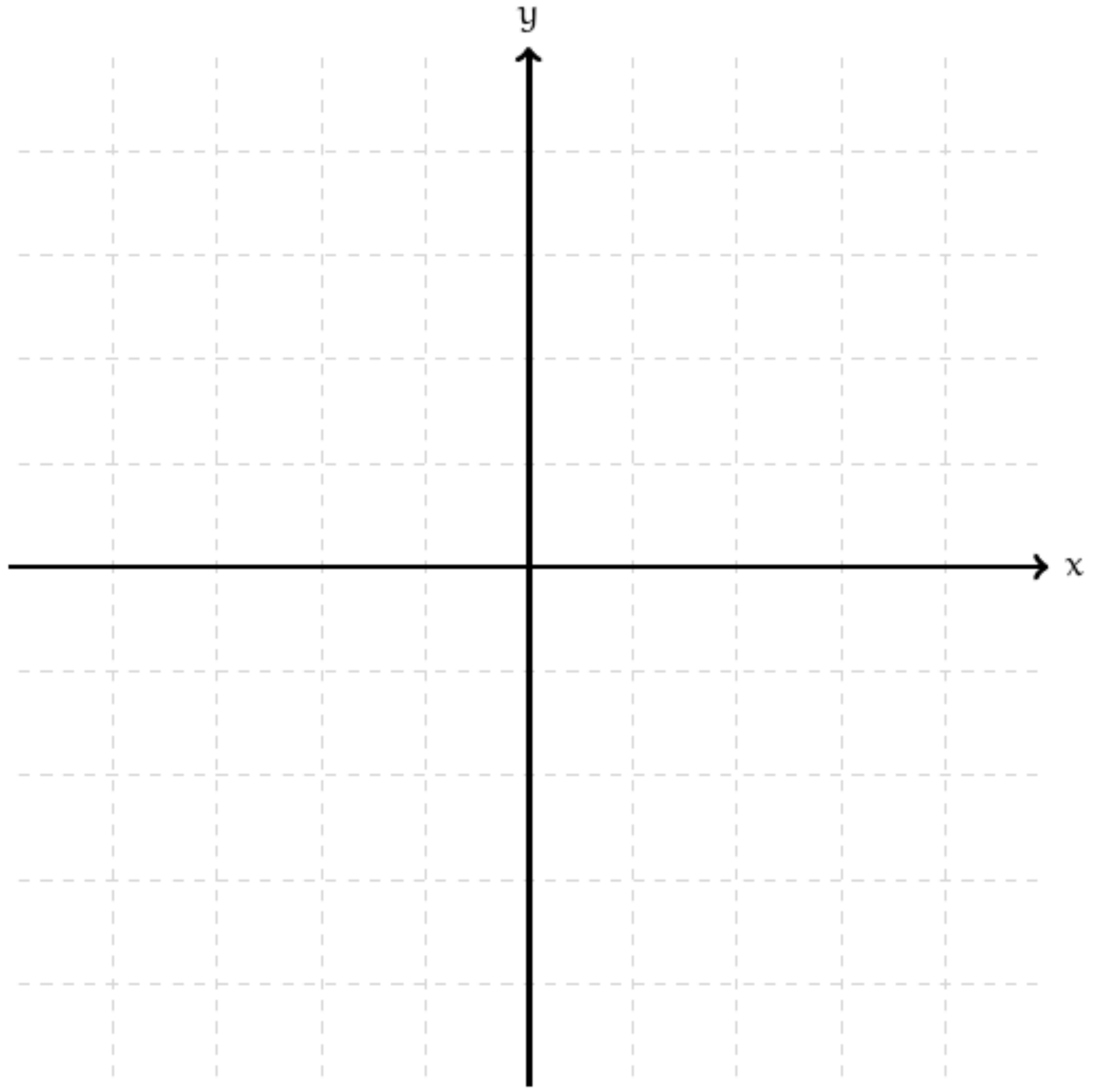

Summary

Matrices can be viewed as linear transformations.

Matrix transformations change the "shape" of points sets.

Linear transformations behave well with respect to linear combinations.