

Matrix Inverses

Geometric Algorithms

Lecture 11

Practice Problem(s)

1. Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

2. Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Connect everything(!)

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1-1 Correspondence

`numpy.linalg.inv`

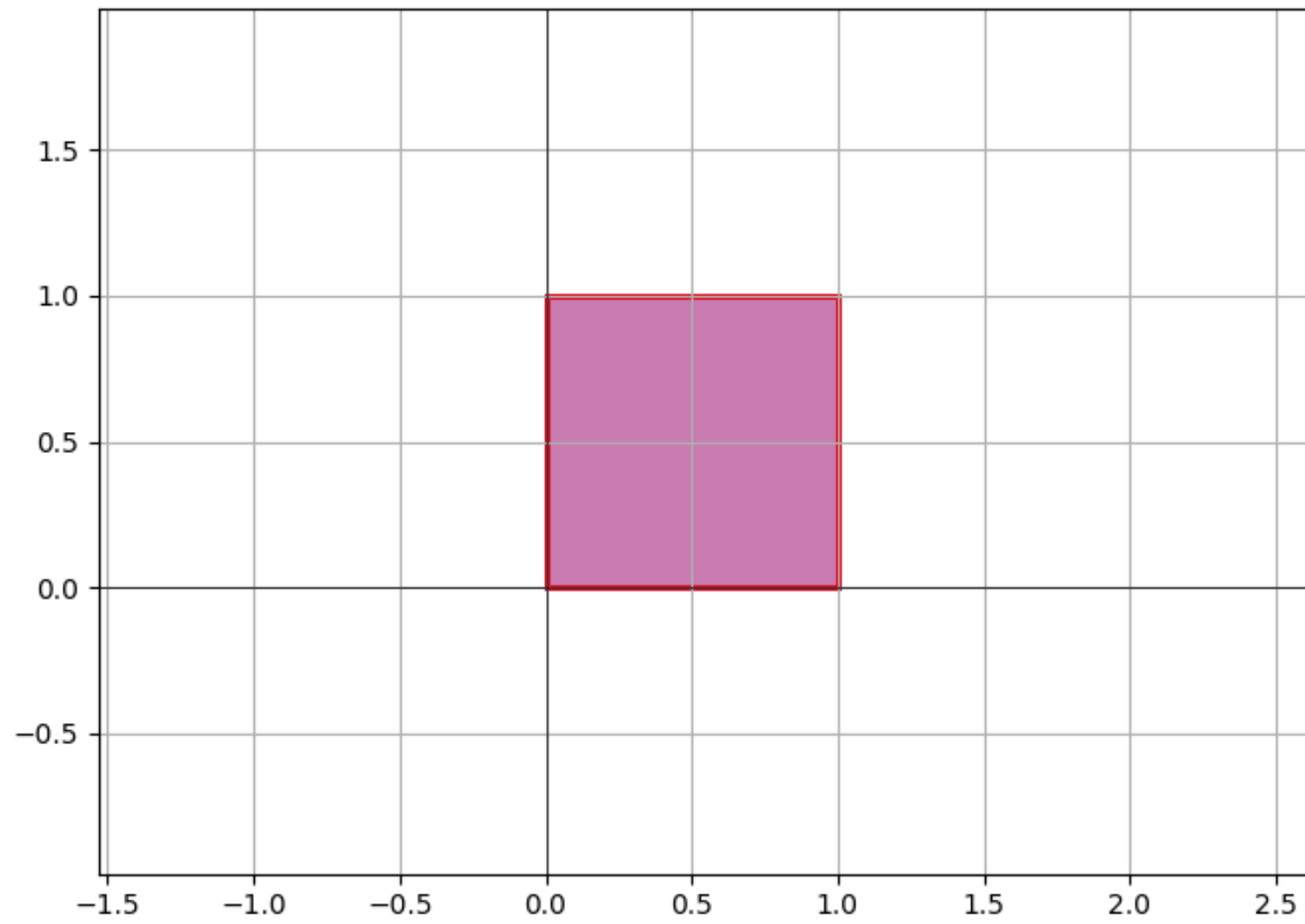
determinant

Invertible Matrix Theorem

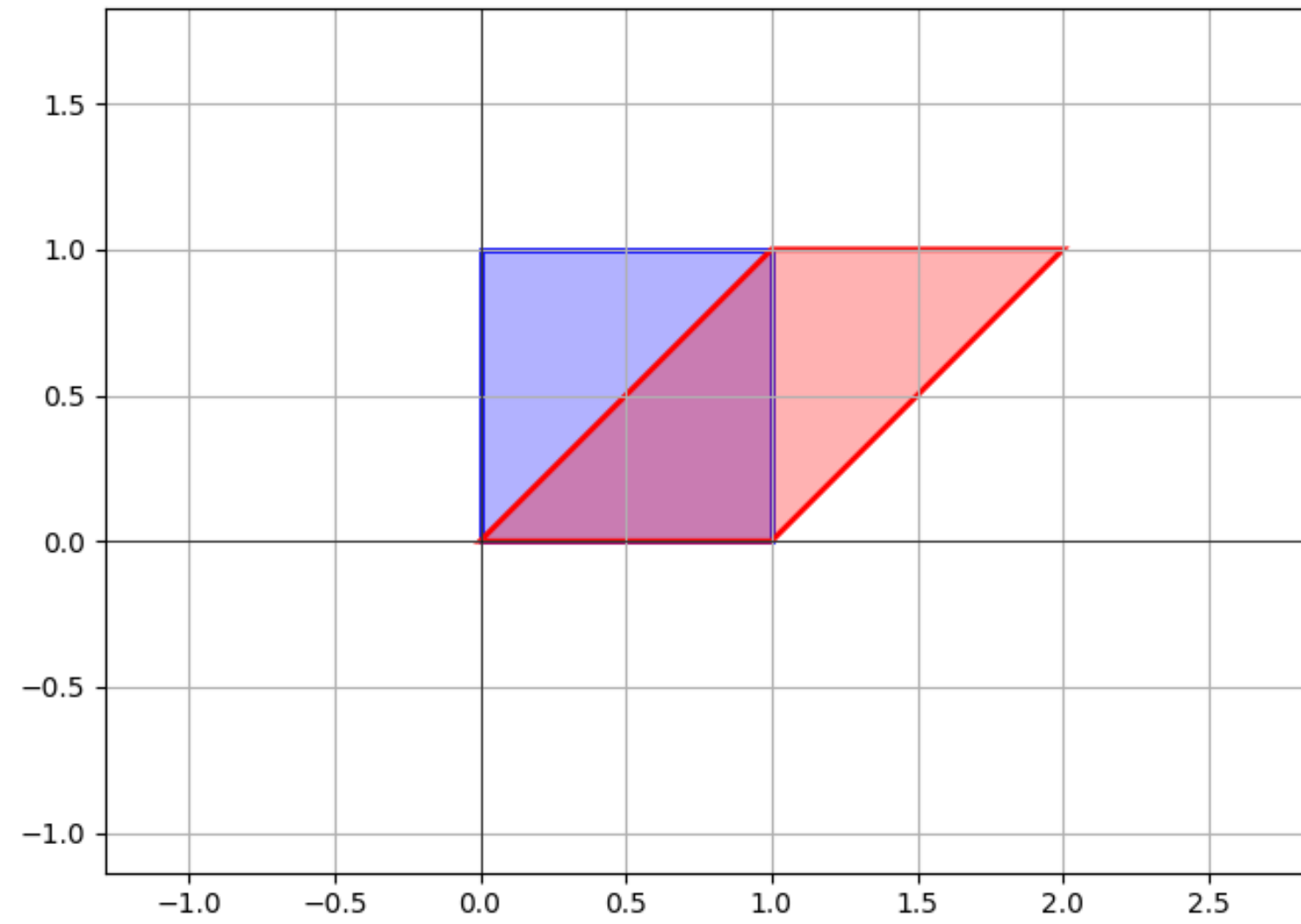
Recap: Matrix Multiplication

Recall: Composition

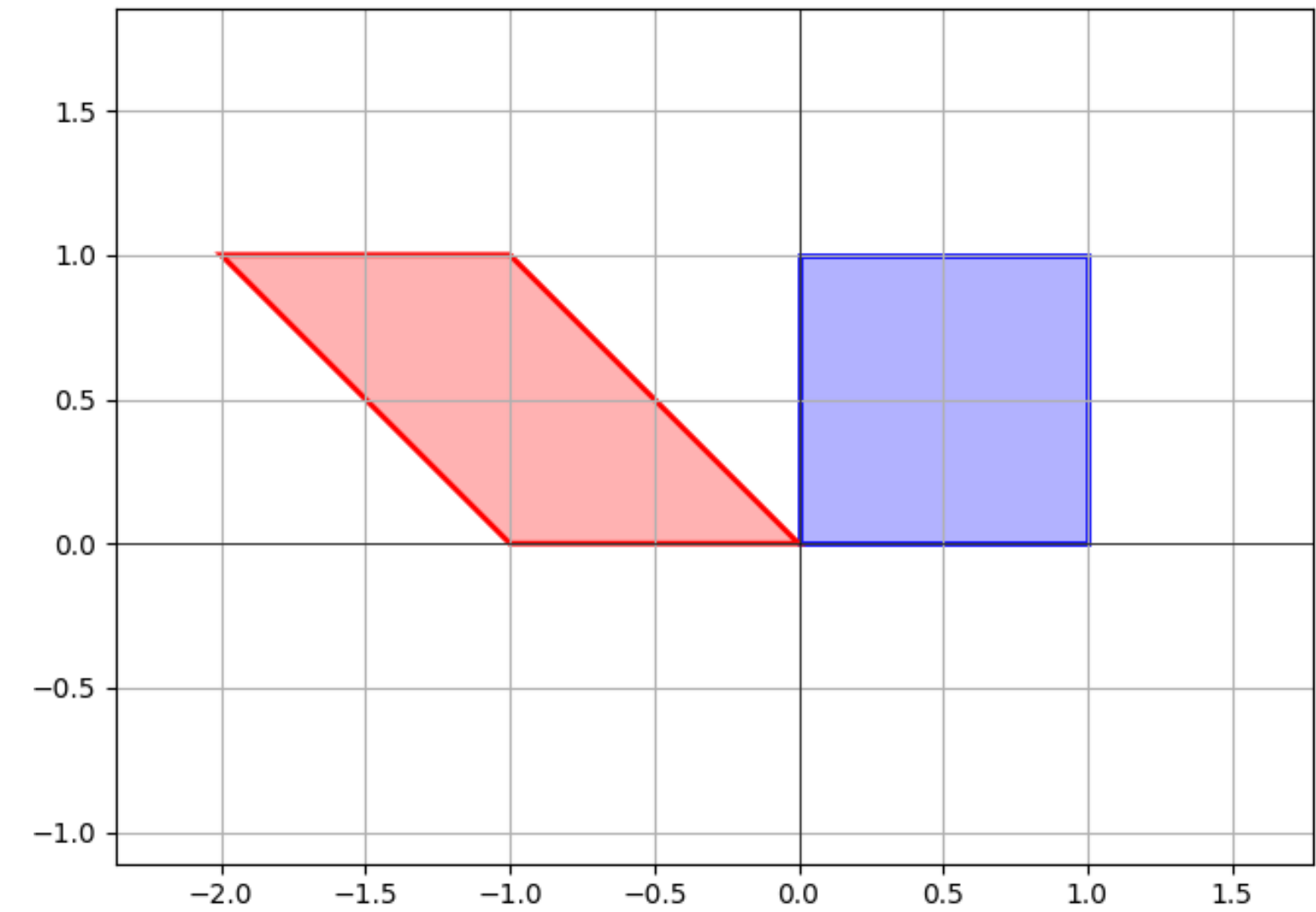
2D Matrix Transformations



2D Matrix Transformations



2D Matrix Transformations



shear



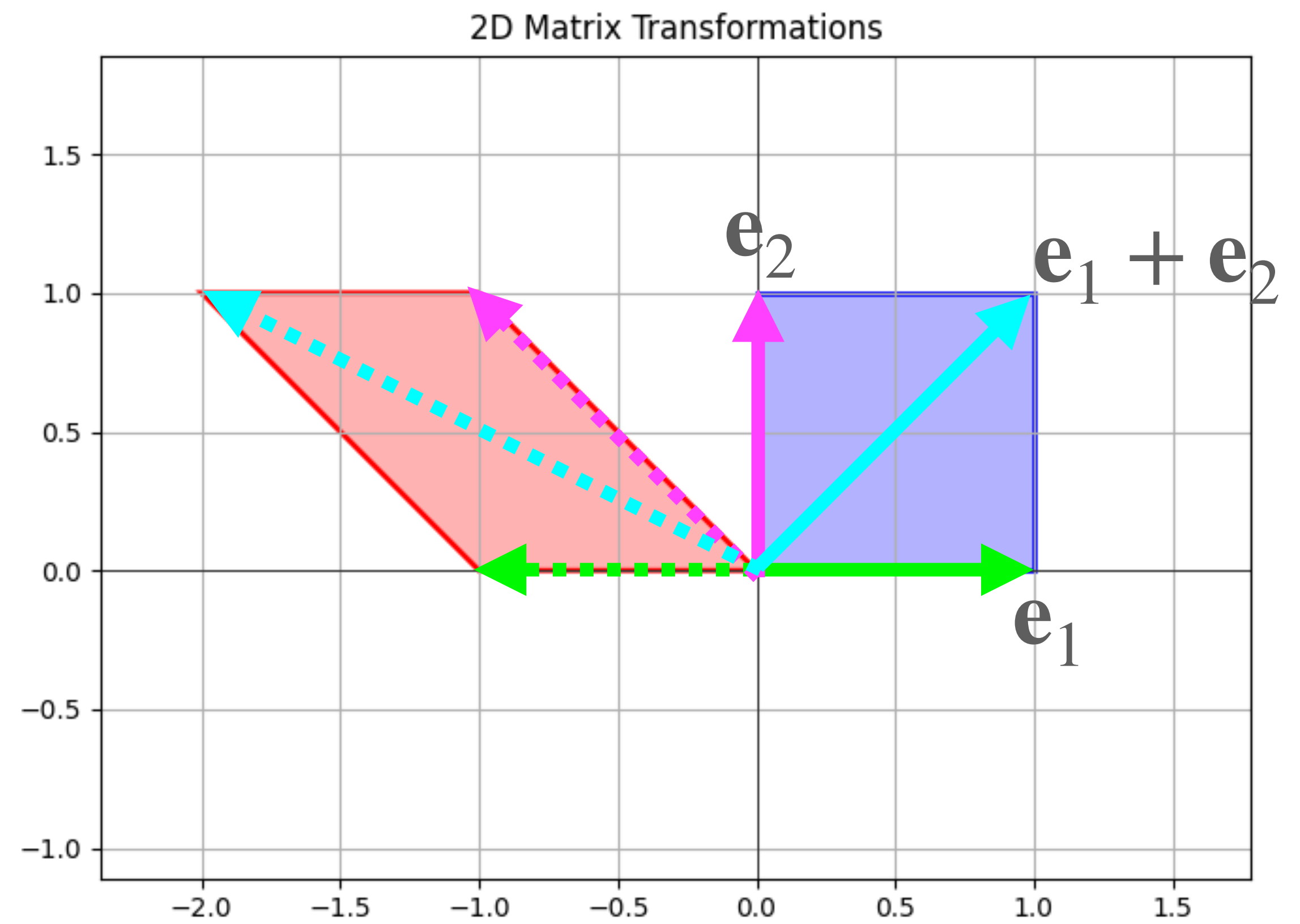
reflect

Recall: Composition

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates the dimensions of matrix multiplication. It shows three matrices:

- A left matrix with 5 rows and 3 columns, labeled m (rows) and n (columns). Its dimensions are $(m \times n)$.
- A middle matrix with 3 rows and 4 columns, labeled n (rows) and k (columns). Its dimensions are $(n \times k)$.
- An equals sign followed by a right matrix with 5 rows and 4 columns, labeled m (rows) and k (columns). Its dimensions are $(m \times k)$.

The diagram uses colored lines and labels to track the dimensions: a blue vertical line for m , a red horizontal line for n , and a purple horizontal line for k .

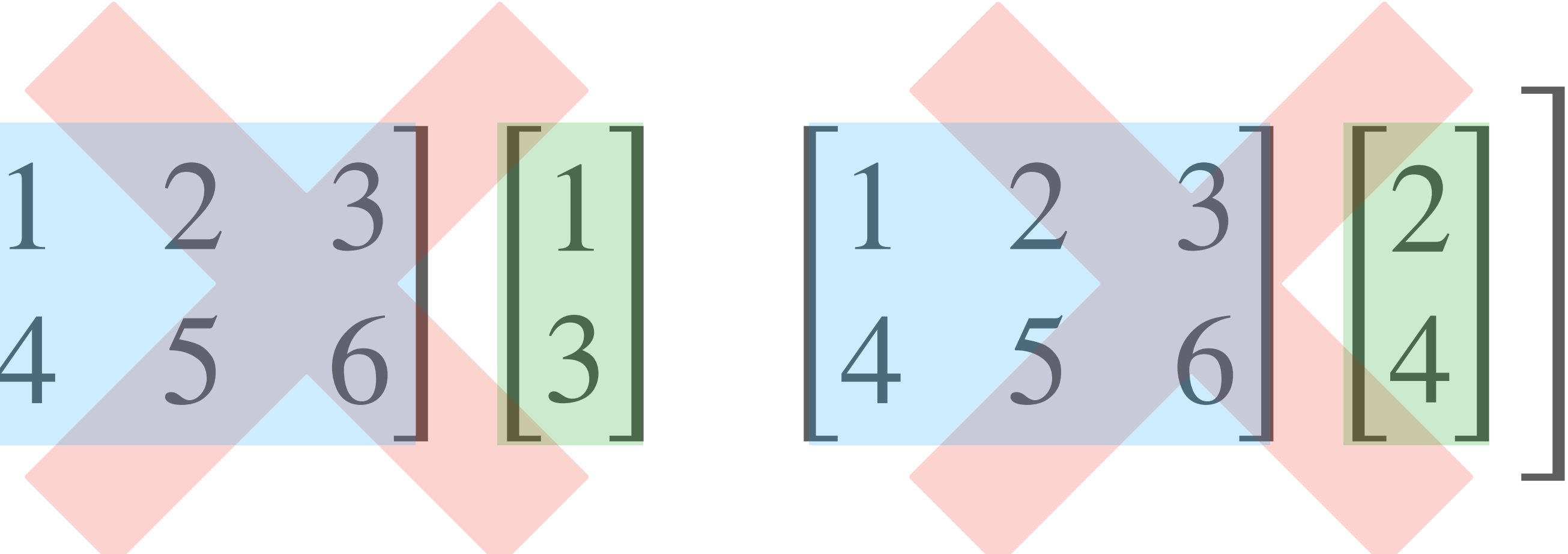
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{array} \right]$$


These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Example

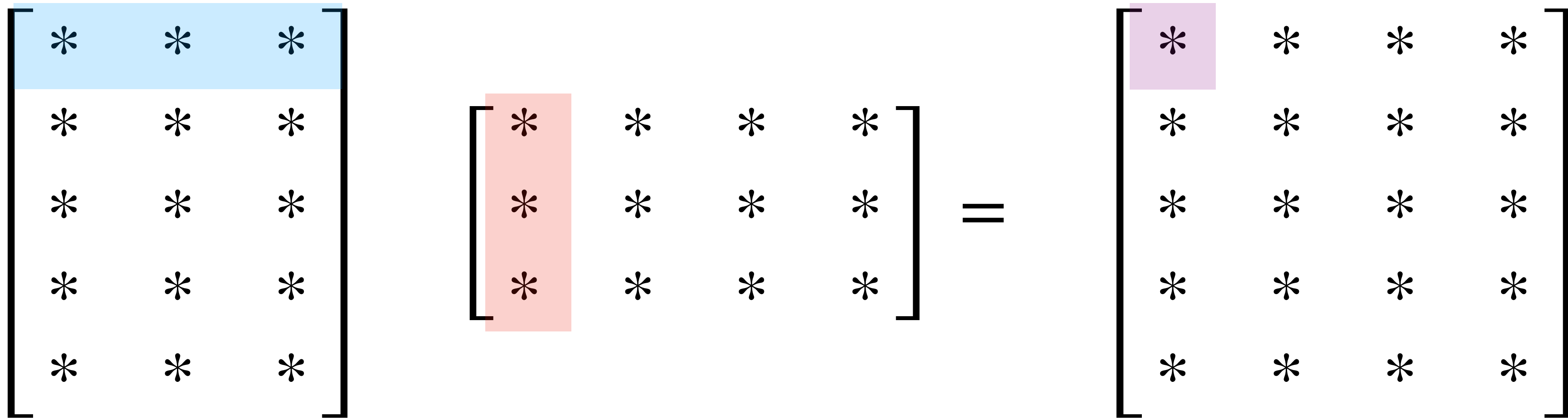
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

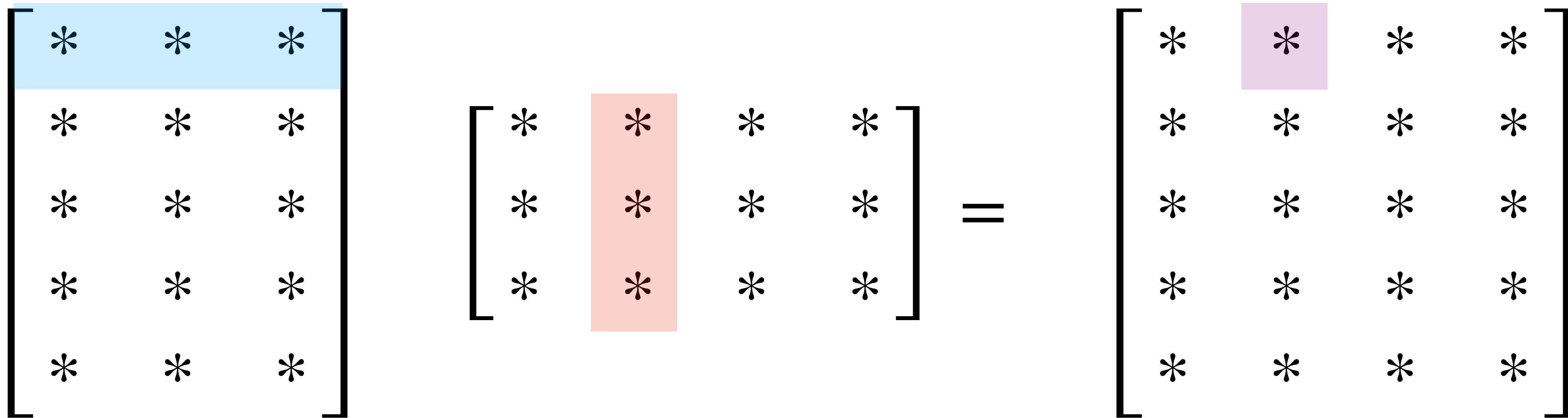
Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above.

Row-Column Rule (Pictorially)



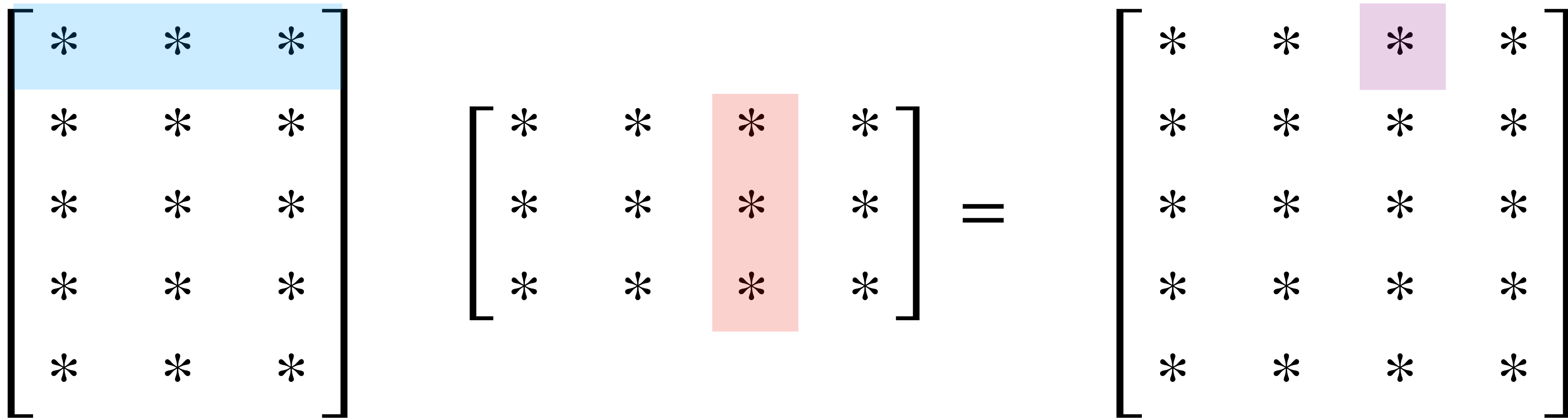
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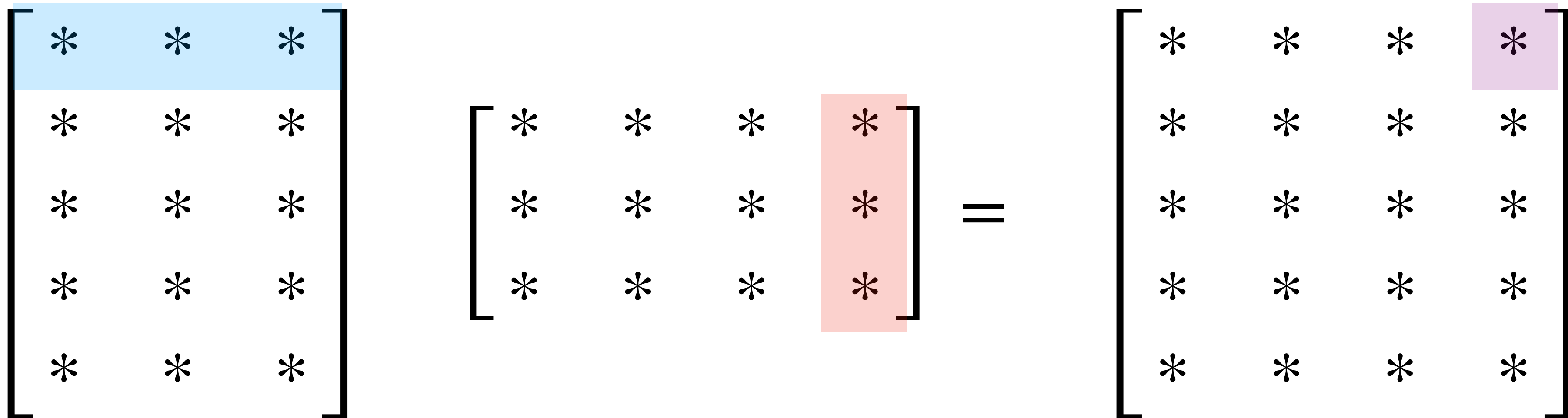
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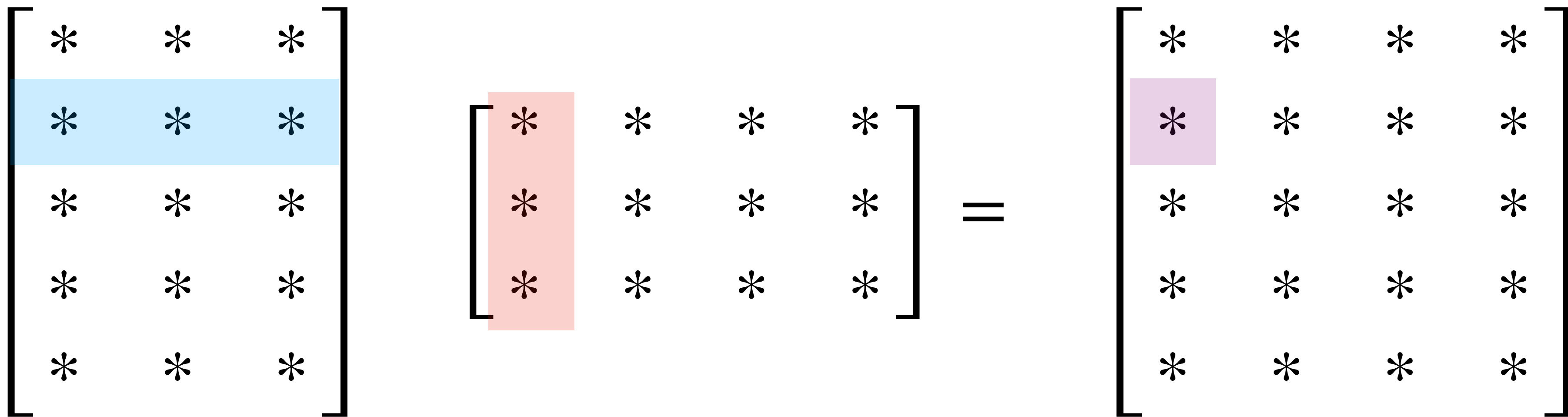
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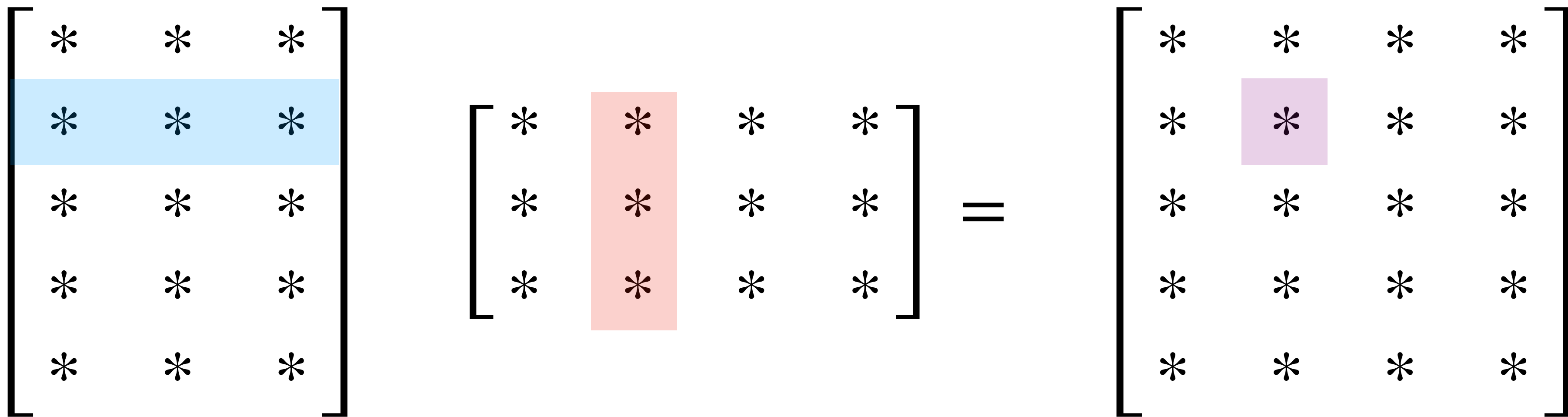
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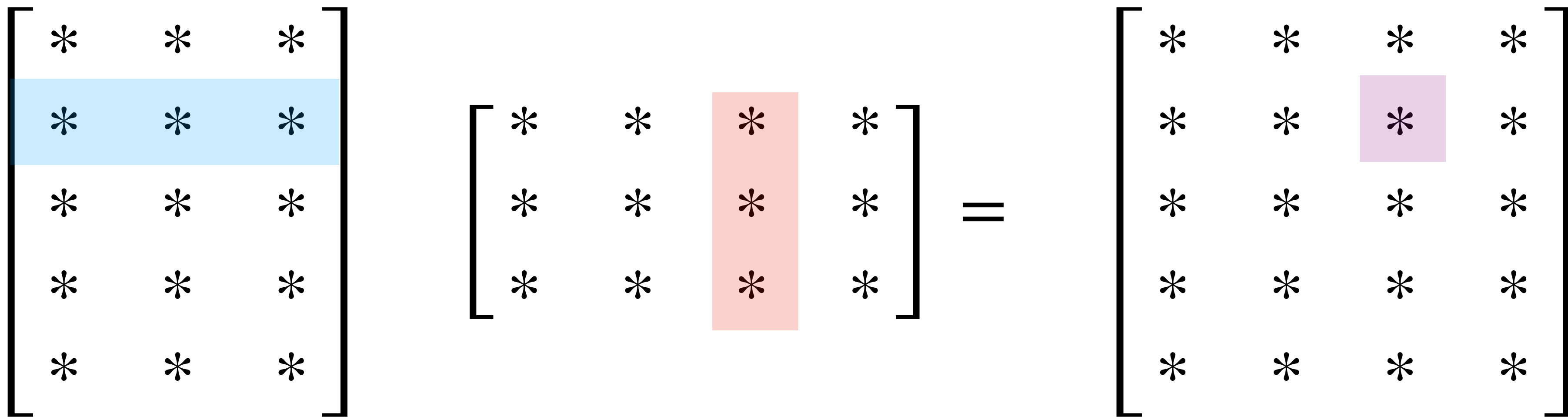
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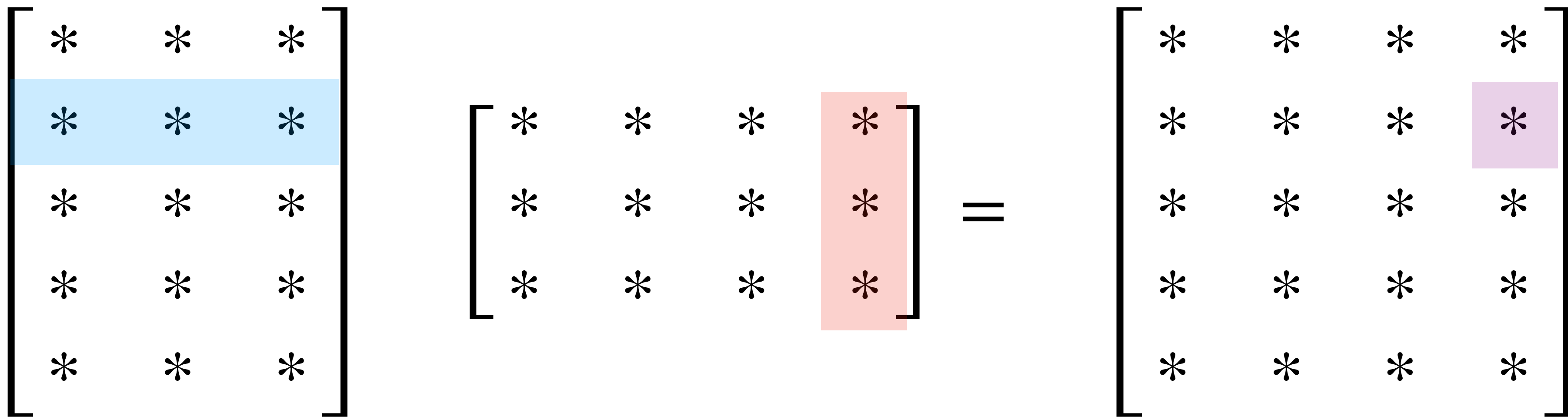
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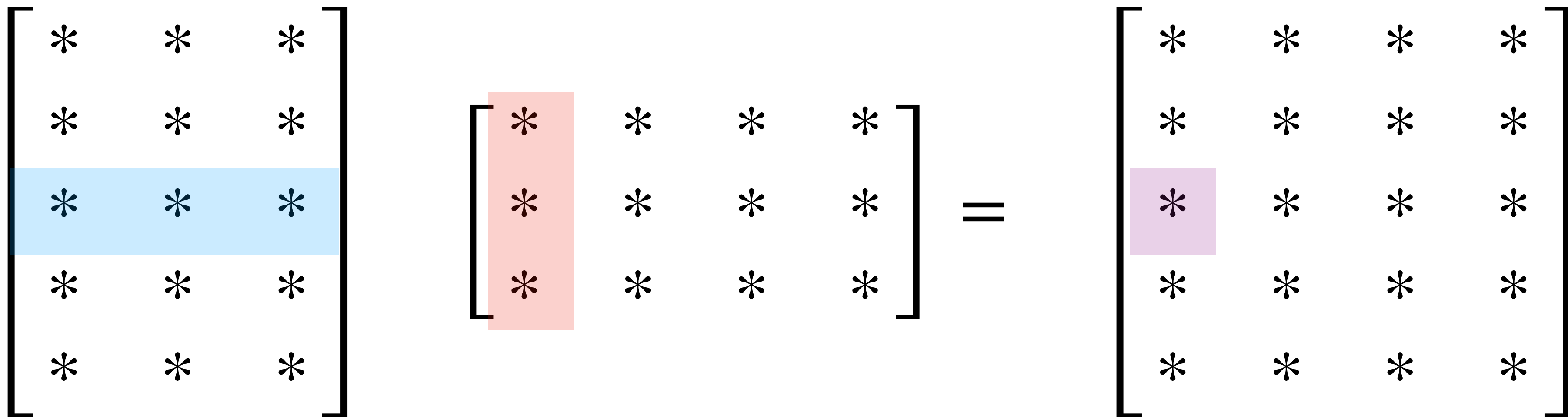
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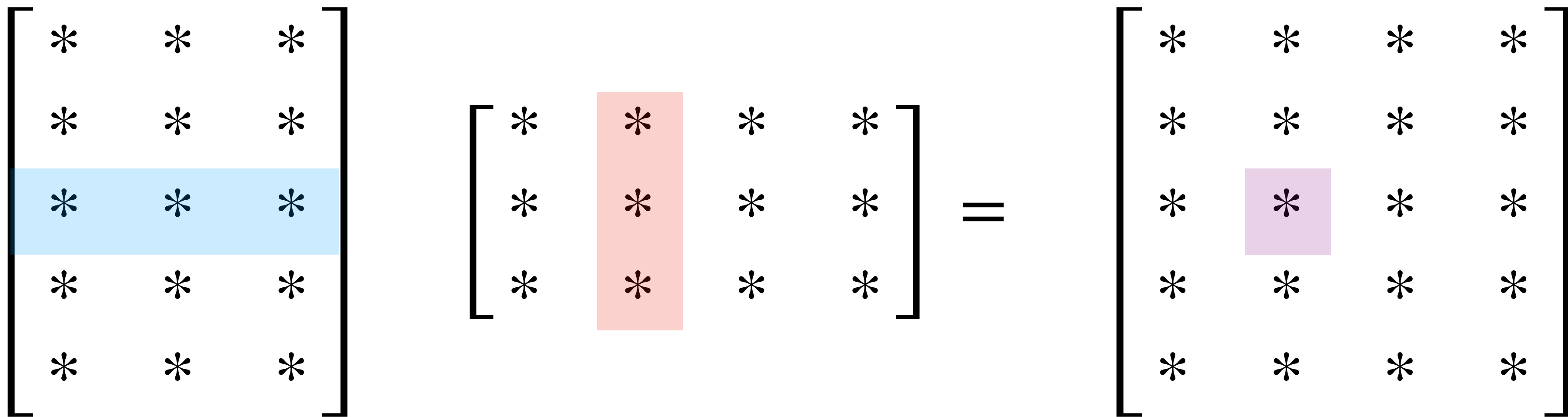
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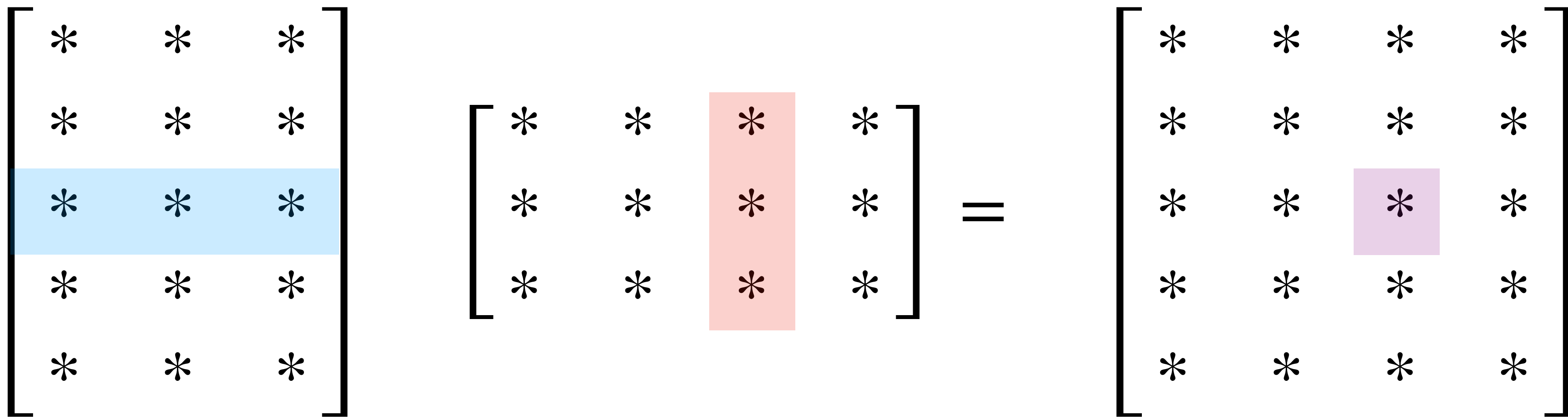
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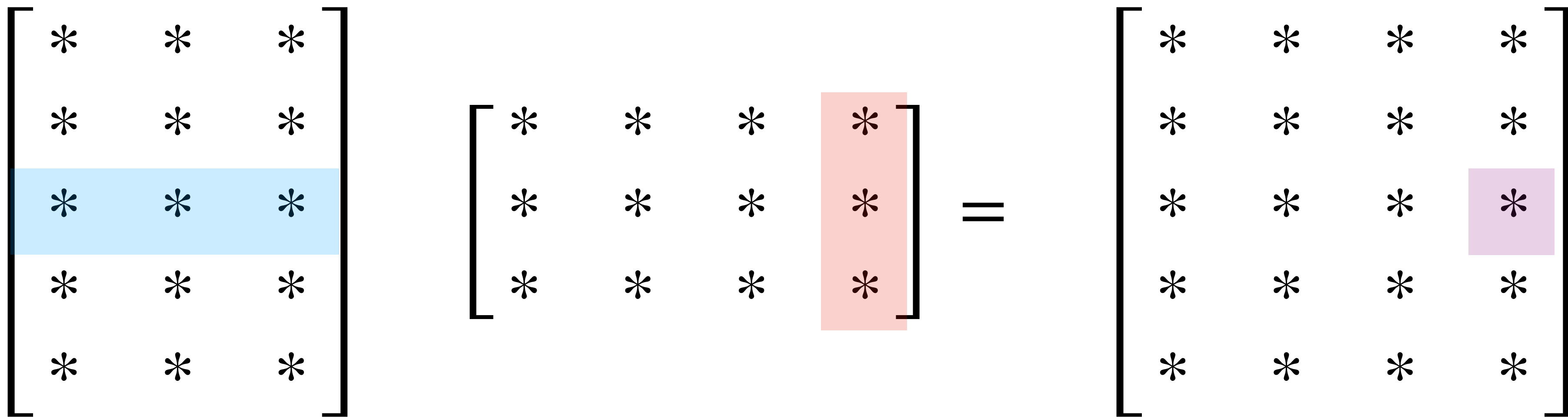
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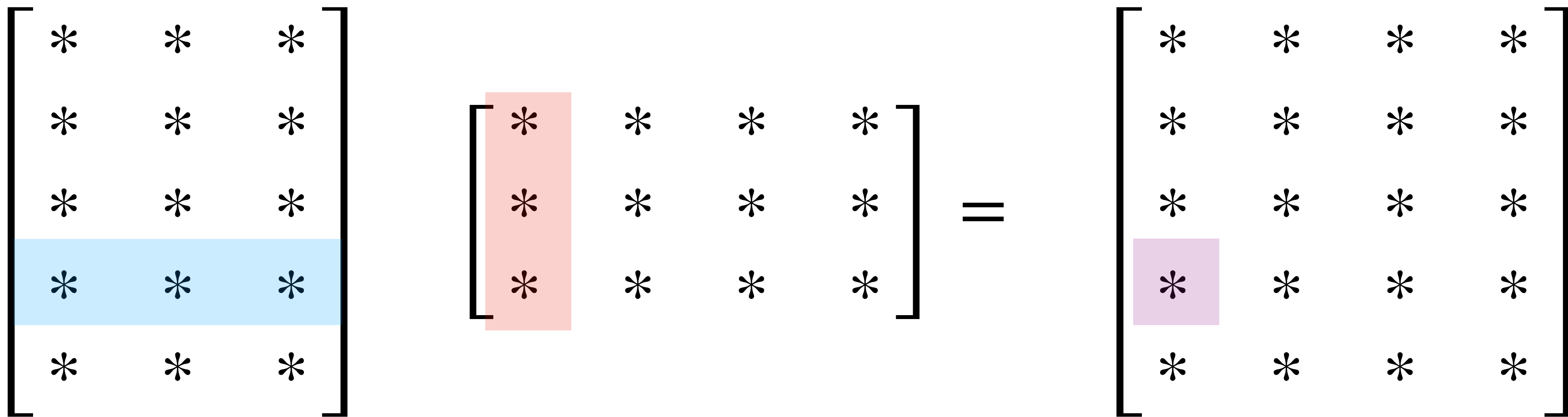
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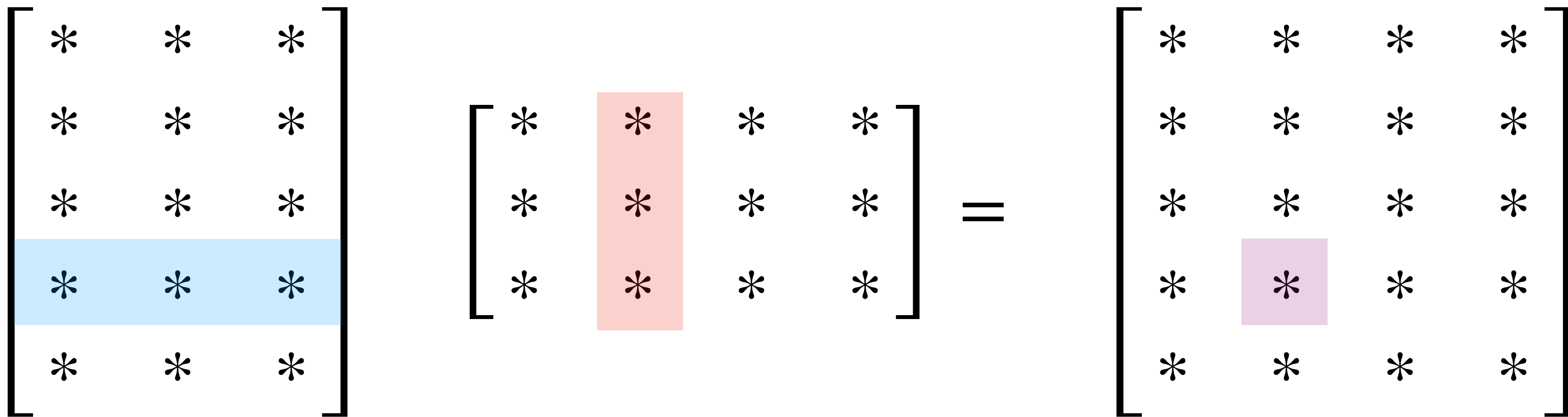
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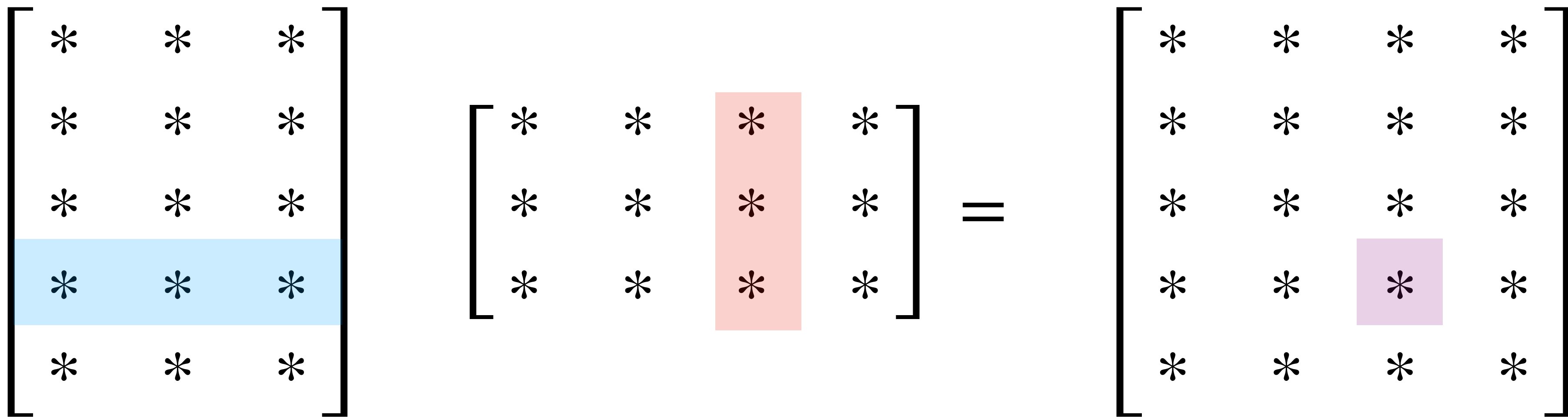
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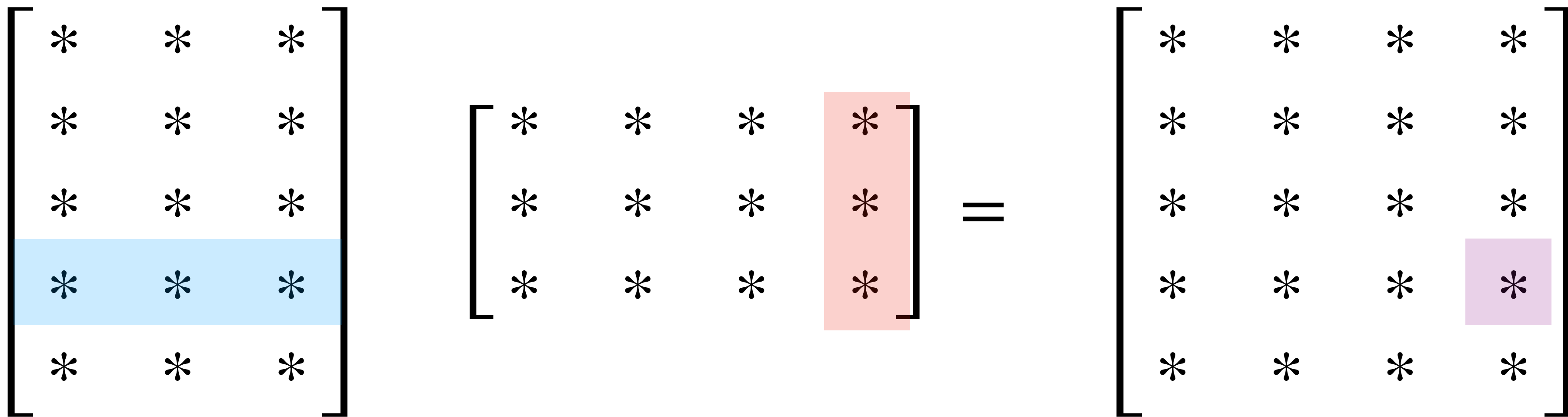
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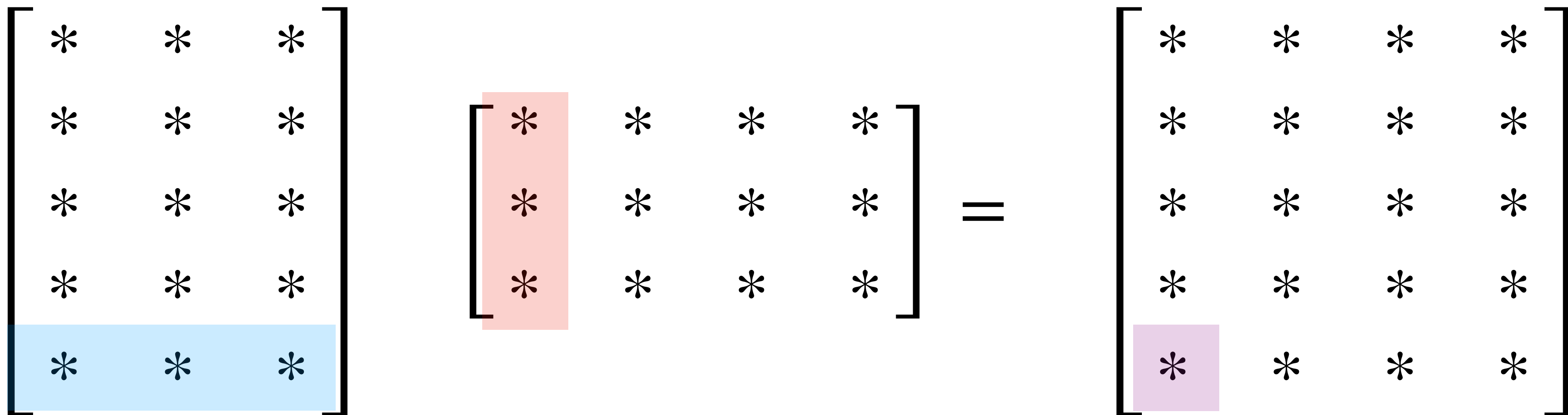
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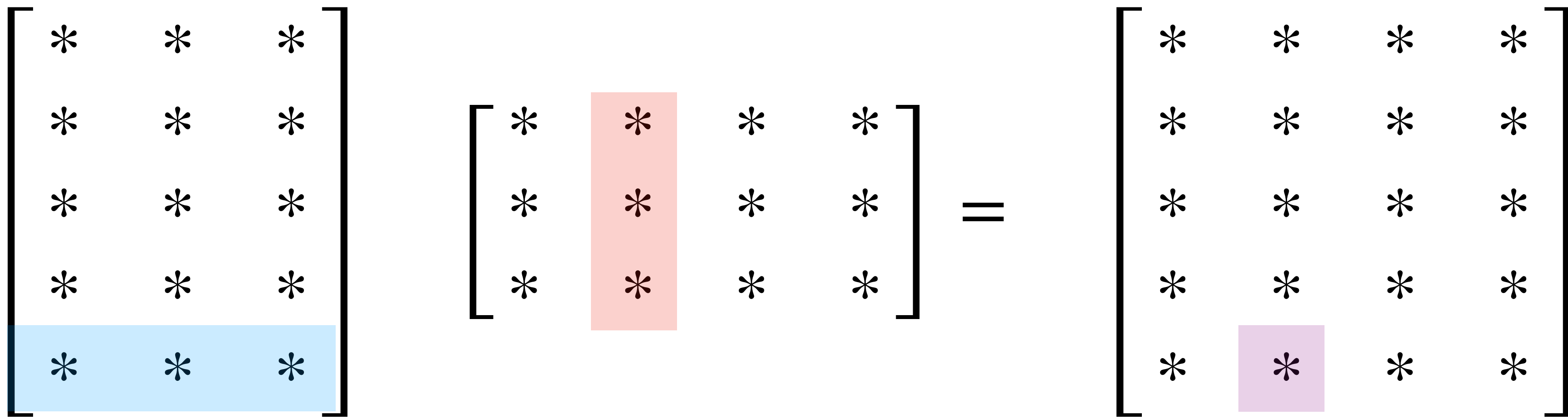
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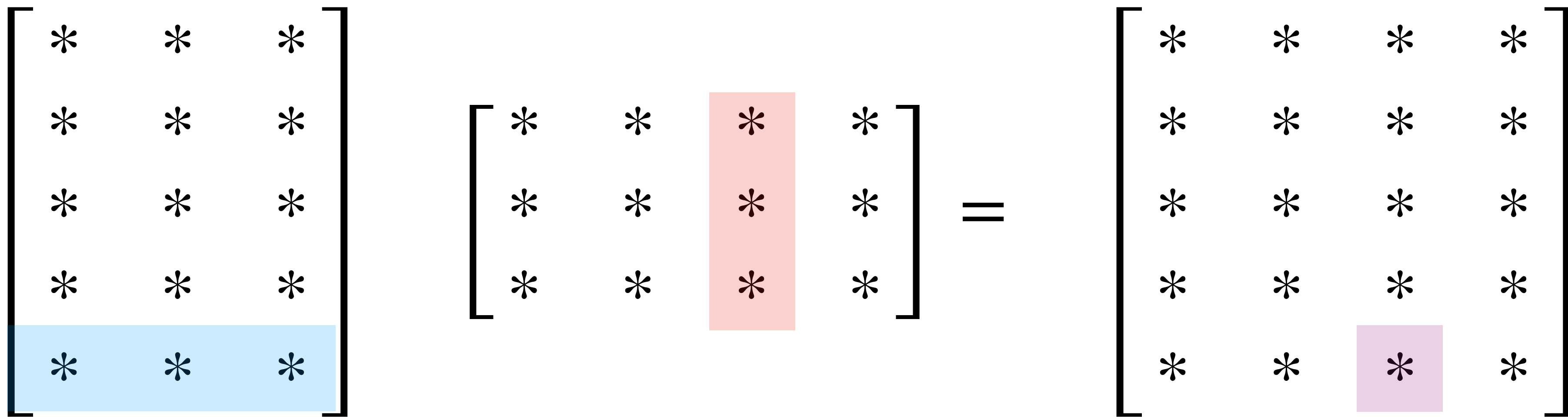
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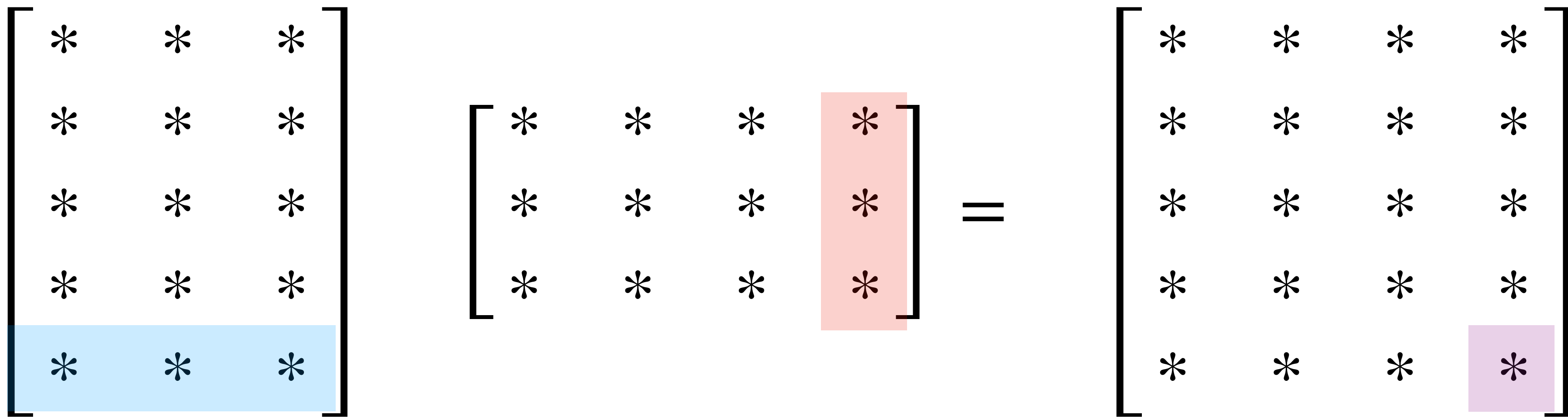
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Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Matrix Operations

Connection with Matrix-Vector Multiplication

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What about when the right matrix is a single column?

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We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

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These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

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This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

We need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

We need to know/memorize these.

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Example

More Matrix Operations

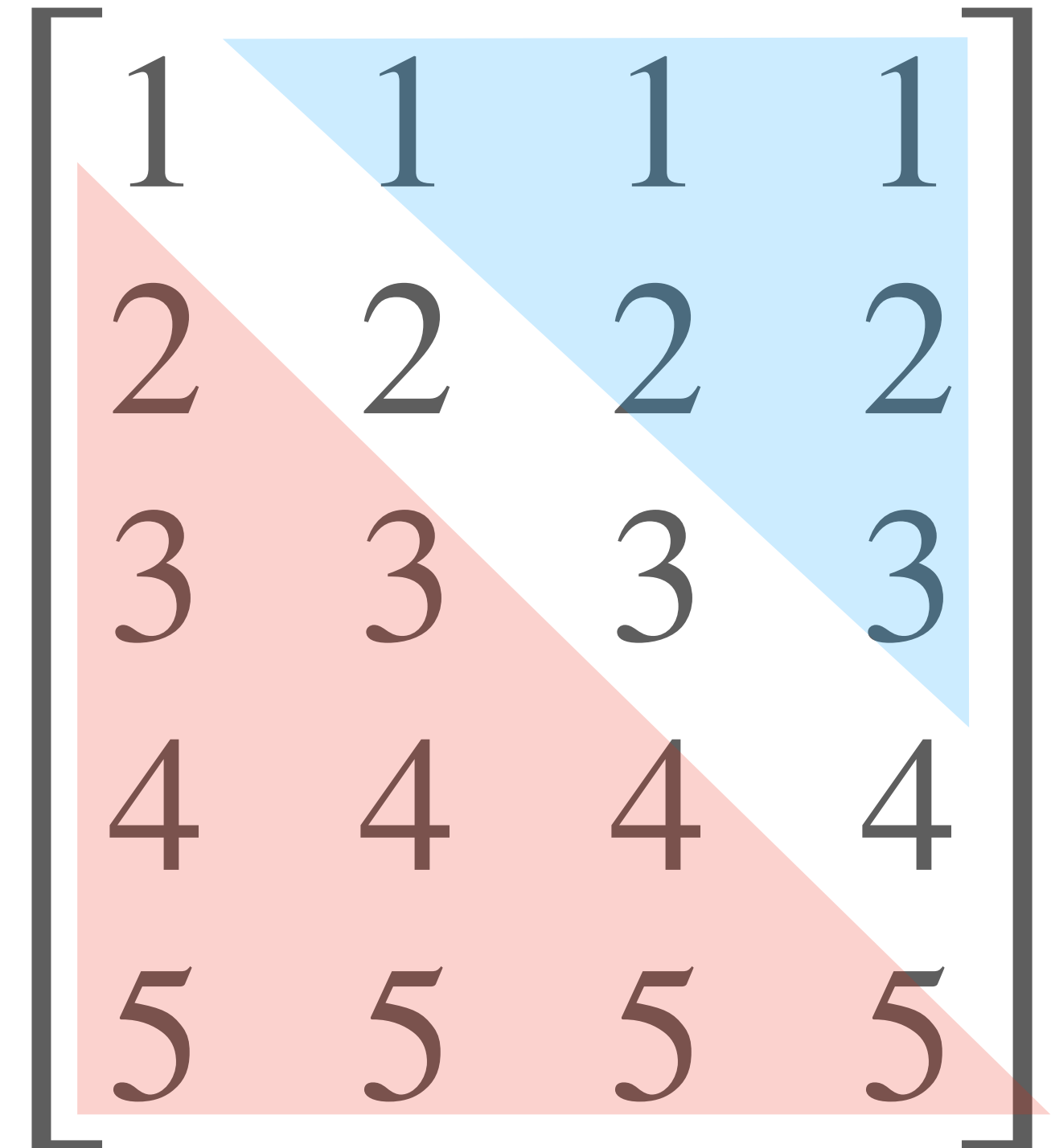
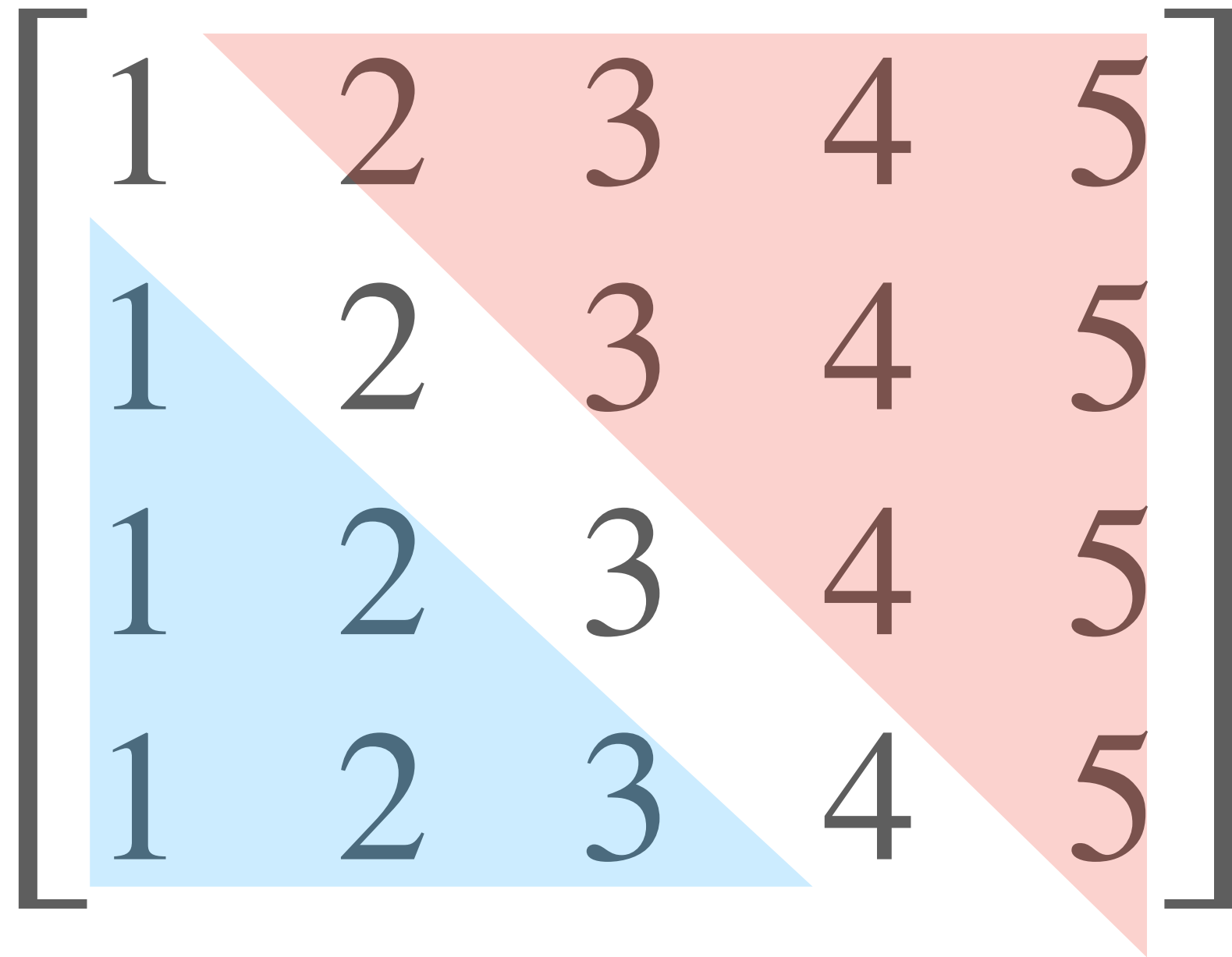
Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)



Transpose

Definition. For a $m \times n$ matrix A , the **transpose** of A , written A^T , is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = cA^T \text{ (where } c \text{ is a scalar)}$$

$$(AB)^T = B^T A^T$$

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$$(AB)^T = B^T A^T \text{ Important: the order reverses!}$$

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^T A^T$.

Example: $\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^T$

Transposes and Inner Products

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$1 \times n$

$n \times 1$

1×1

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$[u_1 \quad u_2 \quad u_3 \quad u_4]$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

=

?

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$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = ?$$

Transposes and Inner Products

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$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Definition. The **inner product** of two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

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(we want $A^0 A^k = A^{0+k} = A^k$)

Matrix Powers (Computationally)

We can use `numpy.linalg.matrix_power`

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16}$$

Why? :

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2. If $AB = AC$ then it is not necessary that $B = C$.
3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. *Choose A and B such that they have all nonzero entries.*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

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transpose

A^T

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transpose

$$A^T$$

scaling

$$cA$$

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addition (subtraction)

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$$A + (-1)B = A - B$$

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multiplication (powers)

$$AB$$

$$A^k$$

So Far: Matrix Operations

transpose

$$A^T$$

scaling

$$cA$$

addition (subtraction)

$$A + B$$

$$A + (-1)B = A - B$$

multiplication (powers)

$$AB$$

$$A^k$$

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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It is the "1" of matrices. For any A

$$IA = AI = A$$

Recall: The Identity Matrix

The identity matrix implements the "do nothing" transformation. For any \mathbf{v} ,

$$I\mathbf{v} = \mathbf{v}$$

It is the "1" of matrices. For any A

$$IA = AI = A$$

These may be different sizes

Recall: The Identity Matrix

$$\begin{array}{ccccccc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \\ 2 \times 2 & 2 \times 4 & & 2 \times 4 & 4 \times 4 & & 2 \times 4 \end{array}$$

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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How do we solve this equation?

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Divide on both sides by 2 to get $x = 5$.

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$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ a.k.a. 2^{-1} .

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Basic Algebra

$$1x = 5$$

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Multiply each side by $\frac{1}{2}$ a.k.a. 2^{-1} .

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Basic Algebra

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$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

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Multiply each side by \mathbf{A}^{-1} to get $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

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Do all matrices have
inverses?

Do all matrices have
inverses?

No. If they did, then every linear system would have a solution.

When does a matrix have
an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

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» whose columns can have full span and be linearly independent.

» that can have inverses.

Matrix Inverses

Matrix Inverses

Definition. For a $n \times n$ matrix A , an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1 -axis in \mathbb{R}^2 is its own inverse.

Verify:

Example: No inverse

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

If A is invertible, then we write A^{-1}
for *the* inverse of A .

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A , if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of \mathbf{b} .

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» exactly one solution for any choice of \mathbf{b}

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If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

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Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» T is onto

» T is one-to-one

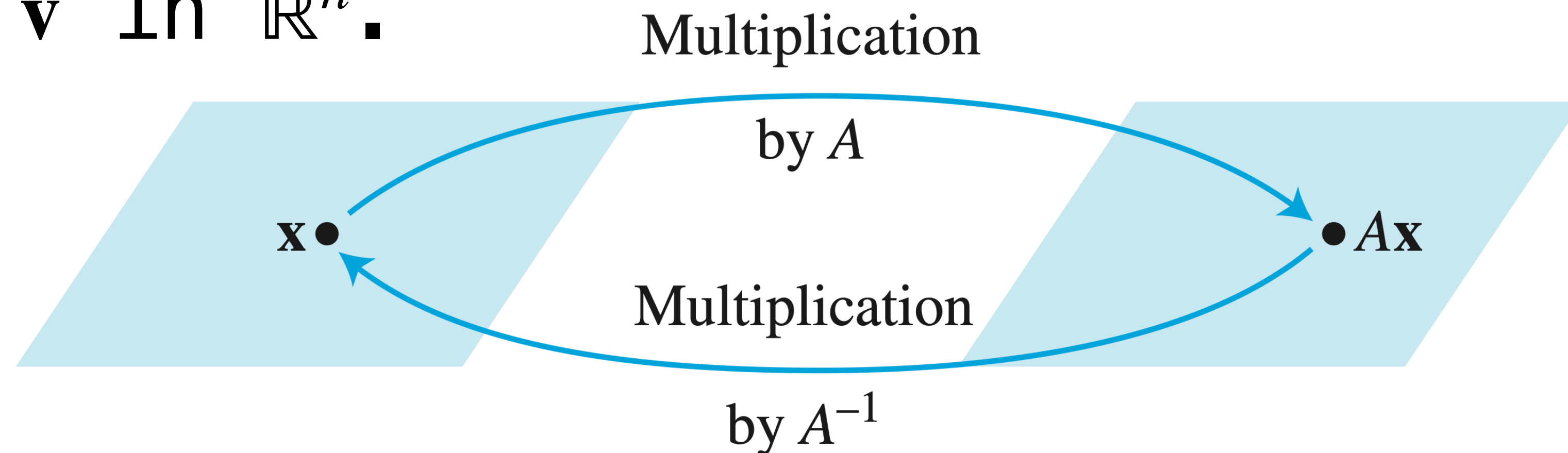
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n .



Connection to Transformations

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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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Non-Example. Projection onto the x_1 -axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

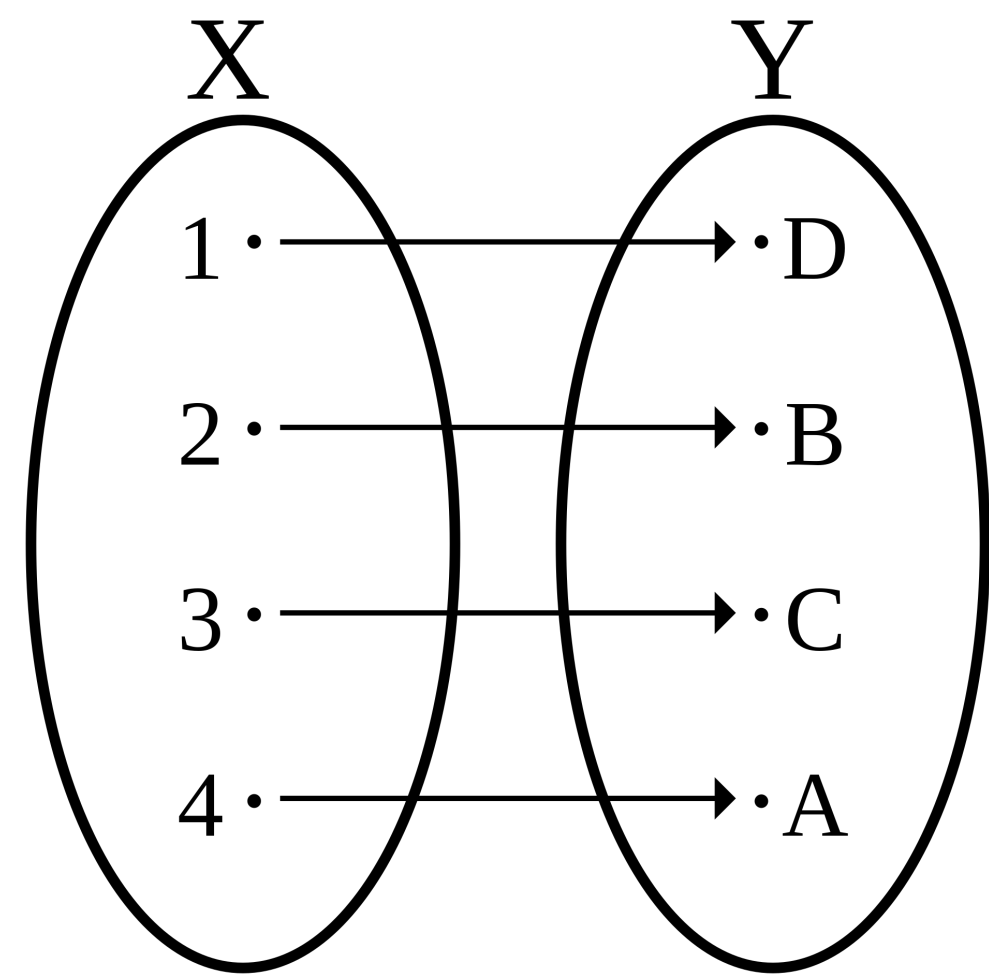
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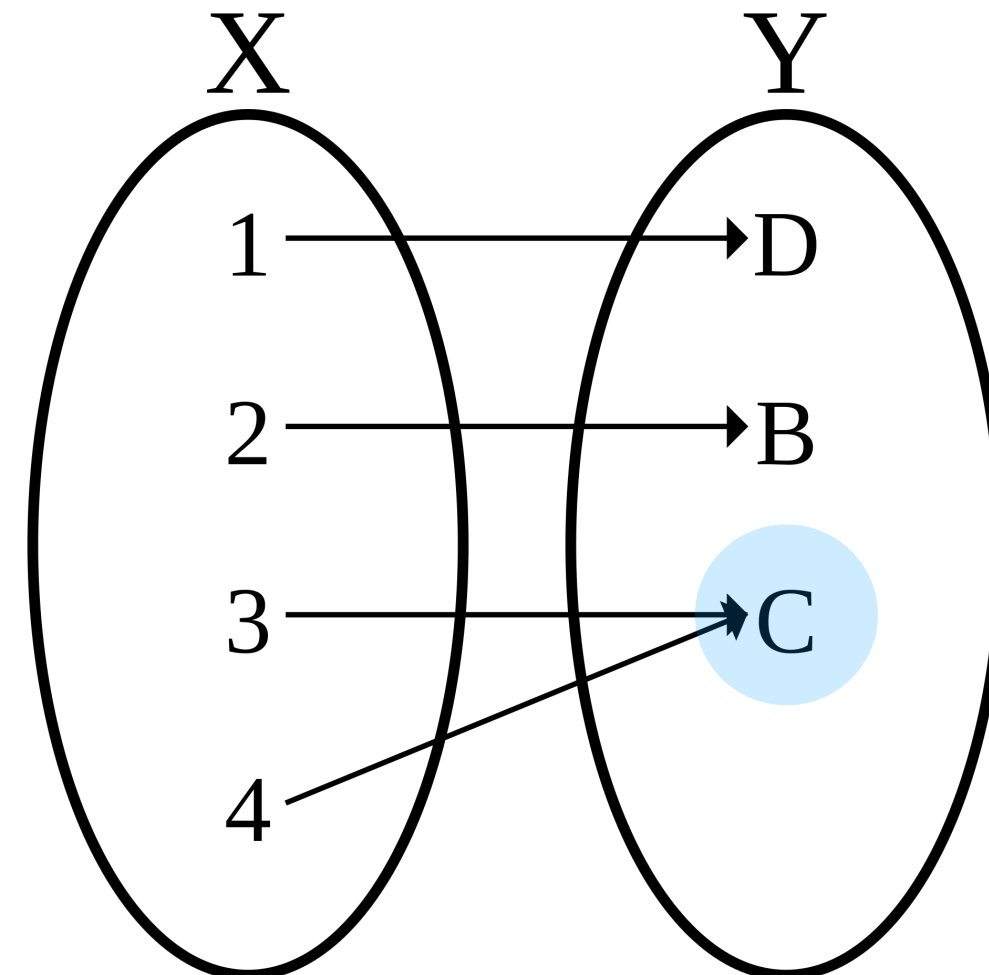
Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



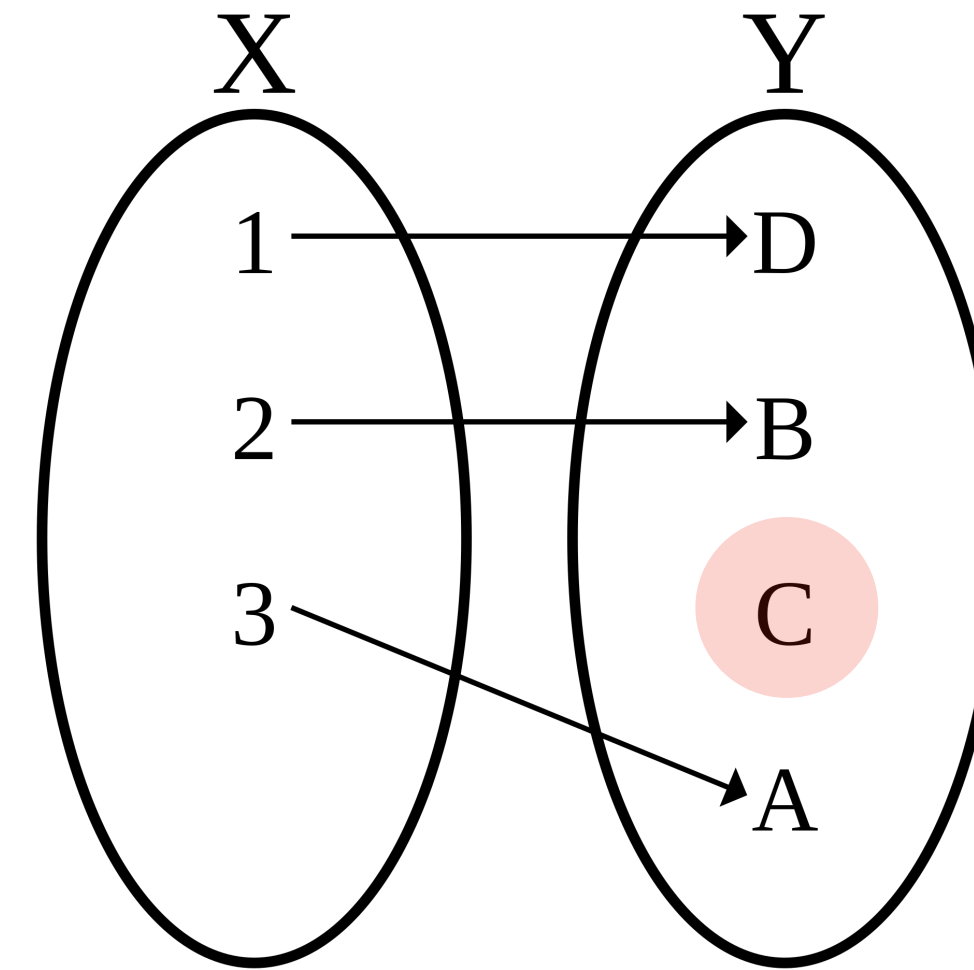
1-1 correspondence

collision



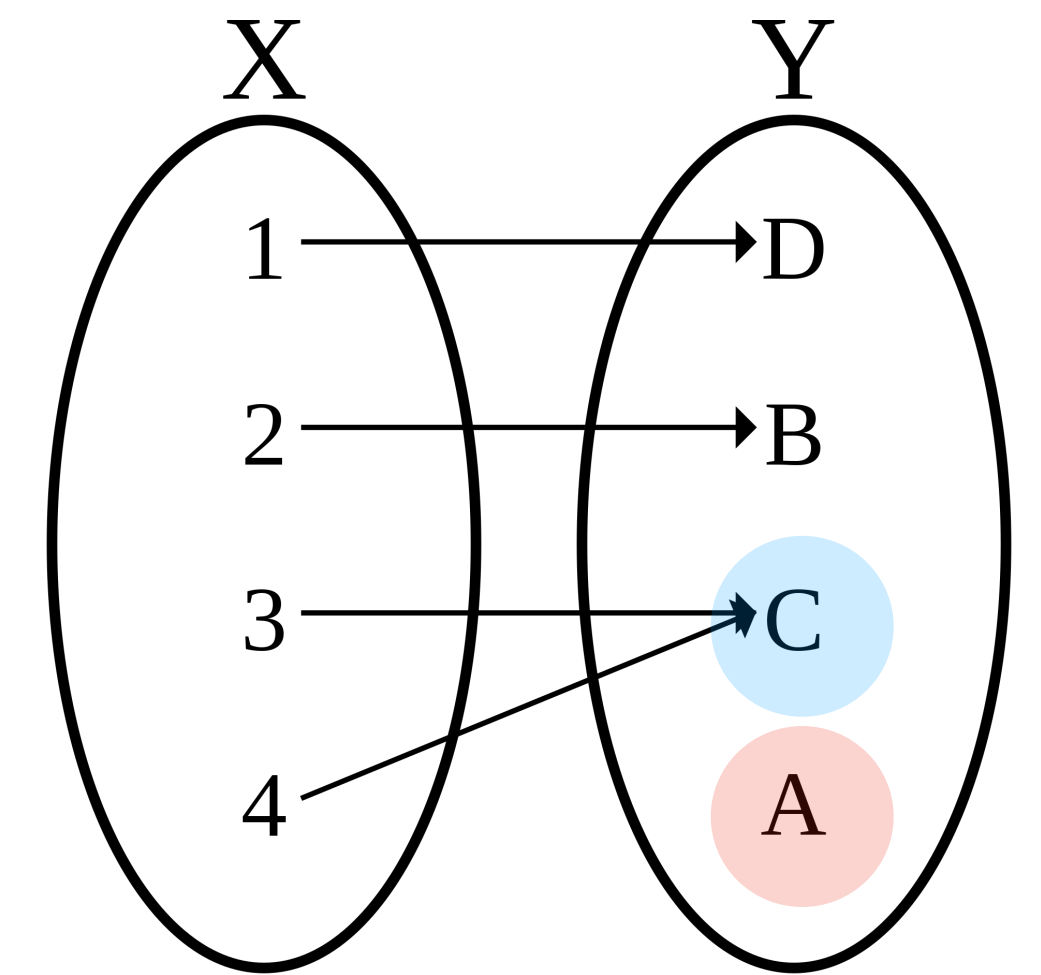
onto, not 1-1

not covered



1-1 not onto

not covered
collision



not 1-1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each \mathbf{b}_i ?:

In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

Recall: In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \dots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Row reduce the matrix $[A \ I]$ to a matrix $[I \ B]$. Then B is the inverse of A .

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrix Inverses

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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The inverse is defined only if the determinant is nonzero.

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(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C , demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B , the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix.

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen.

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every \mathbf{b}

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every column
6. A has a pivot in every row
7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span \mathbb{R}^n

Verify:

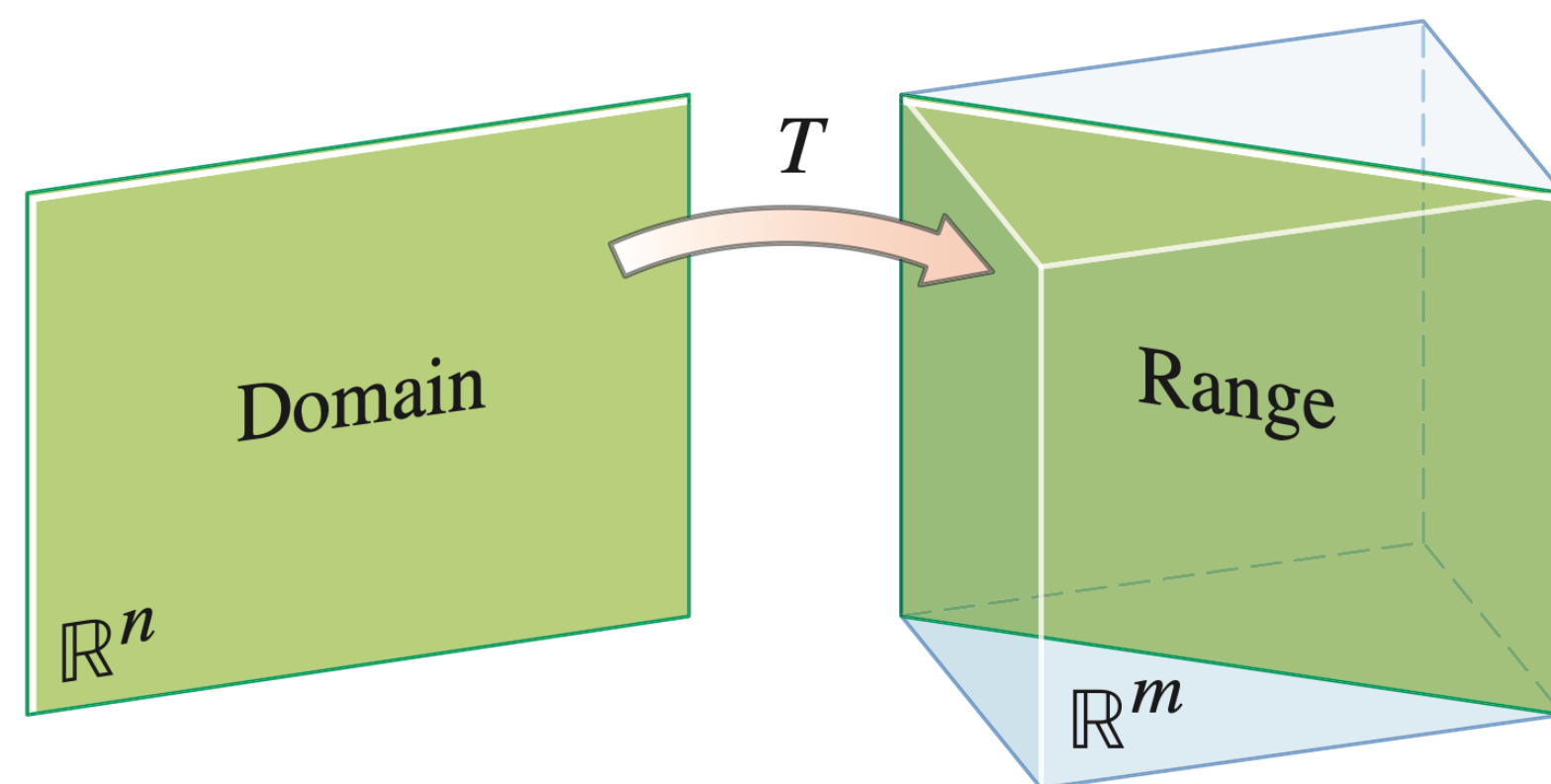
Recall: Onto Transformations

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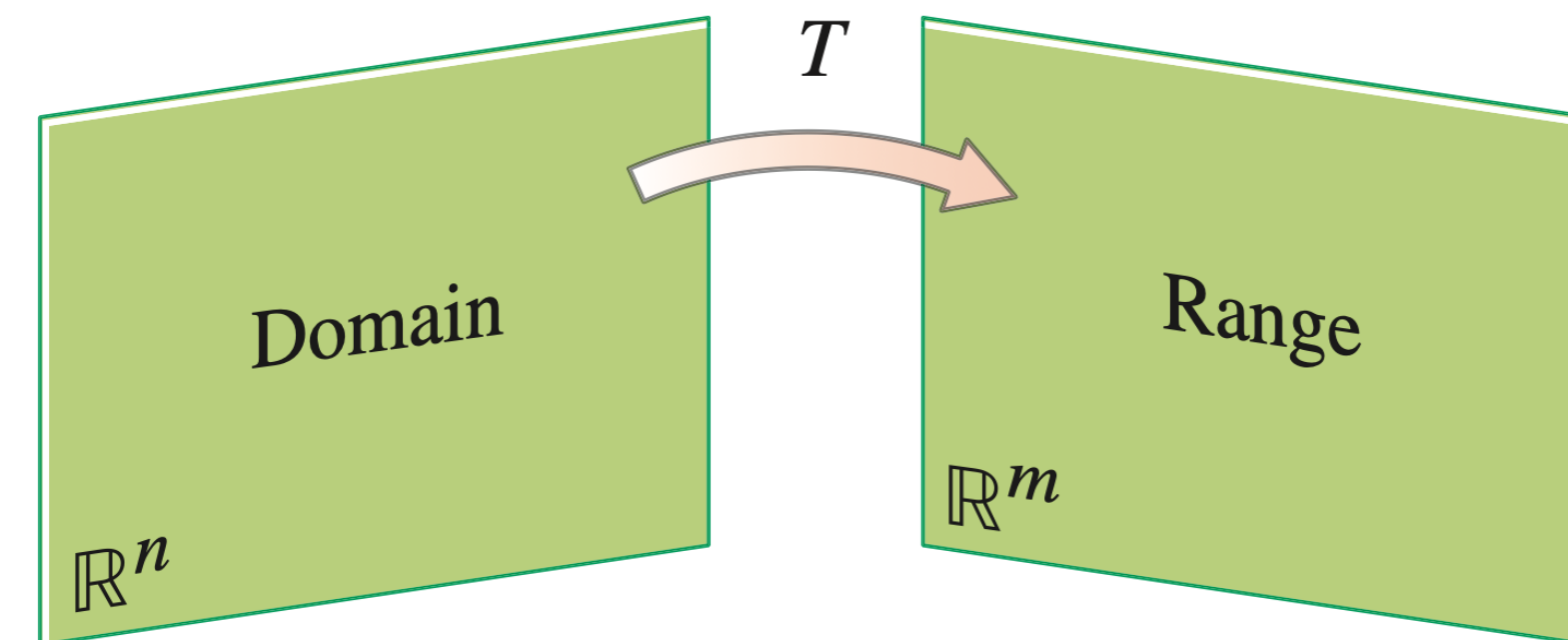
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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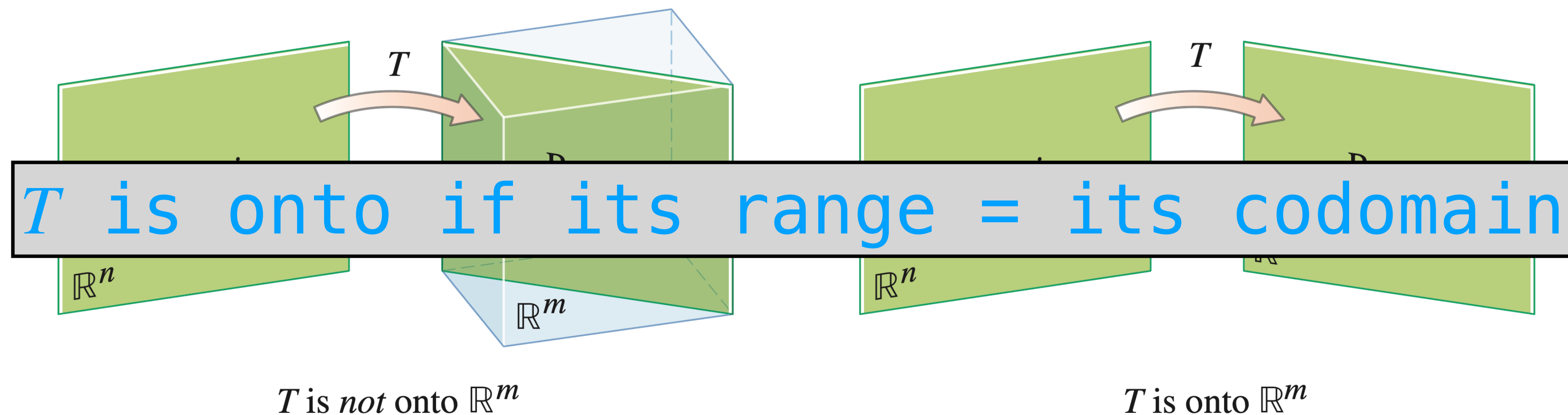
T is not onto \mathbb{R}^m



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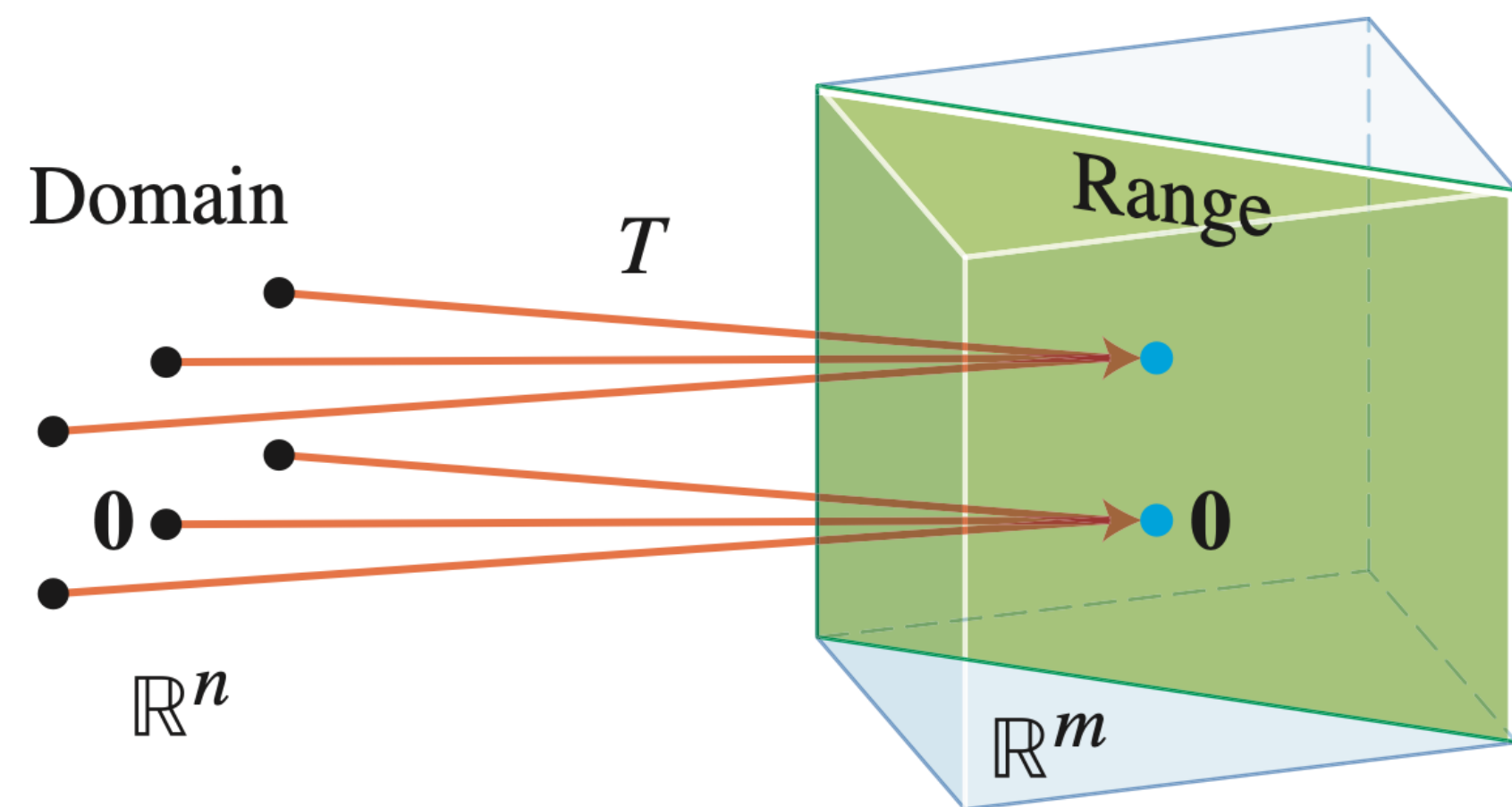
Recall: One-to-one Transformations

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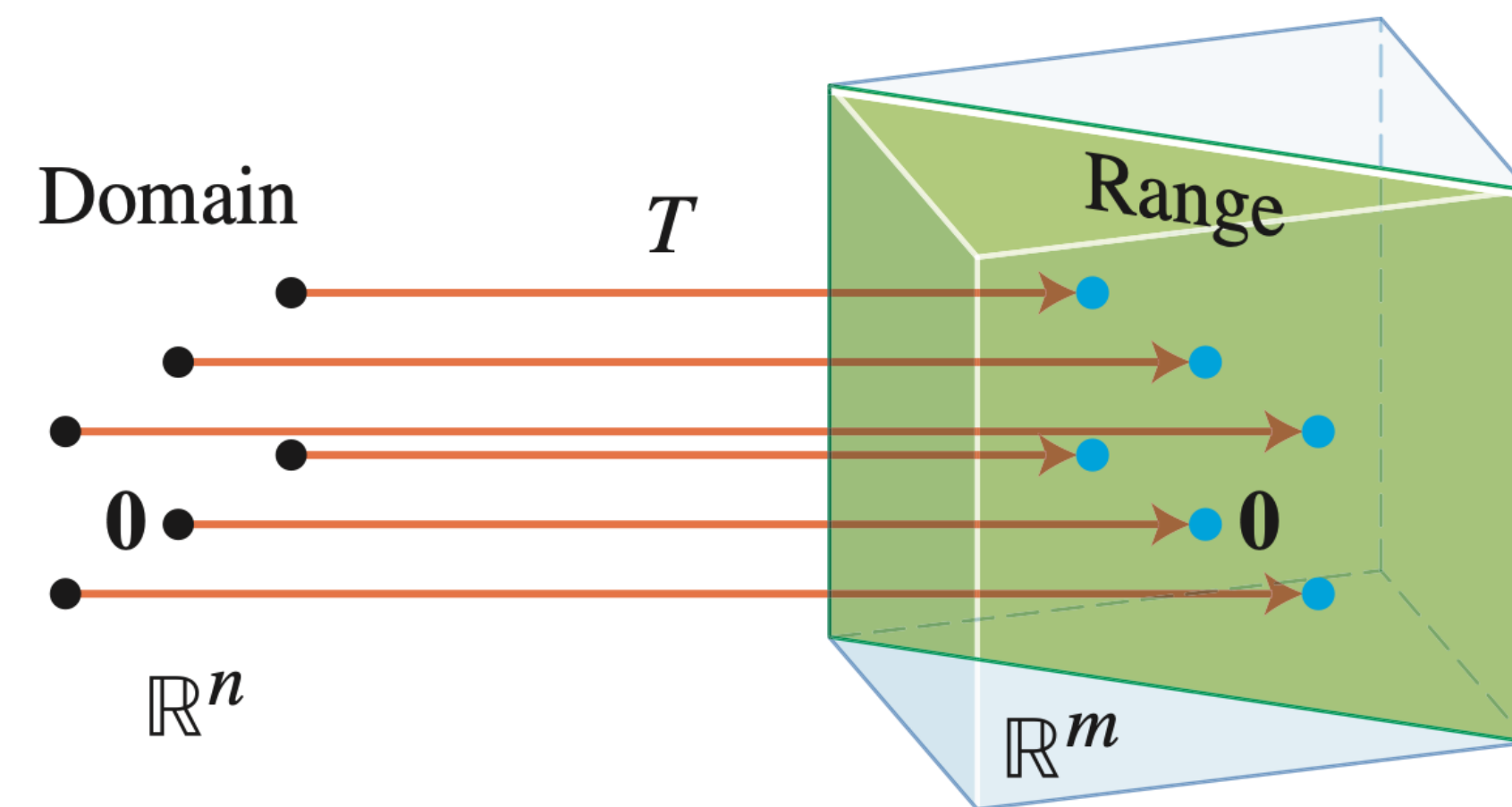
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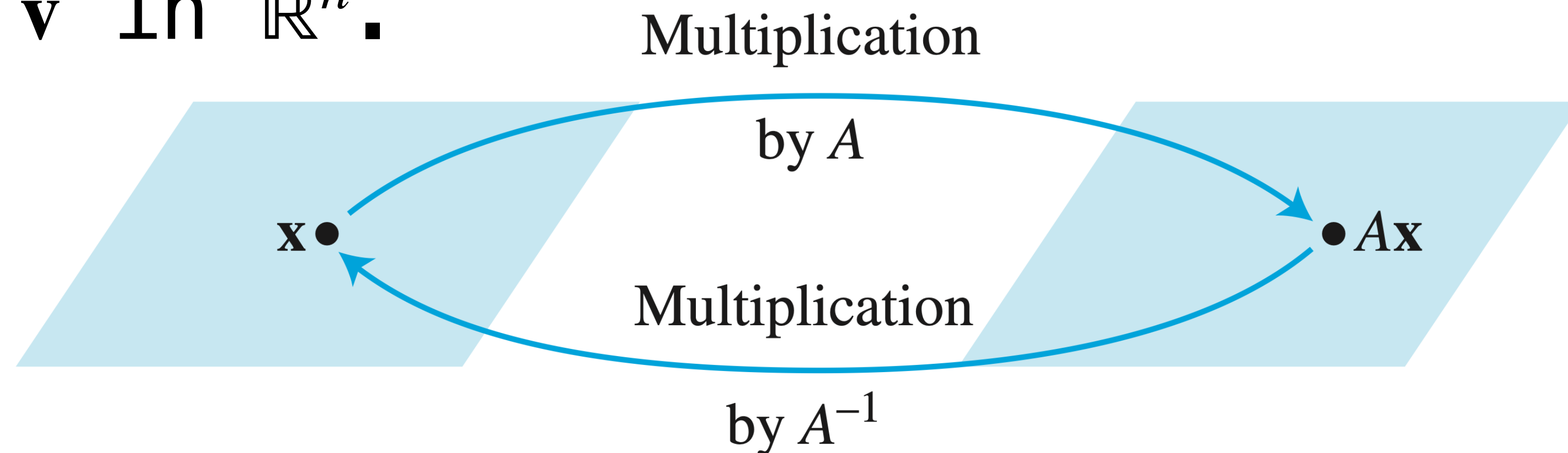
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Recall: Invertible Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

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Recall: One-to-One Correspondence

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Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any \mathbf{b}
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span \mathbb{R}^n
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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1-1** if and only if A **is onto**

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We only need to check one of these.

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Theorem. If A is square, then

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We only need to check one of these.

Warning. Remember this only applies square matrices.

We get a lot of information for free

We get a lot of information for free

Theorem. If A is square, then

$$A \text{ is invertible} \quad \equiv \quad Ax = 0 \text{ implies } x = 0$$

We get a lot of information for free

Theorem. If A is square, then

$$A \text{ is invertible} \quad \equiv \quad Ax = 0 \text{ implies } x = 0$$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True or False: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), then B is also invertible.

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ is invertible, then is $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]$ also invertible? Justify your answer.

Answer

Consider $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$. We can get to $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]^T$ by row operations

Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.