

Invertible Matrix Theorem + Algebraic Graph Theory

**Geometric Algorithms
Lecture 12**

Objectives

1. Recap matrix inverses ~~(it's been a while)~~
2. Finish up the algebra of matrix inverses
3. Connect everything we've talked about so far via the Invertible Matrix Theorem (IMT)
4. Connect linear algebra to graph theory

Keywords

matrix inverses

invertible matrix theorem

directed/undirected graphs

weighted/unweighted graphs

adjacency matrices

symmetric matrices

triangle counting

Recap: Matrix Inverse

Motivation

$$A\mathbf{x} = \mathbf{b}$$

When can we solve a matrix equation
by "*dividing on both sides by A?*"

Motivation

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

When can we solve a matrix equation
by "*dividing on both sides by A?*"

Motivation

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by A?*"

Recall: Matrix Inverses

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Definition. For a $n \times n$ matrix A , an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ (and } BA = I_n \text{)}$$

Recall: Matrix Inverses

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition. For a $n \times n$ matrix A , an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ (and } BA = I_n)$$

A is **invertible** if it has an inverse. Otherwise it is **singular**.

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

$$\begin{aligned} B &= B I = B(AC) = (BA)C \\ &= I C \\ &= C \end{aligned}$$

Inverses are Unique

Theorem. If B and C are inverses of A , then $B = C$.

Verify:

If A is invertible, then we write A^{-1}
for *the* inverse of A .

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A , if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of \mathbf{b} .

Verify: $A \left(A^{-1} \vec{b} \right) = \vec{b}$

at least
one solution

$$A \vec{v} = \vec{b}$$

$$A^{-1} A \vec{v} = A^{-1} \vec{b}$$

$$\vec{v} = A^{-1} \vec{b}$$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» exactly one solution for any choice of \mathbf{b}

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» at most one solution for any choice of \mathbf{b}

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of \mathbf{b} , then it has

» T is onto

» T is one-to-one

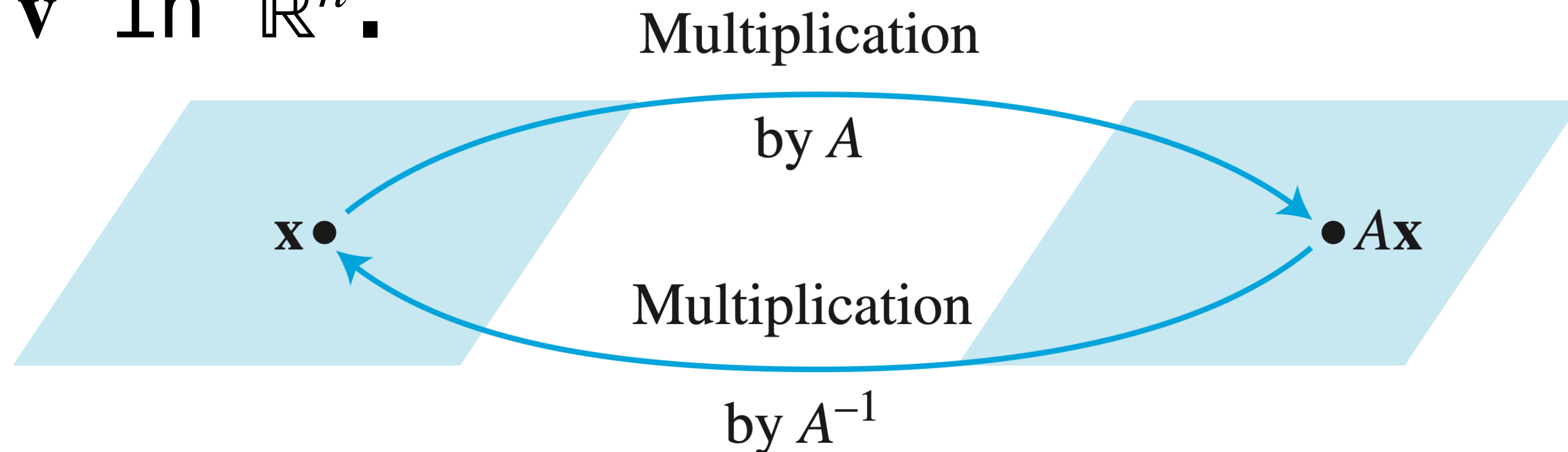
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any \mathbf{v} in \mathbb{R}^n .



Connection to Transformations

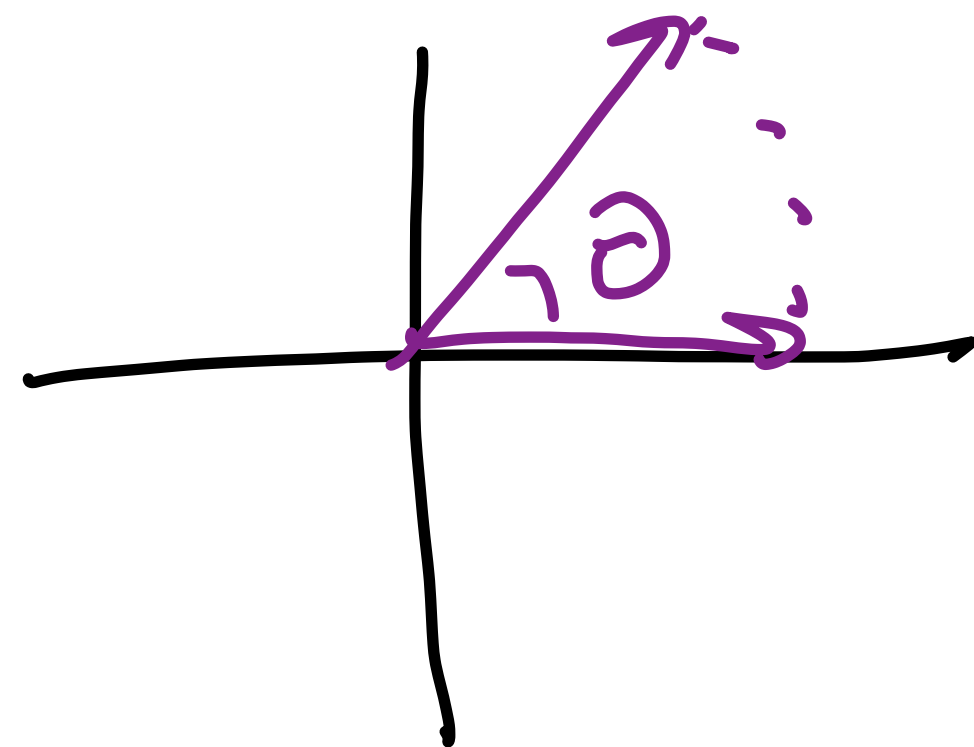
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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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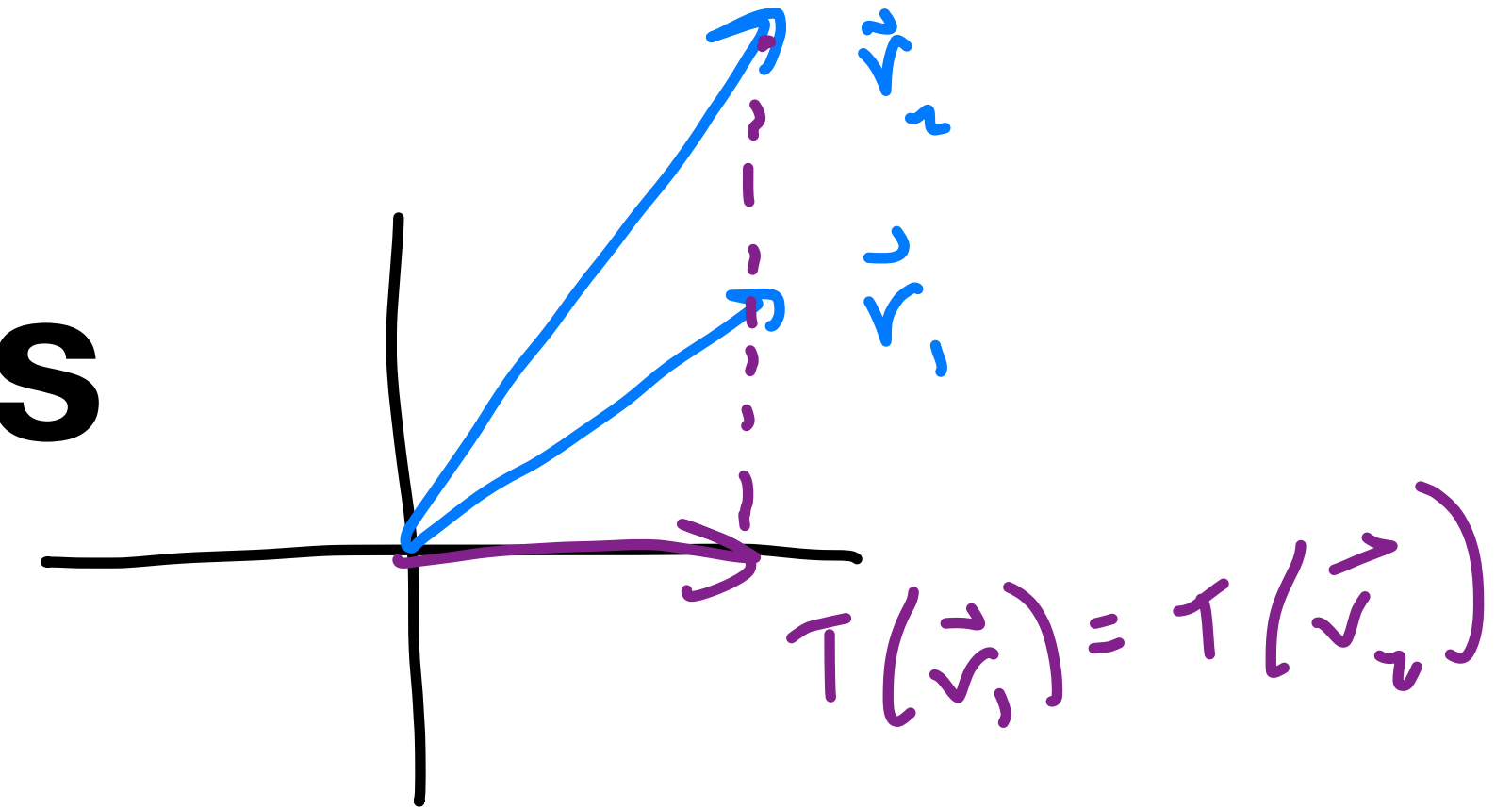
Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

A matrix is invertible if it's possible to "undo" its transformation without "losing information".



$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Connection to Transformations



Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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Non-Example. Projection onto the x_1 -axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector \mathbf{b} in \mathbb{R}^n is the **image of exactly one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

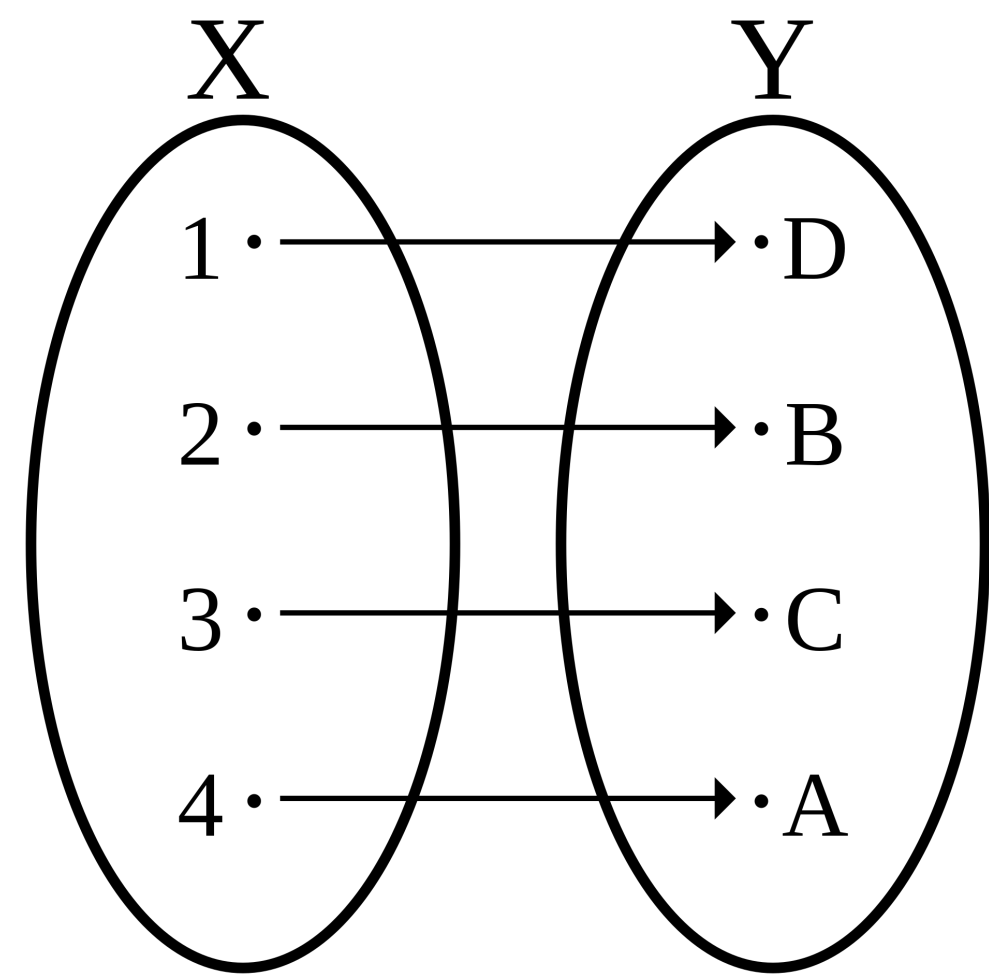
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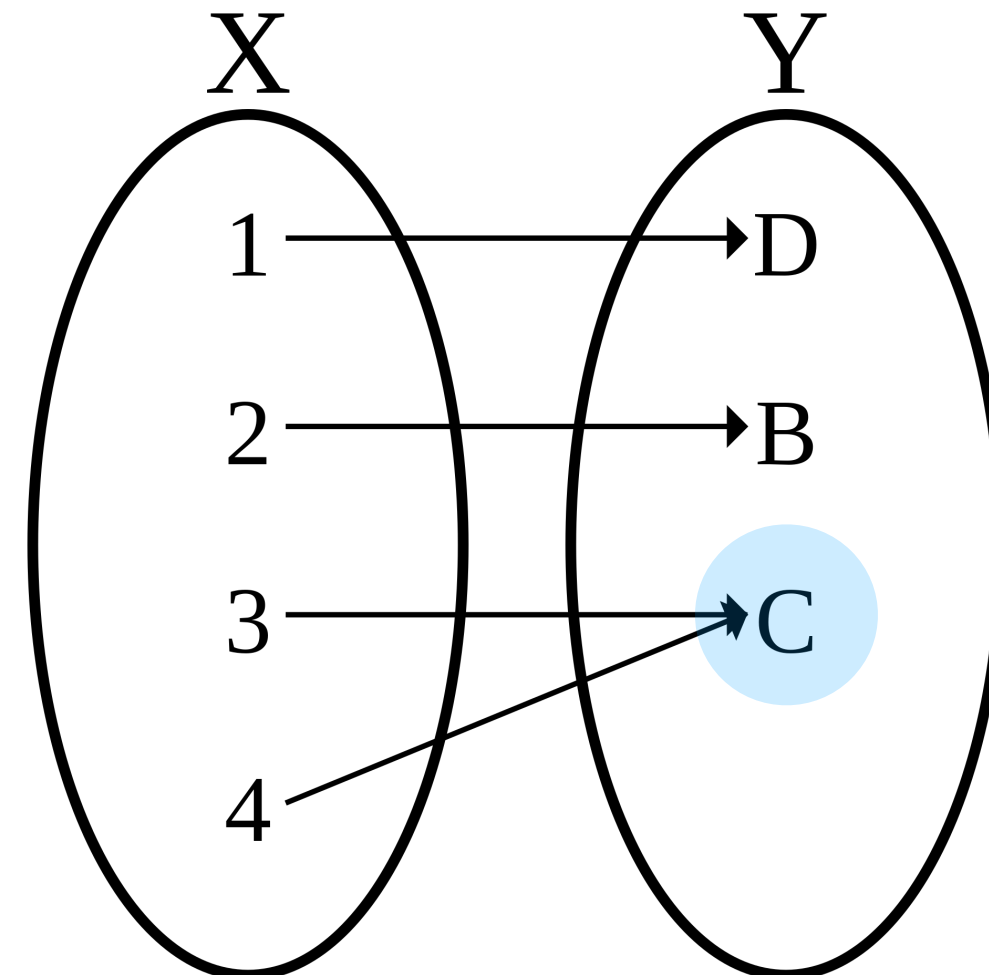
Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



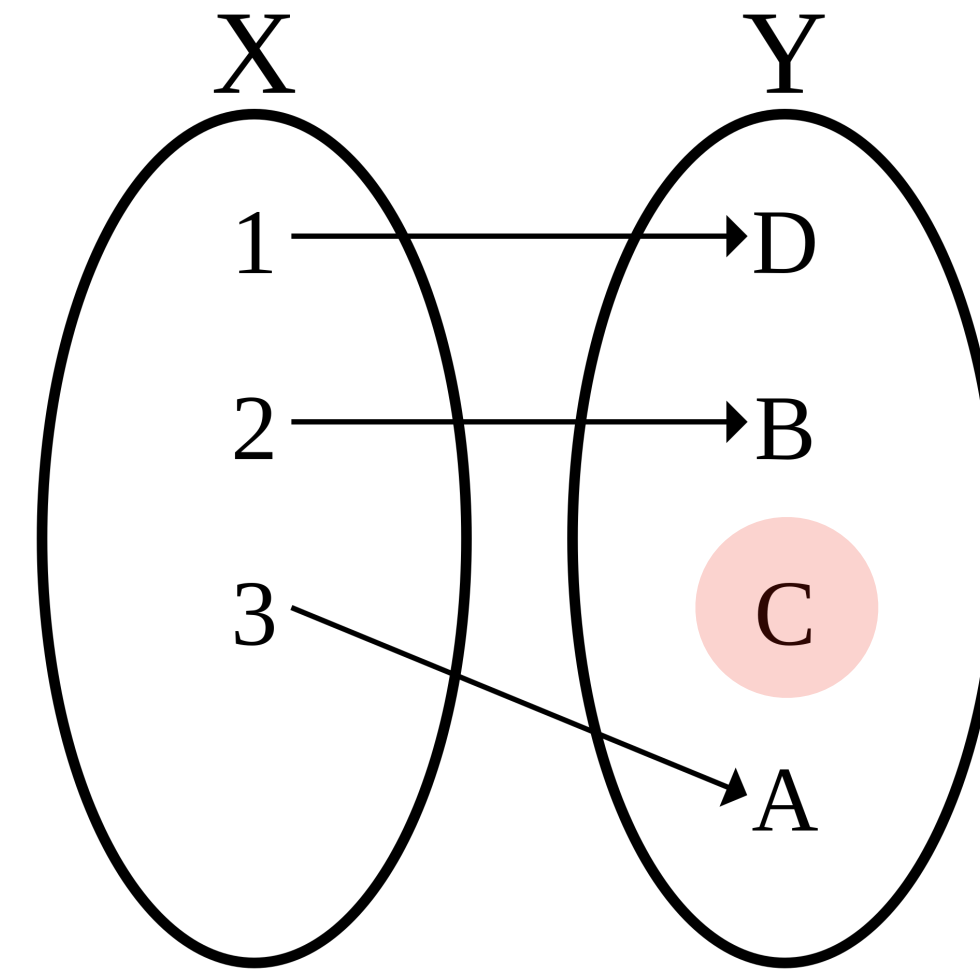
1-1 correspondence

collision



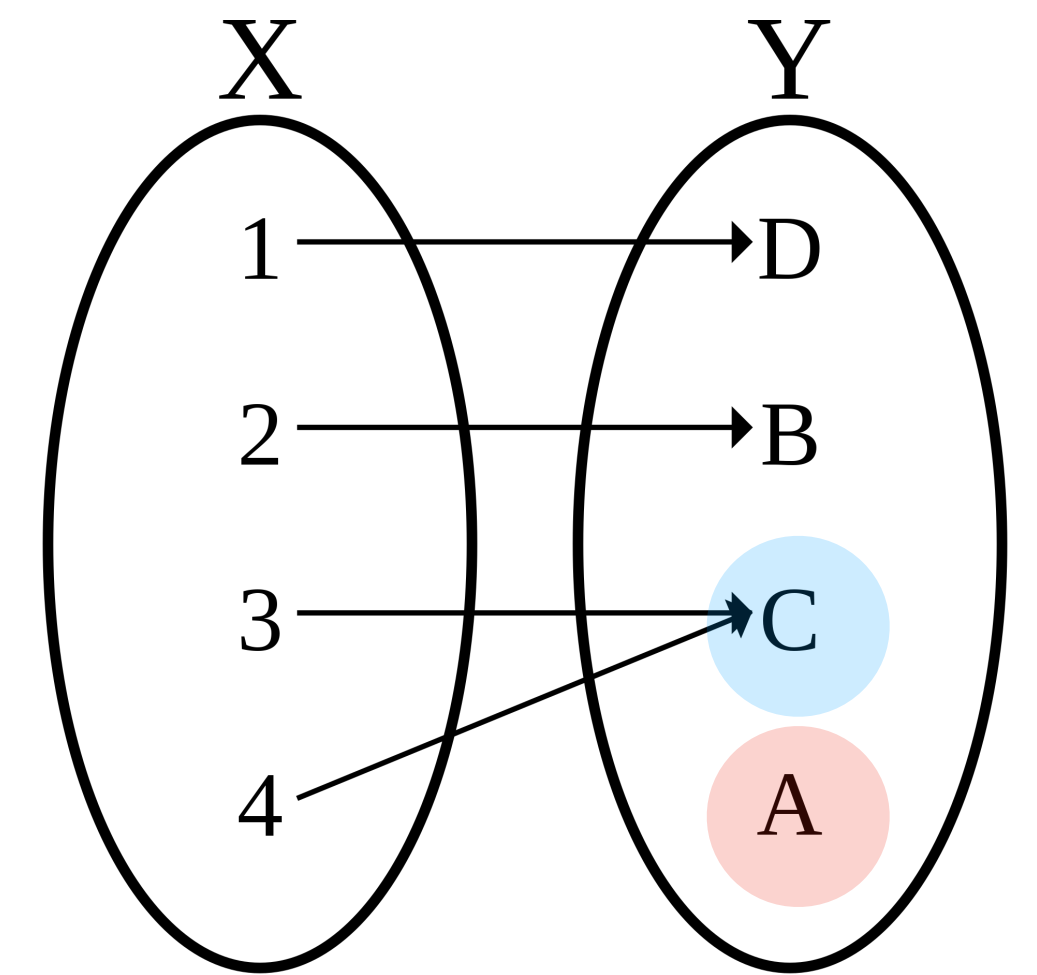
onto, not 1-1

not covered



1-1 not onto

not covered
collision



not 1-1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$AB = I$$

$$BA = I$$

$$A \begin{matrix} 3 \times 3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{matrix} = I = \begin{matrix} 3 \times 3 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Can we solve for each \mathbf{b}_i ?:

$$A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

Recall: In General

$$[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3] = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each \mathbf{b}_i ?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \dots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A .

Solution. Row reduce the matrix $[A \ I]$ to a matrix $[I \ B]$. Then B is the inverse of A .

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrix Inverses

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

Example

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C , demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

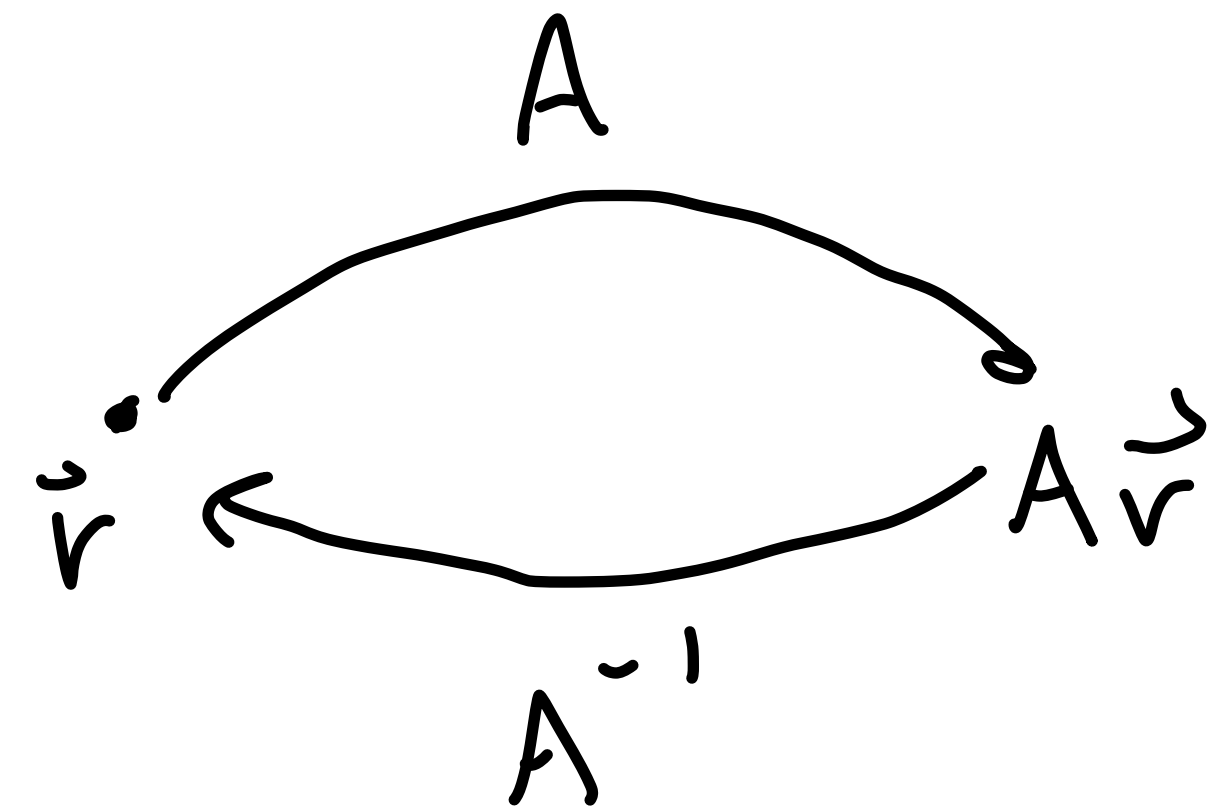
Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

$$A(A^{-1}) = \underline{I}$$



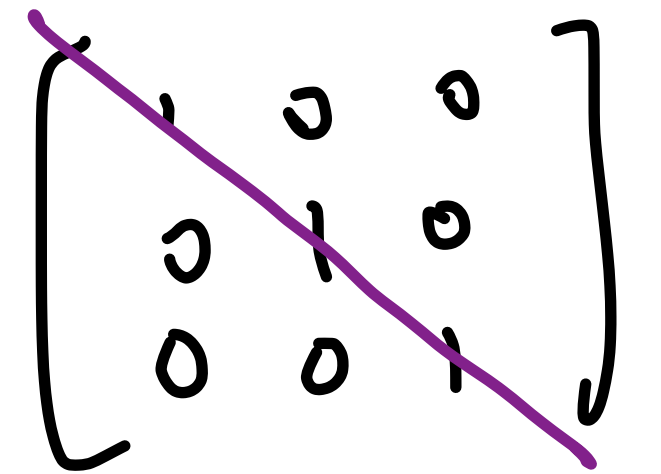
Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A , the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

$$(A^{-1})^T A^T = (A A^{-1})^T = I^T = I$$


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algebraic Properties (Matrix Inverses)

$$B^{-1} \times B \neq X$$

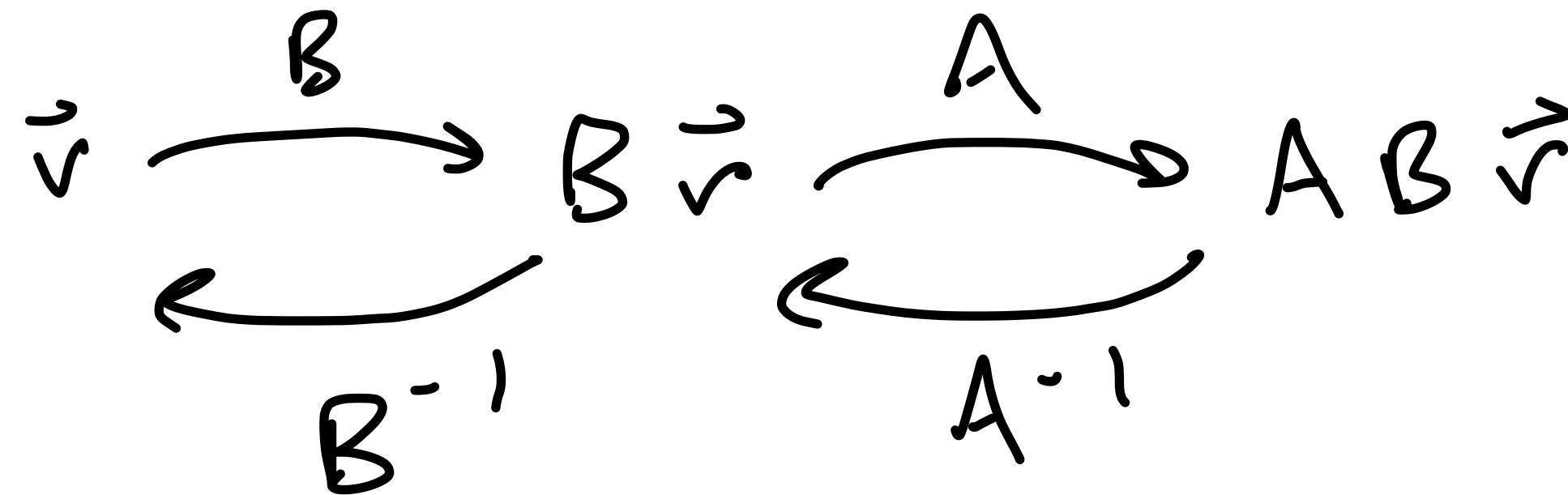
Theorem. For a $n \times n$ invertible matrices A and B , the matrix AB is invertible and

Composition \rightarrow $(AB)^{-1} = B^{-1}A^{-1}$

Verify:

$$B^{-1}A^{-1}AB = B^{-1}IB = \cancel{B^{-1}}\cancel{B} = I$$

$$B^{-1} \boxed{A^{-1}A} B$$



Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix.

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen.

Answer: B^T

$$A(BA^{-1})^T =$$

$$A(A^{-1})^T B^T =$$

$$A(A^T)^{-1} B^T =$$

$$\cancel{A} \cancel{A^{-1}} B^T = B^T$$

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix.
Then the following hold.

1. A^T is invertible

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
3. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every \mathbf{b}

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every column
6. A has a pivot in every row
7. A is row equivalent to I_n

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

9. The columns of A are linearly independent

10. The columns of A span \mathbb{R}^n

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any \mathbf{b}
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any \mathbf{b}
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
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These all express the
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(this is a stronger statement than
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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1-1** if and only if A **is onto**

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We only need to check one of these.

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We only need to check one of these.

Warning. Remember this only applies square matrices.

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Theorem. If A is square, then

$$A \text{ is invertible} \quad \equiv \quad Ax = 0 \text{ implies } x = 0$$

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Theorem. If A is square, then

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Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True or False: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), then B is also invertible.

Answer: True

Row reductions don't change the number of pivots.

Question

If $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ is invertible, then is $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]$ also invertible? Justify your answer.

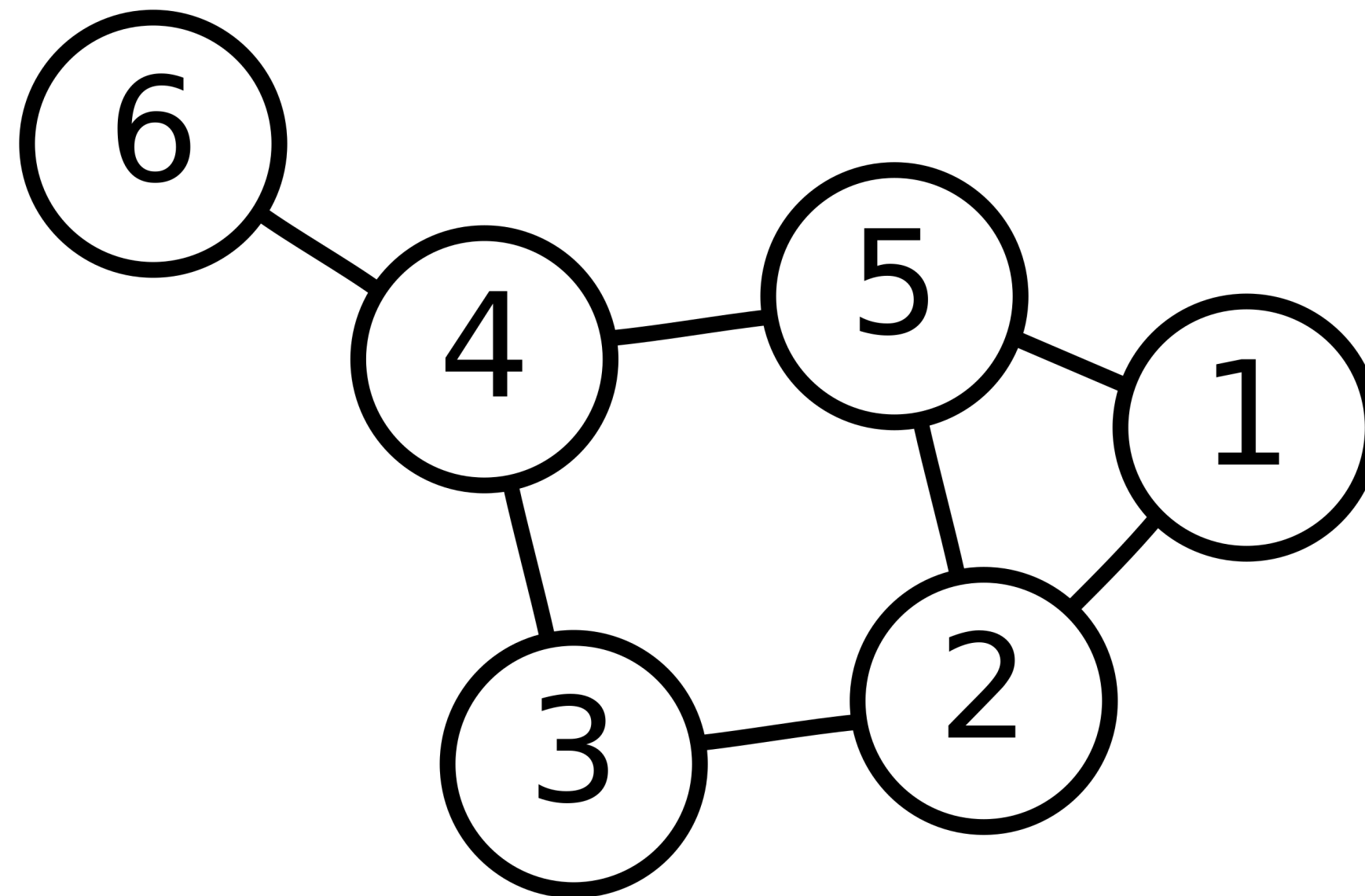
Answer

Consider $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$. We can get to $[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \ (\mathbf{a}_2 + 5\mathbf{a}_3) \ \mathbf{a}_3]^T$ by row operations

Algebraic Graph Theory

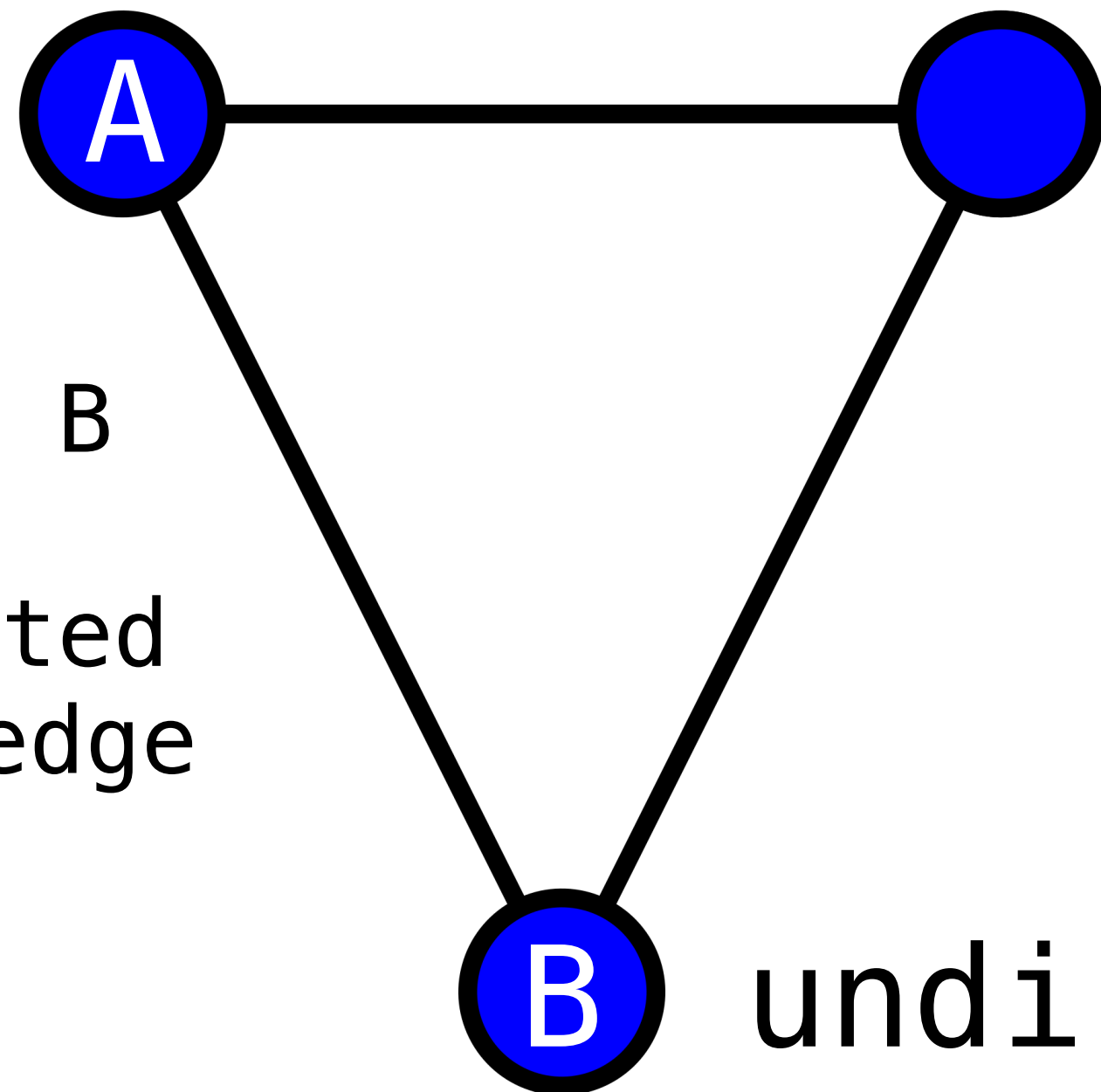
Graphs

Definition (Informal). A graph is a collection of nodes with edges between them.



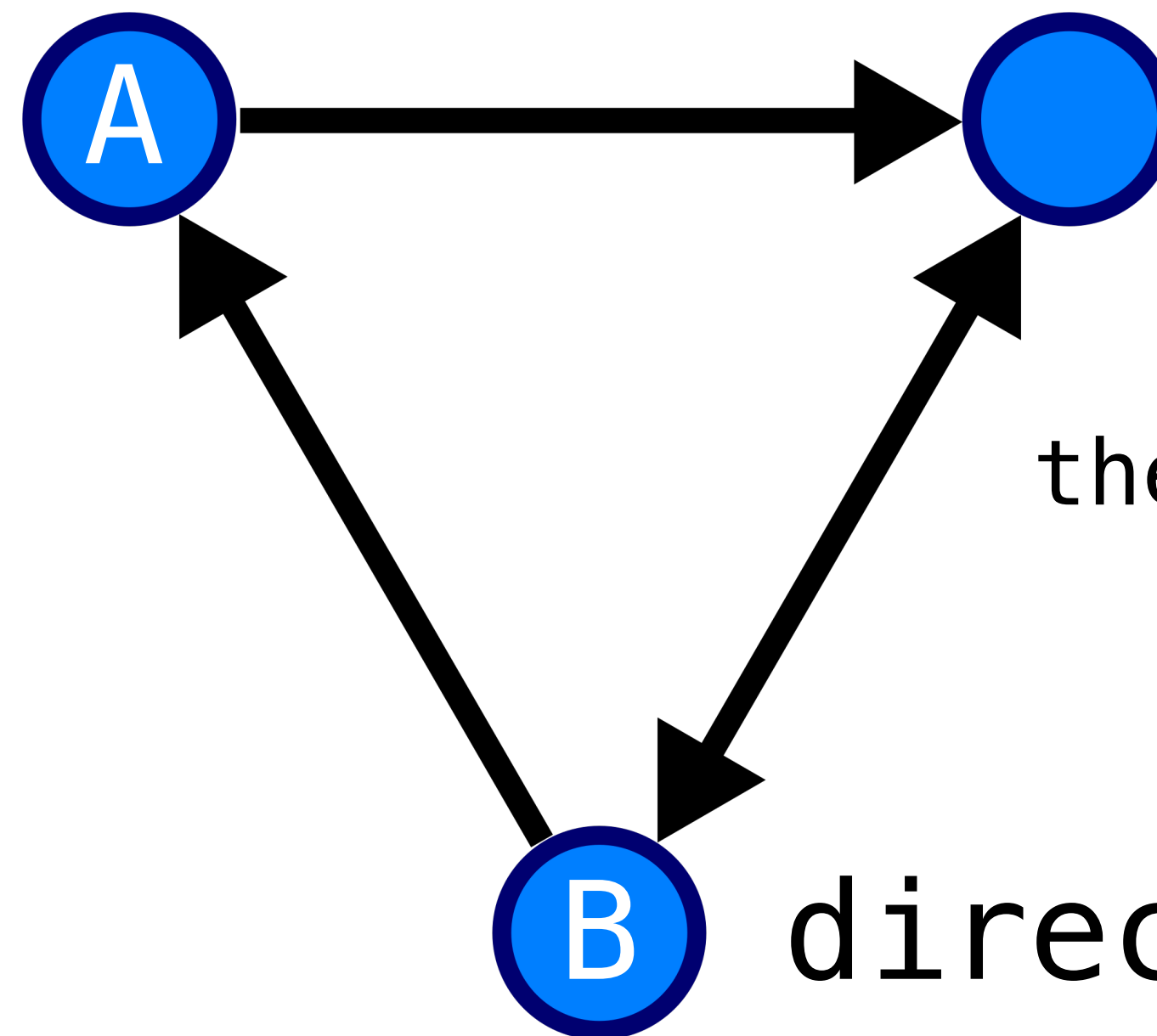
Directed vs. Undirected Graphs

A graph is **directed** if its edges have a direction.



A and B
are
connected
by an edge

undirected

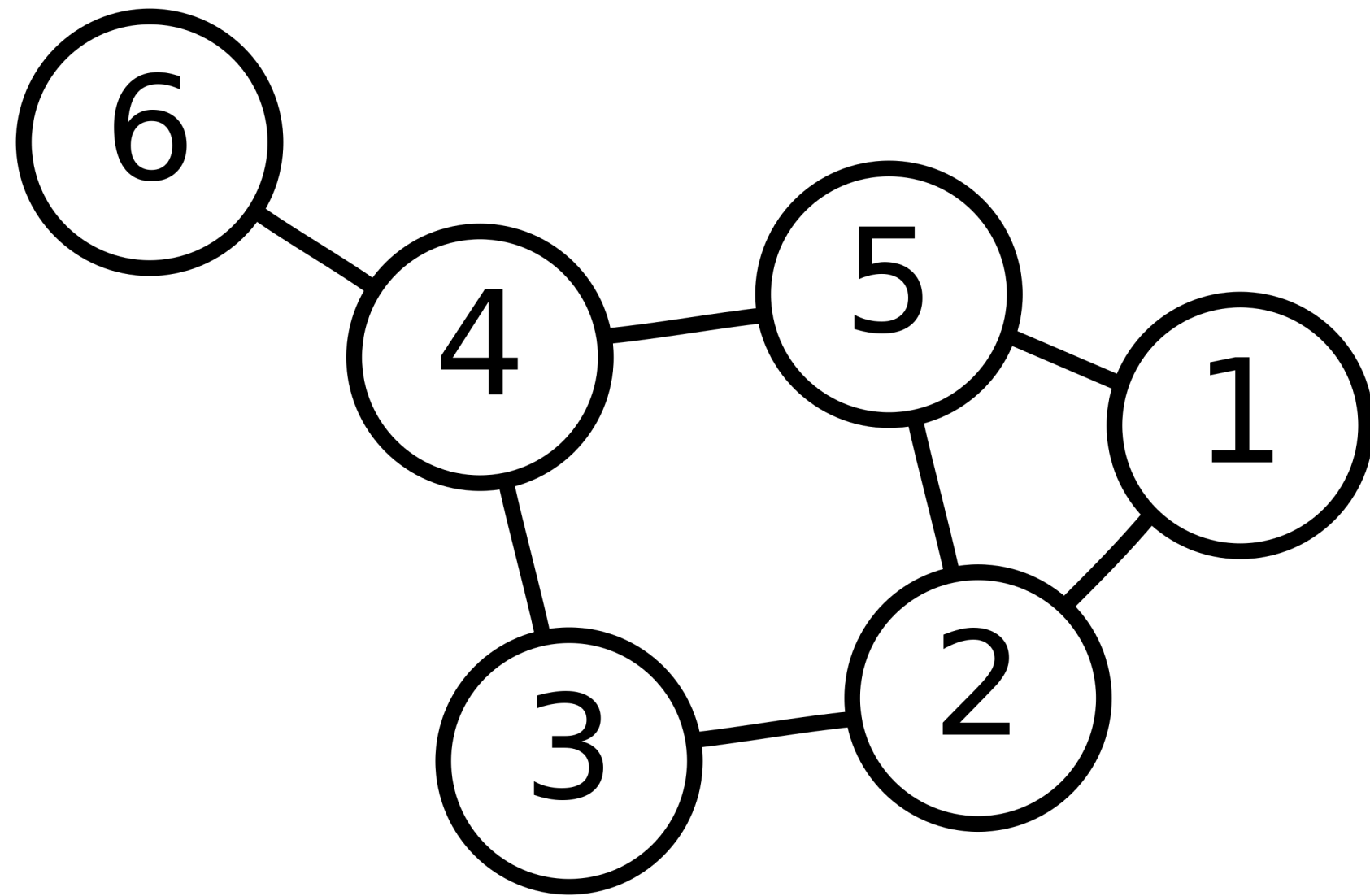


there is an edge
from B to A

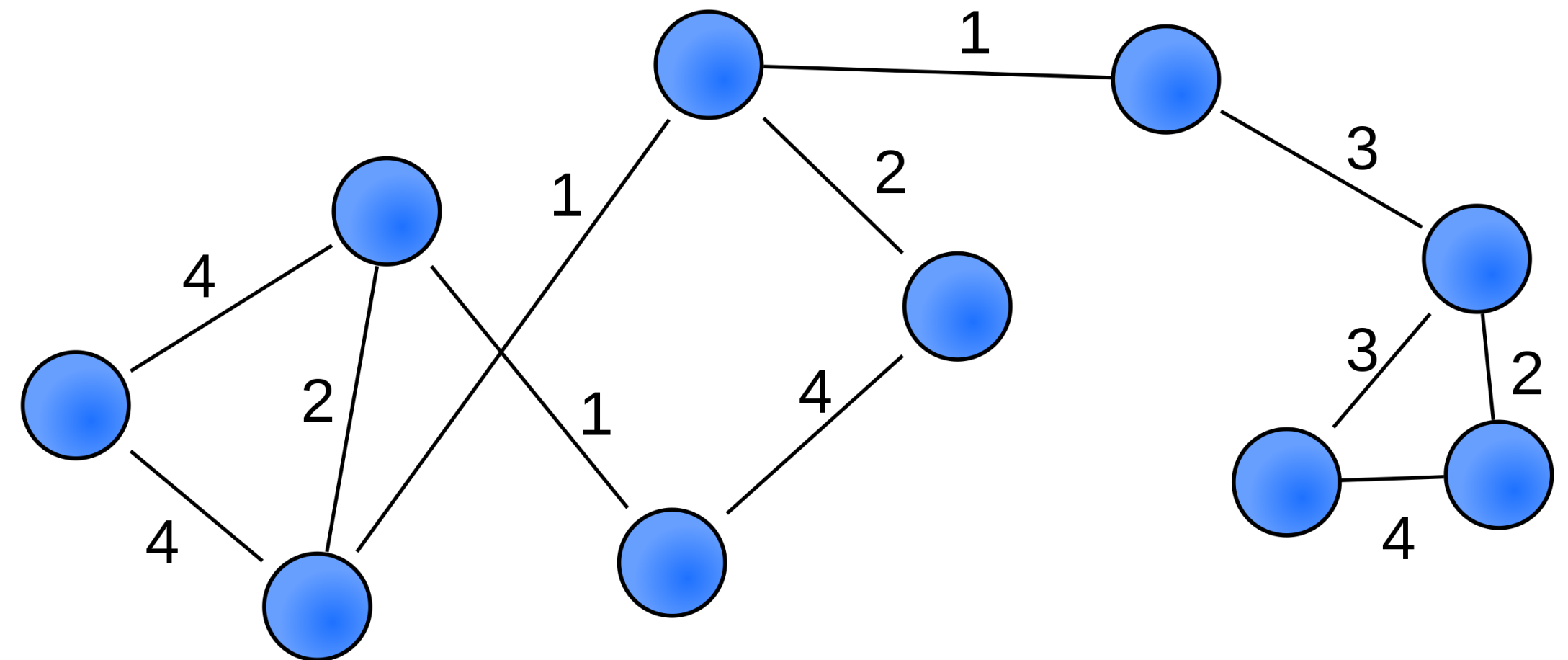
directed

Weighted vs Unweighted graphs

A graph is **weighted** if its edges have associated values.



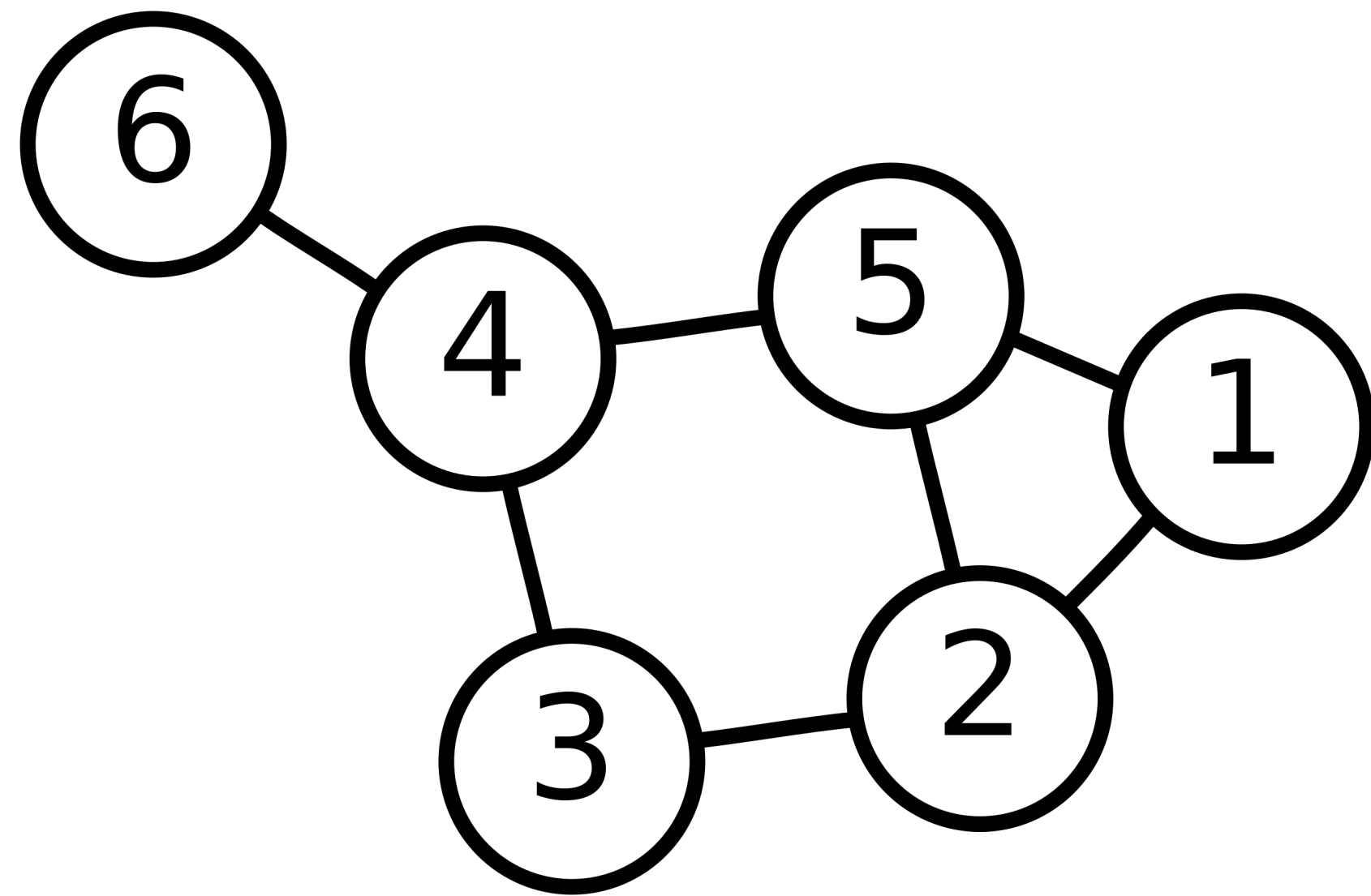
unweighted



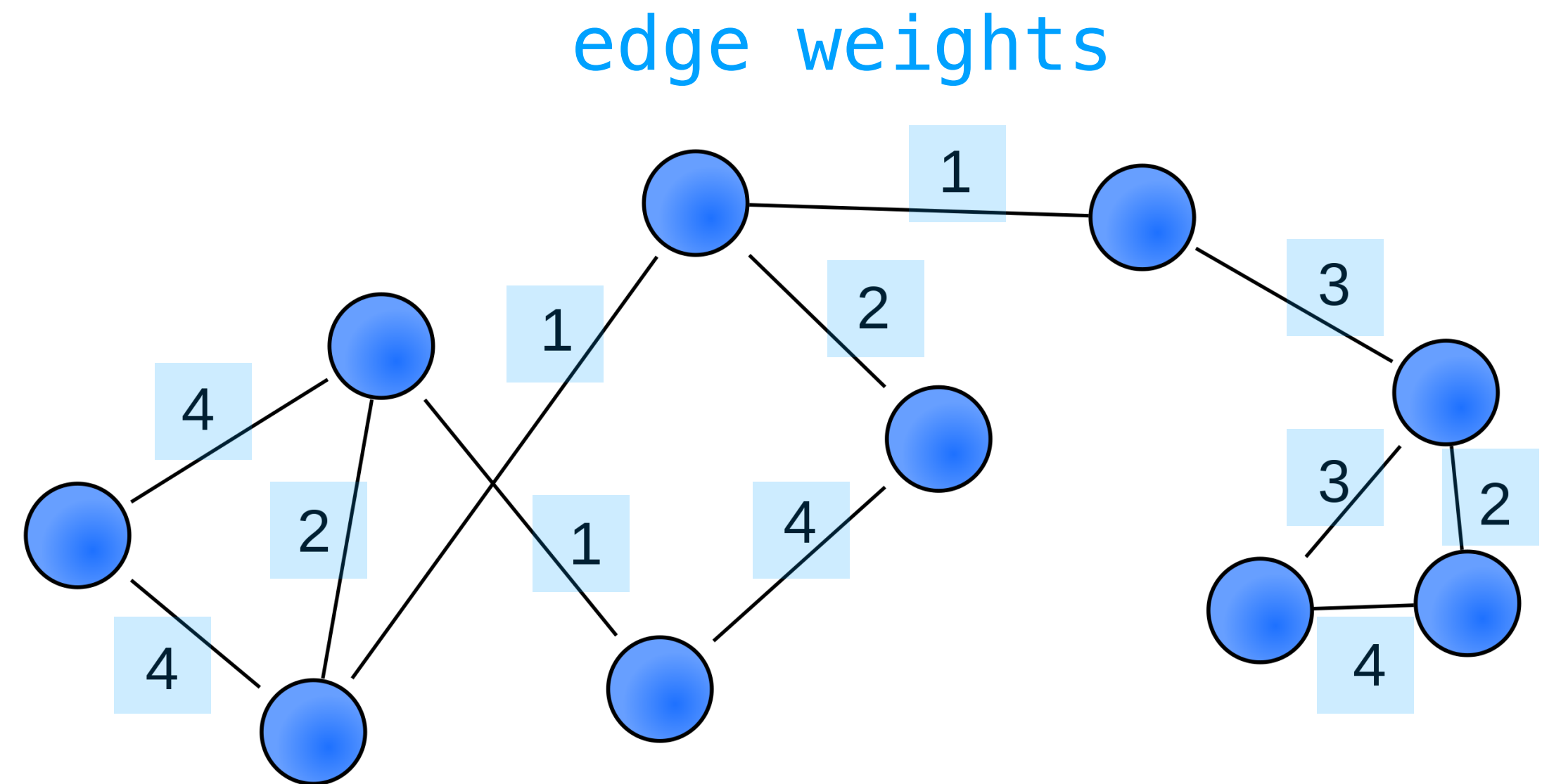
weighted

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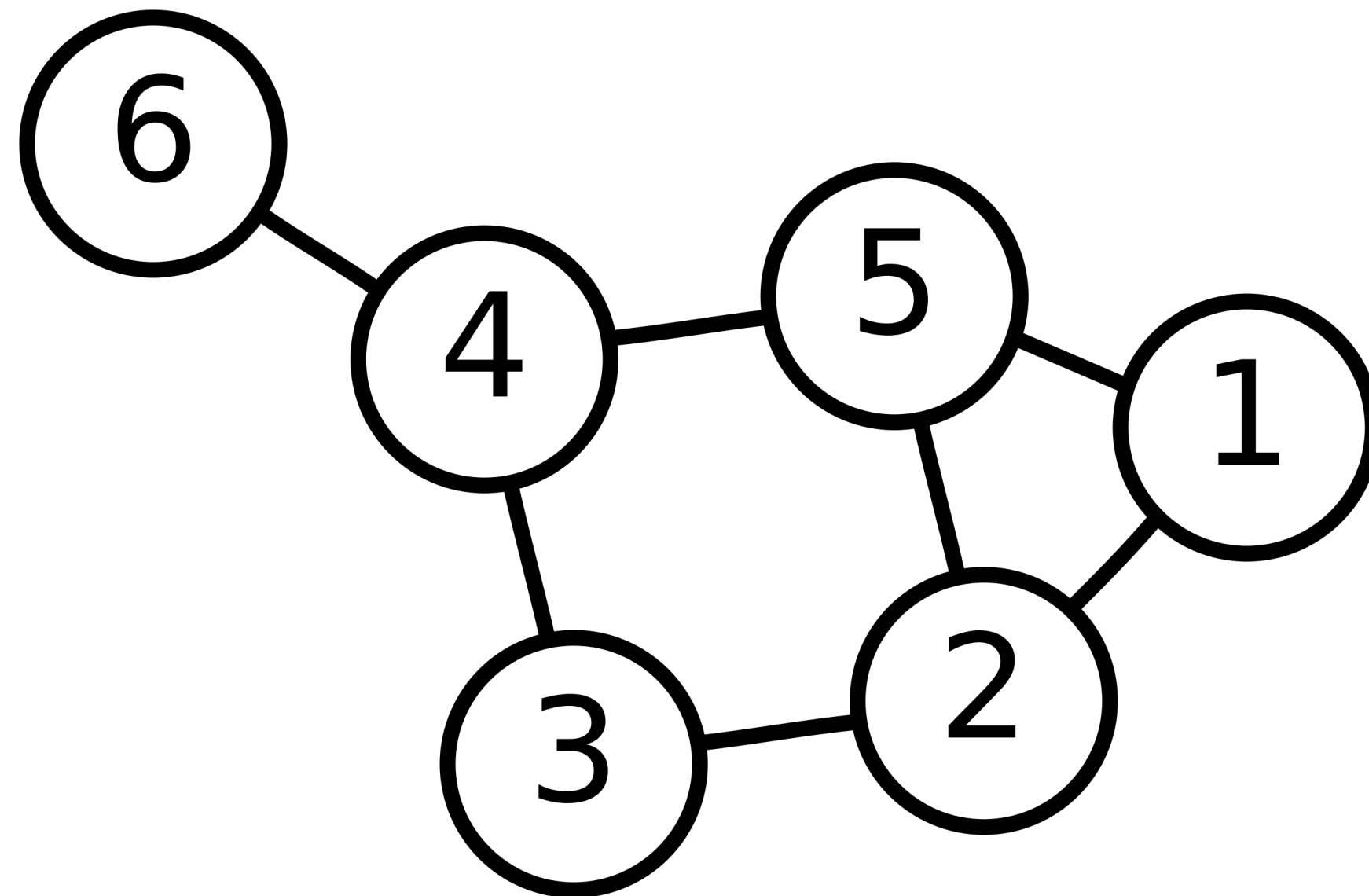
unweighted



weighted

Simple Graphs

A graph is **simple** if it is undirected, has no self loops, and no multi-edges.



Four Kinds of Graphs

directed

undirected

weighted

nodes are traffic lights
edges are streets
weights are number of lanes

nodes are musicians
edges are collaborations
weights are number of collaborations

unweighted

nodes are instagram users
edges are follows

nodes are bodies of land
edges are pedestrian bridges

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Today

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Markov Chains

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Today

Fundamental Question

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How do we represent a graph formally in a computer?

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There are a couple ways, but one way is to use matrices.

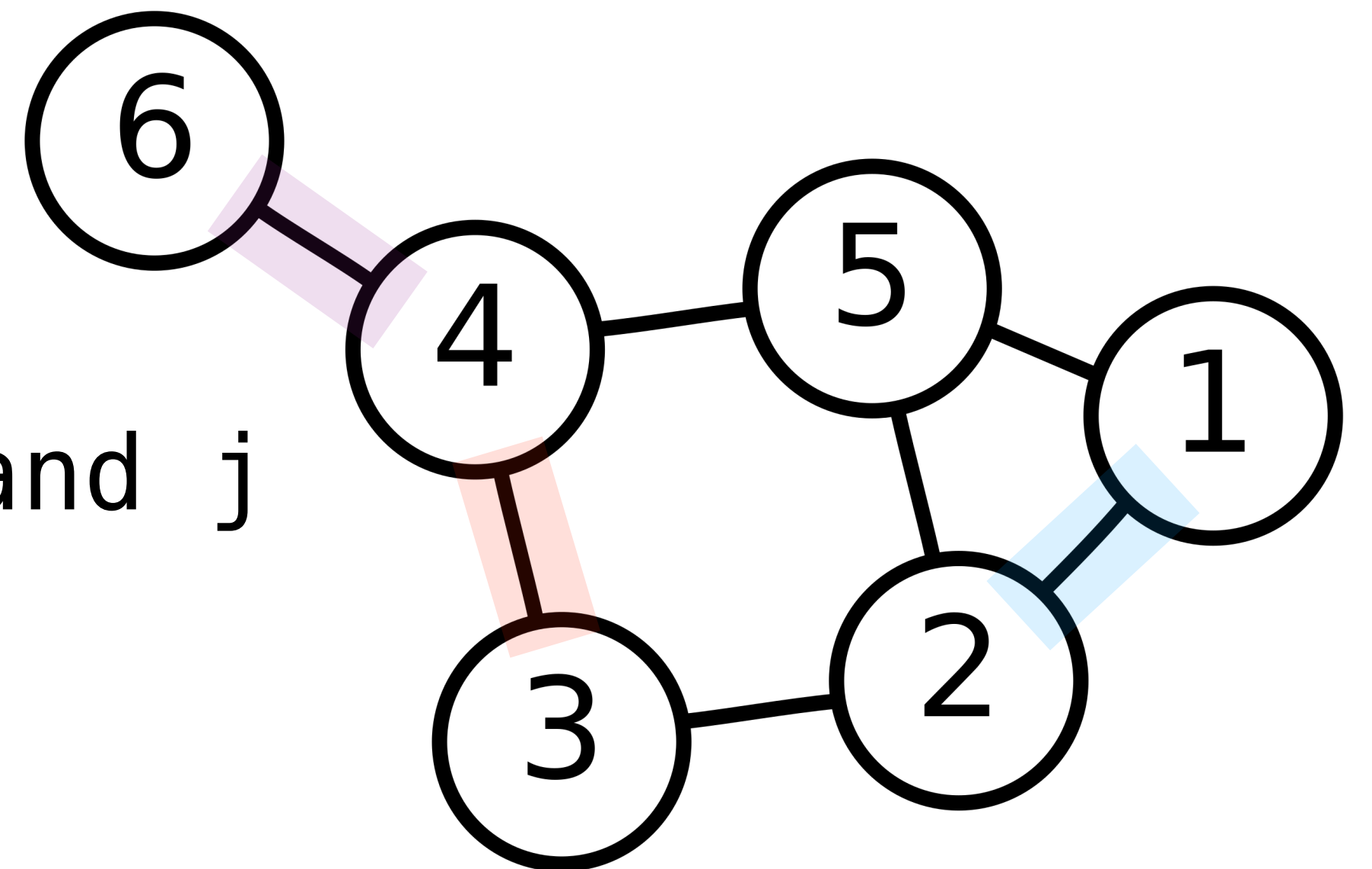
Adjacency Matrices

Let G be an simple graph with its nodes labeled by numbers 1 through n .

We can create the **adjacency matrix** A for G as follows.

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{matrix} & & A_{12} & & A_{34} & & A_{46} \\ A_{21} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$



Symmetric Matrices

Definition. A $n \times n$ matrix is **symmetric** if

$$A^T = A$$

Example.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Algebraic Graph Theory

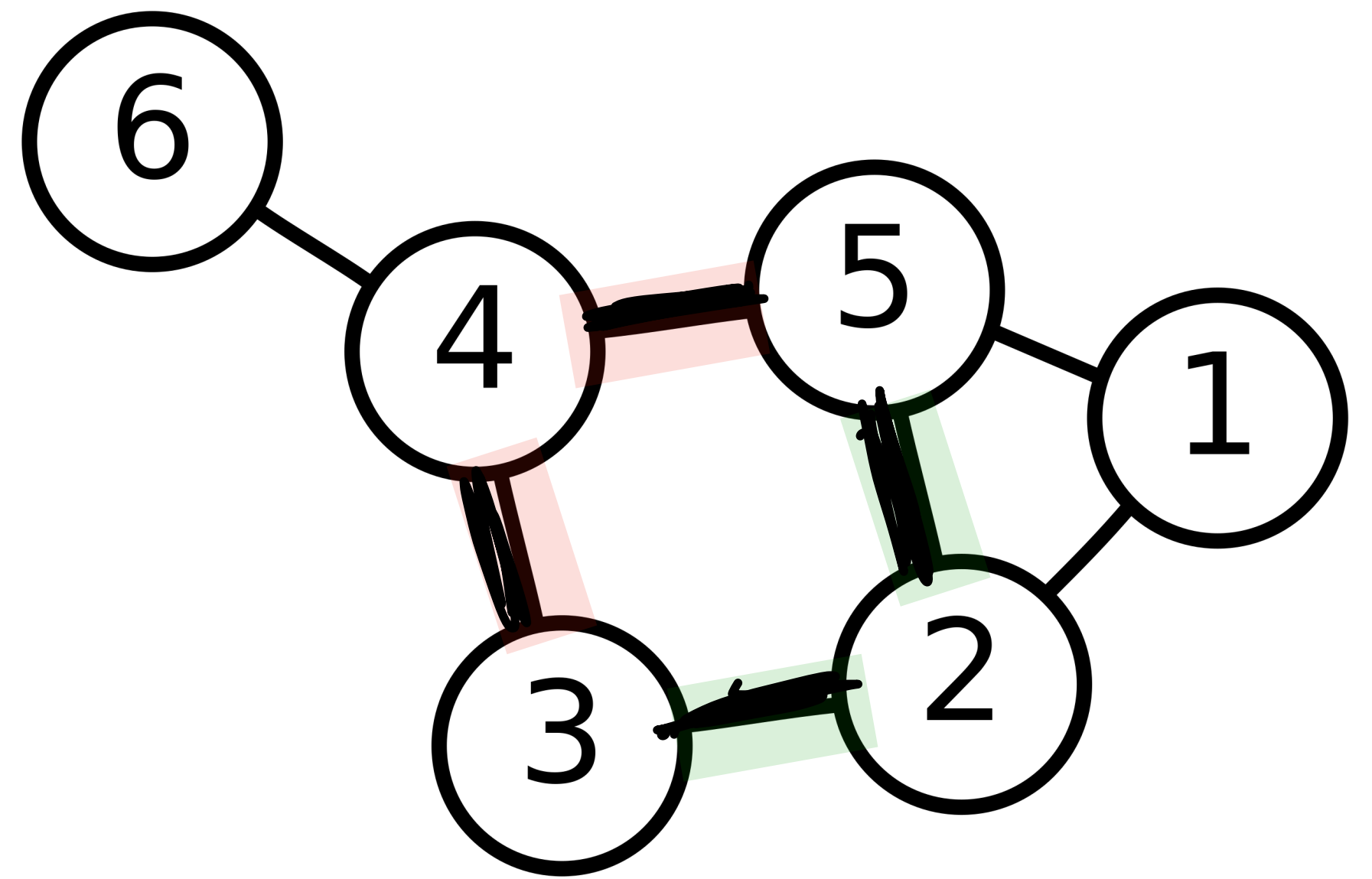
Once we have an adjacency matrix, we can do linear algebra on graphs.

Example: Squared Adjacency Matrices

Given an adjacency matrix A , can we interpret anything meaningful from A^2 ?

Example: Squared Adjacency Matrices

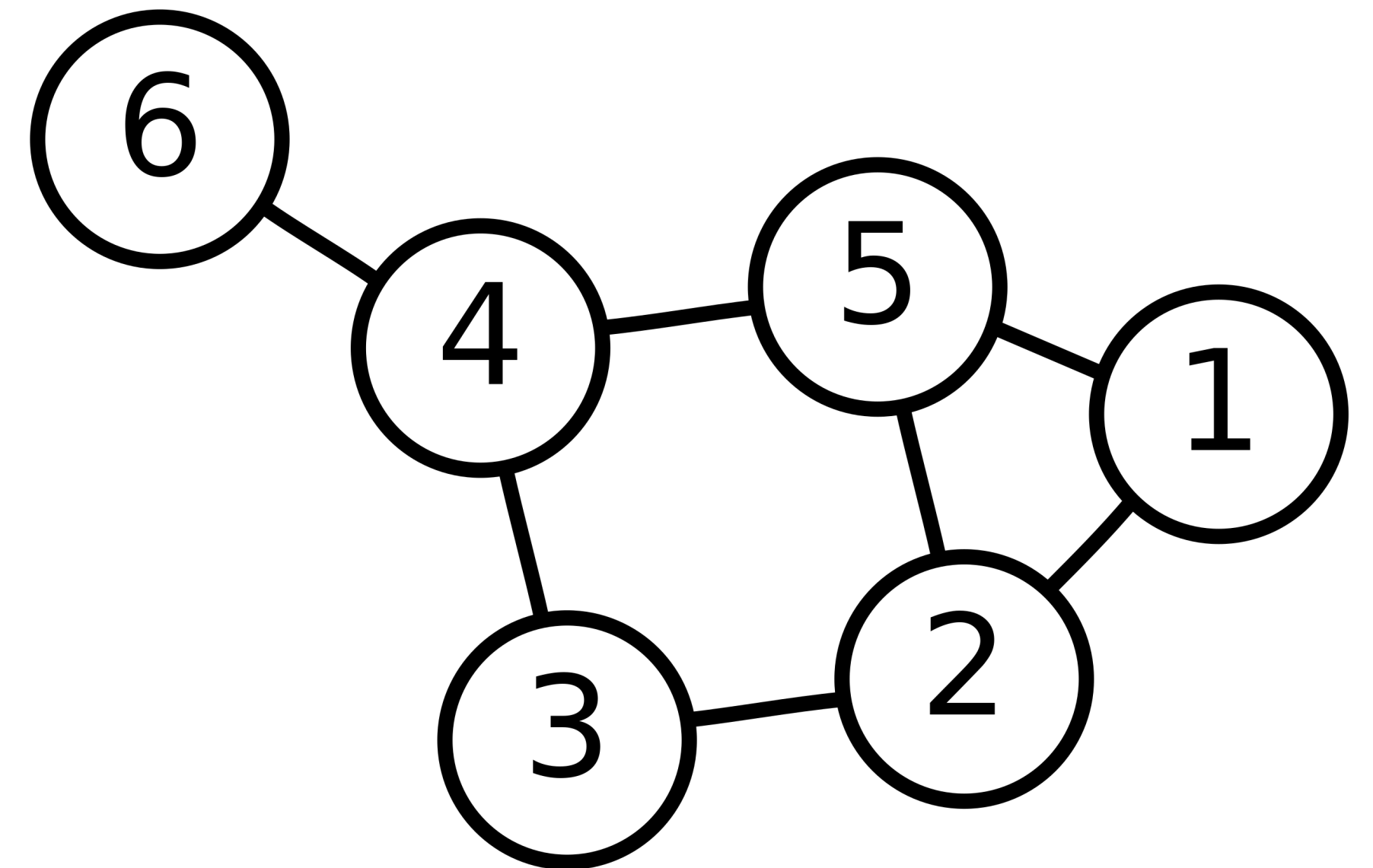
$$\begin{array}{c}
 \begin{matrix} & & & & 3 & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 5 & & & & & & \\ & & & & & & \end{matrix}
 \begin{bmatrix}
 0 & 1 & 0 & 0 & 1 & 0 \\
 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0
 \end{bmatrix}
 \begin{matrix}
 & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{matrix}
 \begin{bmatrix}
 0 & 1 & 0 & 0 & 1 & 0 \\
 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 \\
 1 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0
 \end{bmatrix}
 \end{array}$$



$$(A^2)_{53} = 1(0) + 1(1) + 0(0) + 1(1) + 0(0) + 0(0) = 2$$

Example: Squared Adjacency Matrices

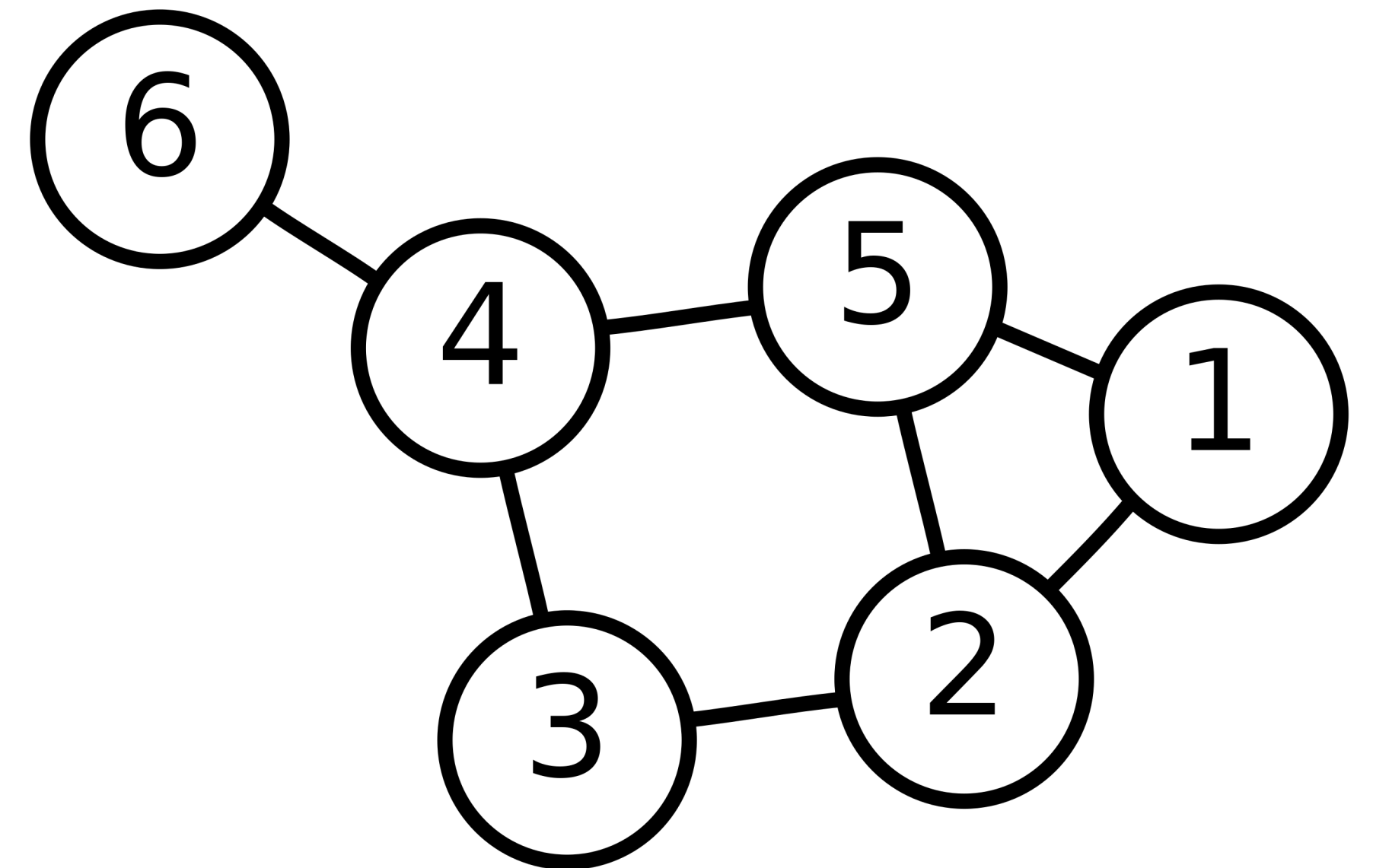
$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \dots + A_{in}A_{nj}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \dots + A_{in}A_{nj}$$

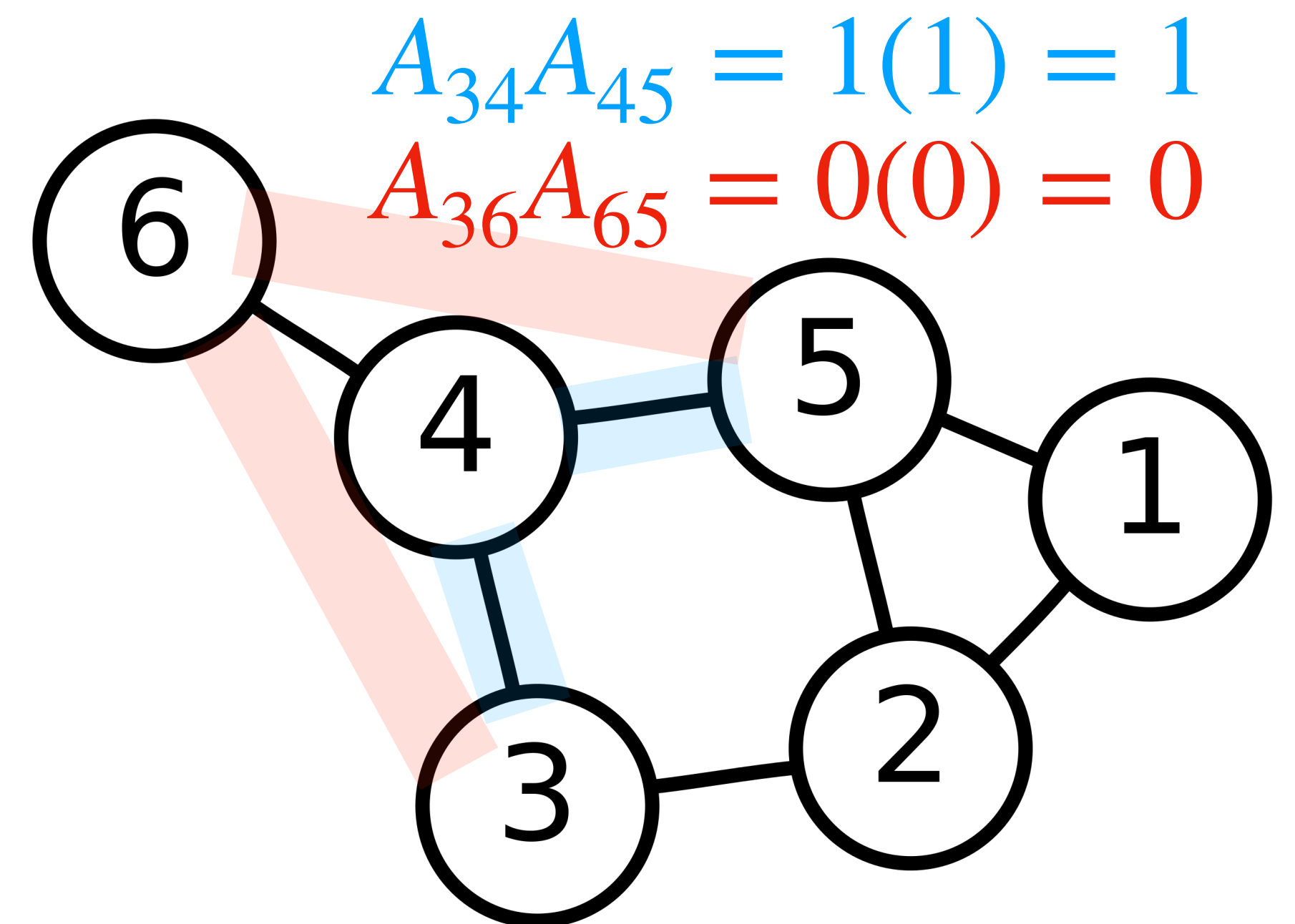
$$A_{ik}A_{kj} = \begin{cases} 1 & \text{there are edges } i \text{ to } k \text{ and } k \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \dots + A_{in}A_{nj}$$

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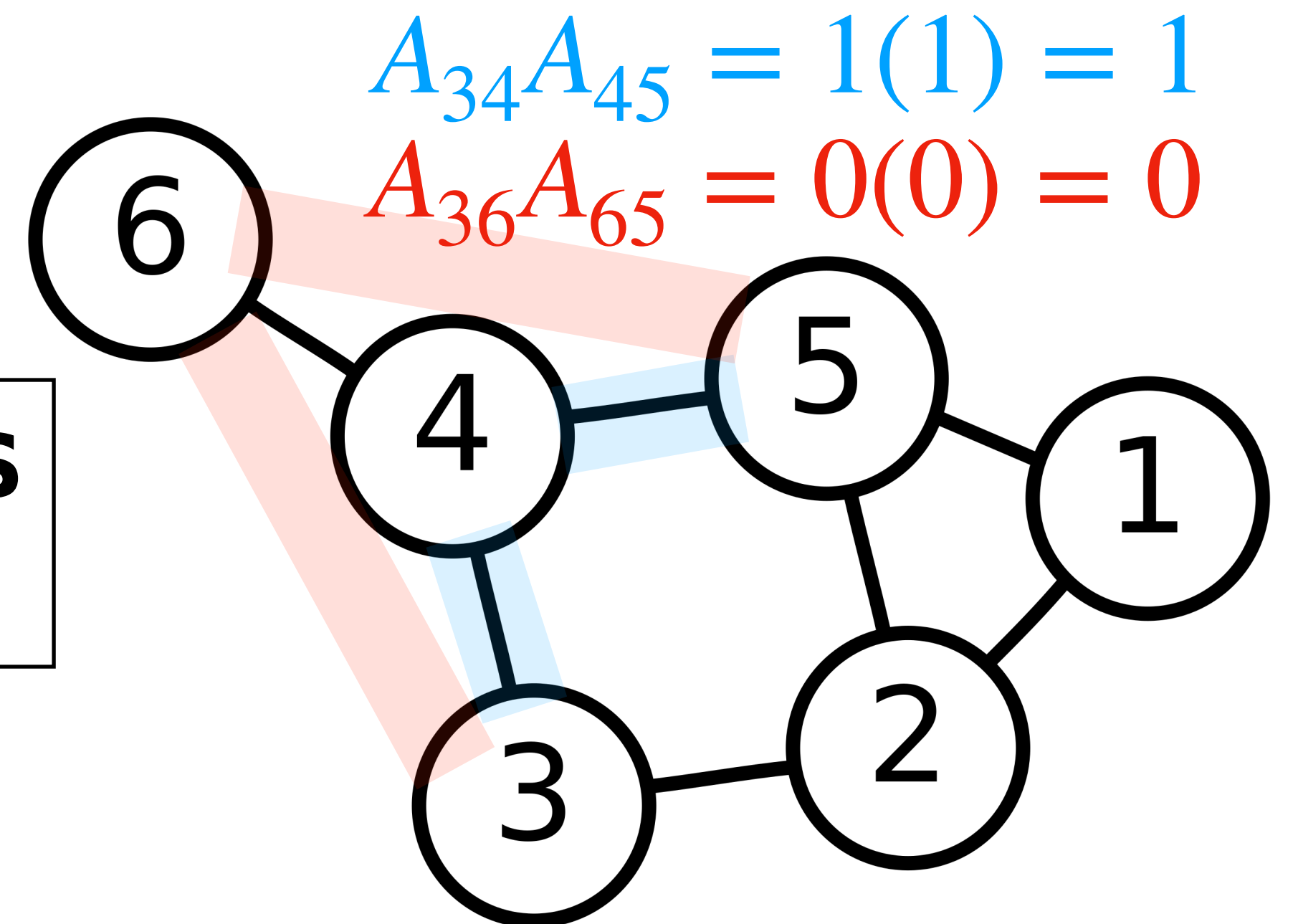


Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \dots + A_{in}A_{nj}$$

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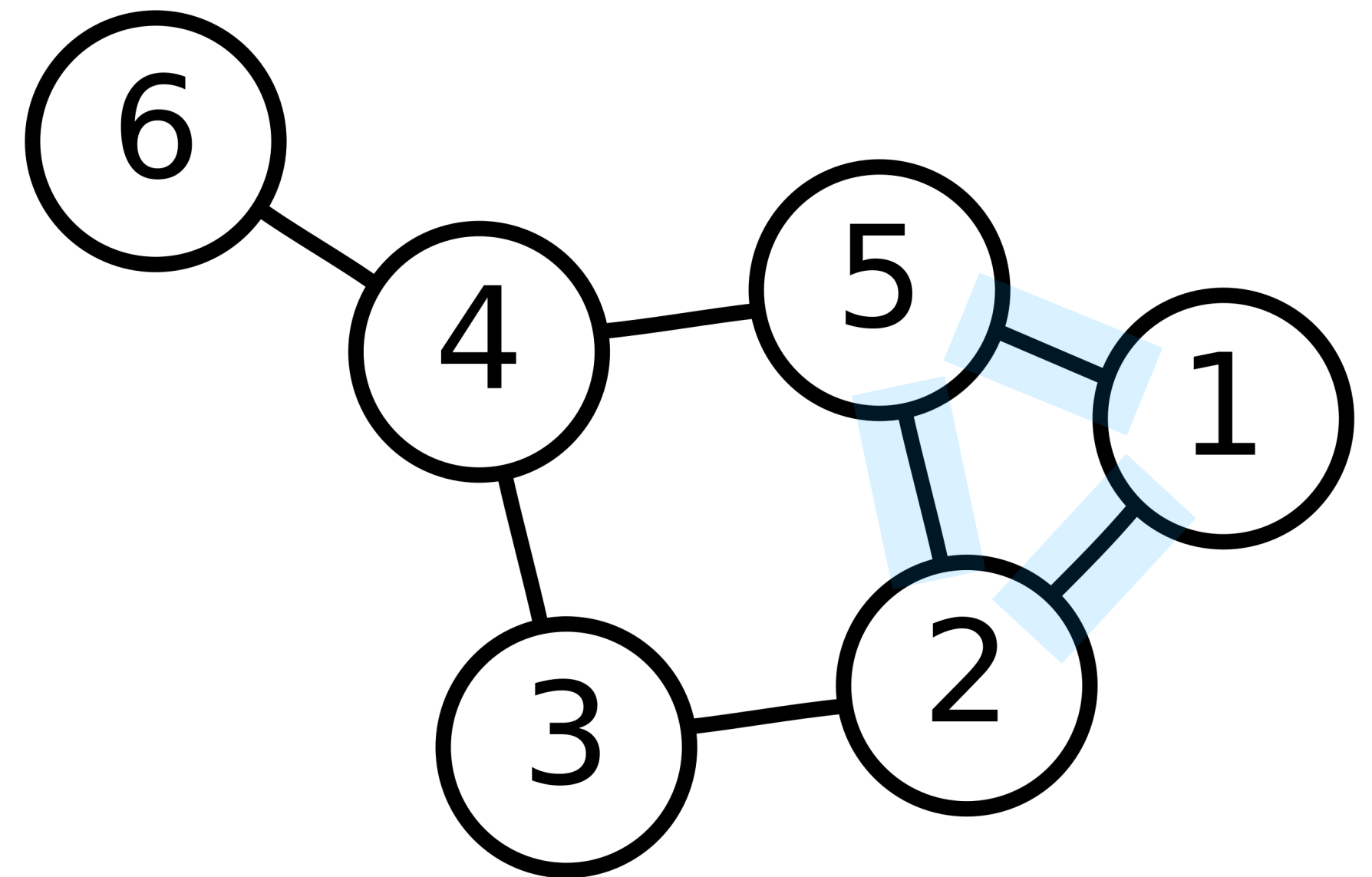
$$(A^2)_{ij} = \text{number of 2-step paths from } i \text{ to } j$$



Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)



Application: Triangle Counting (Naive)

```
FUNCTION tri_count_naive(A):
```

```
    count = 0
```

```
    for i from 1 to n:
```

```
        for j from i + 1 to n:
```

```
            for k from j + 1 to n:
```

```
                if  $A_{ij} = 1$  and  $A_{jk} = 1$  and  $A_{ki} = 1$ : # an edge between each pair
```

```
                    count += 1:
```

```
RETURN count
```

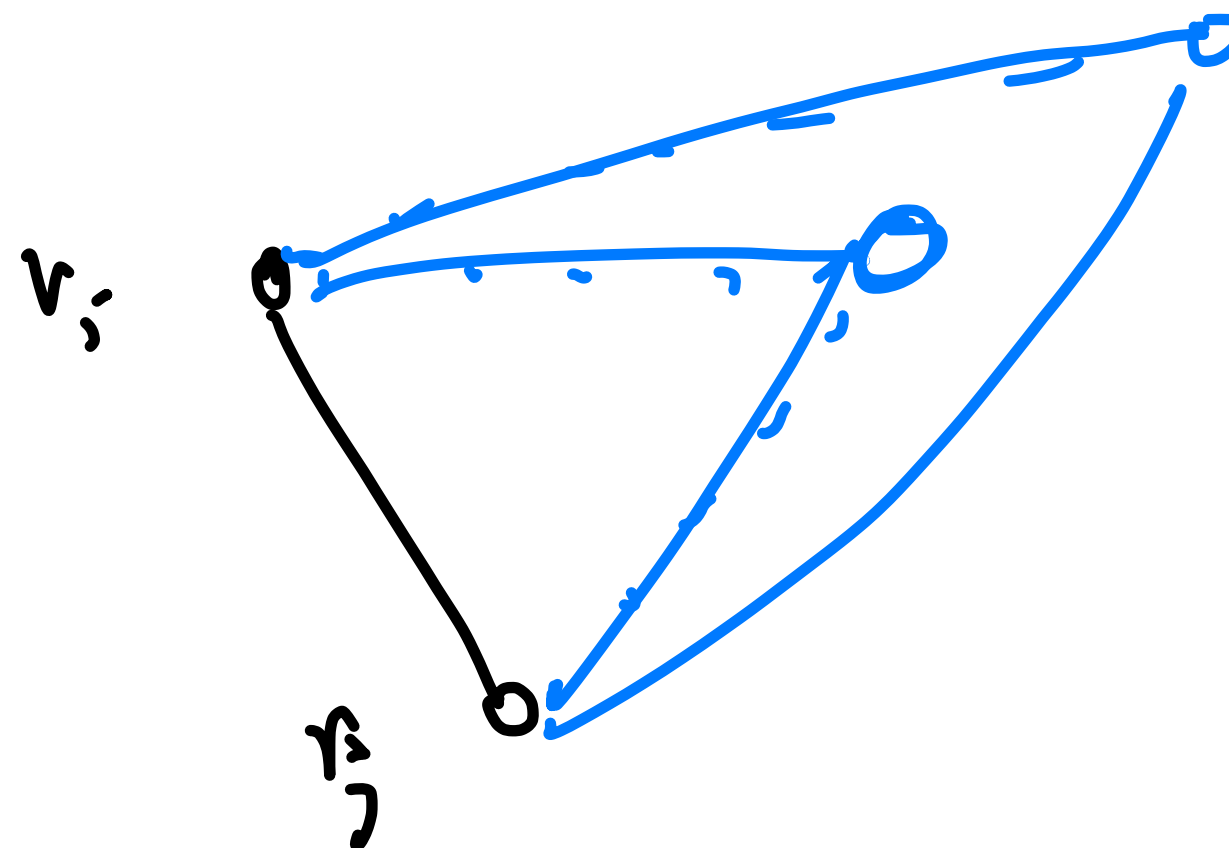

Application: Triangle Counting

Theorem. For an adjacency matrix A , the number of triangle containing the edge (i,j) is

know why
paths

$$(A^2)_{ij} * A_{ij}$$

Verify:



Application: Triangle Counting

FUNCTION tri_count(A):

compute A^2

count \leftarrow sum of $(A^2)_{ij} * A_{ij}$ for all distinct i and j

RETURN count / 6 # why divided by 6?

Application: Triangle Counting

```
FUNCTION tri_count(A):
```

```
# in NumPy '*' is entry-wise multiplication
```

```
#      also called the HADAMARD PRODUCT
```

```
count ← sum of the entries of  $A^2 * A$ 
```

```
RETURN count / 6
```

Application: Triangle Counting

```
FUNCTION tri_count(A):
```

```
# in NumPy '*' is entry-wise multiplication
```

```
# also called the HADAMARD PRODUCT
```

```
# and 'np.sum' sums the entry of a matrix
```

```
RETURN np.sum((A @ A) * A) / 6
```

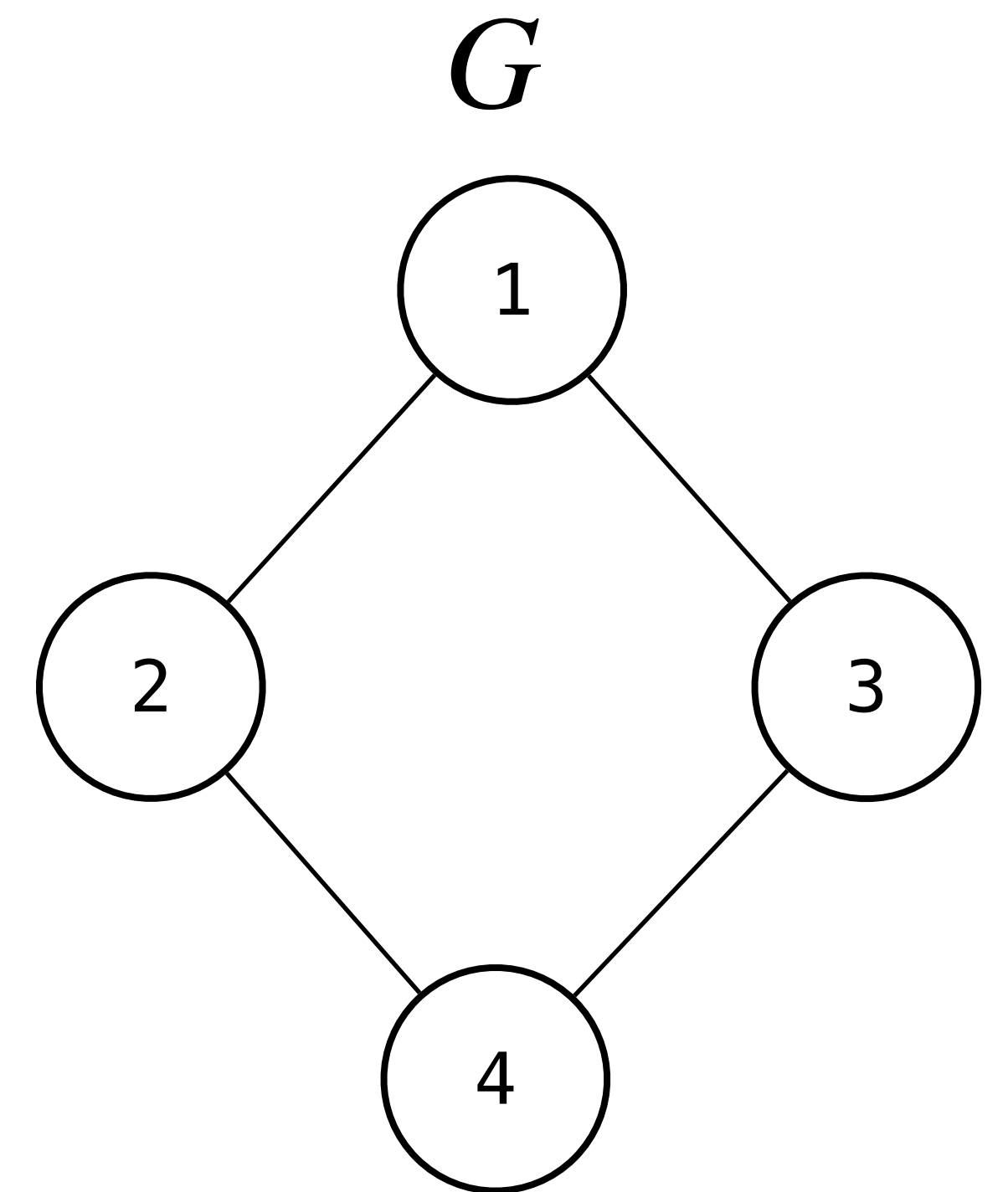
demo

Another Application: Reachability

Question: If A^2 gives us information about length 2 paths, then what about A^k ?

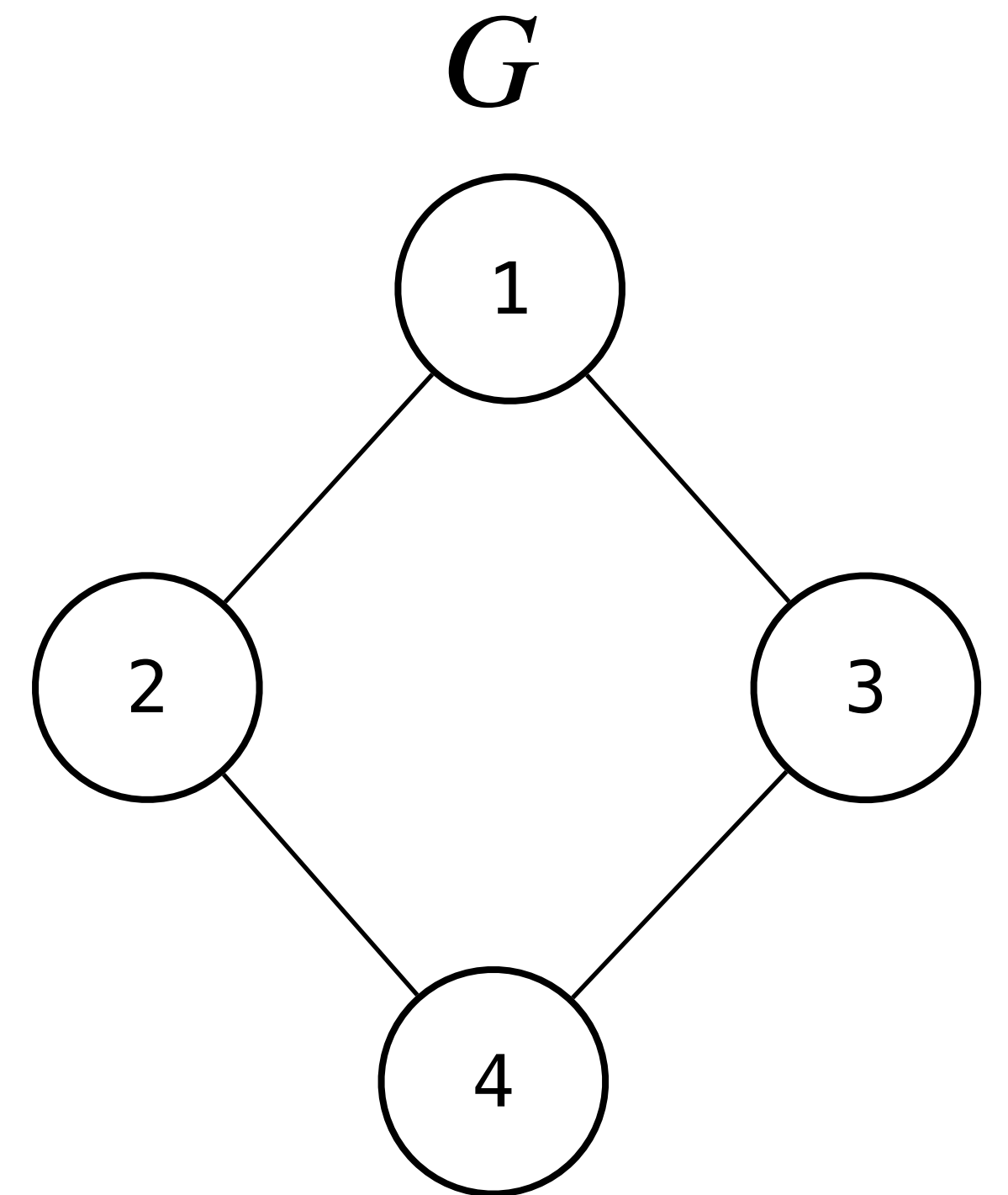
A^k gives us information about k -length paths.

Example



Example

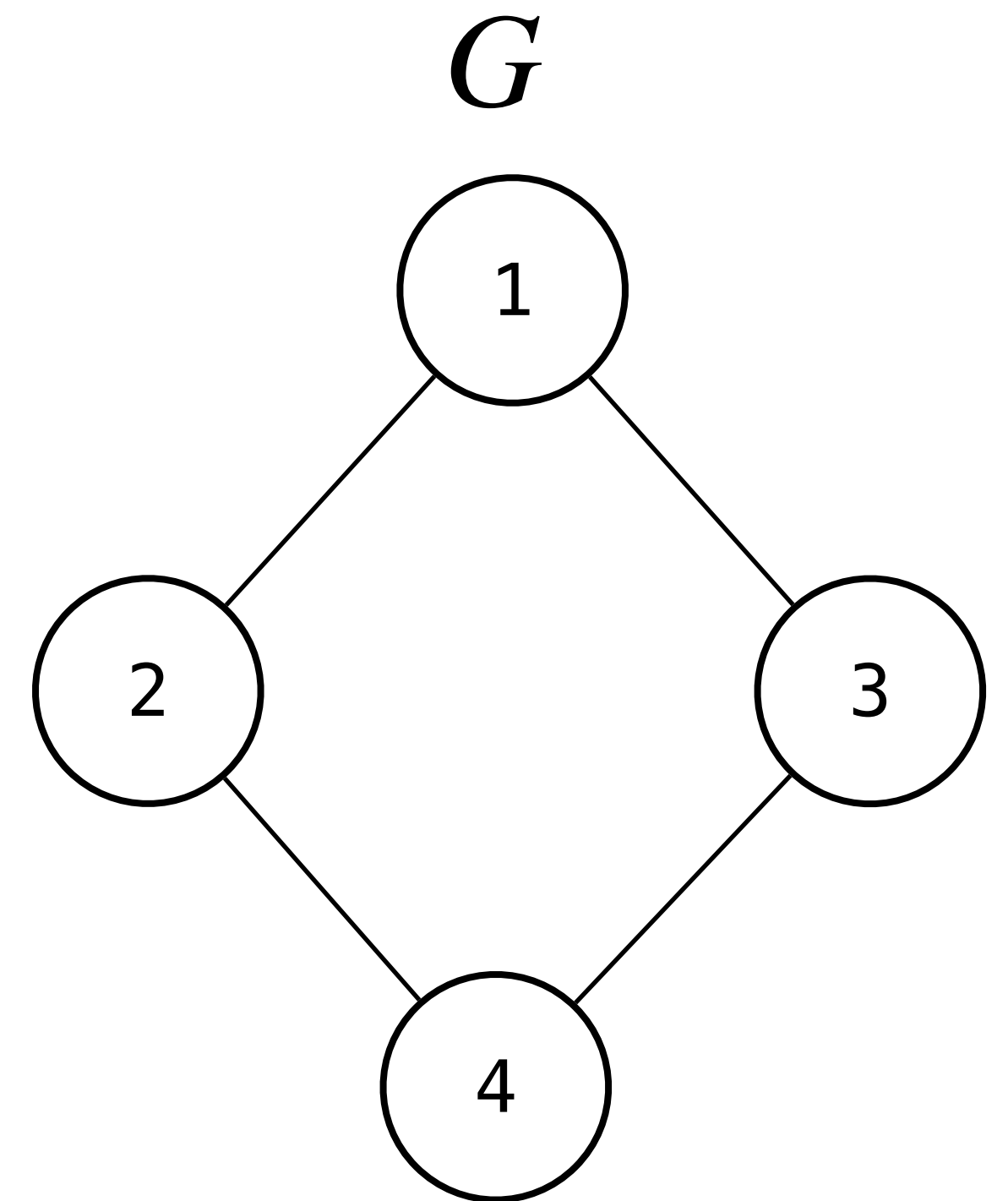
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$



Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

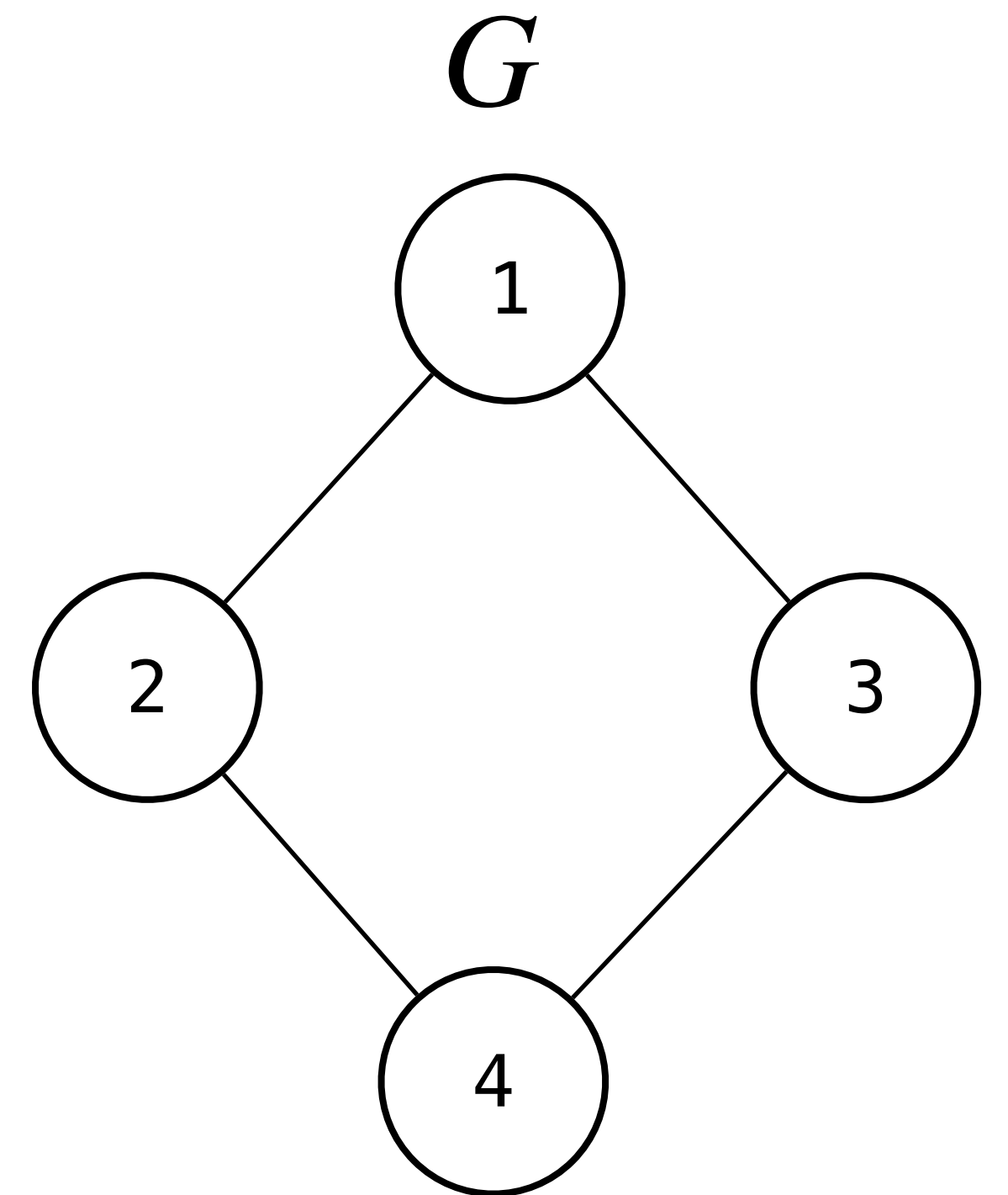


Example

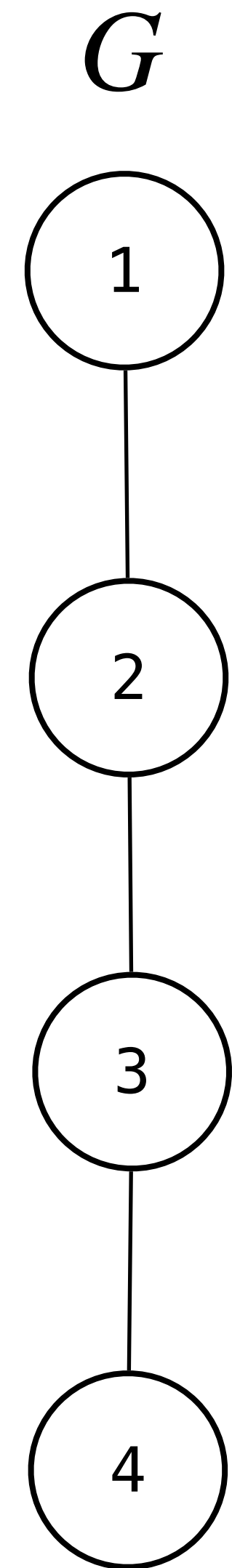
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

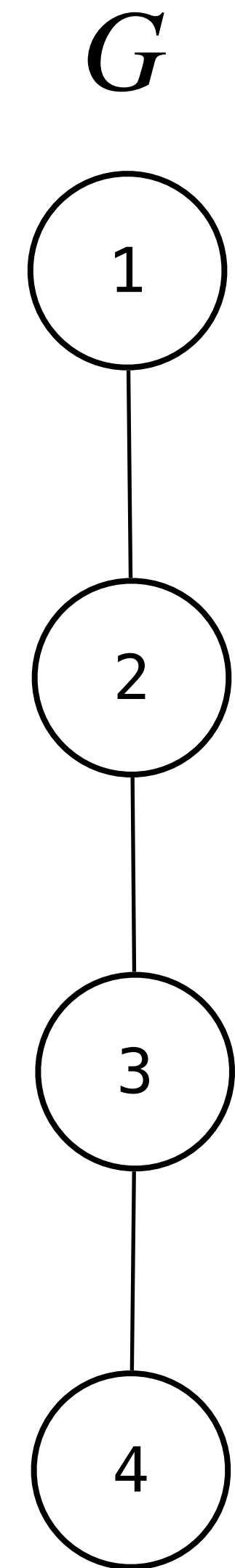


Example



Example

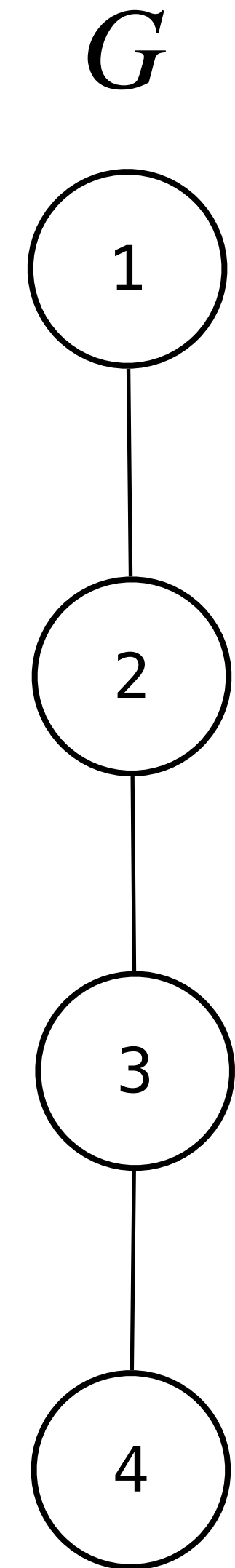
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$



Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

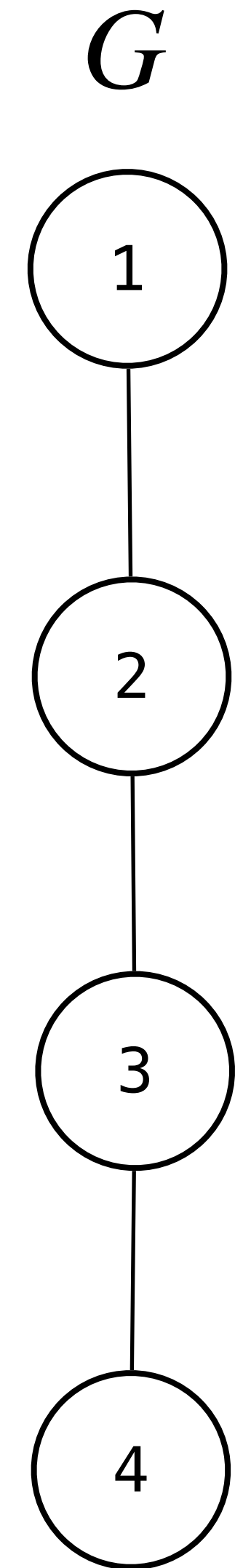


Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

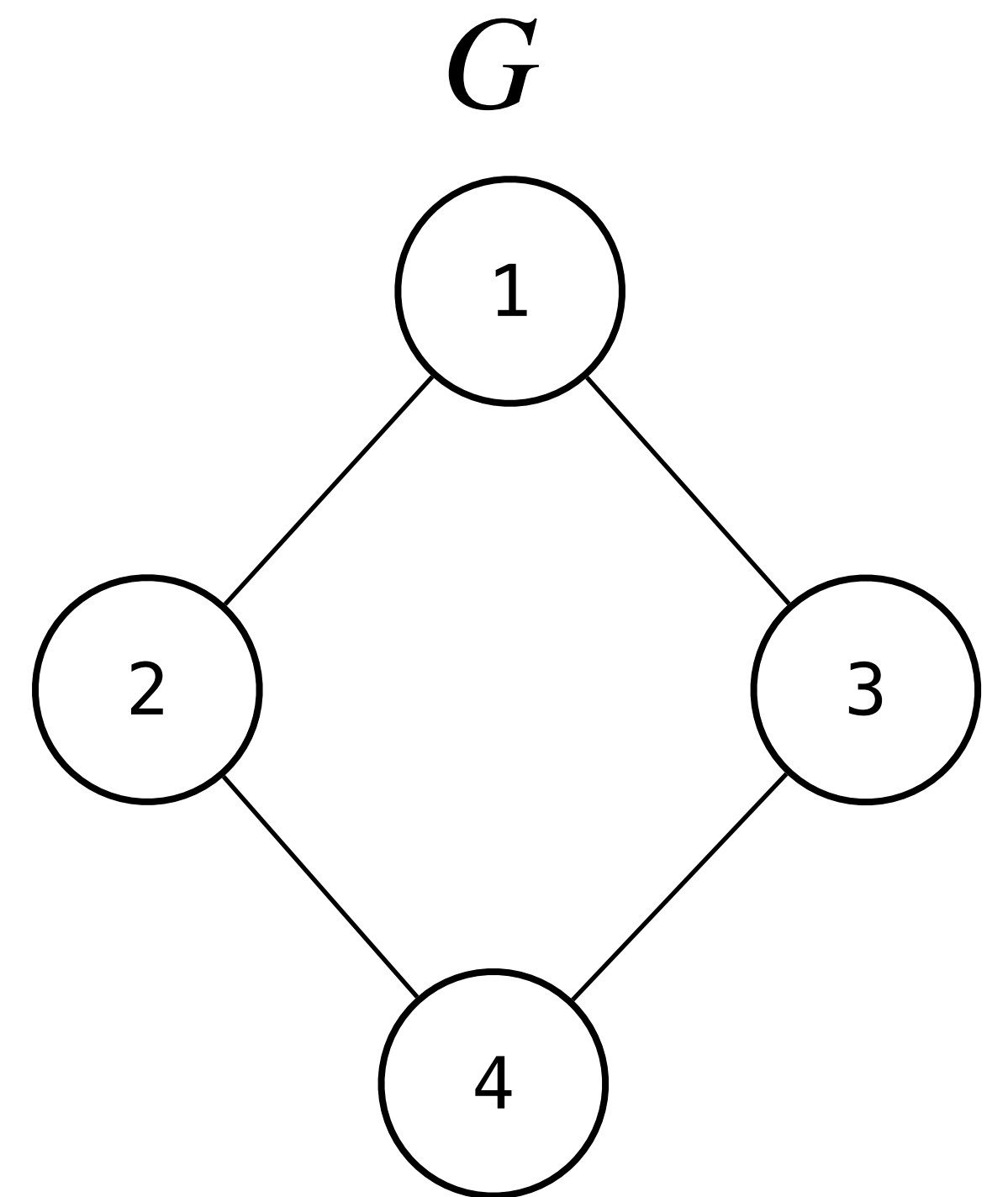


Another Application: Reachability

Theorem: Let G be a simple graph.

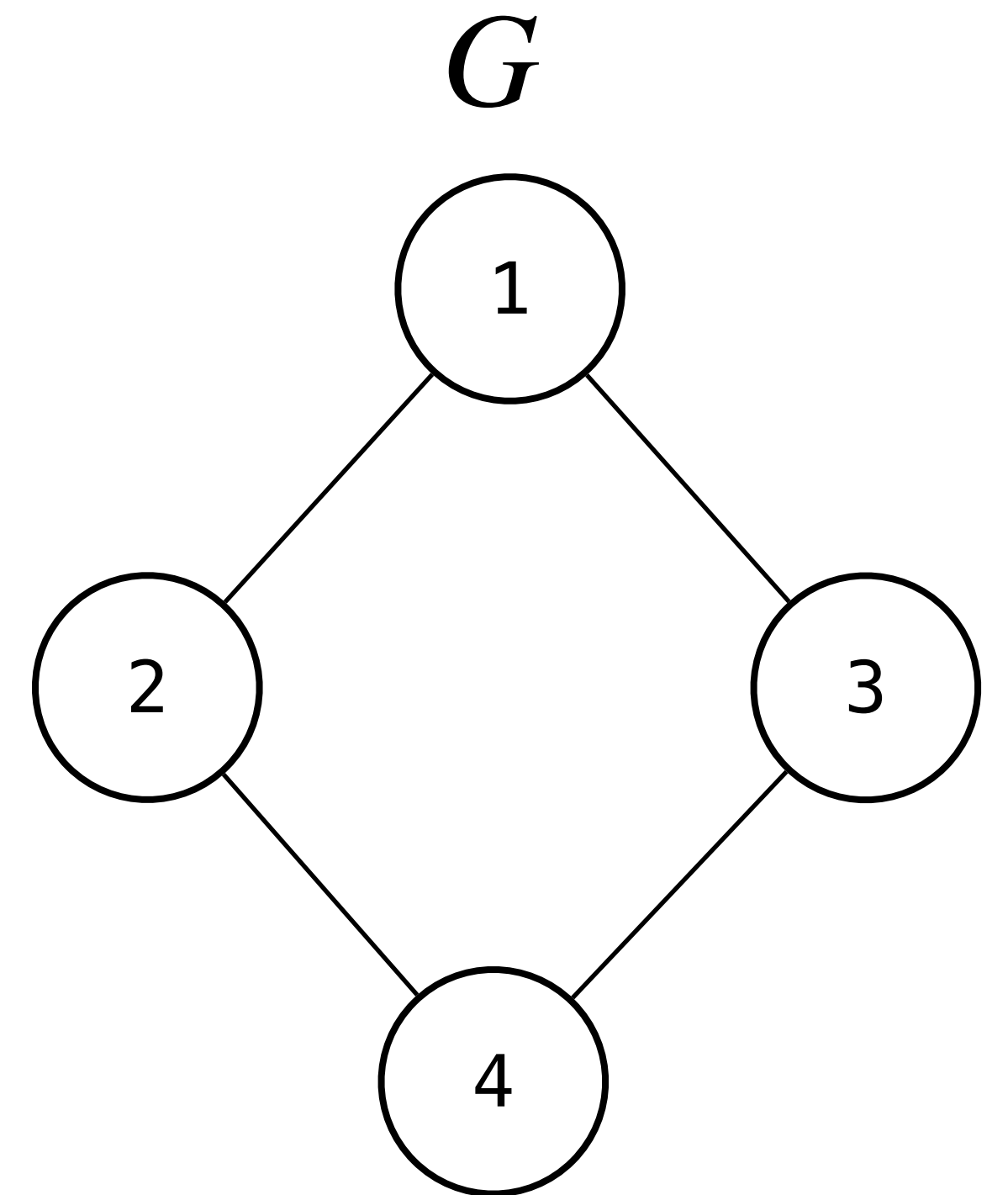
- $(A_G^k)_{ij}$ is the number of paths of length **exactly** k from v_i to v_j .
- ~~$((A_G + I)^k)_{ij}$ is the number of paths of length **at**~~
~~**most** k from~~ v_i is nonzero if and only if there
is a path of length at most k from v_i to
 v_j

Example



Example

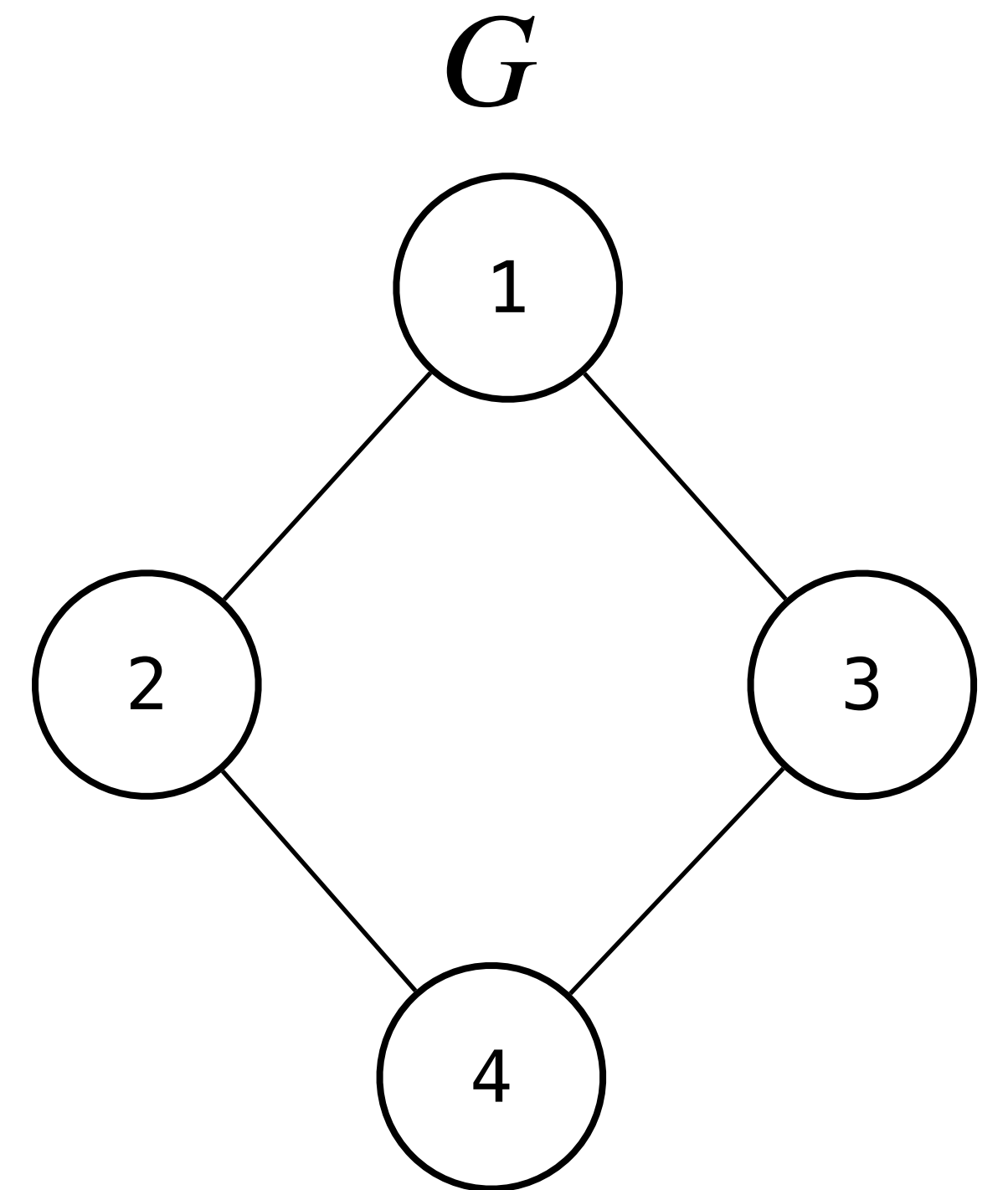
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = (\text{adjacency matrix for } G) + I$$



Example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = (\text{adjacency matrix for } G) + I$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

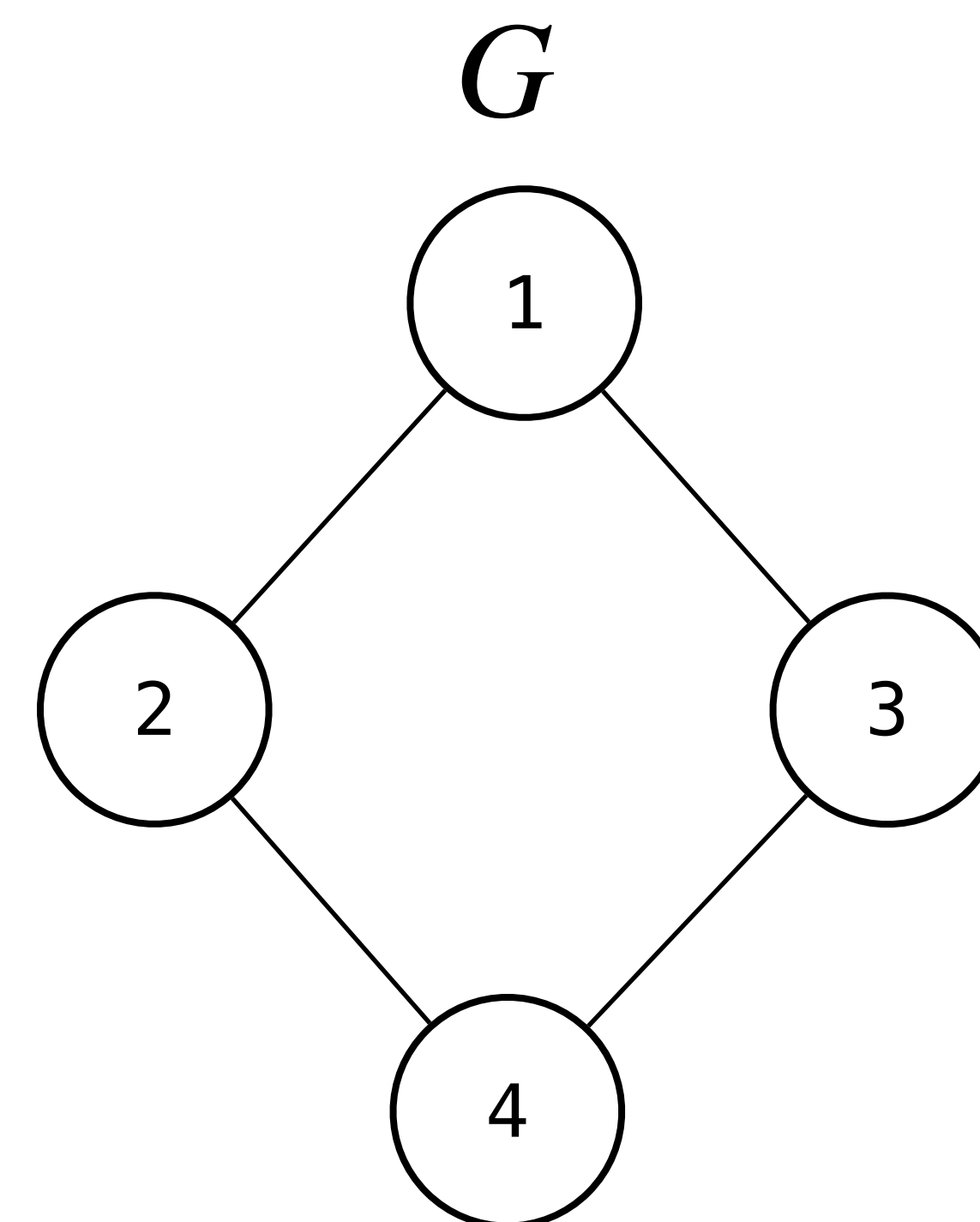


Example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = (\text{adjacency matrix for } G) + I$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 & 7 \\ 6 & 6 & 7 & 6 \\ 6 & 7 & 6 & 6 \\ 7 & 6 & 6 & 6 \end{bmatrix}$$



How To: Reachability

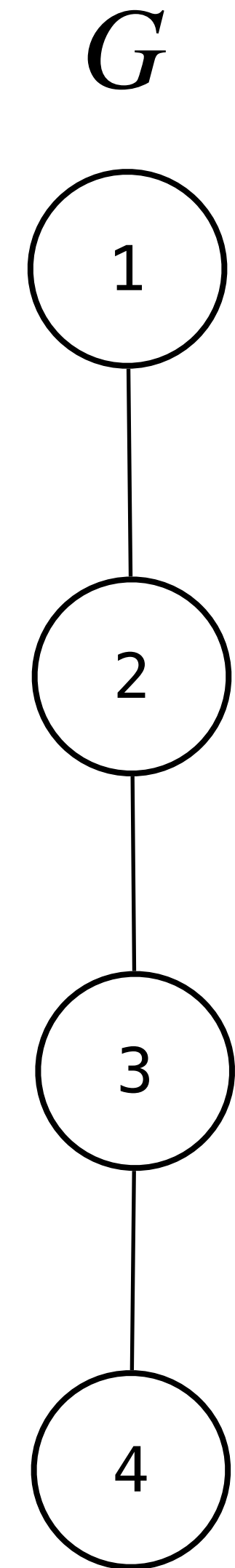
Question: Given a simple graph G determine how many nodes, v_i can reach in at least k steps.

Answer: Find $(A_G + I)^k$ and count the number of nonzero elements in column i .

(This could be useful for homework 6.)

Question

Determine the $(A_G + I)^2$ and $(A_G + I)^3$ and interpret the results.



Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.

Adjacency matrices are linear algebraic representations of graphs.