Matrix Inverses

Geometric Algorithms

Lecture 11

CAS CS 132

Practice Problem(s)

1. *Compute* $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

2. *Find a pair of 2D linear transformations* T_1 *and* T_2 *such that* T_1 *followed by* T_2 *is not the same as* T_2 *followed by* T_1*.*

Answer

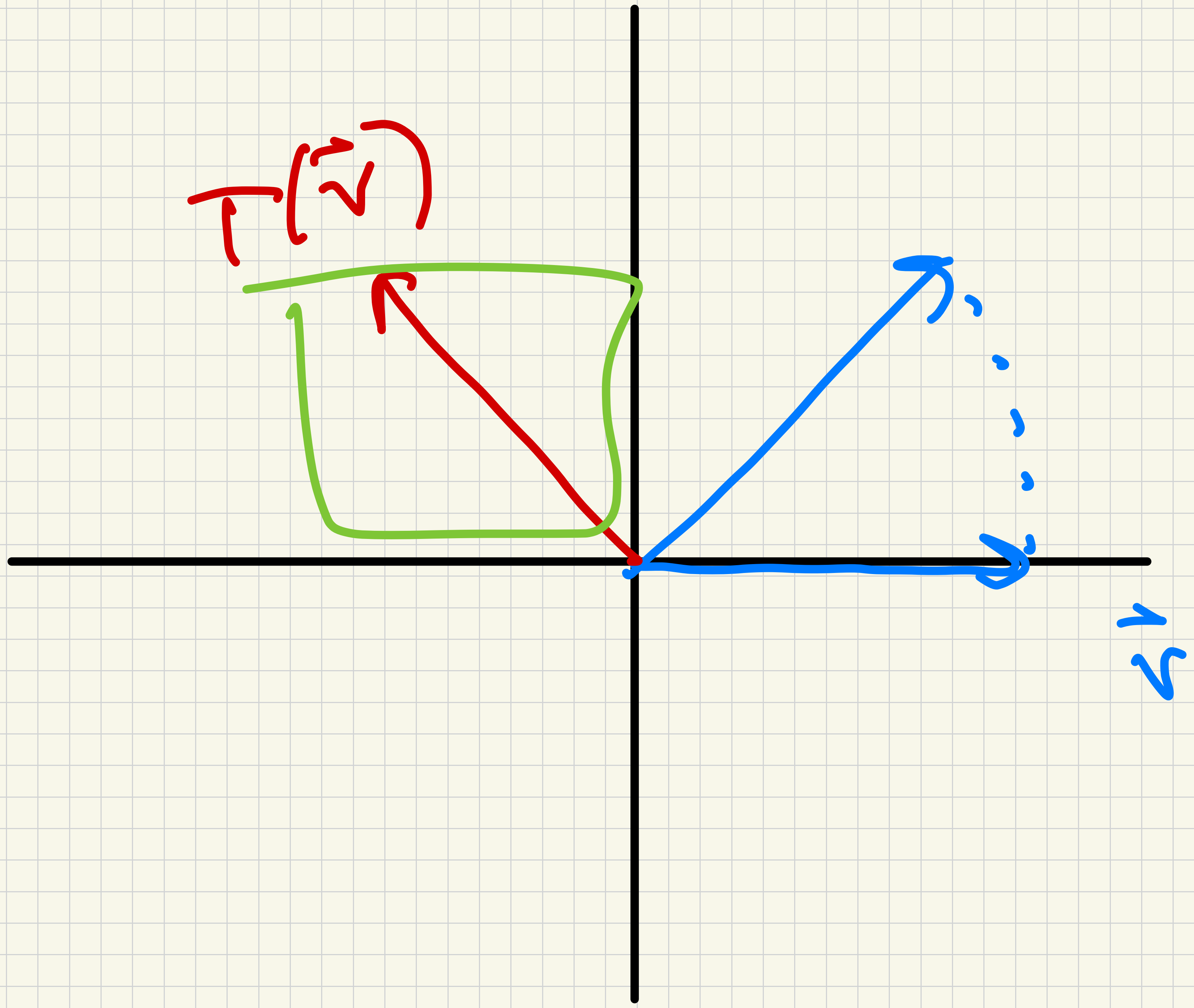
$$A[\vec{b}_1, \vec{b}_2] = [A\vec{b}_1, A\vec{b}_2]$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$



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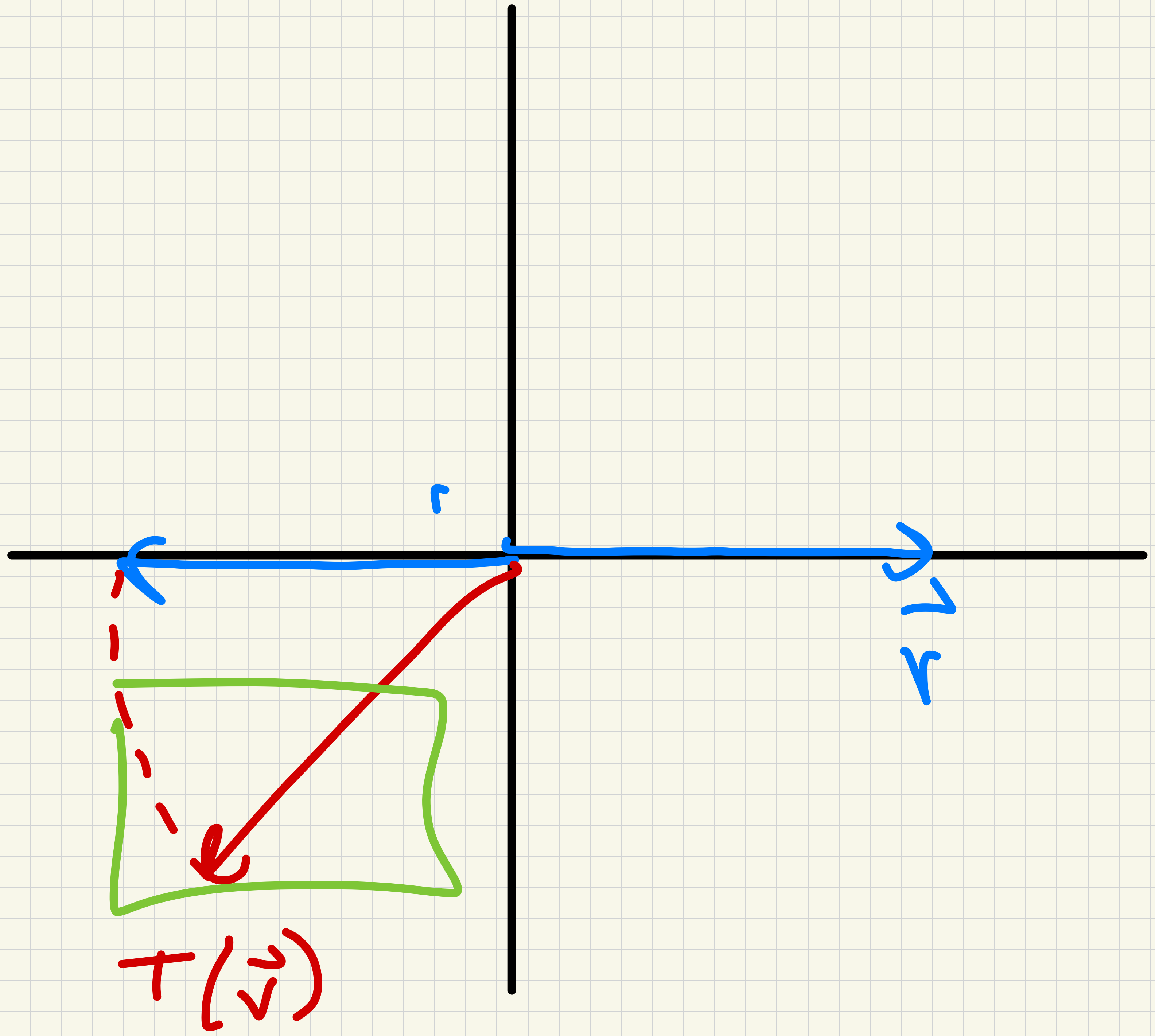

rotate then reflect $\neq$
reflect then rotatio

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Connect everything(!)

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

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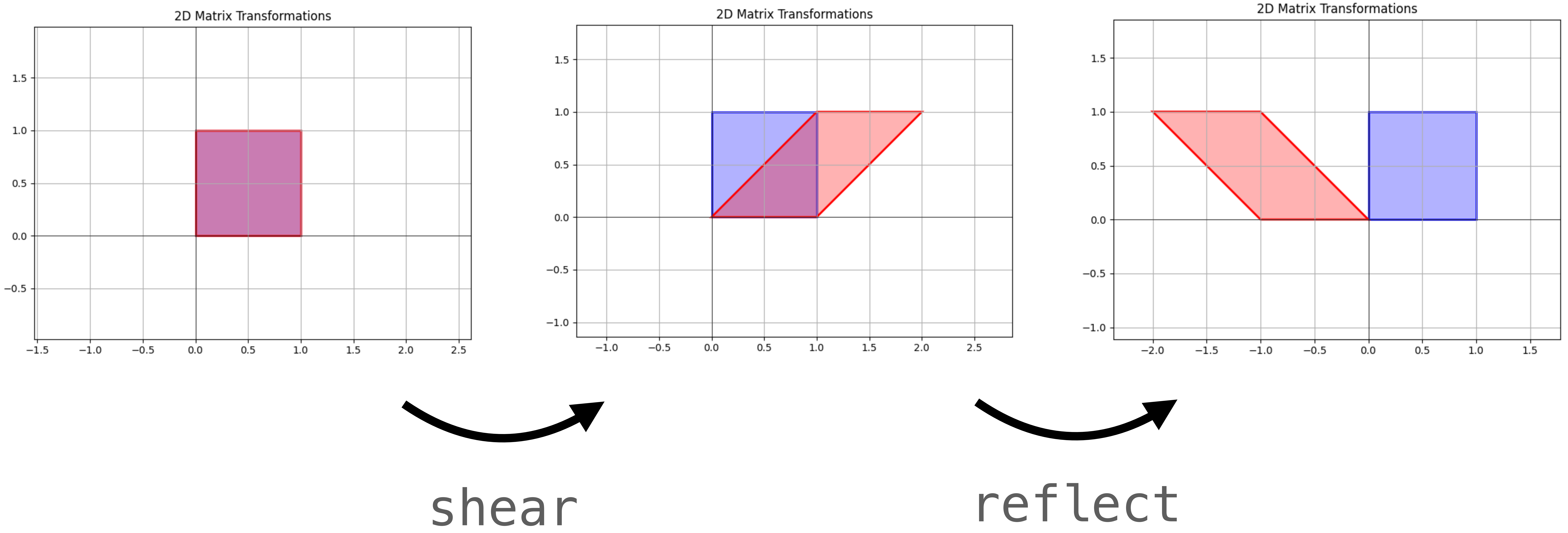
Invertible Matrix Theorem

Recap: Matrix Multiplication

Recall: Composition

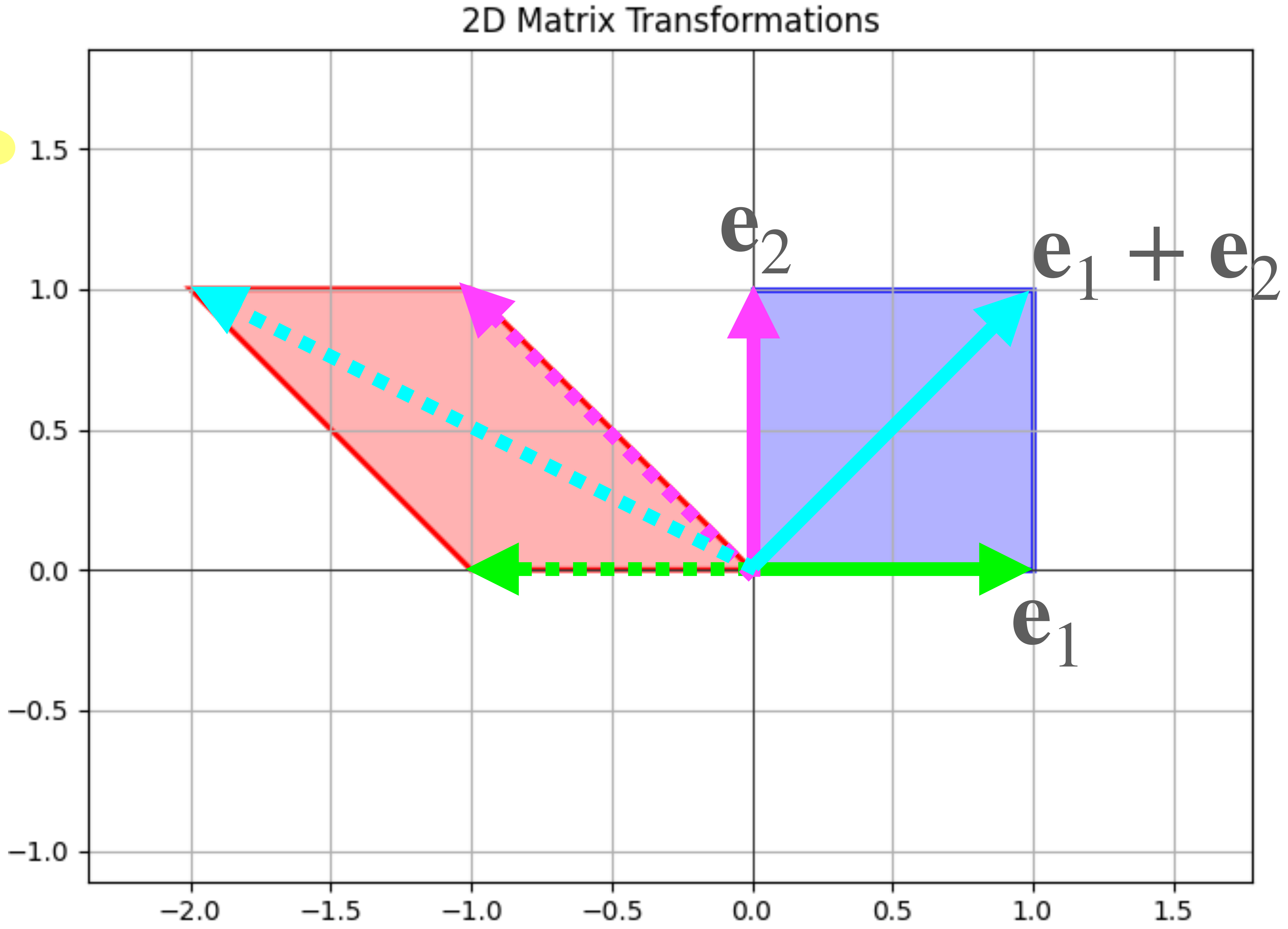


Recall: Composition

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



General Composition (2D)

$$A\left(\underline{\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}}\,\underline{\begin{bmatrix}x_1\\x_2\end{bmatrix}}\right) =$$

$$A\left(x_1\vec{b}_1 + x_2\vec{b}_2\right) =$$

$$x_1\boxed{A\vec{b}_1} + x_2\boxed{A\vec{b}_2} = \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

vectors

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p\end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

`this only works if the number of columns of the left matrix matches the number of rows of the right matrix`

$$m\,\overbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}^{n} \; n\,\overbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}^{k} = m\,\overbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}^{k}$$

$$(m \times n) \qquad (n \times k) \qquad (m \times k)$$

Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \left[\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} \quad \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\right]$$

$2 \times \textcircled{3}$ $\textcircled{2} \times 2$

X

2×3 $\mathbb{R}^2$

Non-Example

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \begin{bmatrix}\xcancel{\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}} & \xcancel{\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}}\end{bmatrix}$$

These are not defined.

Example

$2 \times 2 \quad \mathbb{R}^2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]$$

$\boxed{2} \times \textcircled{2} \quad \textcircled{2} \times \boxed{3}$

2×3

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector $\mathbf{v}$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

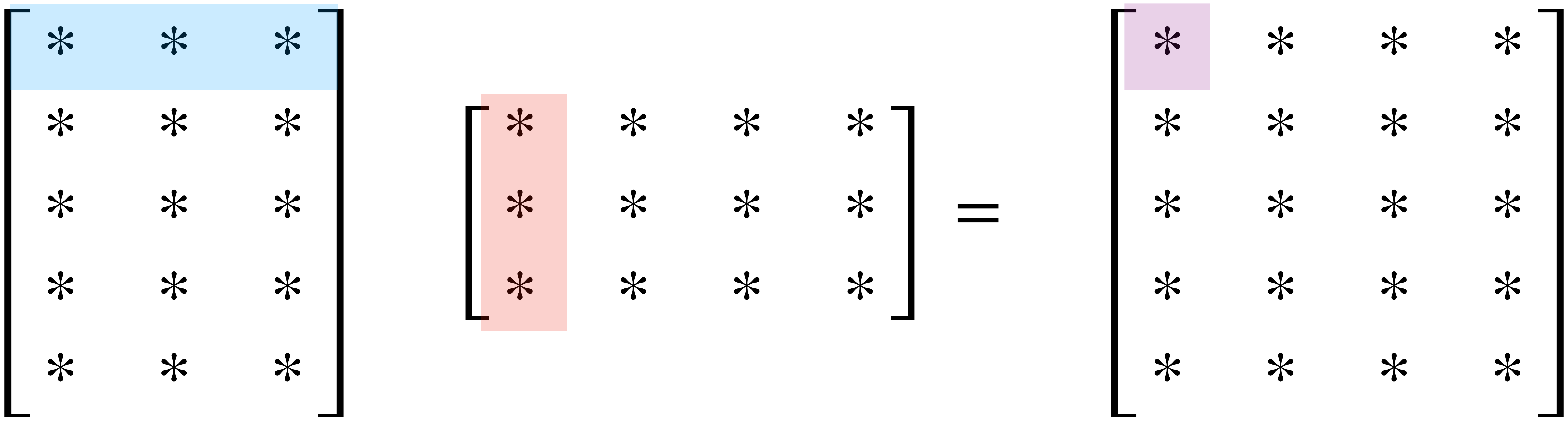
$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B, the entry in row i and column j of AB is defined above.

Row-Column Rule (Pictorially)



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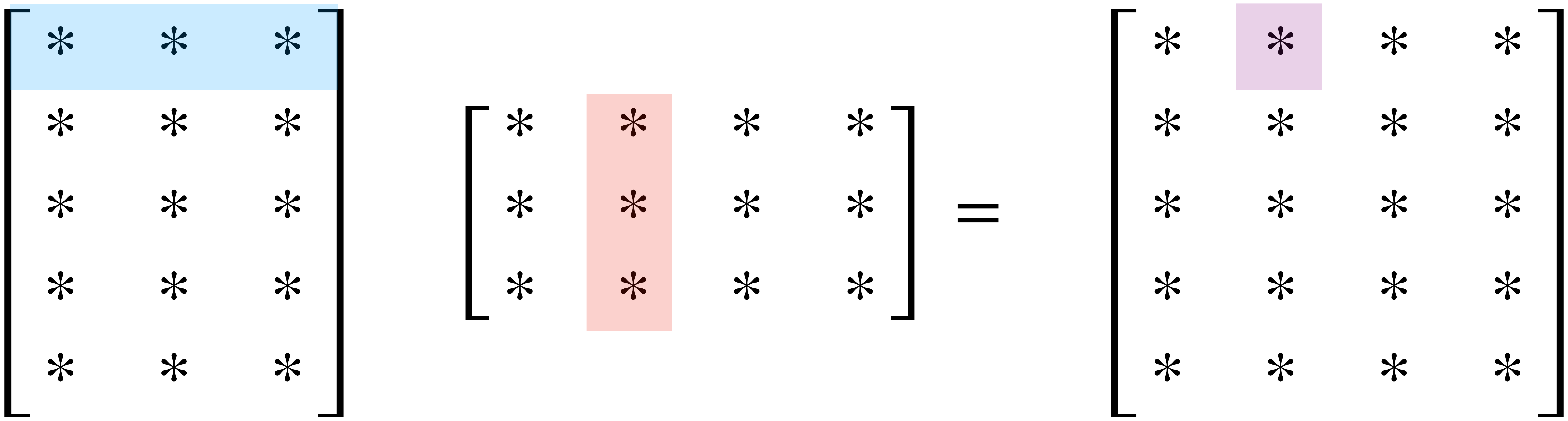

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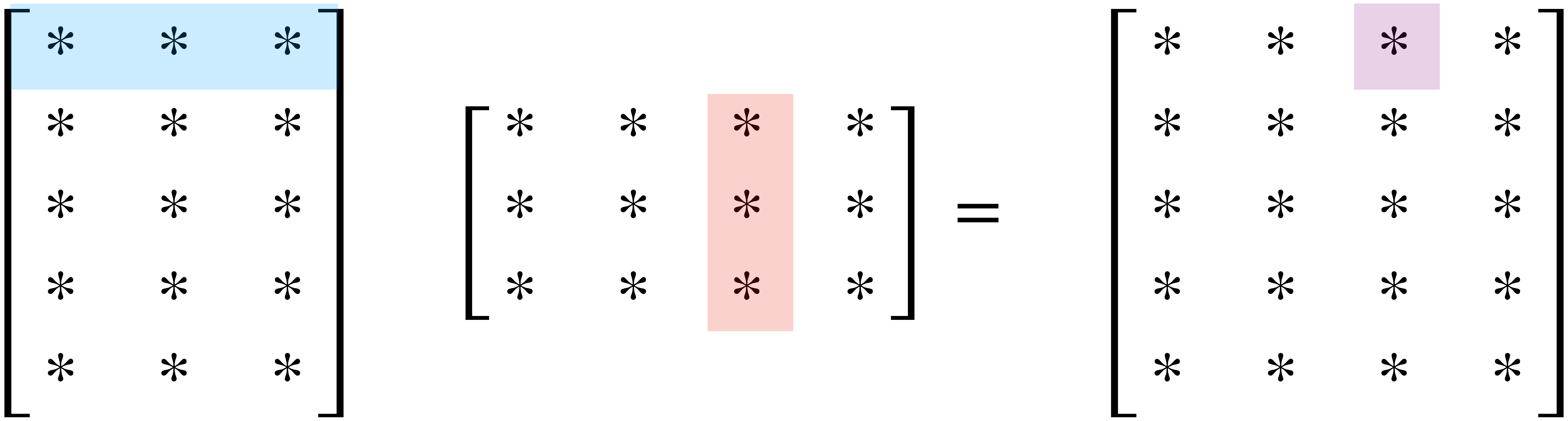

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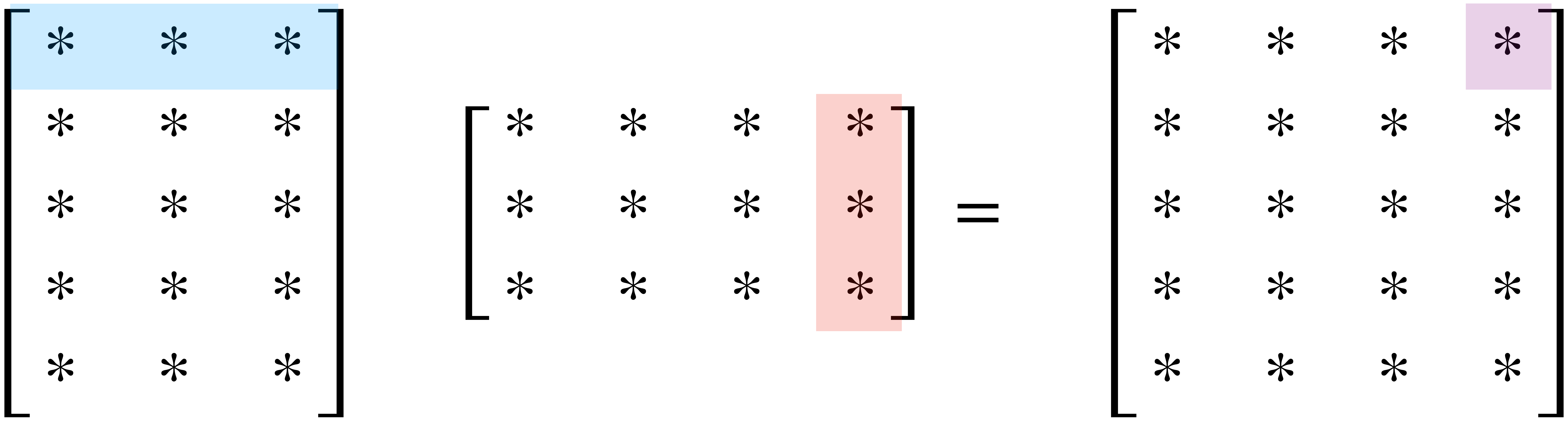

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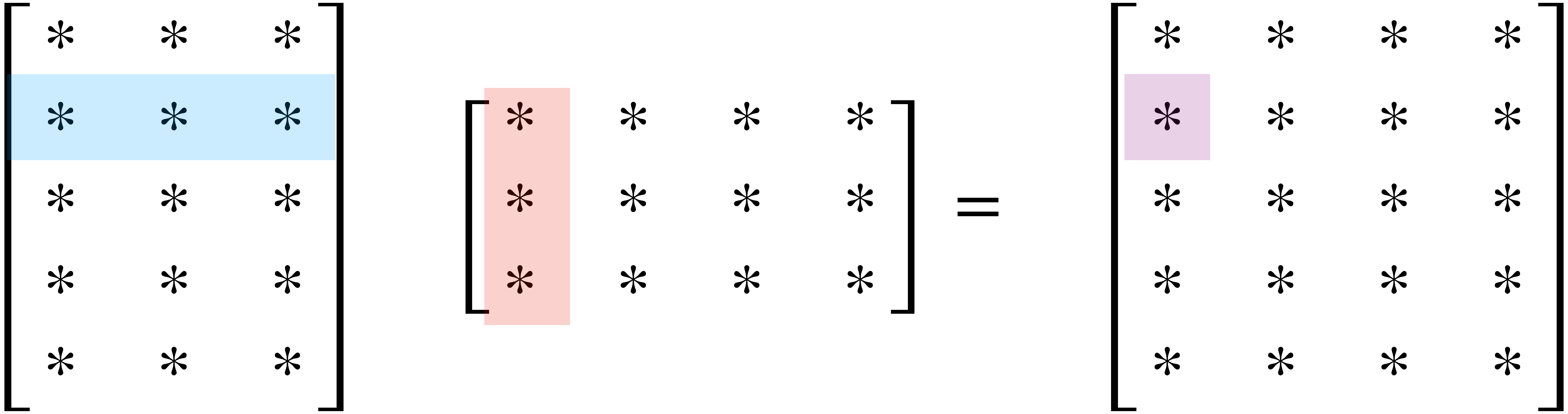

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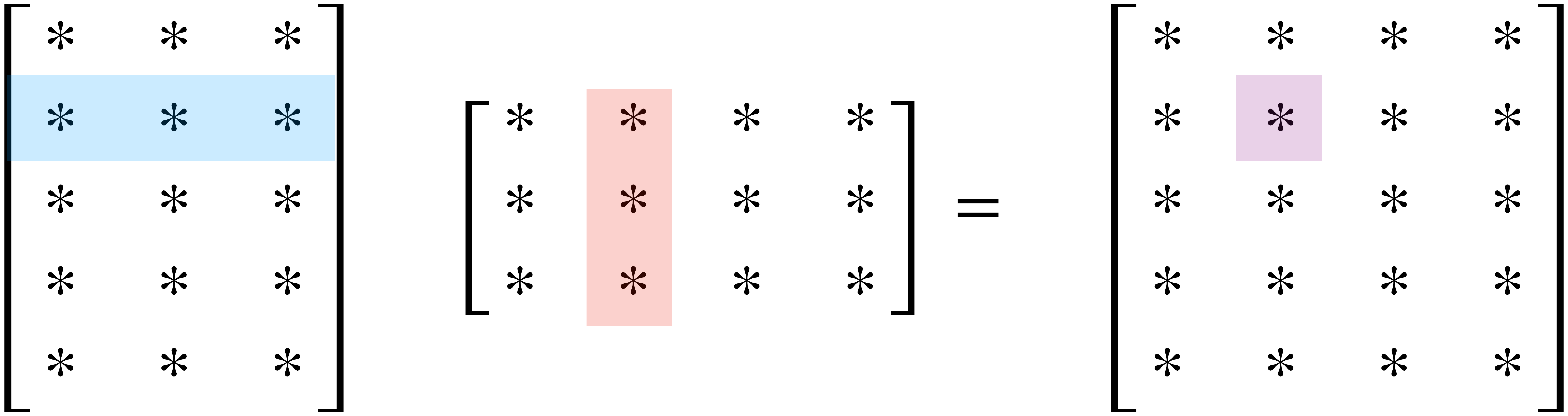

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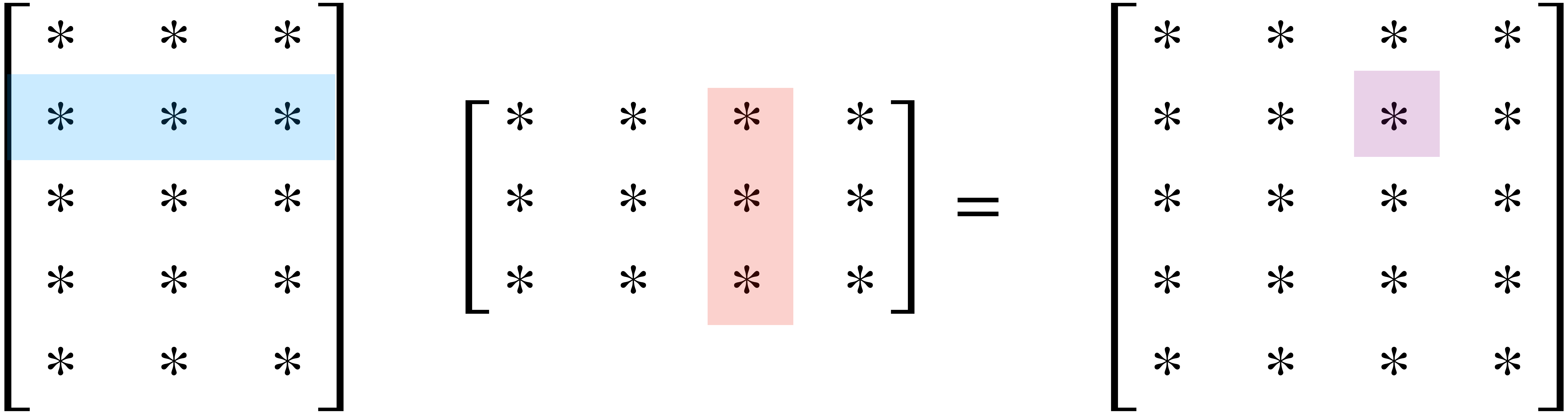

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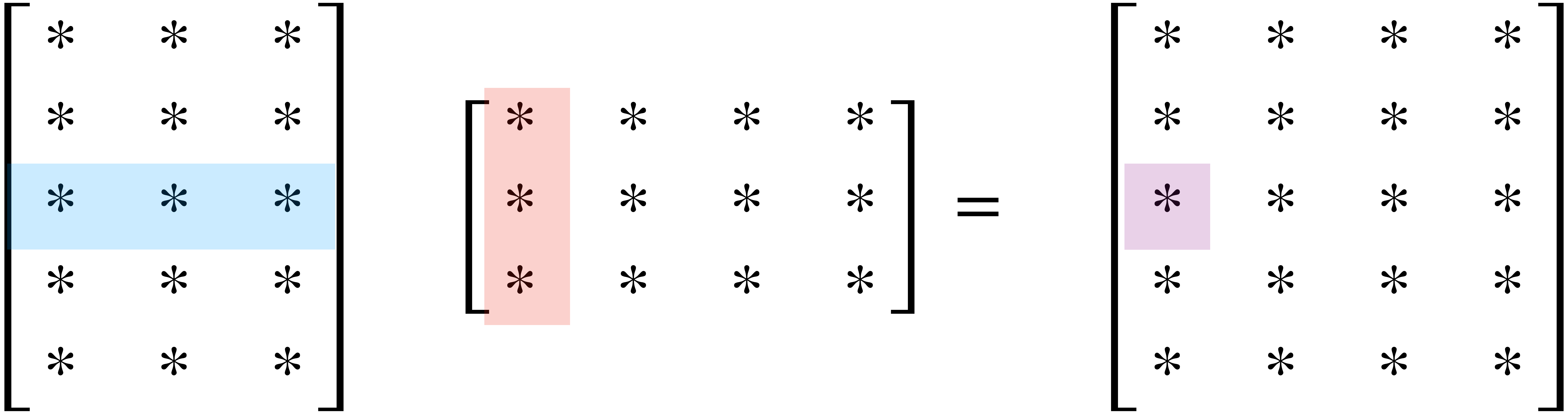

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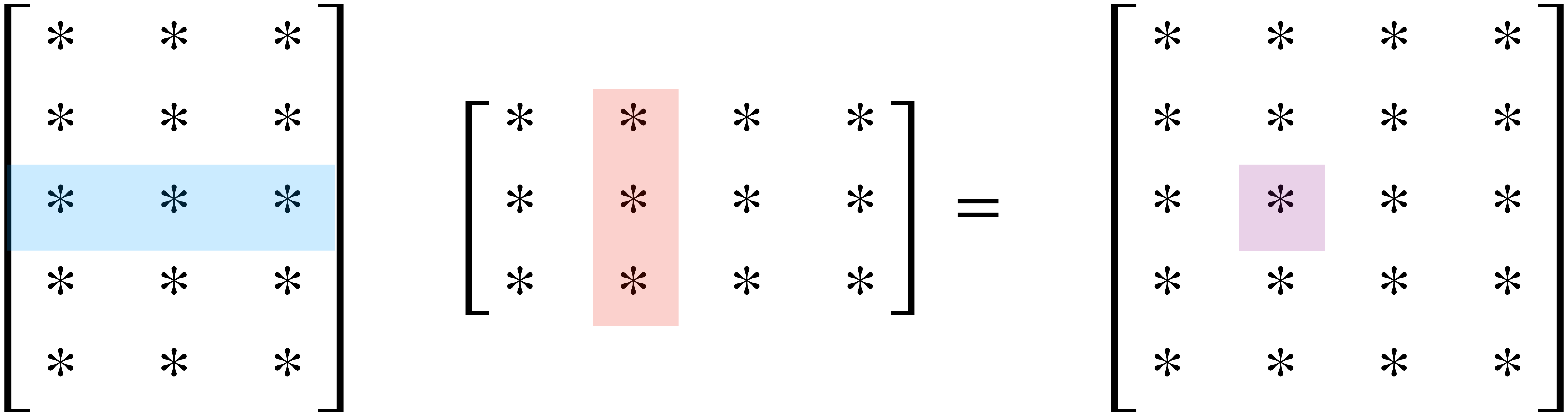

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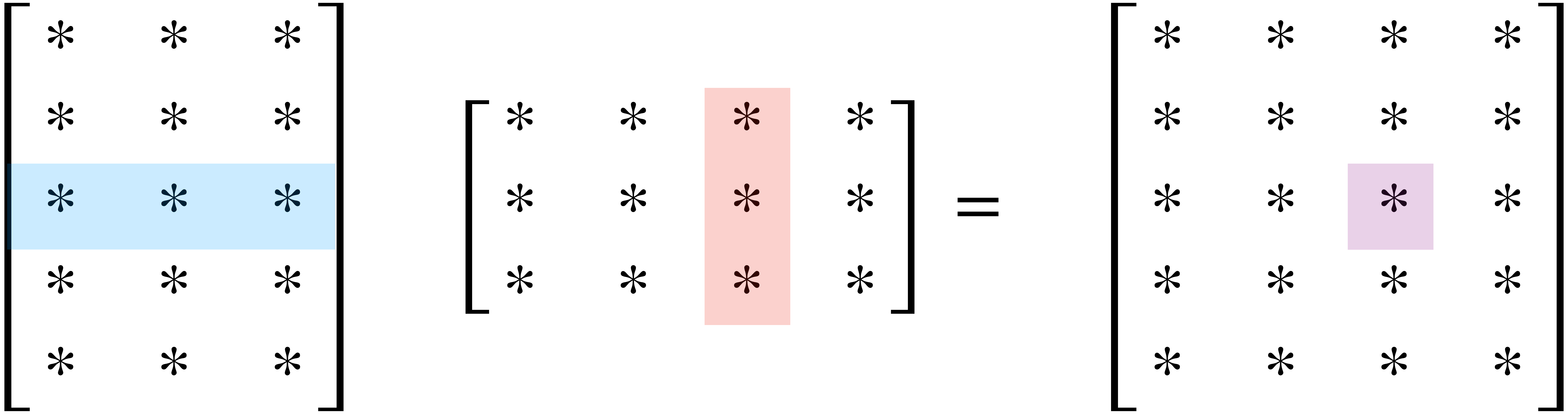

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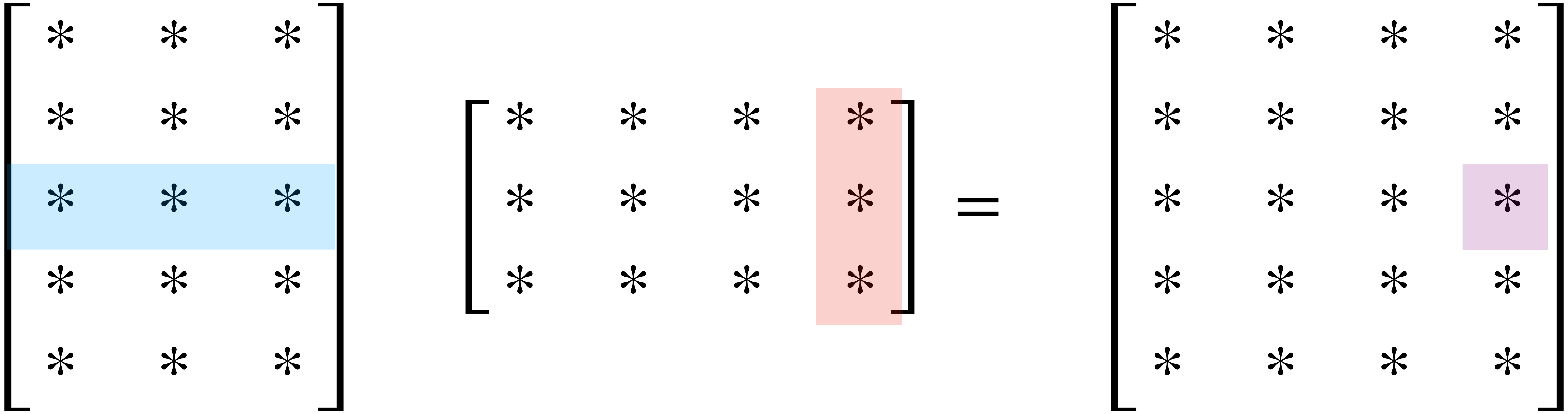

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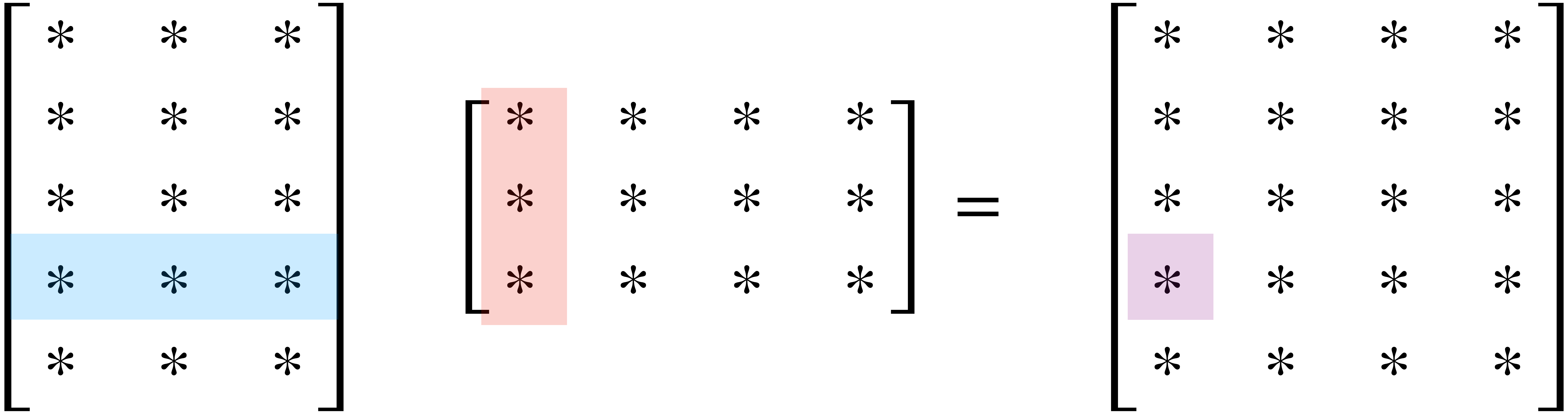

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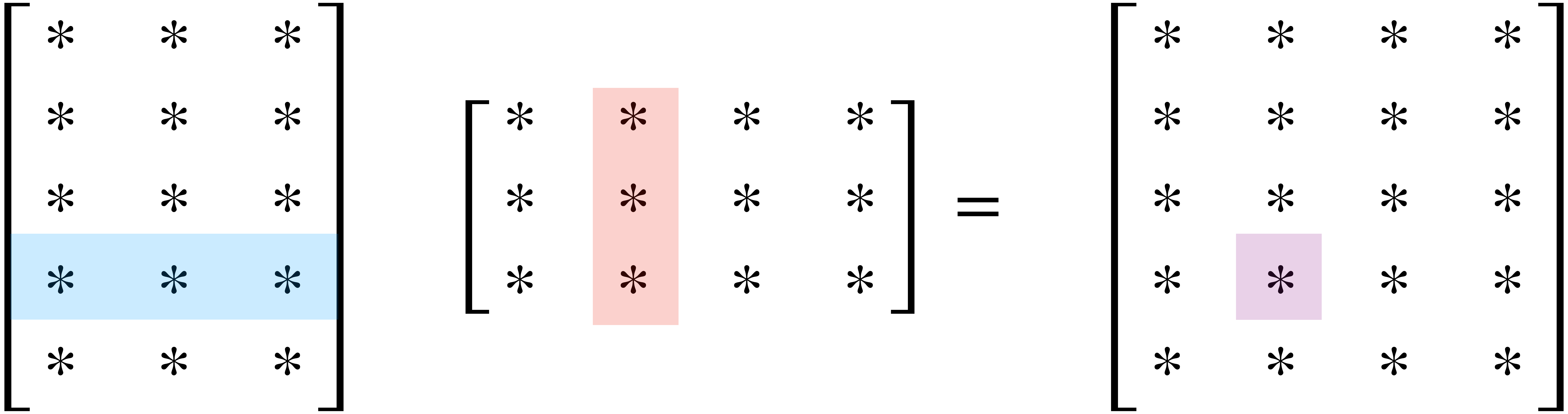

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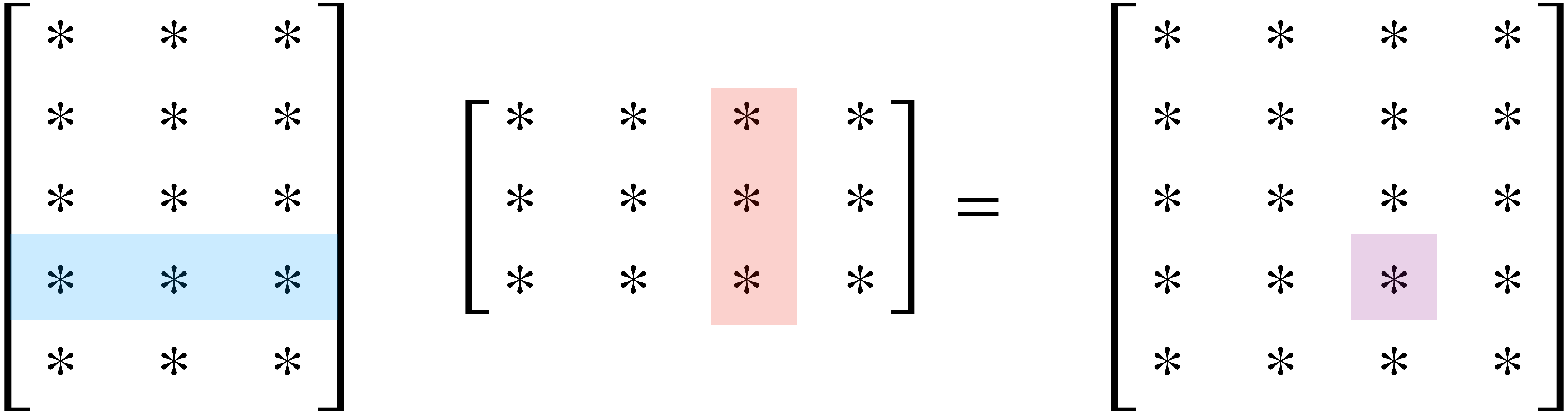

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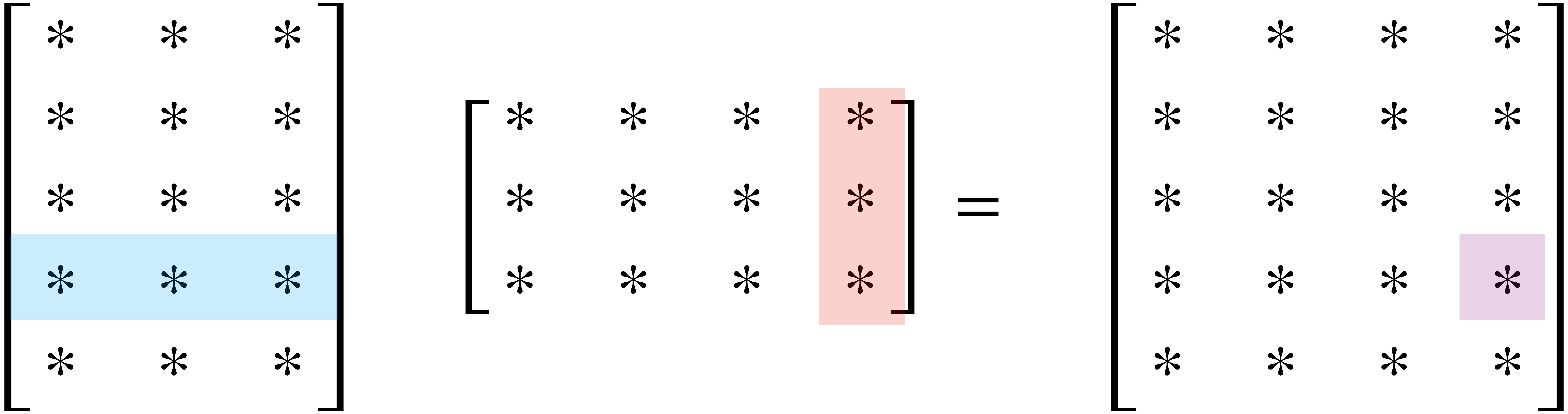

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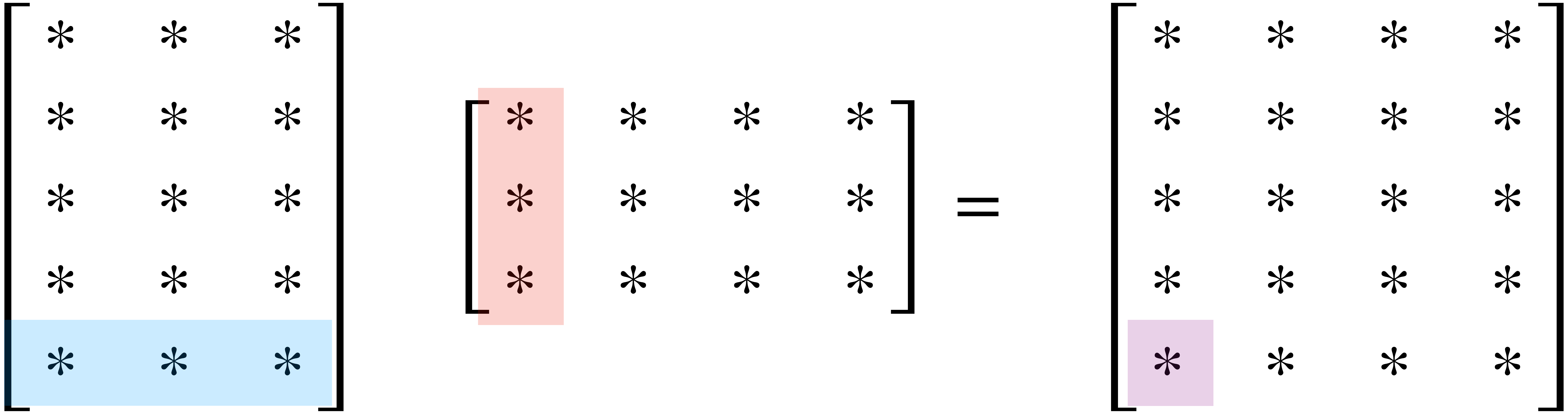

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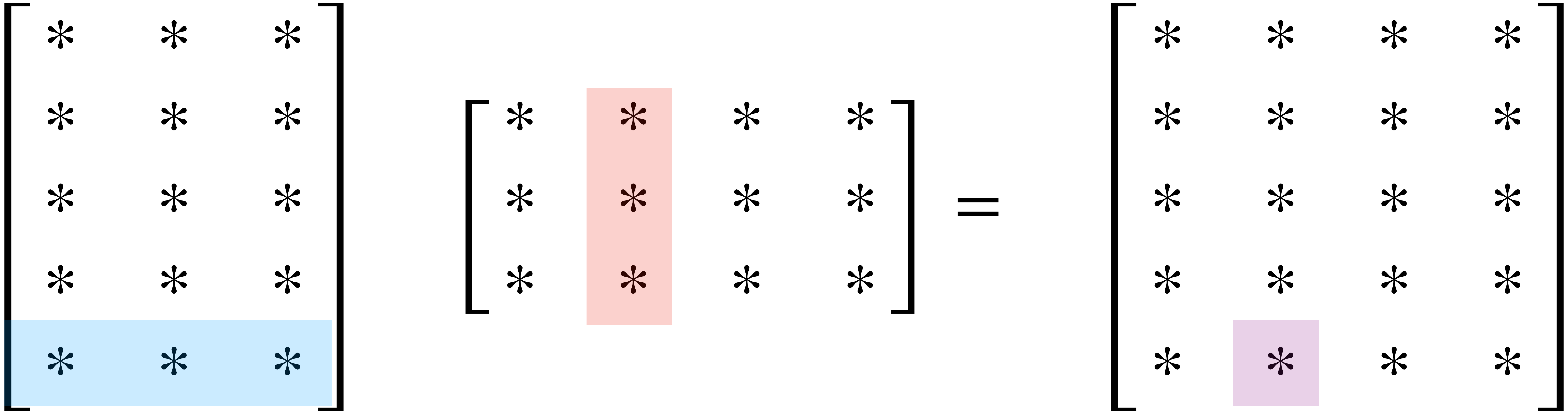

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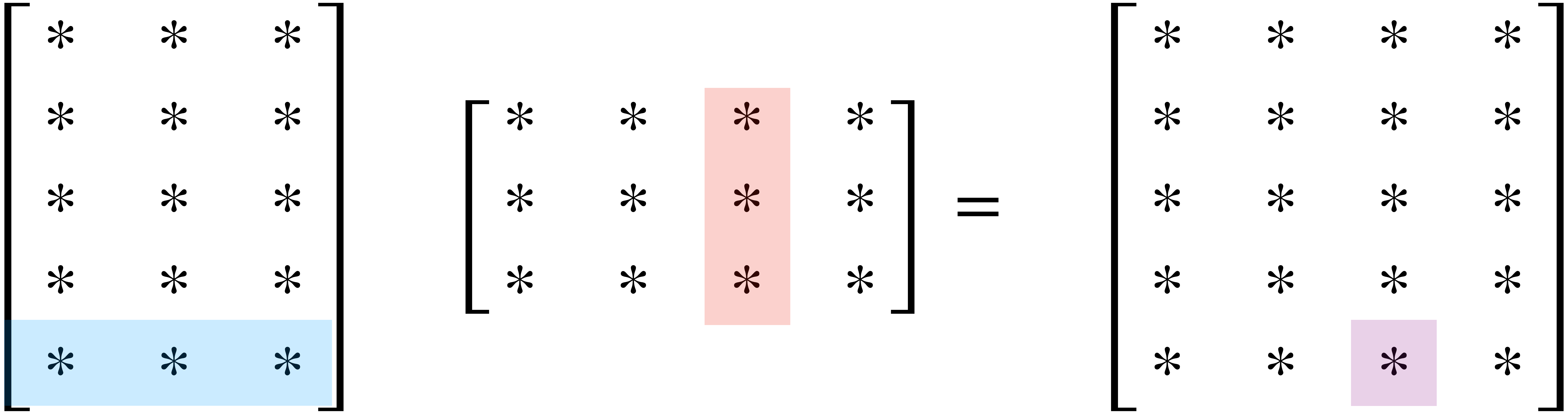

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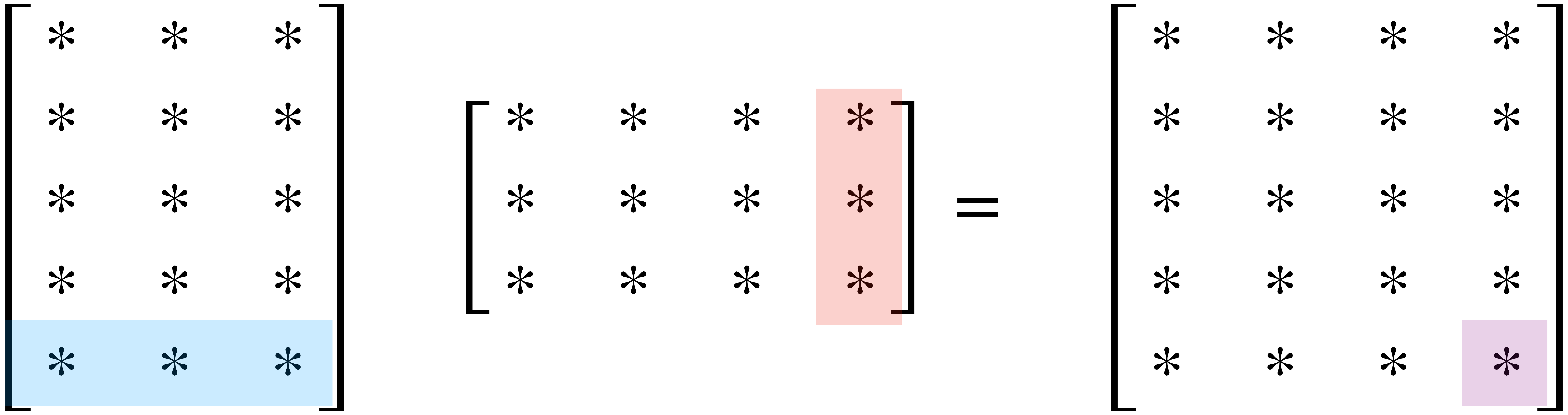

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Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$2 \times \textcircled{2}$ √ $\textcircled{2} \times 2$

2 2

$$\begin{bmatrix} (-1)(1) + 0(0) & (-1)(1) + 0(1) \\ 0(1) + 1(0) & 0(1) + 1(1) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

`What about when the right matrix is a single column?`

Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

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This is just vector multiplication.

Connection with Matrix-Vector Multiplication

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This is just vector multiplication.

We can think of $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix}$ as collection of simultaneous matrix–vector multiplications

Matrix "Interface"

multiplication	what does AB mean when A and B are matrices?
addition	what does $A + B$ mean when A and B are matrices?
scaling	what does cA mean when A is matrix and c is a real number?

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multiplication	what does AB mean when A and B are matrices?
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These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

$m \times n$ $m \times n$

Addition is done column-wise (or equivalently, element-wise)

e.g. $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

e.g. $2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$

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This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the same size and r and s are scalars ($\mathbb{R}$)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = \underset{BA}{\cancel{BC}} + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_mA = A = AI_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Example

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\neq$$

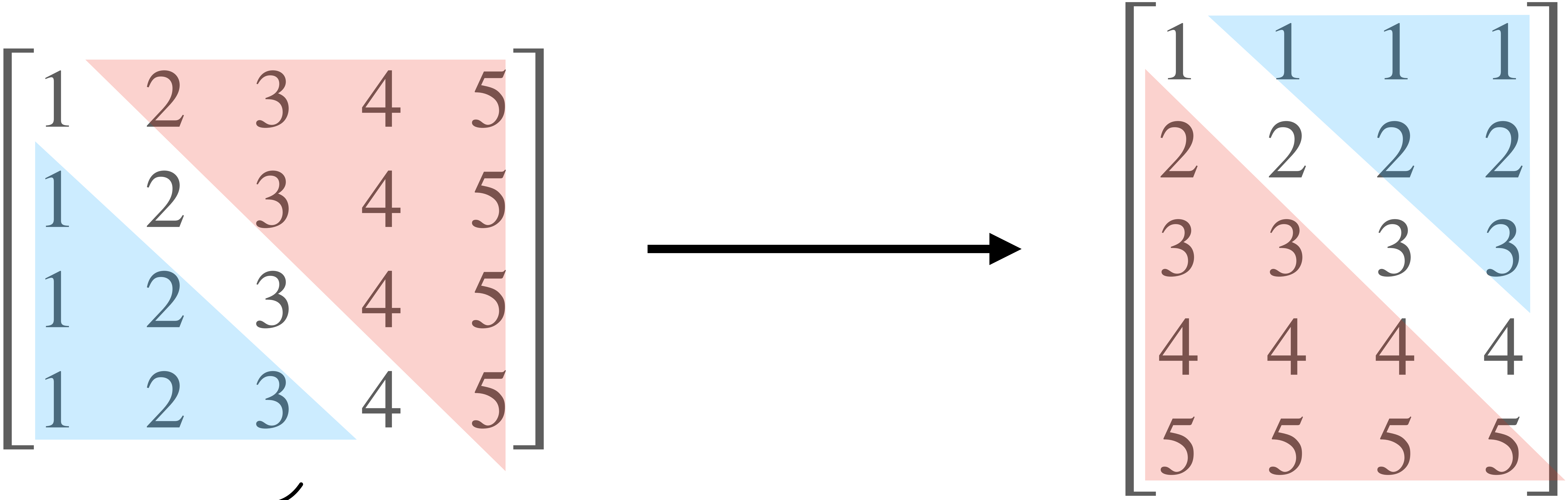
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

More Matrix Operations

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)



$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

4 × 5

5 × 4

Transpose

python:

a_t[i][j] = a[j][i

Definition. For a $m \times n$ matrix A, the **transpose of** A, written A^T, is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ `(where` c `is a scalar)`

$(AB)^T = B^T A^T$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$ Important: the order reverses!

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^TA^T$.

Example: $\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right)^T$

$$\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}^T = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$$

$$\left(\sum_{k=1}^{n} A_{jk}B_{ki}\right)$$

$$\|$$

$$(AB)^T_{ij} = (AB)_{ji}$$

$$= \ldots \leftarrow \text{hard part}$$

$$= (B^TA^T)_{ij}$$

Transposes and Inner Products

Transposes and Inner Products

For a vector $\mathbf{v} \in \overset{\mathbb{R}^{n \times 1}}{\overset{\parallel}{\mathbb{R}^n}}$, what is $\mathbf{v}^T$? $\in \mathbb{R}^{1 \times n}$

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
is $\mathbf{u}^T\mathbf{v}$ defined?

$1 \times n \quad n \times 1$ $\qquad$ 1×1

✓

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
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$$\begin{array}{ccccc} 1 \times n & & n \times 1 & & 1 \times 1 \\ \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} & & \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} & = & ? \end{array}$$

Transposes and Inner Products

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For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
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$$1 \times n \qquad n \times 1 \qquad 1 \times 1$$

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

Transposes and Inner Products

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \boxed{\mathbf{u}^T\mathbf{v}} \longleftarrow \text{matrix mult.}$$

Matrix Powers

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Definition. For a $n \times n$ matrix A, we write A^k for the k–fold product of A with itself.

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(we want $A^0 A^k = A^{0+k} = A^k$)

$I'' A^k = A^k$

Matrix Powers (Computationally)

We can use **numpy.linalg.matrix_power**

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16} = \underbrace{(A^{8})}_{\underbrace{A^{4}}_{\underbrace{A^{2}}_{AA}A^{2}}A^{4}} A^{8}$$

Why?:

Final Warnings about Matrix Multiplication

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$\cancel{2}x = \cancel{2}y \Rightarrow x = y$

Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

$yx = 0 \Rightarrow x = 0 \quad y = 0$

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. *Choose A and B such that they have all nonzero entries.*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So Far: Matrix Operations

So Far: Matrix Operations

`transpose` A^T

So Far: Matrix Operations

`transpose`	A^T
`scaling`	cA

So Far: Matrix Operations

transpose	A^T	
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addition (subtraction)	$A + B$	$A + (-1)B = A - B$
multiplication (powers)	AB	A^k

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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These may be different sizes

Recall: The Identity Matrix

$$\underset{2\times 2}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}\underset{2\times 4}{\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}}=\underset{2\times 4}{\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}}\underset{4\times 4}{\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}}=\underset{2\times 4}{\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}}$$

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \texttt{otherwise} \end{cases}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

Basic Algebra

$$2x = 10$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

`How do we solve this equation?`

`Divide on both sides by` 2 `to get` $x = 5$`.`

`Multiply each side by` $\frac{1}{2}$ *`a.k.a.`* 2^{-1}`.`

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Basic Algebra

$$1x = 5$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

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Basic Algebra

$$x = 5$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

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Wouldn't it be nice...

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Do all matrices have inverses?

Do all matrices have inverses?

No. If they did, then every linear system would have a solution.

When does a matrix have an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

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Why are square matrices special?

They are the only kind of matrices...

» that can have a pivot in every row and every column.

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» whose columns can have full span and be linearly independent.

» that can have inverses.

Matrix Inverses

Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Geometric

Reflection across the x_1–axis in $\mathbb{R}^2$ is it's own inverse.

Verify: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example: No inverse

$$A \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = I$$

`Verify:`

$$\cancel{A\vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \textcircled{1} \end{bmatrix} = \begin{bmatrix} & \cdots & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

$AB = I \quad AC = I$

$AB - AC = I - I = 0$

$A(B - C) = 0$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of $\mathbf{b}$.

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» exactly one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a <u>unique</u> solution for any choice of $\mathbf{b}$, then it has

» <u>at least one</u> solution for any choice of $\mathbf{b}$

» <u>at most one</u> solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

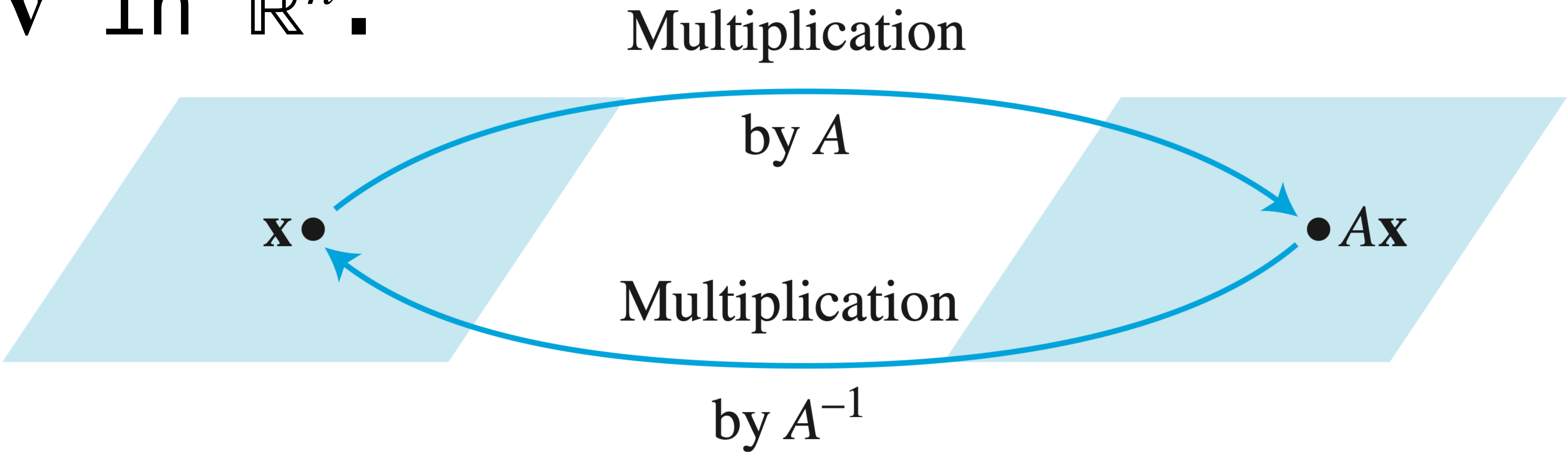
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Non-Example. Projection onto the x_1-axis.

Connection to Transformations

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Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

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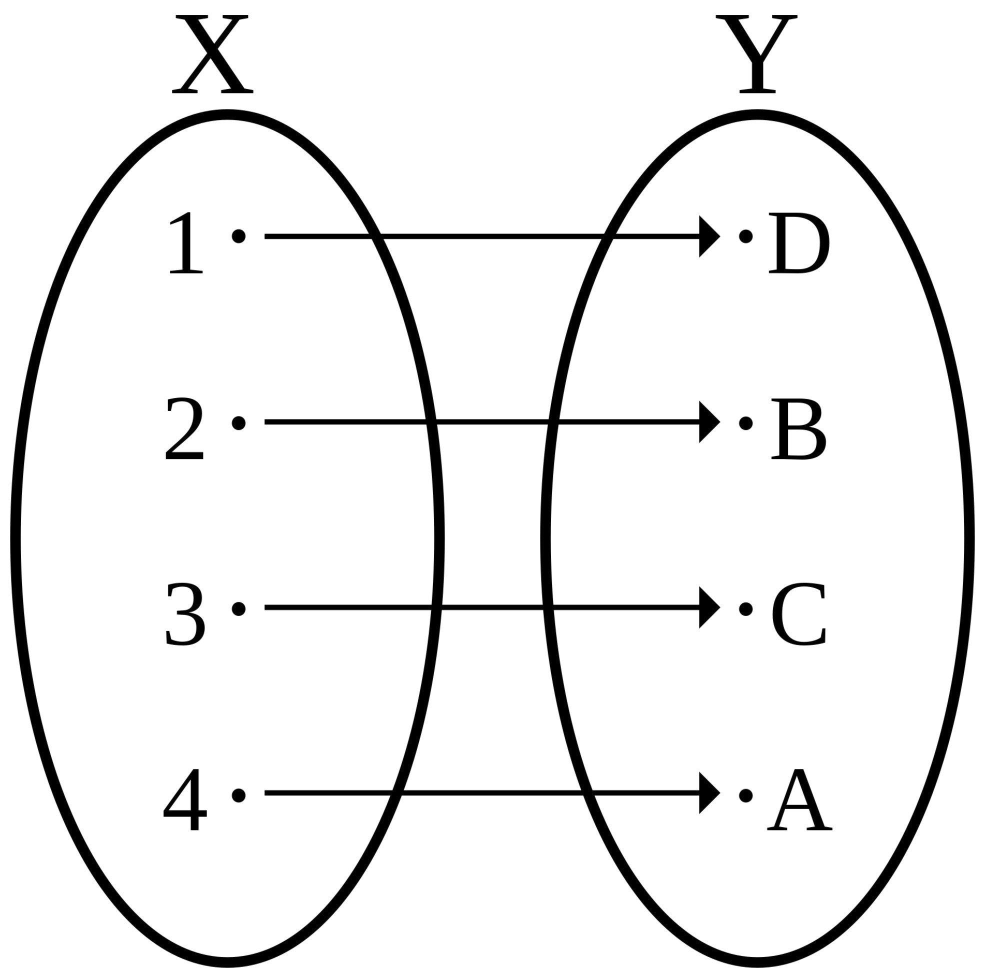
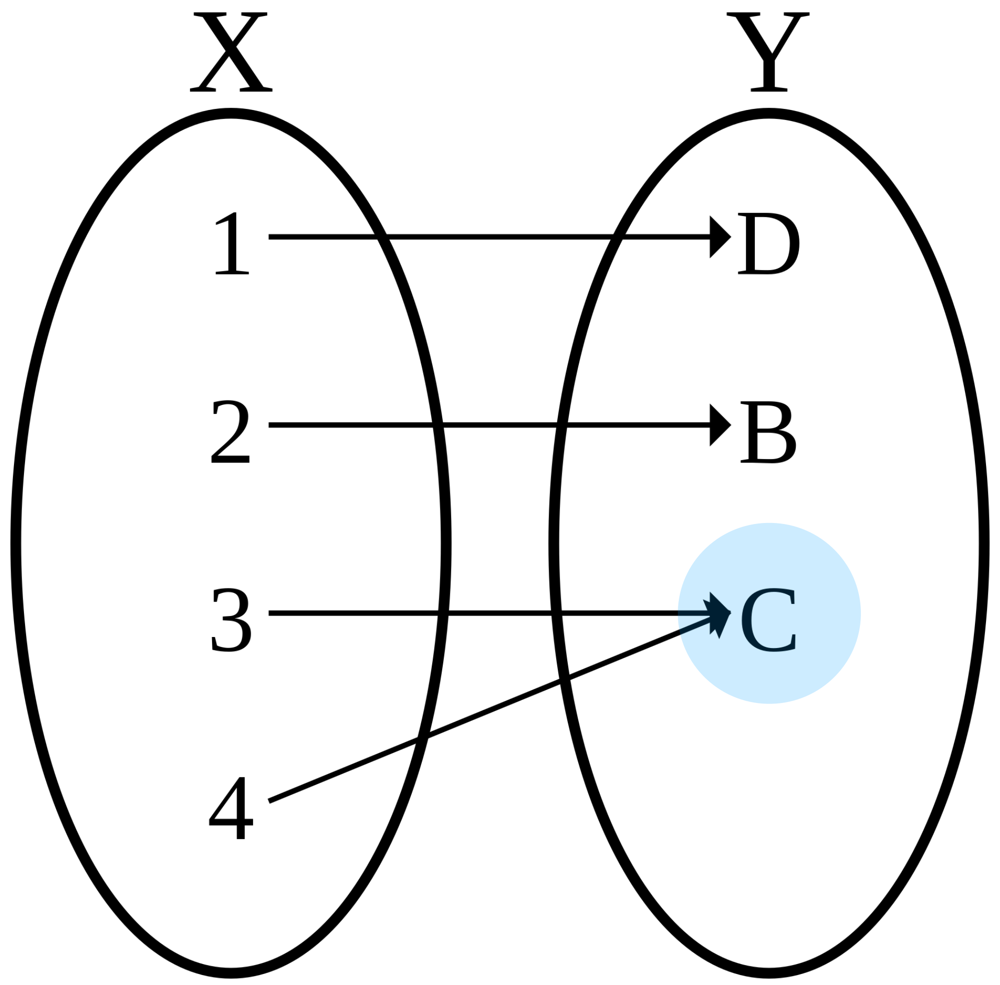
Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



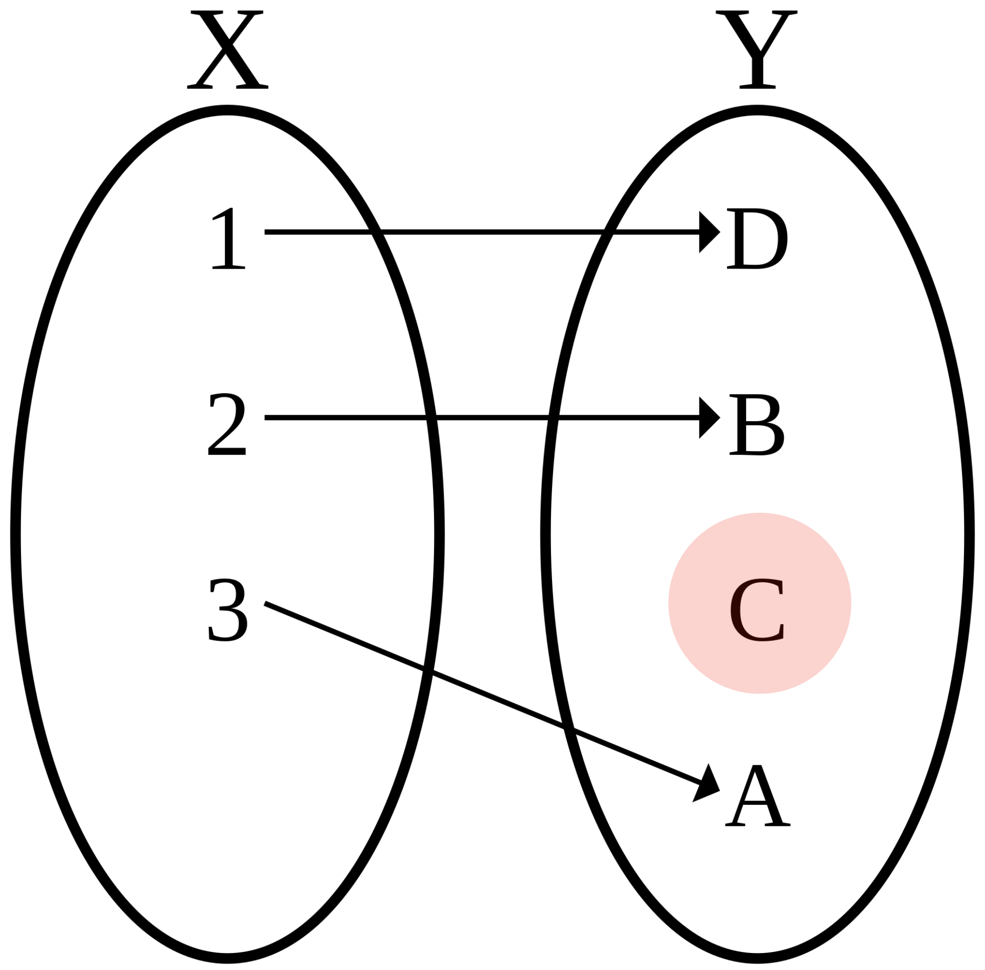
1–1 correspondence

collision



onto, not 1–1

not covered



1–1 not onto

not covered
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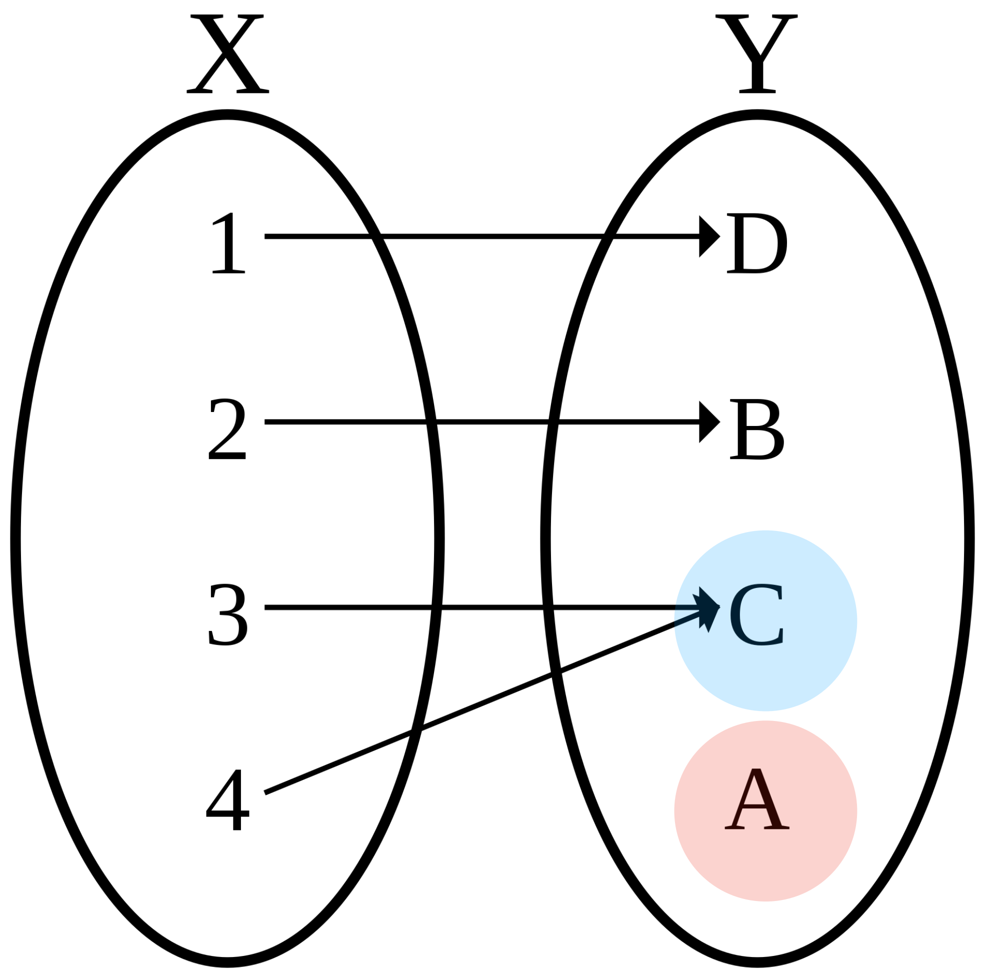

not 1–1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Answer 1: Try to compute it.

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If a matrix has an inverse how do we compute it?

Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

`Can we solve for each` $\mathbf{b}_i$? `:`

In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

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$$A\mathbf{b}_1 = \mathbf{e}_1 \qquad A\mathbf{b}_2 = \mathbf{e}_2 \qquad A\mathbf{b}_3 = \mathbf{e}_3$$

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \;\; I]$ to a matrix $[I \;\; B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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(see the notes on linear transformations for more information about determinants)

Example

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C, demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix.*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen.*

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every $\mathbf{b}$

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Verify:

Recall: Onto Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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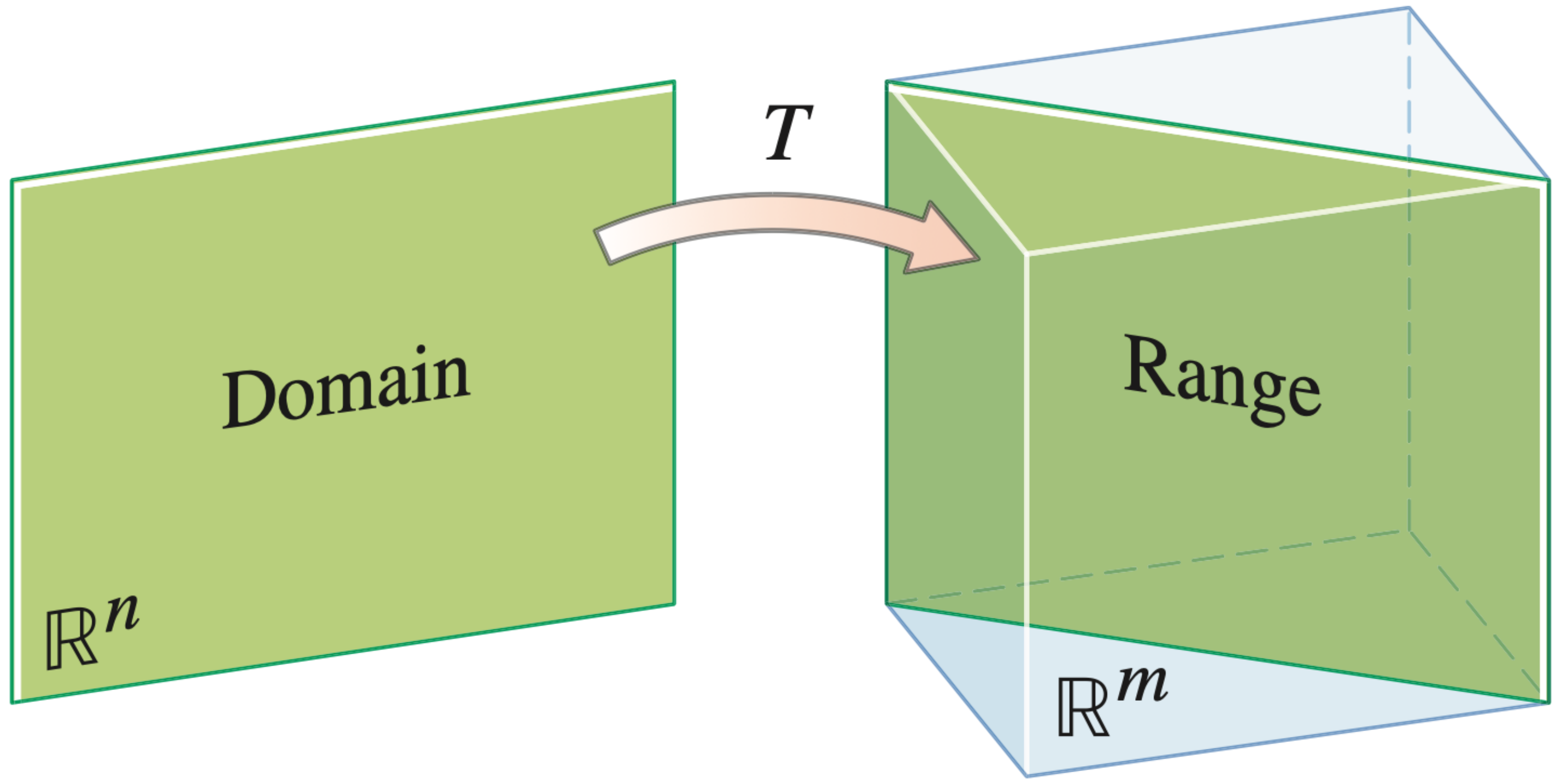

T is *not* onto $\mathbb{R}^m$

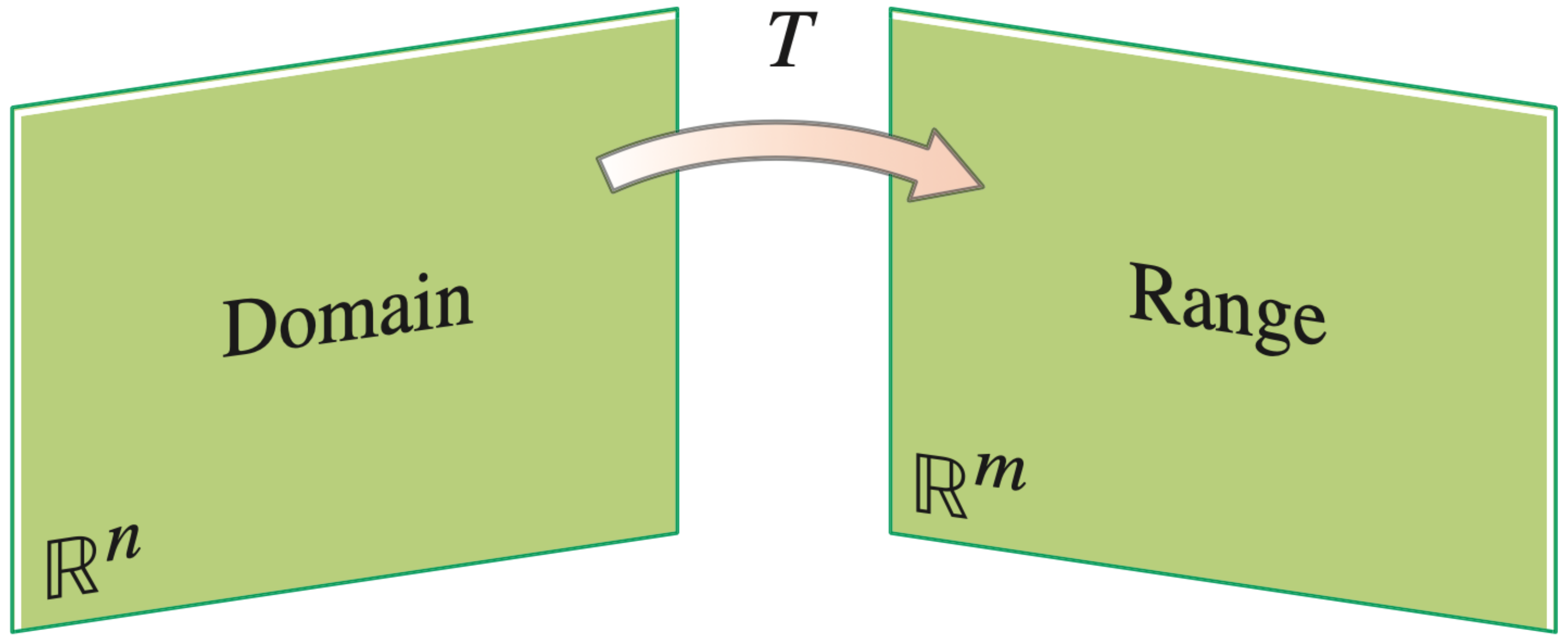

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image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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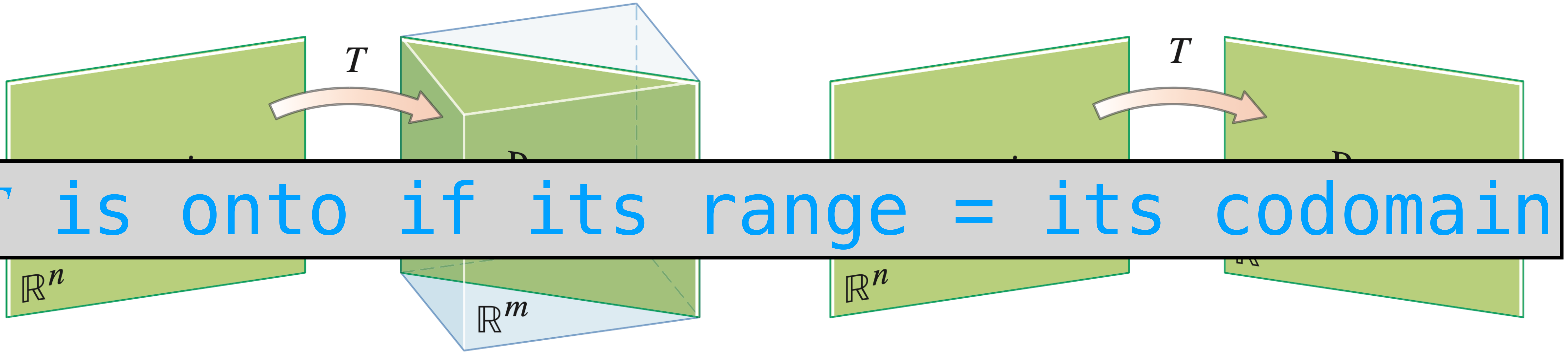

T is onto if its range = its codomain

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Recall: One-to-one Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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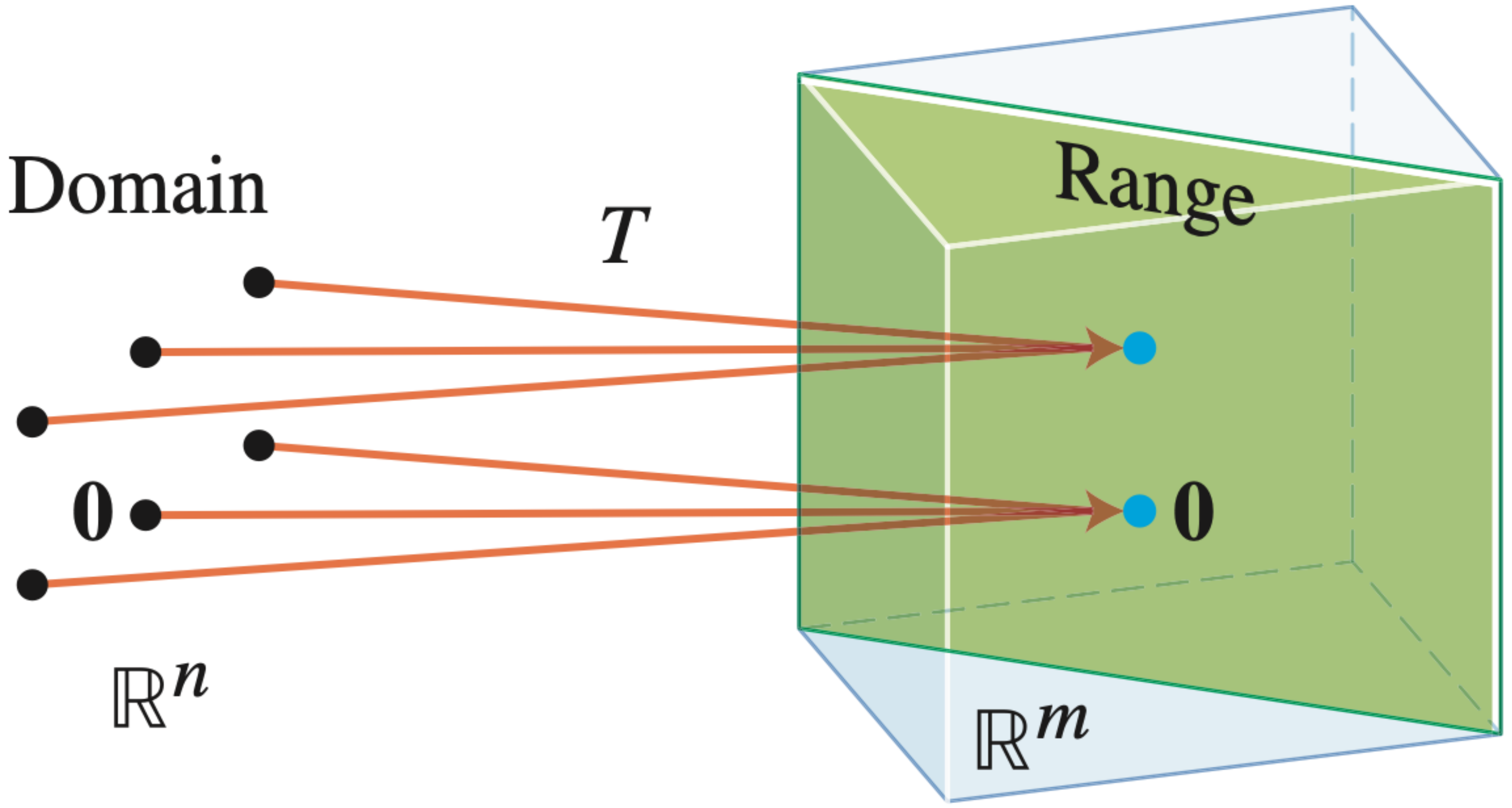

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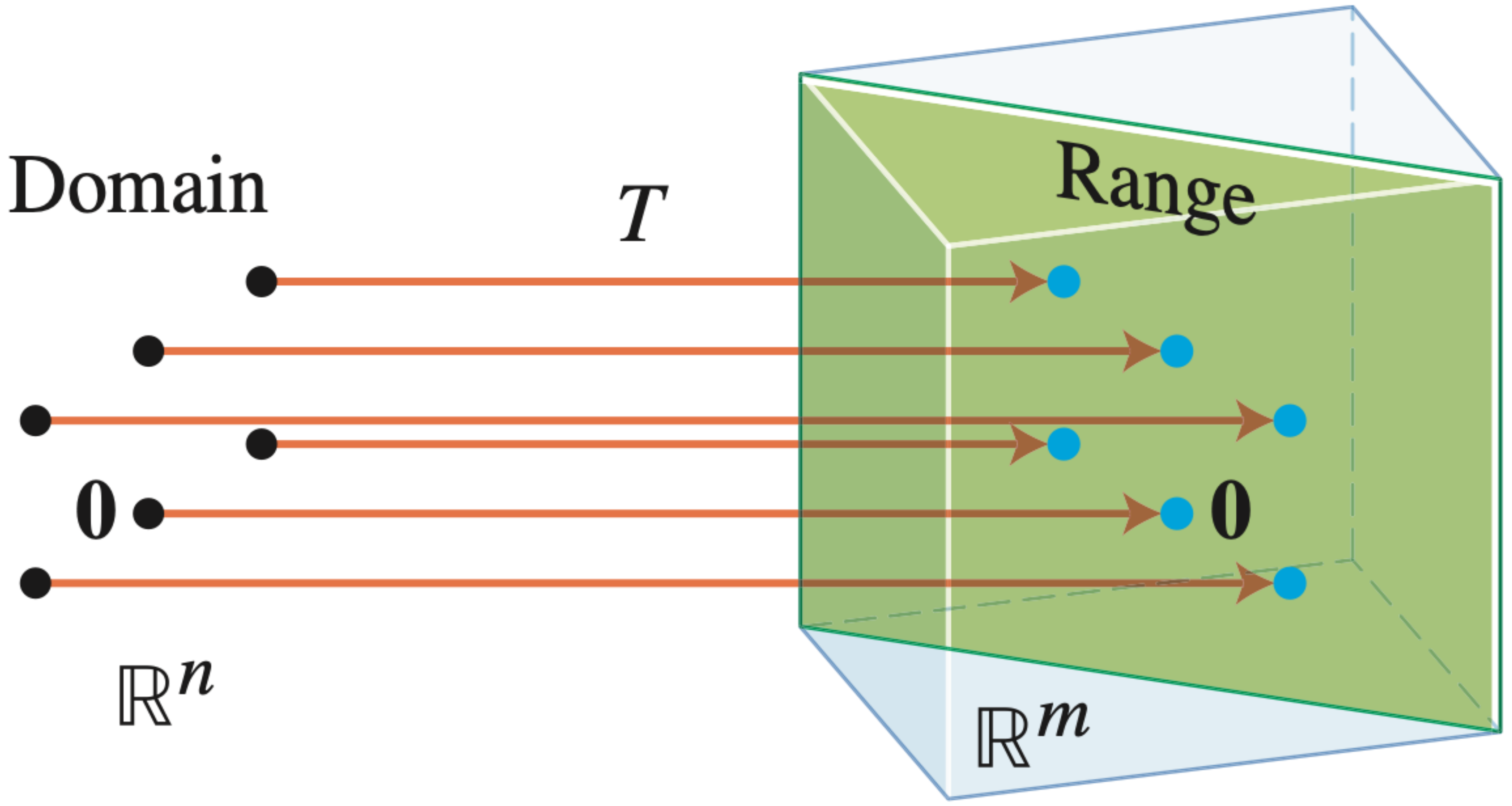

T is one-to-one

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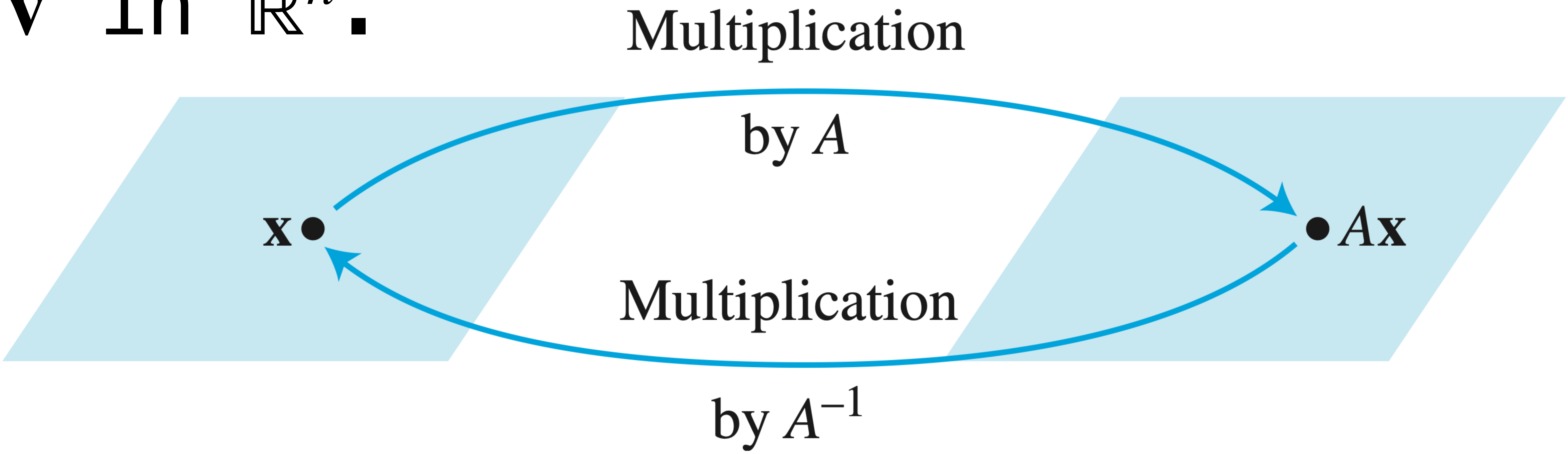
Recall: Invertible Transformations

Definition. A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
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These all express the **same thing**

(this is a stronger statement than we just verified)

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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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Warning. Remember this only applies square matrices.

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Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True *or* ***False:*** *If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), the B is also invertible.*

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3]$ *is invertible, then is* $\left[(\mathbf{a}_1+\mathbf{a}_2-2\mathbf{a}_3) \;\; (\mathbf{a}_2+5\mathbf{a}_3) \;\; \mathbf{a}_3\right]$ *also invertible? Justify your answer.*

Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by <u>row operations</u>

Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.