Invertible Matrix Theorem + Algebraic Graph Theory

Geometric Algorithms
Lecture 12

CAS CS 132

Objectives

1. Recap matrix inverses (it's been a while)
2. Finish up the algebra of matrix inverses
3. Connect everything we've talked about so far via the Invertible Matrix Theorem (IMT)
4. Connect linear algebra to graph theory

Keywords

matrix inverses

invertible matrix theorem

directed/undirected graphs

weighted/unweighted graphs

adjacency matrices

symmetric matrices

triangle counting

Recap: Matrix Inverse

Motivation

$$A\mathbf{x} = \mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by A?*"

Motivation

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

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$$\mathbf{x} = A^{-1}\mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by A?*"

Recall: Matrix Inverses

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Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ (and } BA = I_n\text{)}$$

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A is **invertible** if it has an inverse. Otherwise it is **singular**.

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of $\mathbf{b}$.

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» exactly one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

- at least one solution for any choice of $\mathbf{b}$
- at most one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

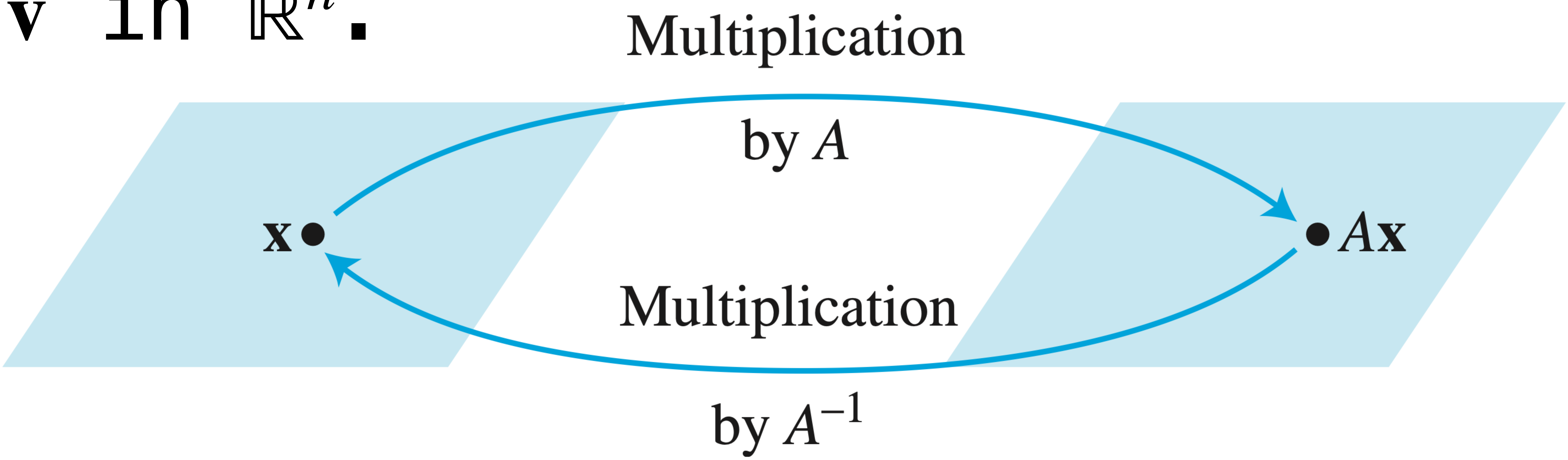
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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Non-Example. Projection onto the x_1-axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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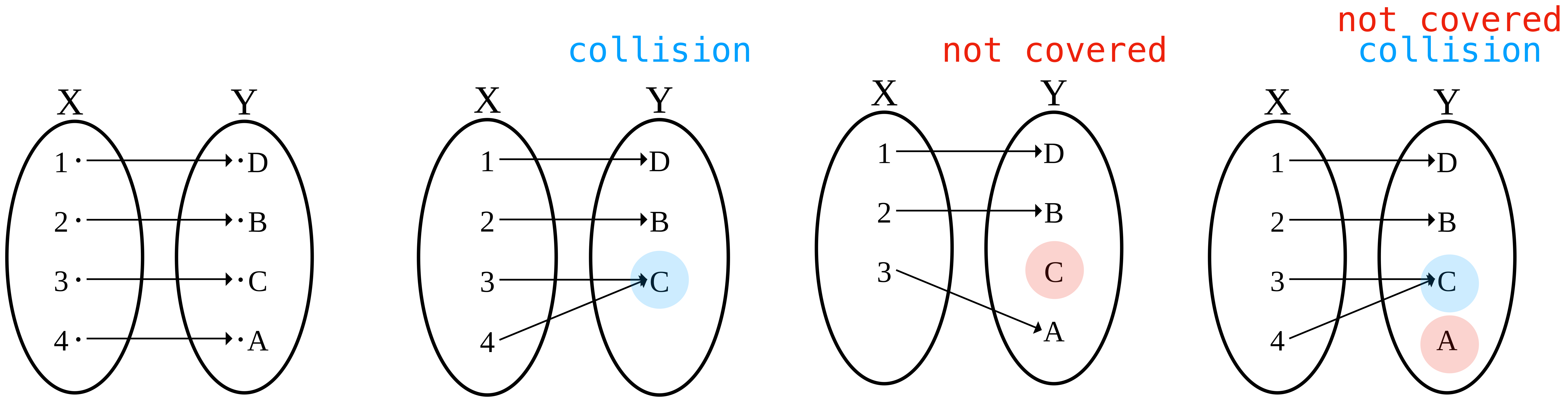
Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



1–1 correspondence   onto, not 1–1   1–1 not onto   not 1–1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$A\mathbf{b}_1 = \mathbf{e}_1 \qquad A\mathbf{b}_2 = \mathbf{e}_2 \qquad A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C, demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix.*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen.*

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$
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4. $A\mathbf{x} = \mathbf{b}$ has at exactly one solution for every $\mathbf{b}$

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
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These all express the **same thing**

(this is a stronger statement than we just verified)

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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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We only need to check one of these.

Warning. Remember this only applies square matrices.

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

We get a lot of information for free

Theorem. If A is square, then

$$A \textbf{ is invertible} \quad \equiv \quad A\mathbf{x} = \mathbf{0} \textbf{ implies } \mathbf{x} = \mathbf{0}$$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

***True** or **False**: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), the B is also invertible.*

Answer: True

```
Row reductions don't change the number of
pivots.
```

Question

If $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ *is invertible, then is* $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]$ *also invertible? Justify your answer.*

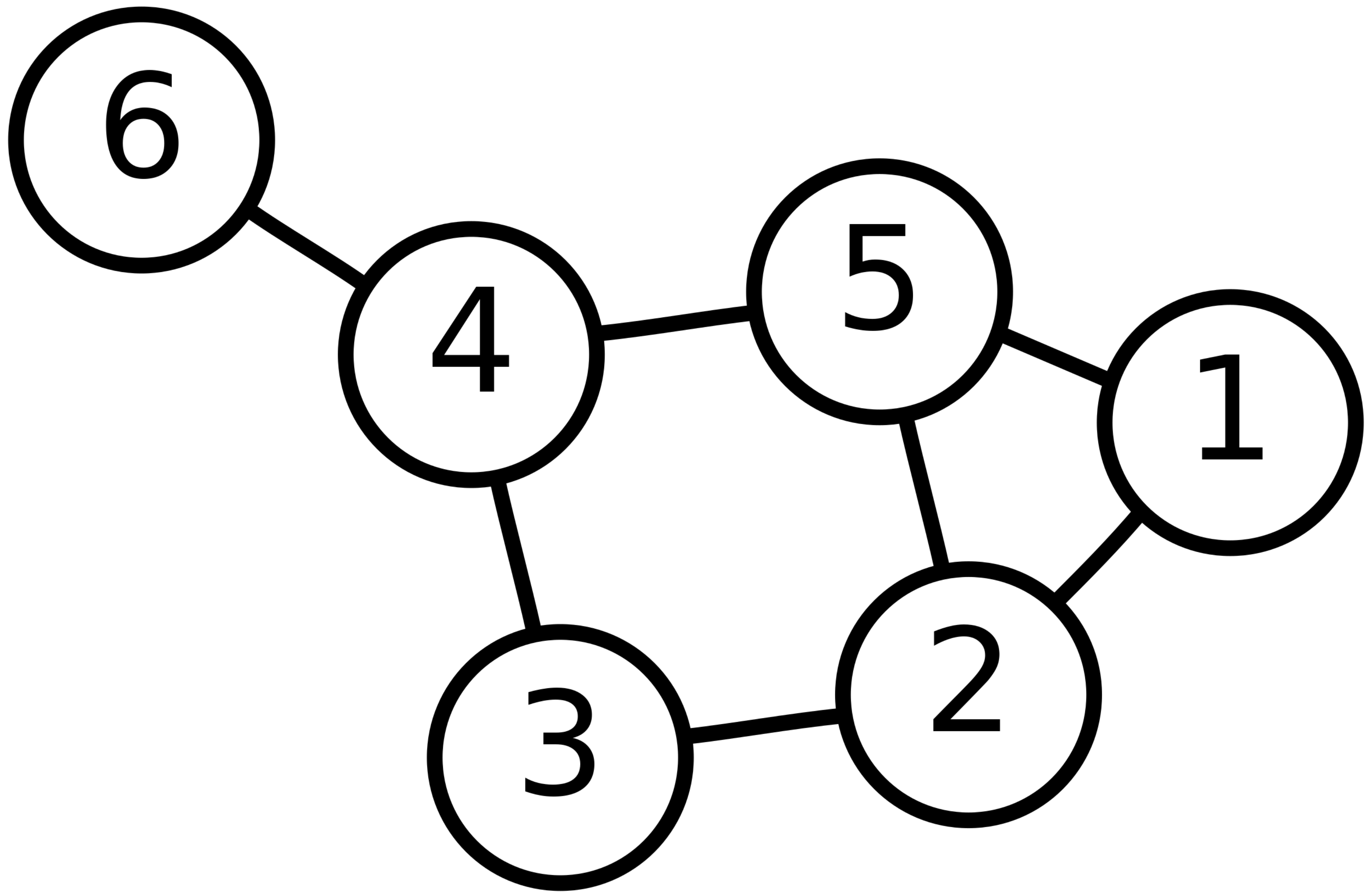
Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by row operations

Algebraic Graph Theory

Graphs

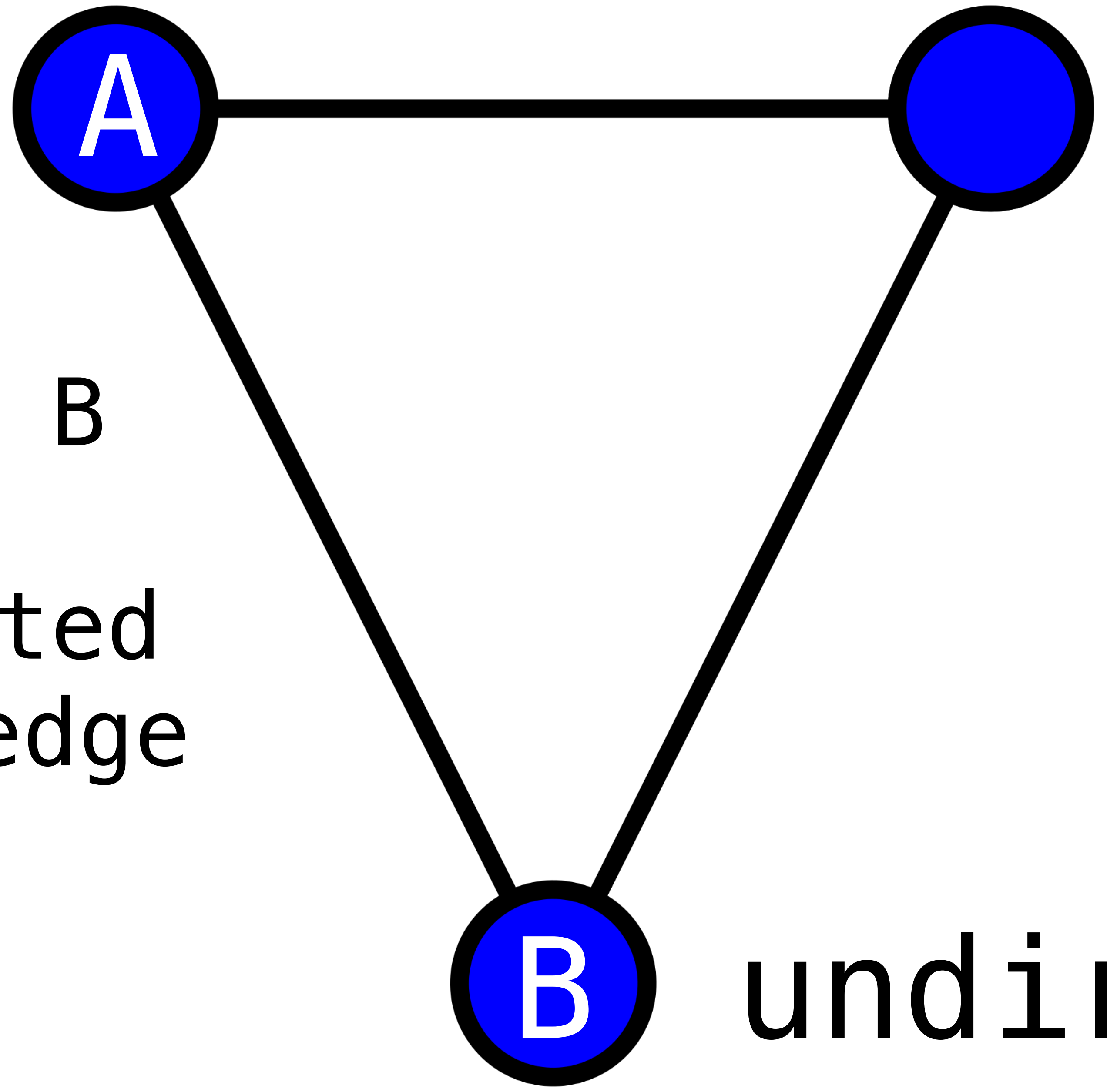
Definition (Informal). A **graph** is a collection of nodes with edges between them.



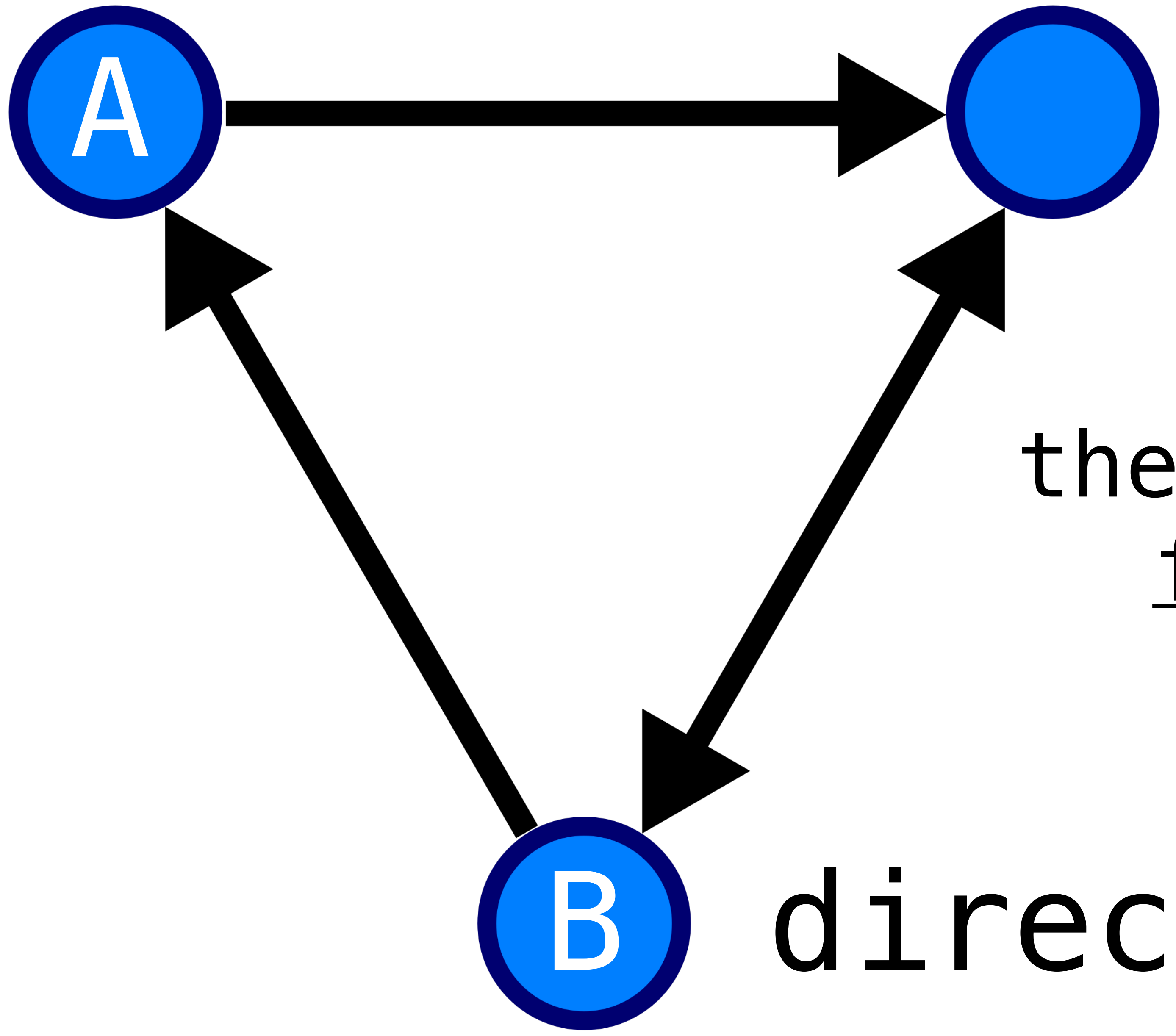
Directed vs. Undirected Graphs

A graph is **directed** if its edges have a direction.

A and B
are
connected
by an edge



undirected

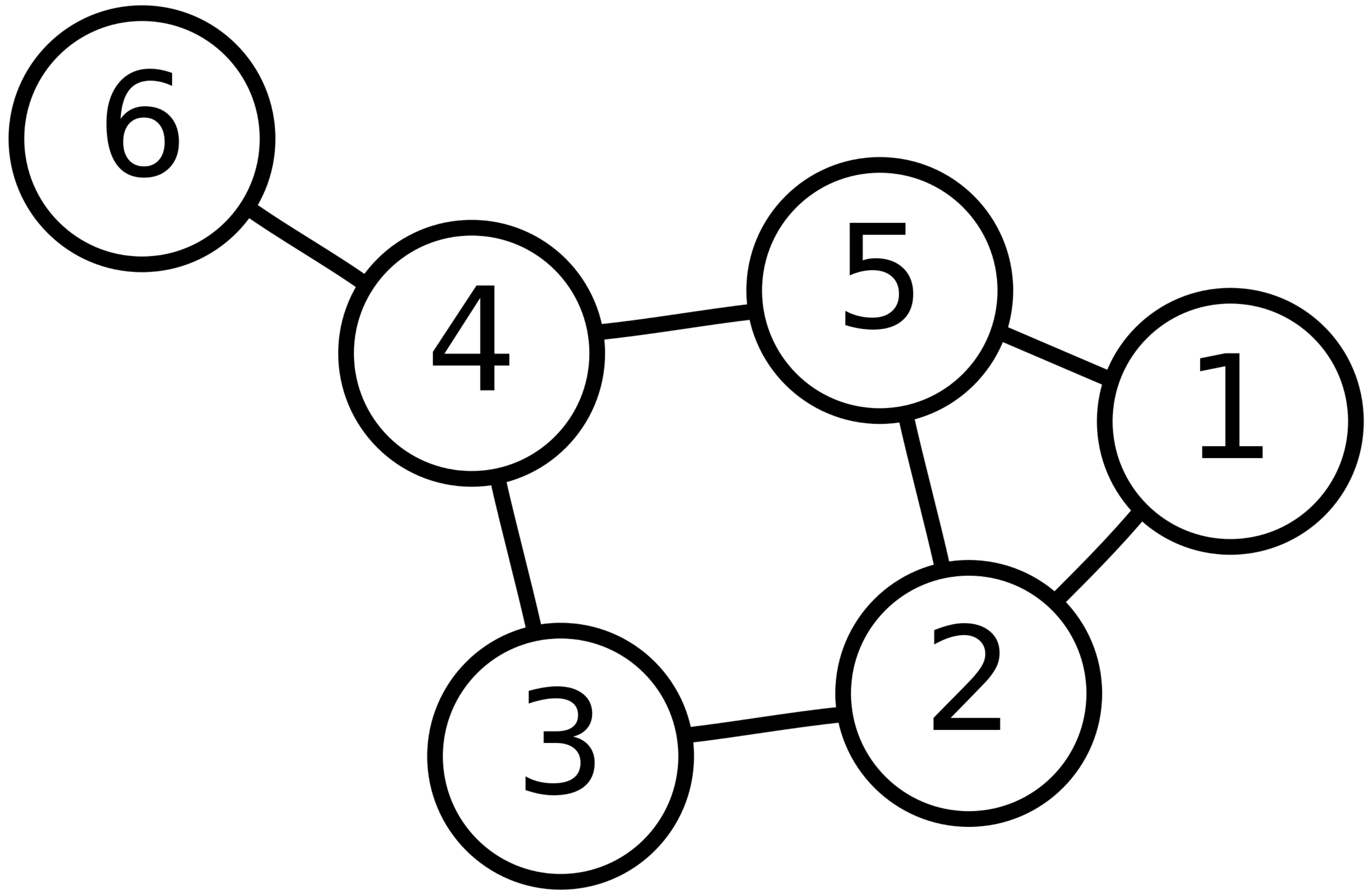


there is an edge
<u>from</u> B <u>to</u> A

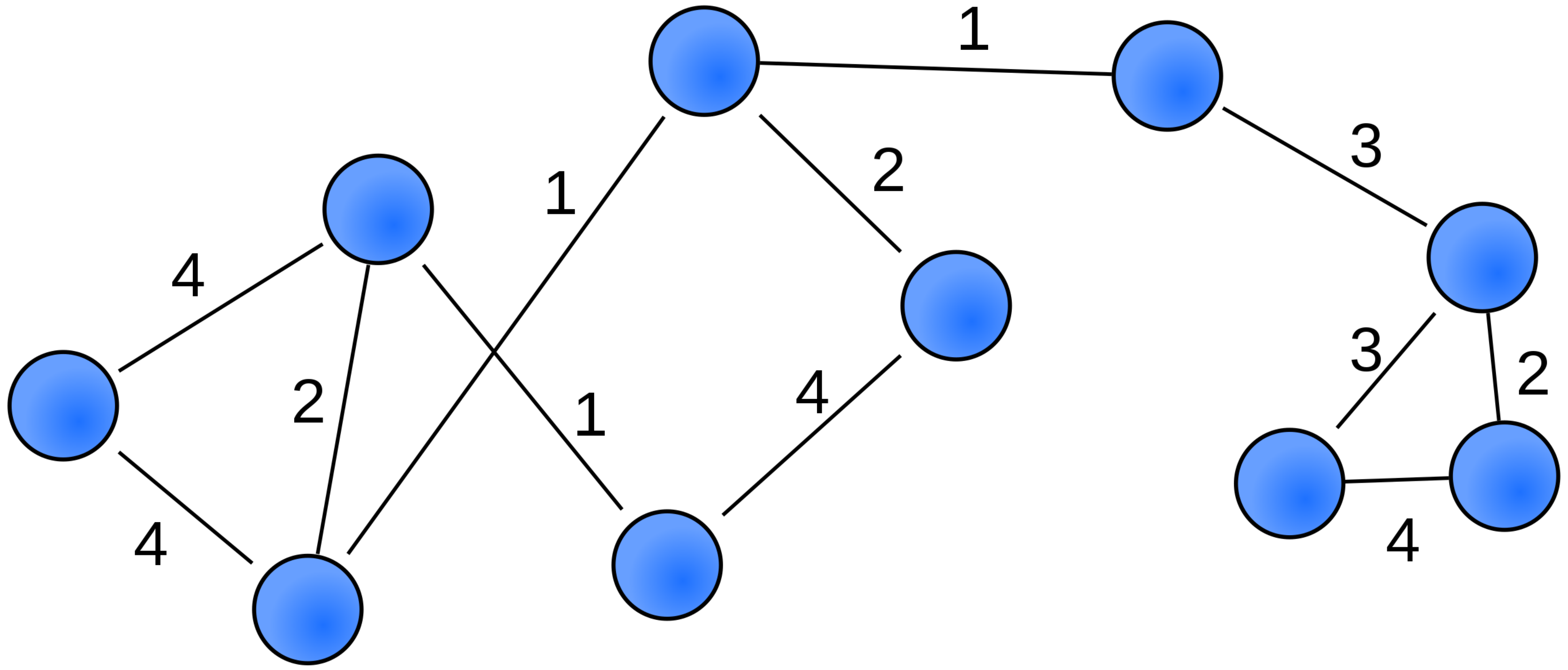
directed

Weighted vs Unweighted graphs

`A graph is **weighted** if its edges have associated values.`



unweighted



weighted

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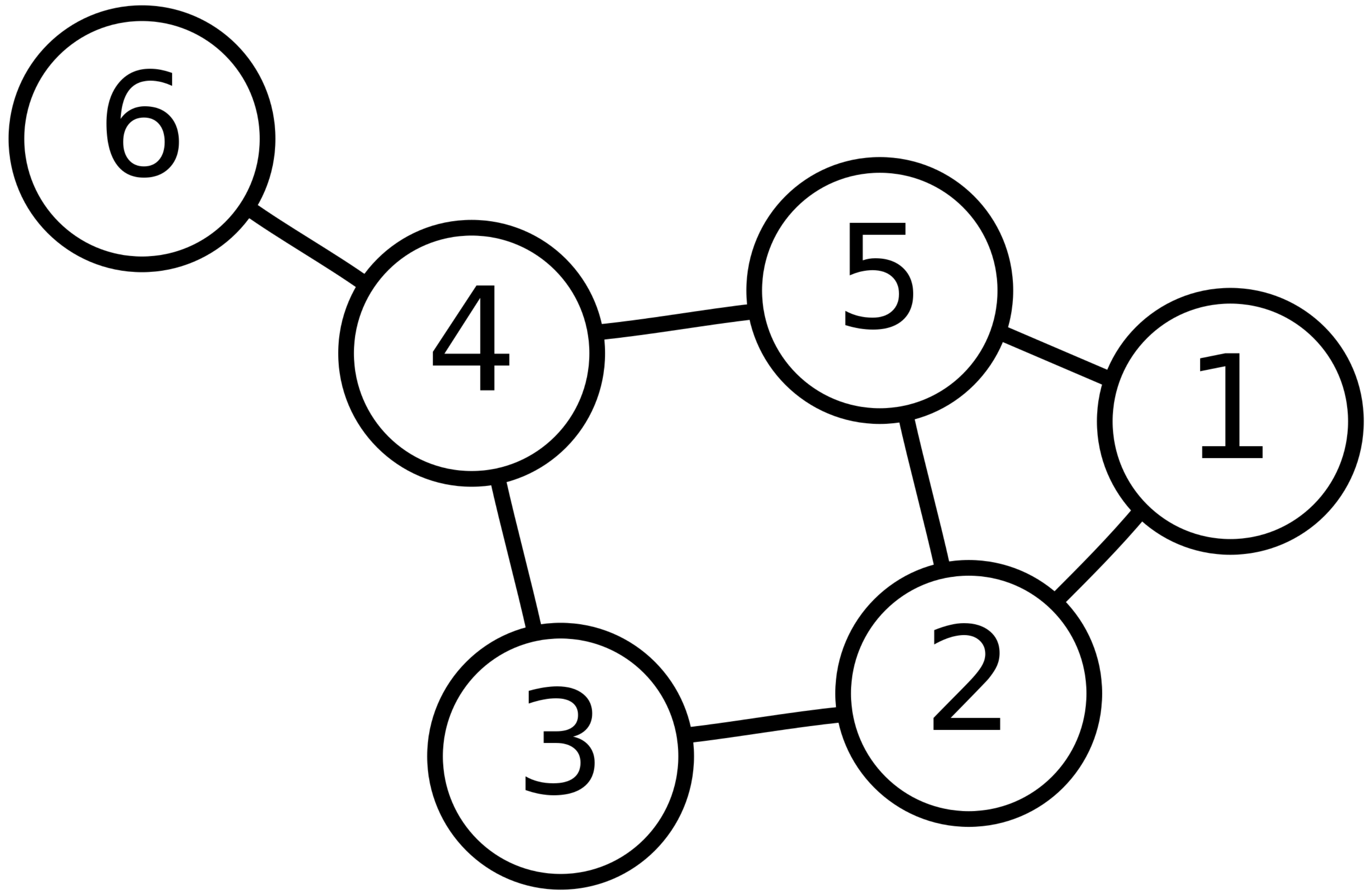
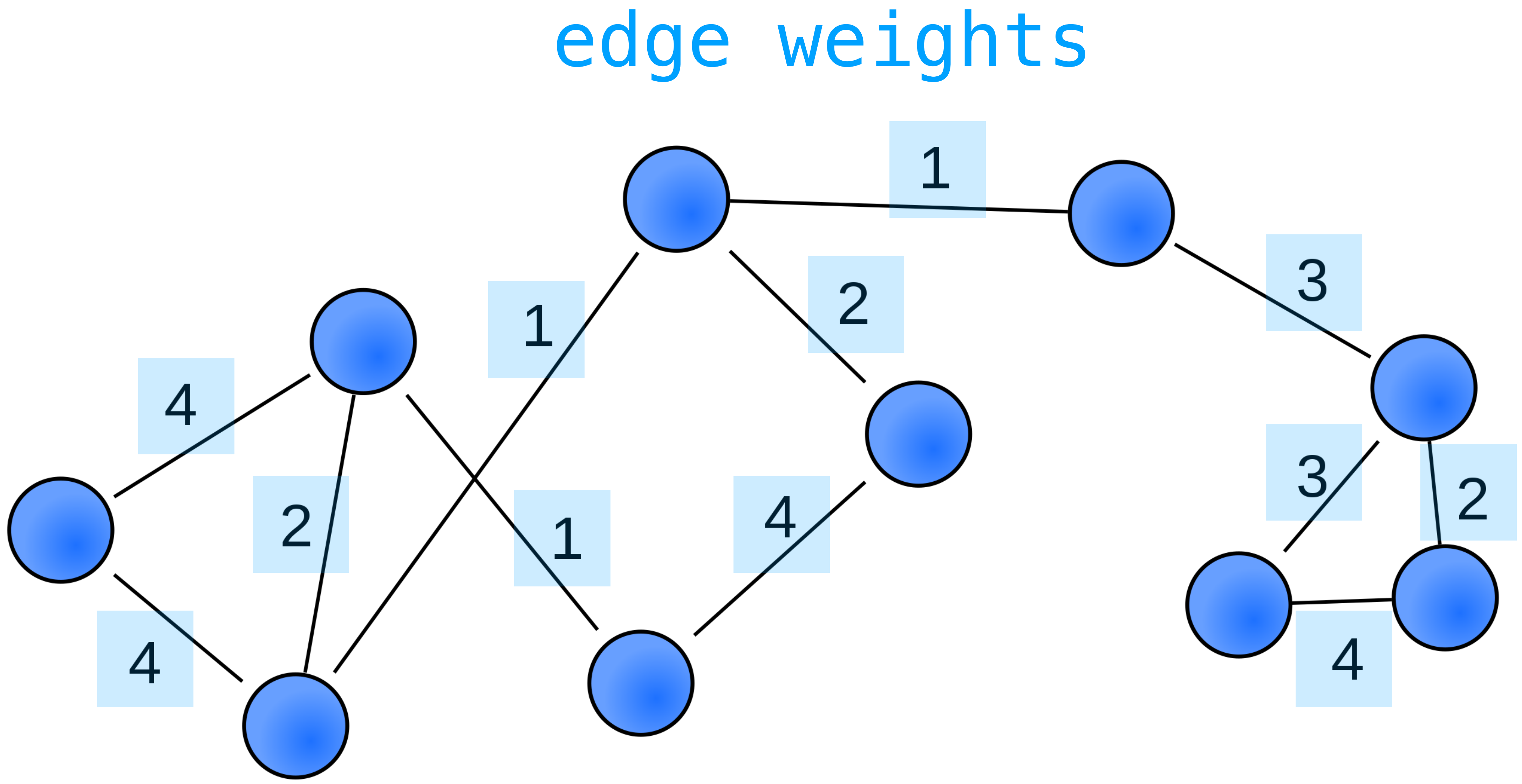


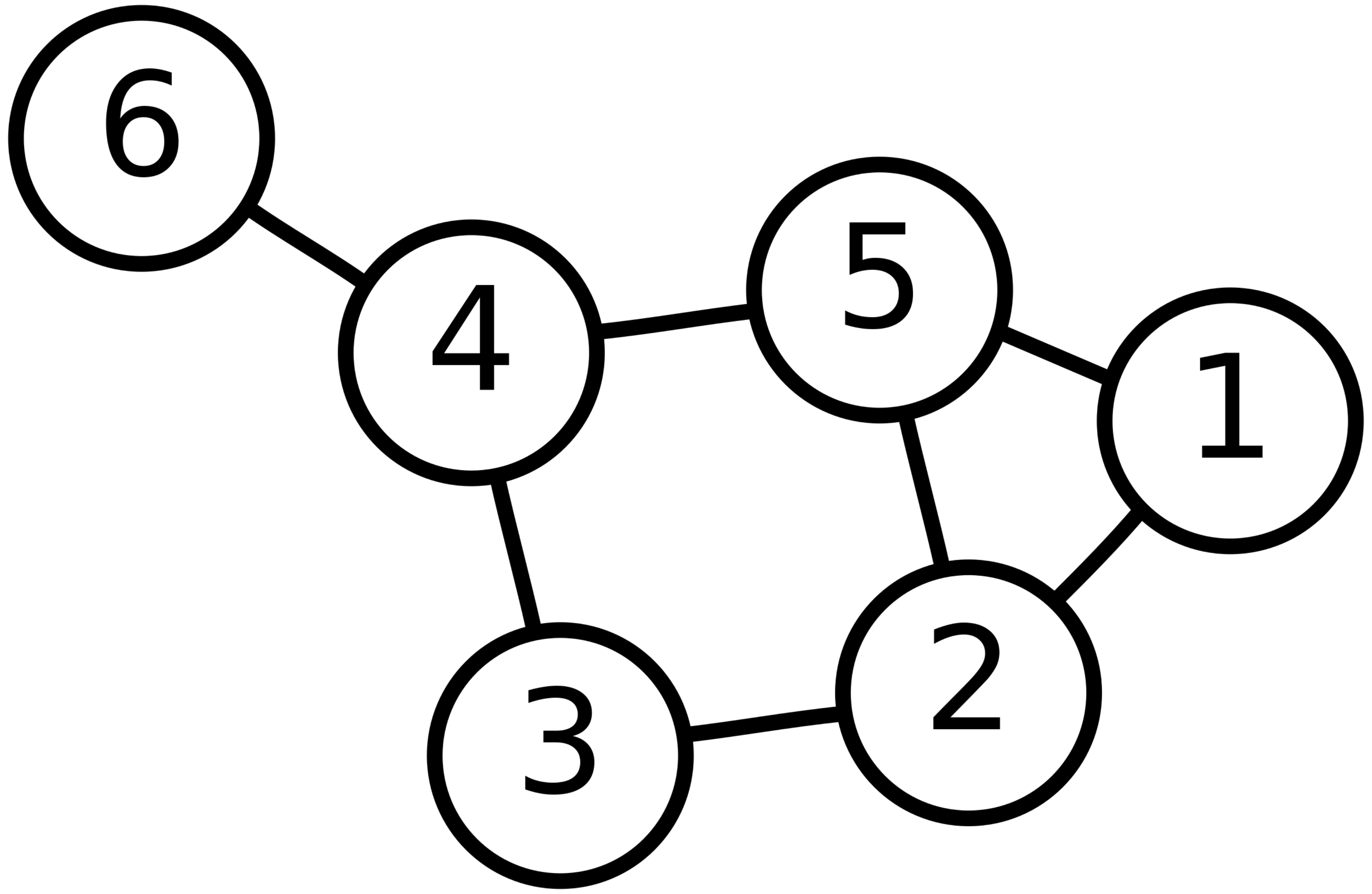
unweighted



weighted

Simple Graphs

A graph is **simple** if it is undirected, has no self loops, and no multi-edges.



Four Kinds of Graphs

	directed	undirected
weighted	nodes are traffic lights edges are streets weights are number of lanes	nodes are musicians edges are collaborations weights are number of collaborations
unweighted	nodes are instagram users edges are follows	nodes are bodies of land edges are pedestrian bridges

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weighted	nodes are traffic lights edges are streets weights are number of lanes Markov Chains	nodes are musicians edges are collaborations weights are number of collaborations
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Fundamental Question

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How do we represent a graph formally in a computer?

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There are a couple ways, but one way is to use matrices.

Adjacency Matrices

Let G be an simple graph with its nodes labeled by numbers 1 through n.

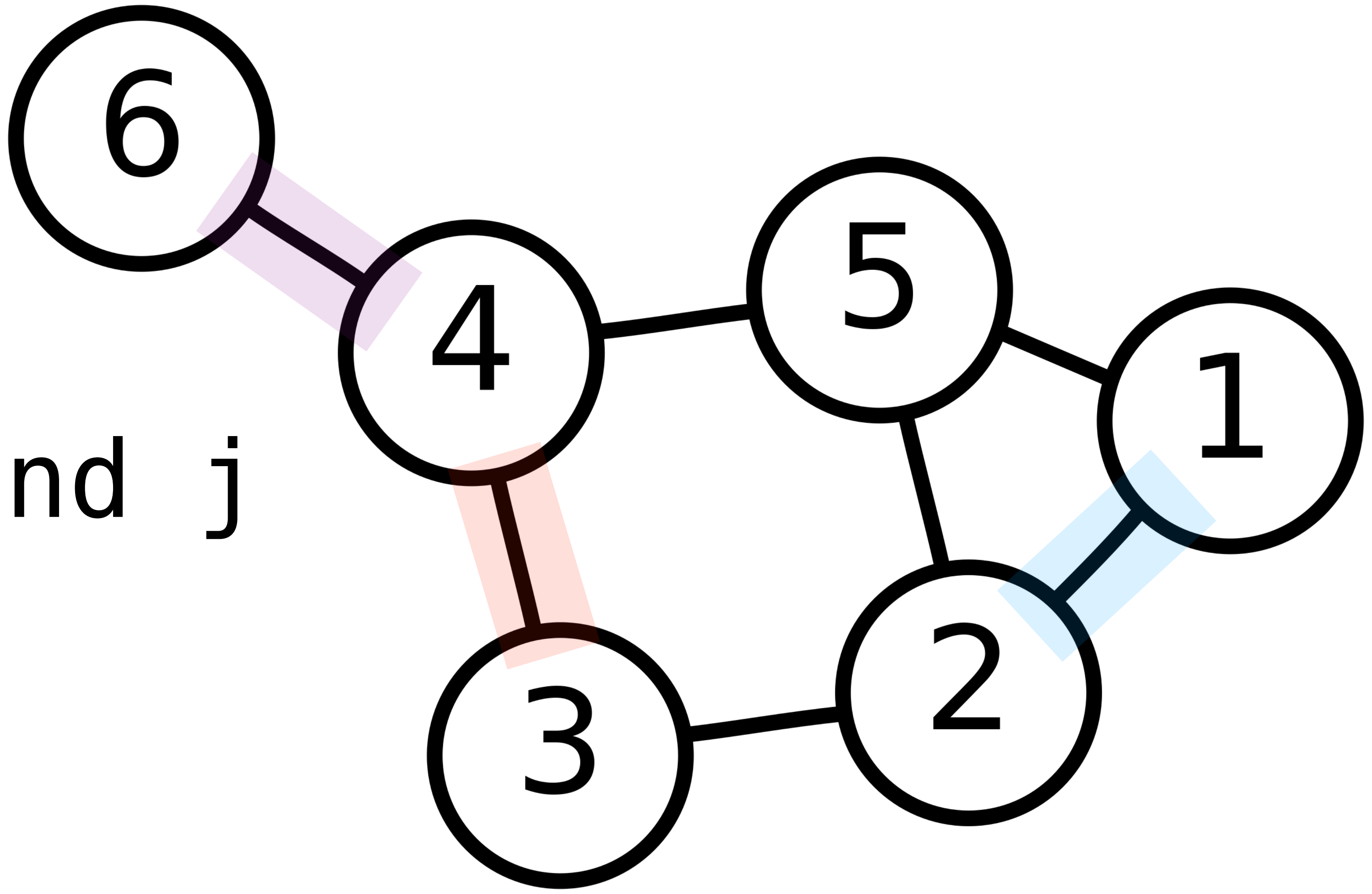
We can create the **adjacency matrix** A for G as follows.

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between i and j} \\ 0 & \text{otherwise} \end{cases}$$

A_{12} A_{34} A_{46}

A_{21} A_{43} A_{64}

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



Symmetric Matrices

Definition. A $n \times n$ matrix is **symmetric** if

$$A^T = A$$

Example.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Algebraic Graph Theory

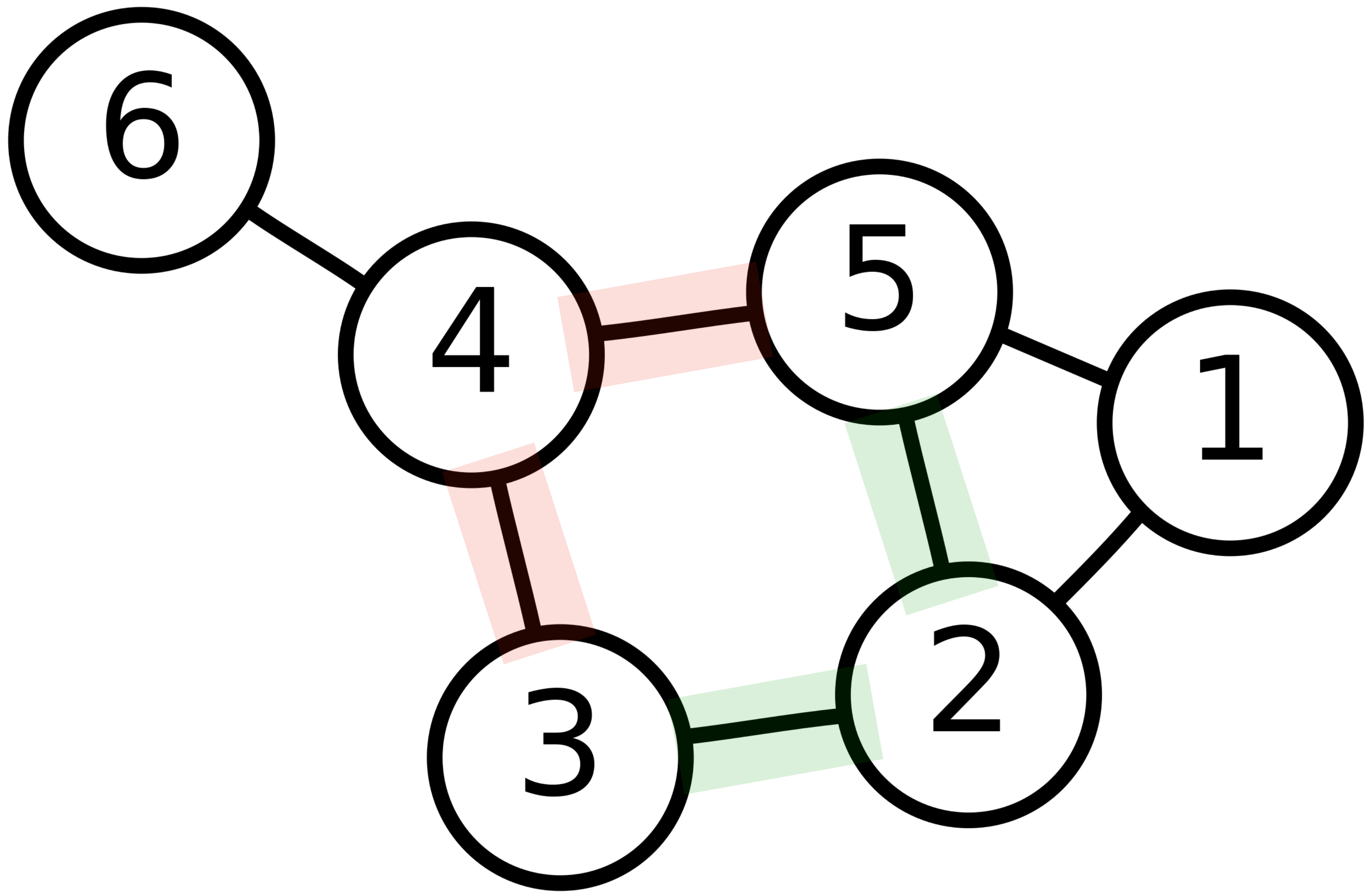
Once we have an adjacency matrix, we can do linear algebra on graphs.

Example: Squared Adjacency Matrices

Given an adjacency matrix A, can we interpret anything meaningful from A^2?

Example: Squared Adjacency Matrices

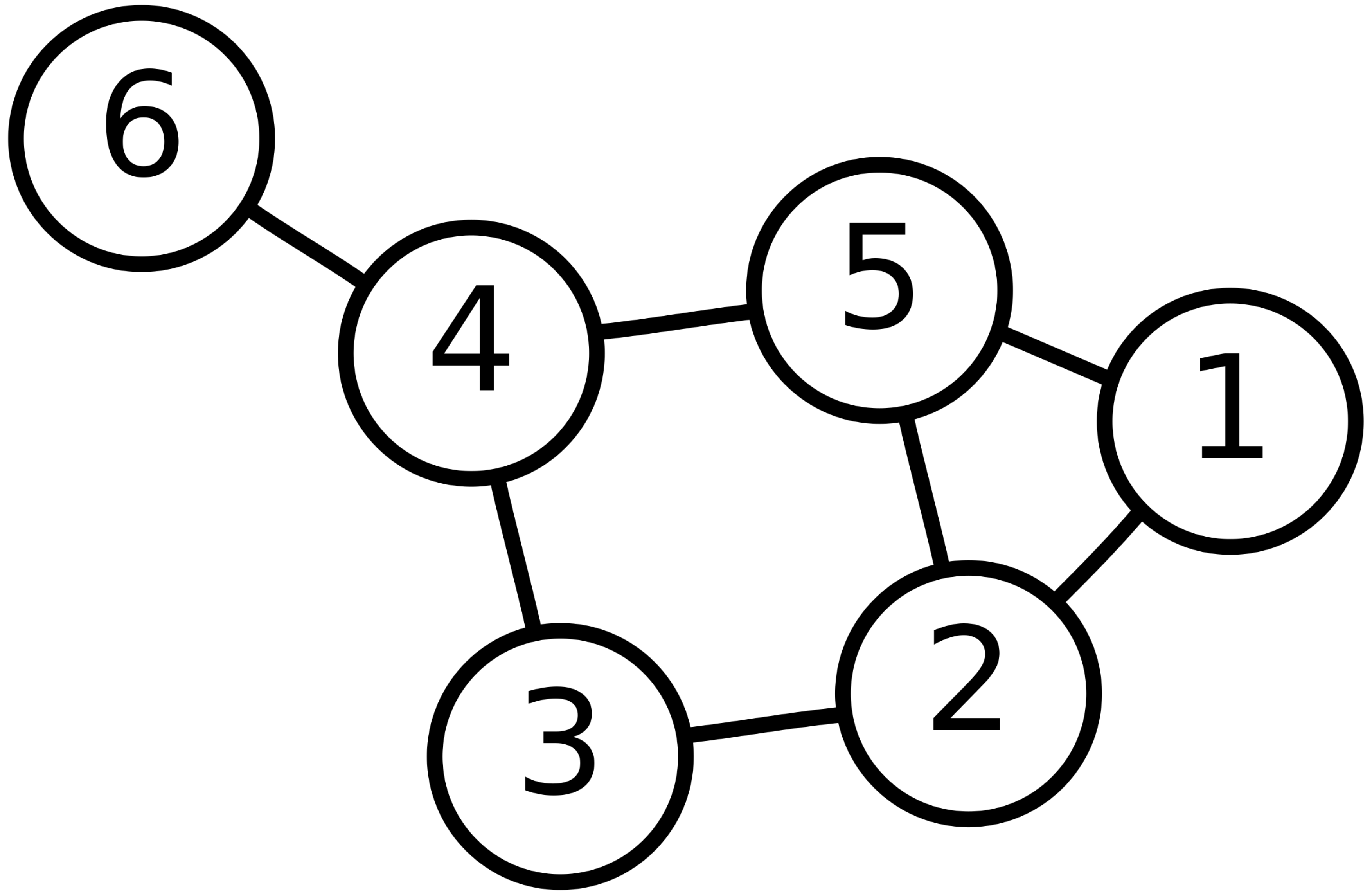
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



$$(A^2)_{53} = 1(0) + 1(1) + 0(0) + 1(1) + 0(0) + 0(0) = 2$$

Example: Squared Adjacency Matrices

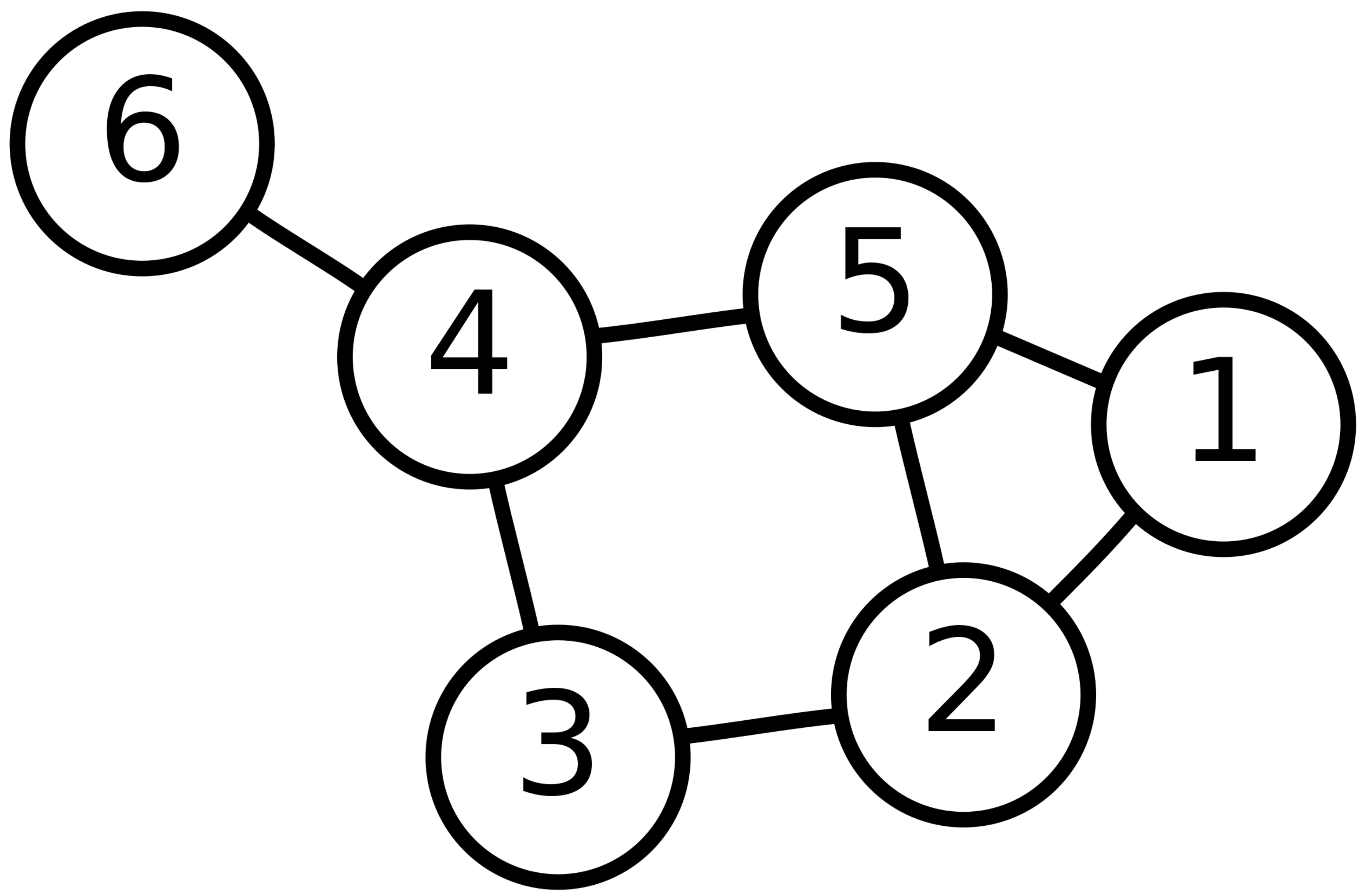
$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$



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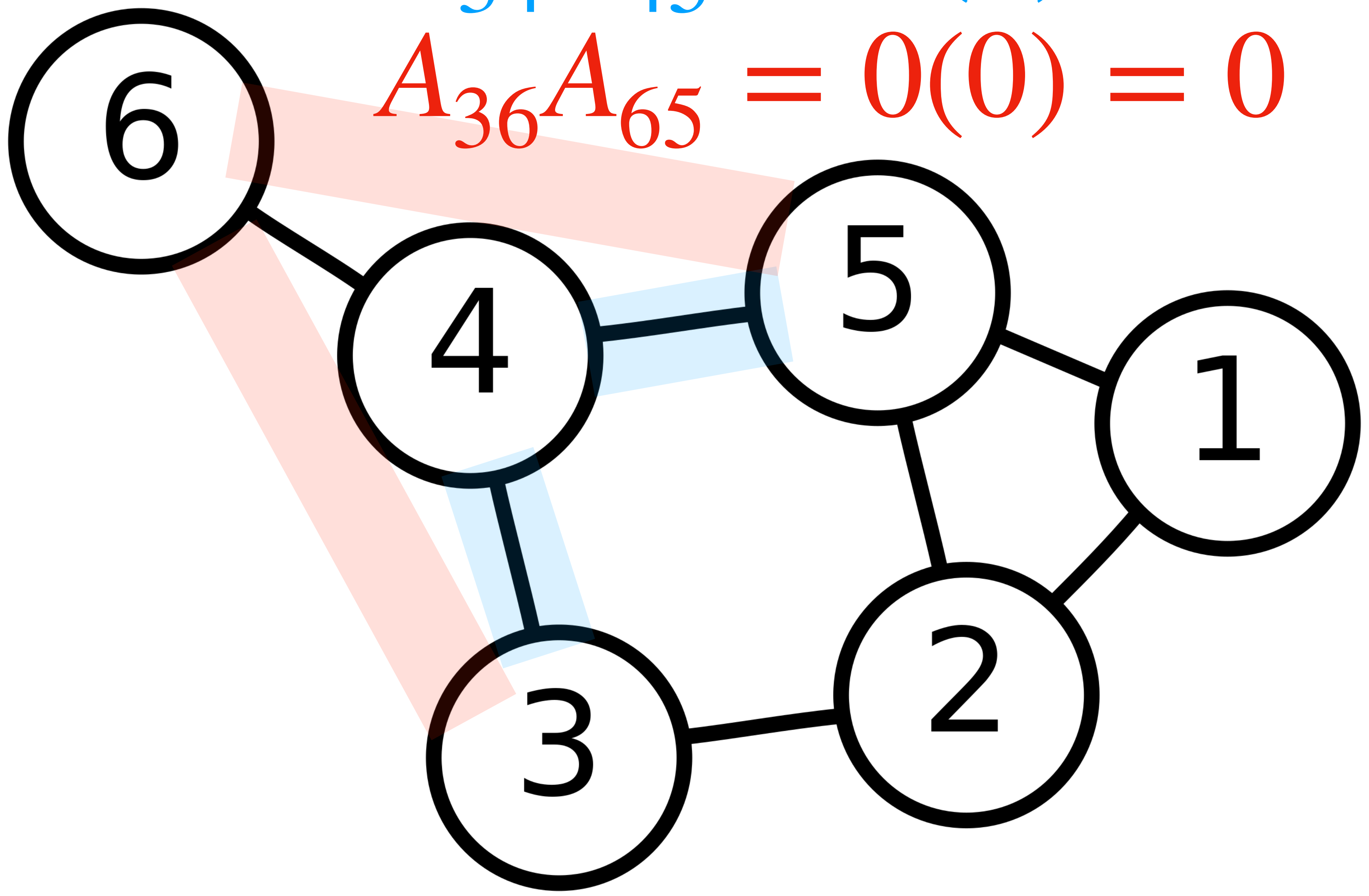
$A_{34}A_{45} = 1(1) = 1$

$A_{36}A_{65} = 0(0) = 0$



Example: Squared Adjacency Matrices

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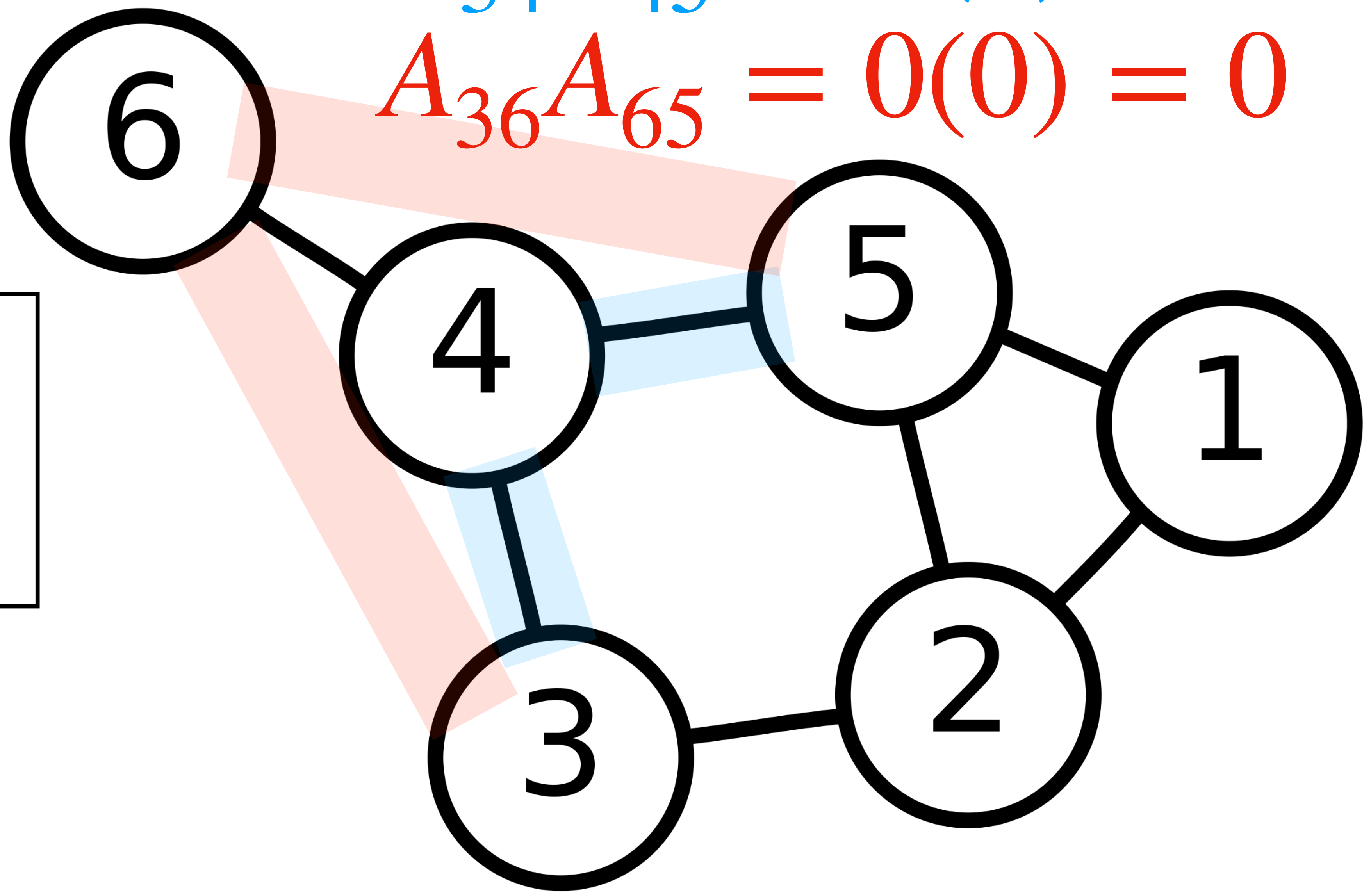
$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$

$$(A^2)_{ij} = \boxed{\texttt{number of 2-step paths from i to j}}$$

$A_{34}A_{45} = 1(1) = 1$

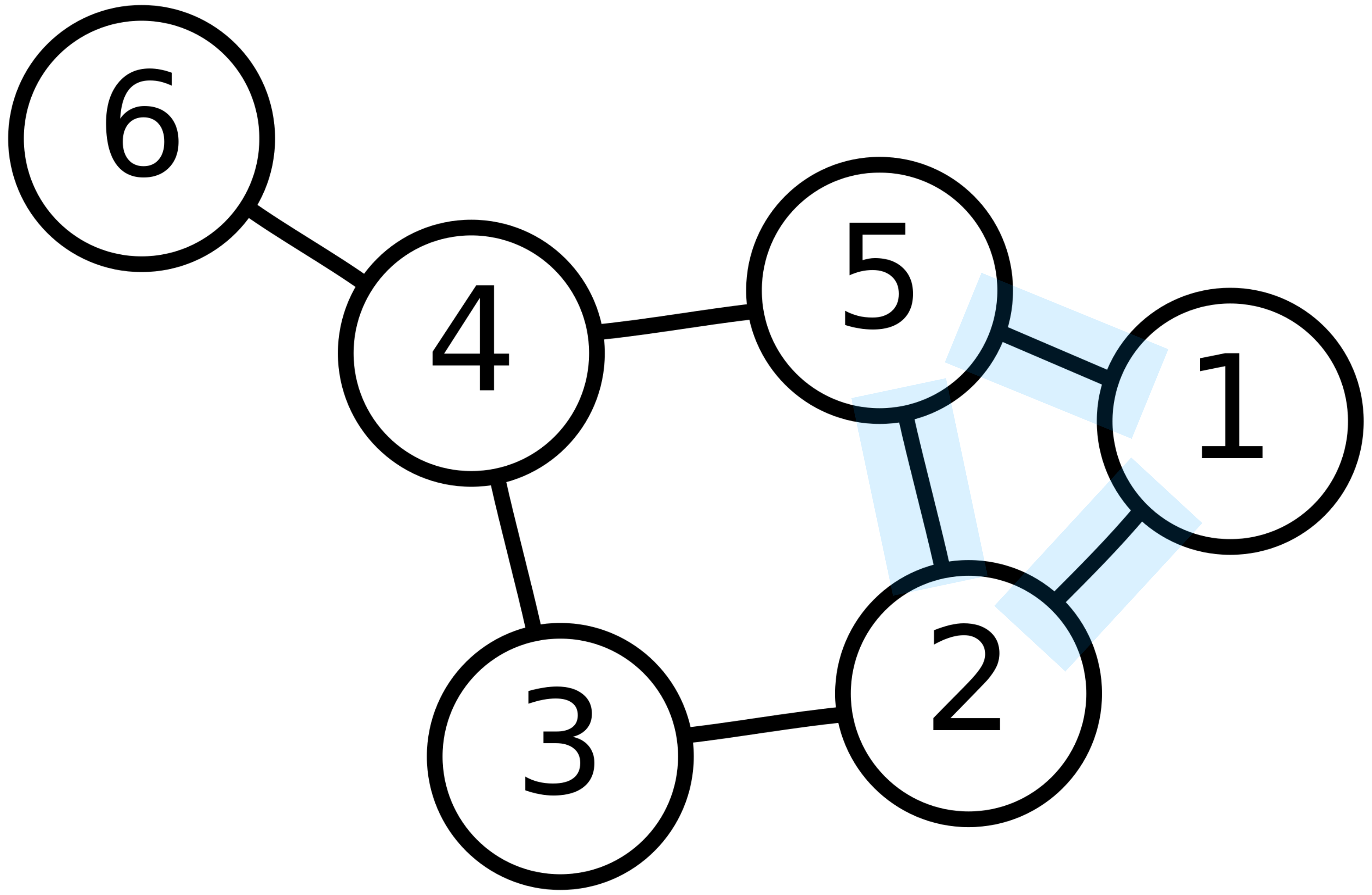
$A_{36}A_{65} = 0(0) = 0$



Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)



Application: Triangle Counting (Naive)

```
FUNCTION tri_count_naive(A):
  count = 0
  for i from 1 to n:
    for j from i + 1 to n:
      for k from j + 1 to n:
        if A_ij = 1 and A_jk = 1 and A_ki = 1: # an edge between each pair
          count += 1:
  RETURN count
```

Application: Triangle Counting

Theorem. For an adjacency matrix A, the number of triangle containing the edge (i,j) is

$$(A^2)_{ij} * A_{ij}$$

Verify:

Application: Triangle Counting

FUNCTION tri_count(A):

compute A^2

count $\leftarrow$ sum of $(A^2)_{ij} * A_{ij}$ for all distinct i and j

RETURN count / 6   # why divided by 6?

Application: Triangle Counting

```
FUNCTION tri_count(A):
    # in NumPy '*' is entry-wise multiplication
    #      also called the HADAMARD PRODUCT
    count ← sum of the entries of A^2 * A
    RETURN count / 6
```

Application: Triangle Counting

```
FUNCTION tri_count(A):
  # in NumPy '*' is entry-wise multiplication
  #      also called the HADAMARD PRODUCT
  # and 'np.sum' sums the entry of a matrix
  RETURN np.sum((A @ A) * A) / 6
```

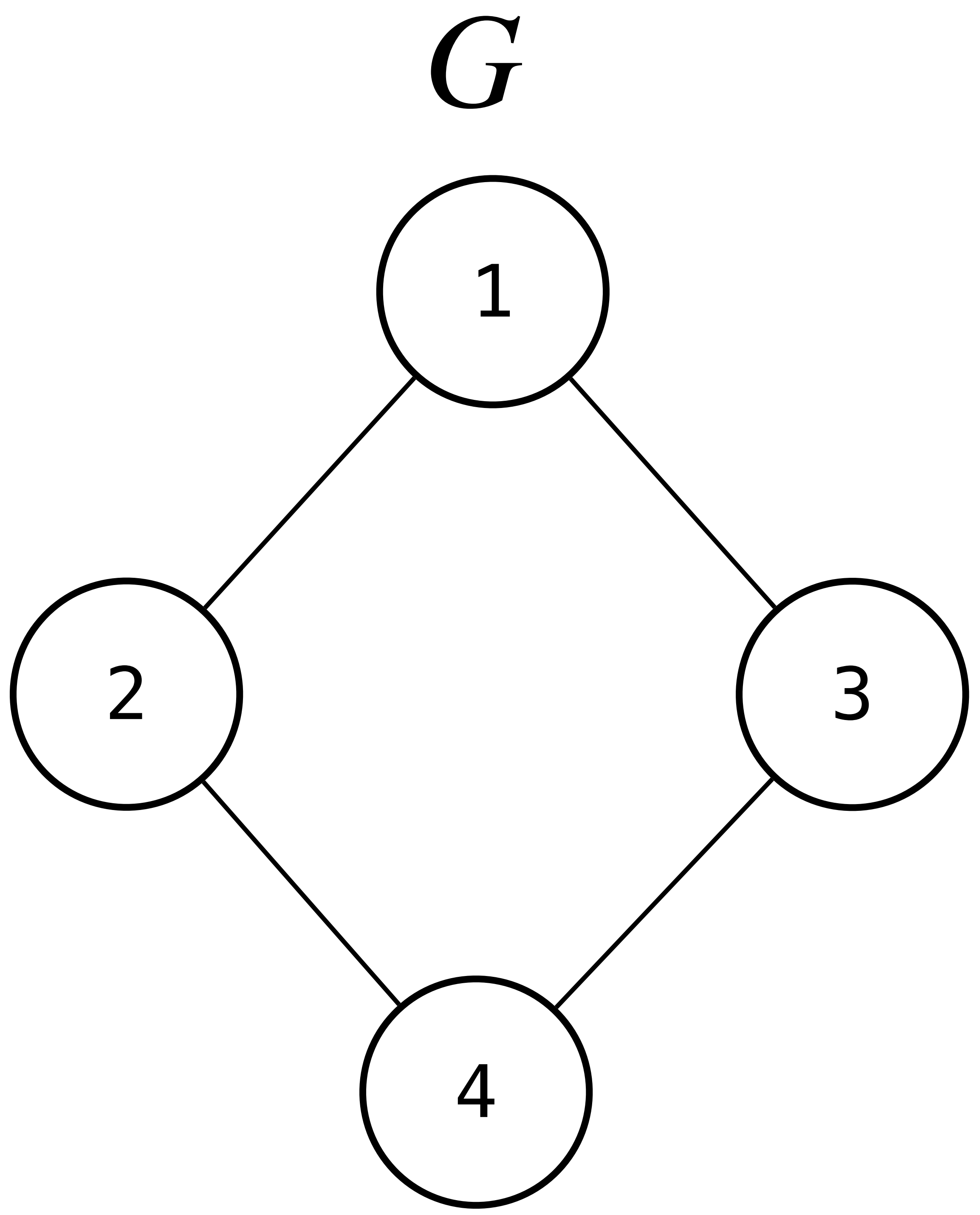
demo

Another Application: Reachability

Question: If A^2 gives us information about length 2 paths, then what about A^k?

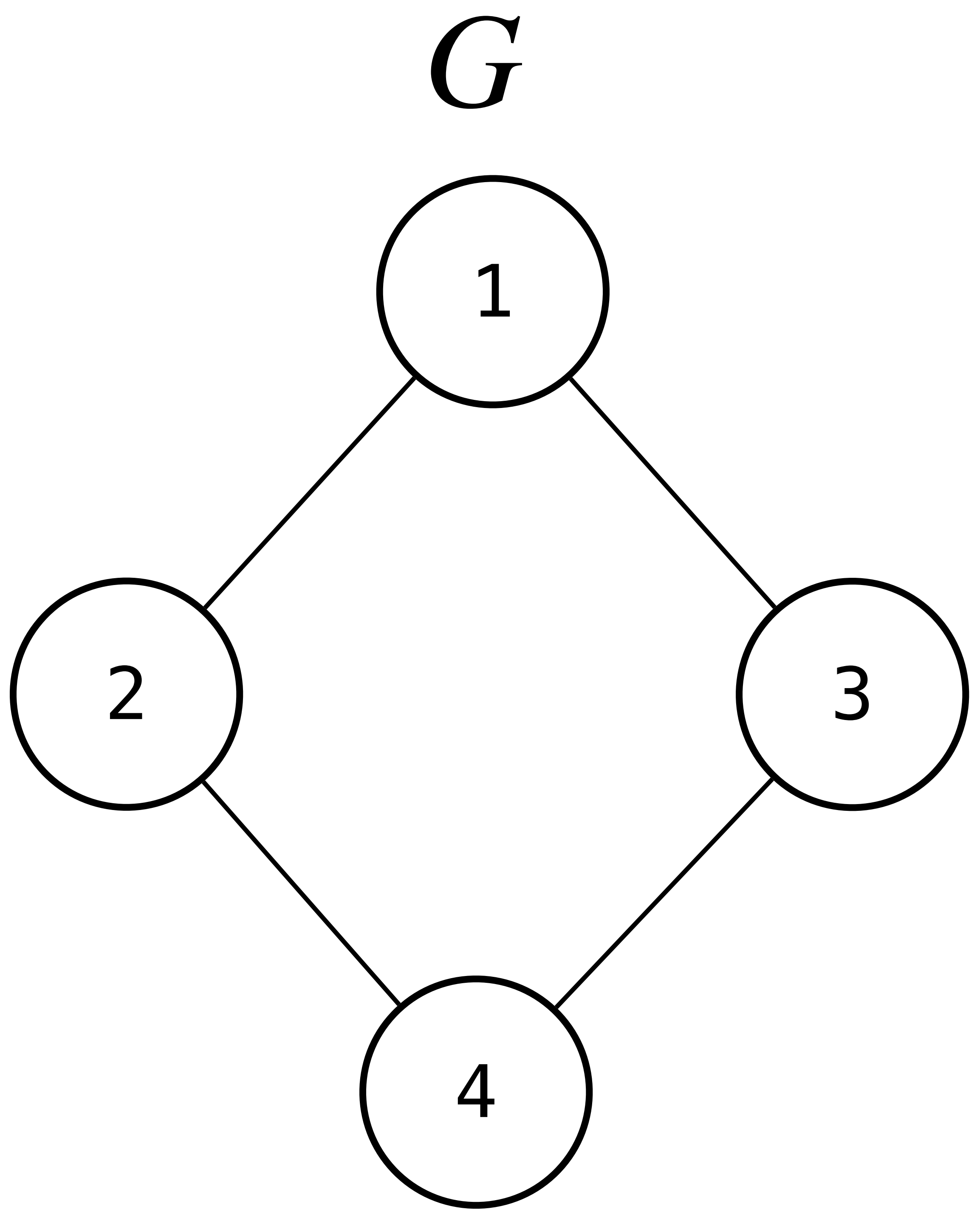
A^k gives us information about k–length paths.

Example



Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

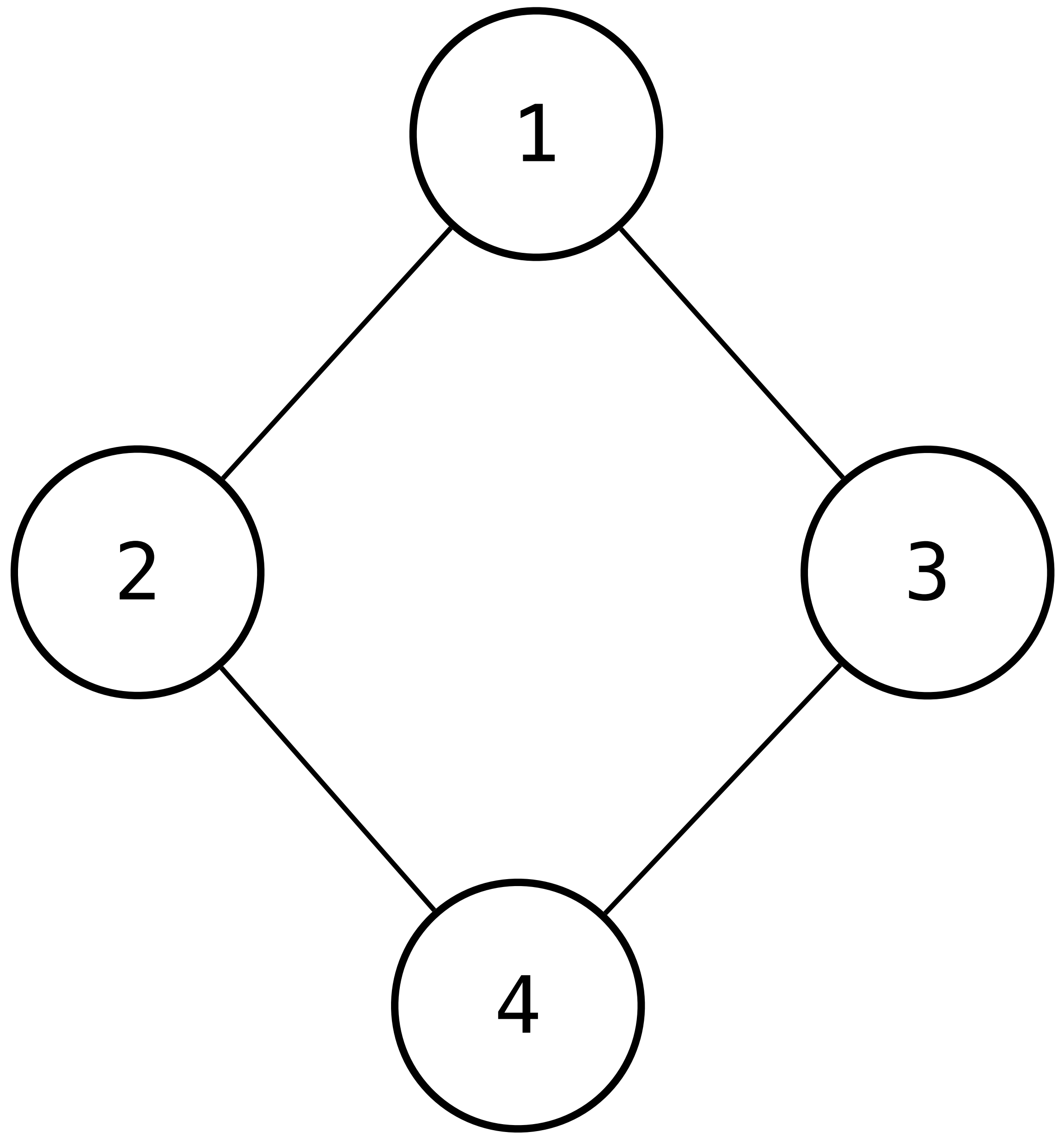


Example

$$\begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix} = \texttt{adjacency matrix for } G$$

$$\begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix}^2 = \begin{bmatrix}2&0&0&2\\0&2&2&0\\0&2&2&0\\2&0&0&2\end{bmatrix}$$

G



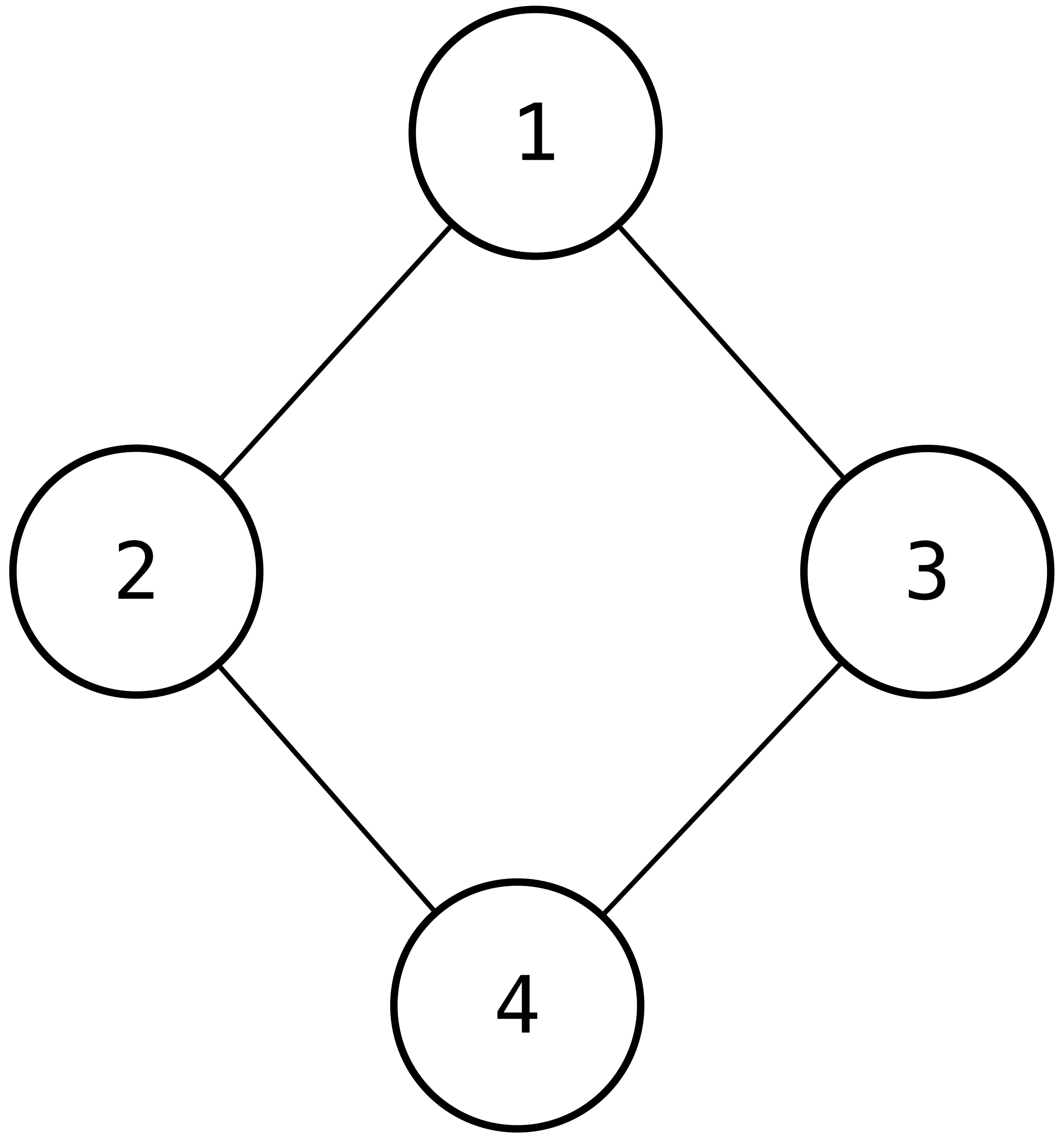
Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

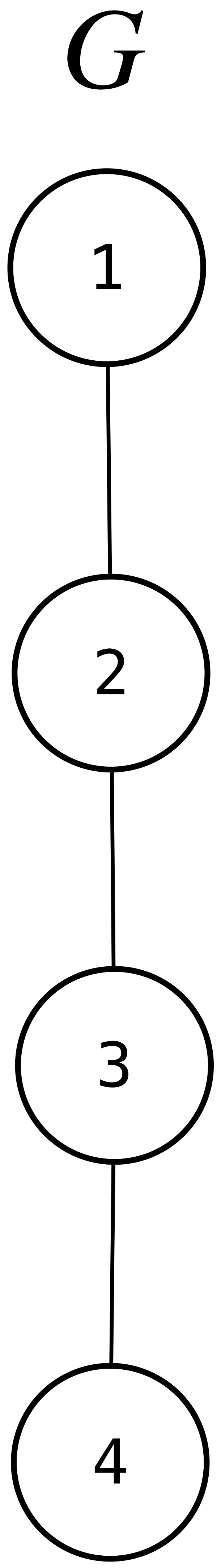
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

G



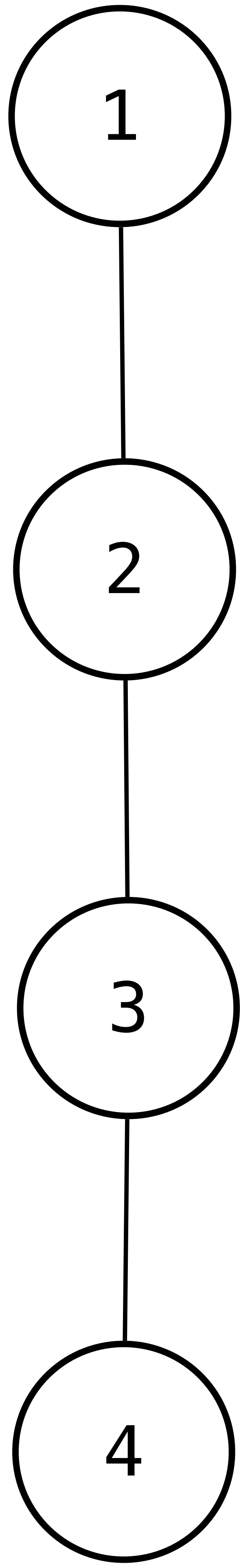
Example



Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

G

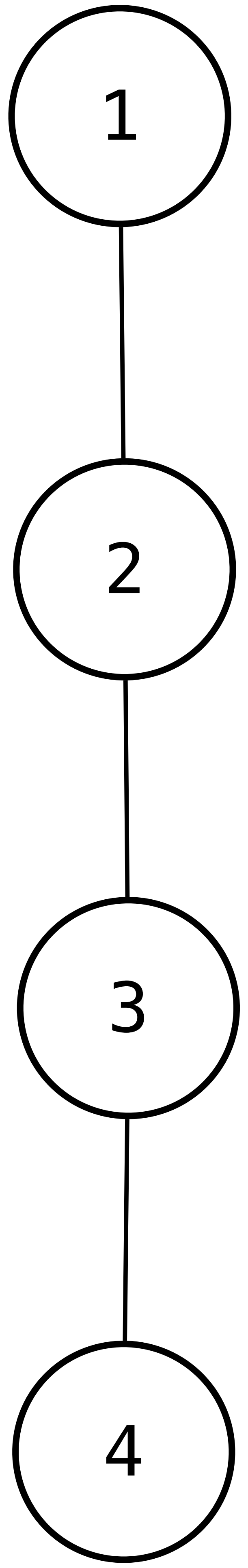


Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

G



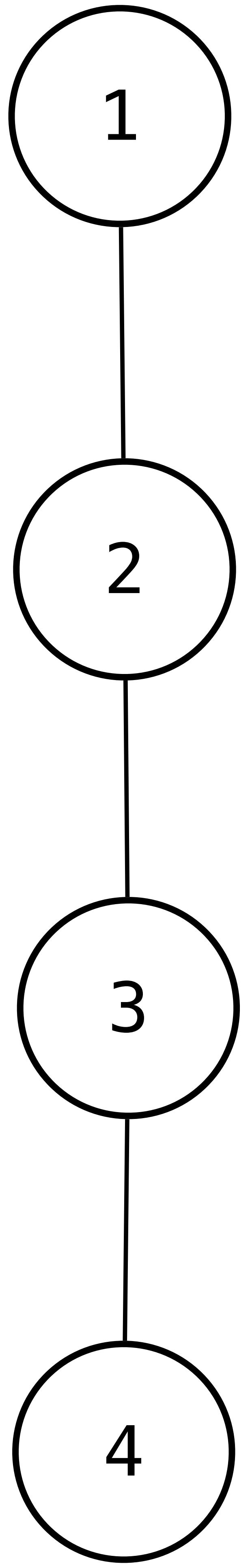
Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \texttt{adjacency matrix for } G$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

G



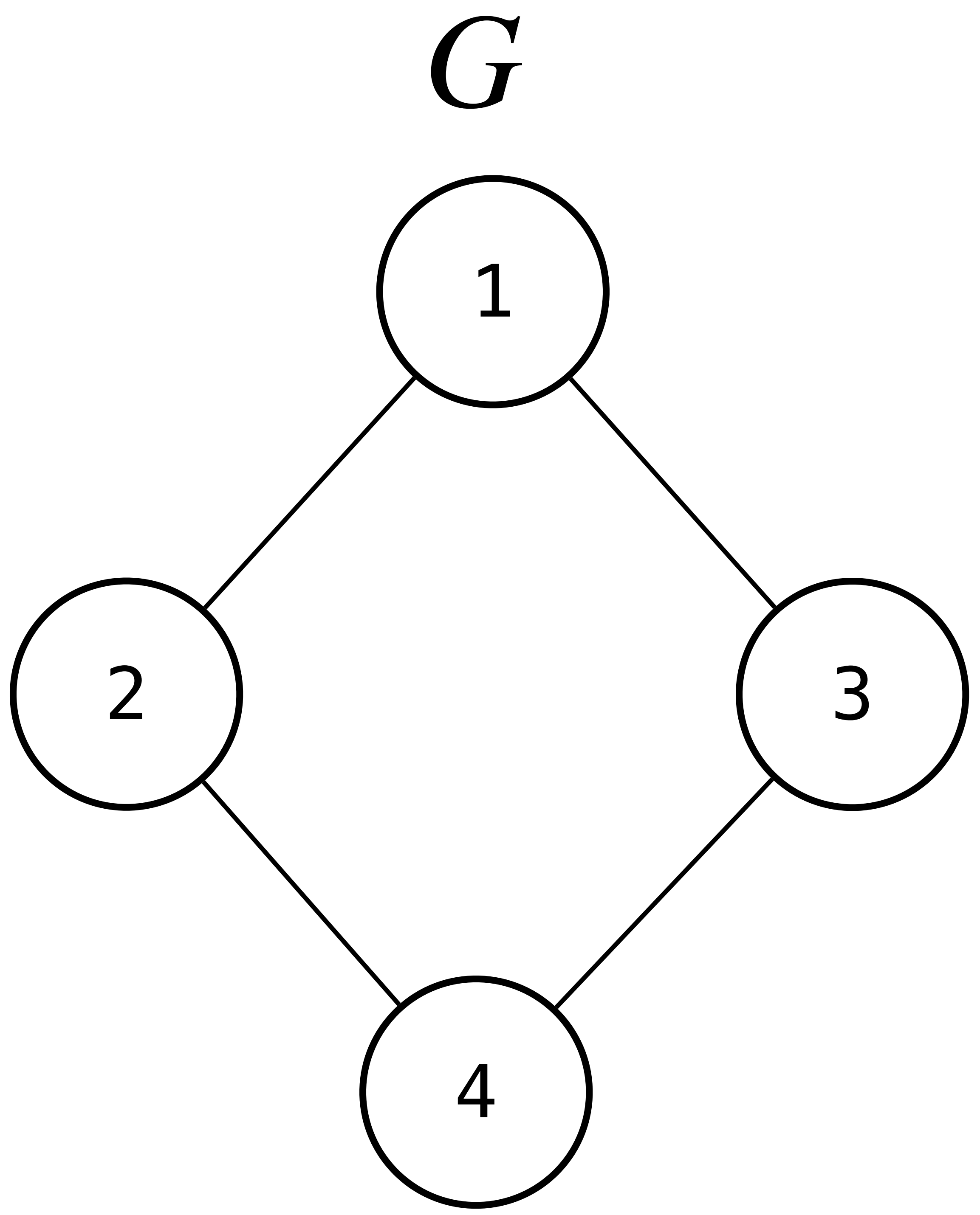
Another Application: Reachability

Theorem: Let G be a simple graph.

- $(A_G^k)_{ij}$ is the number of paths of length exactly k from v_i to v_j.

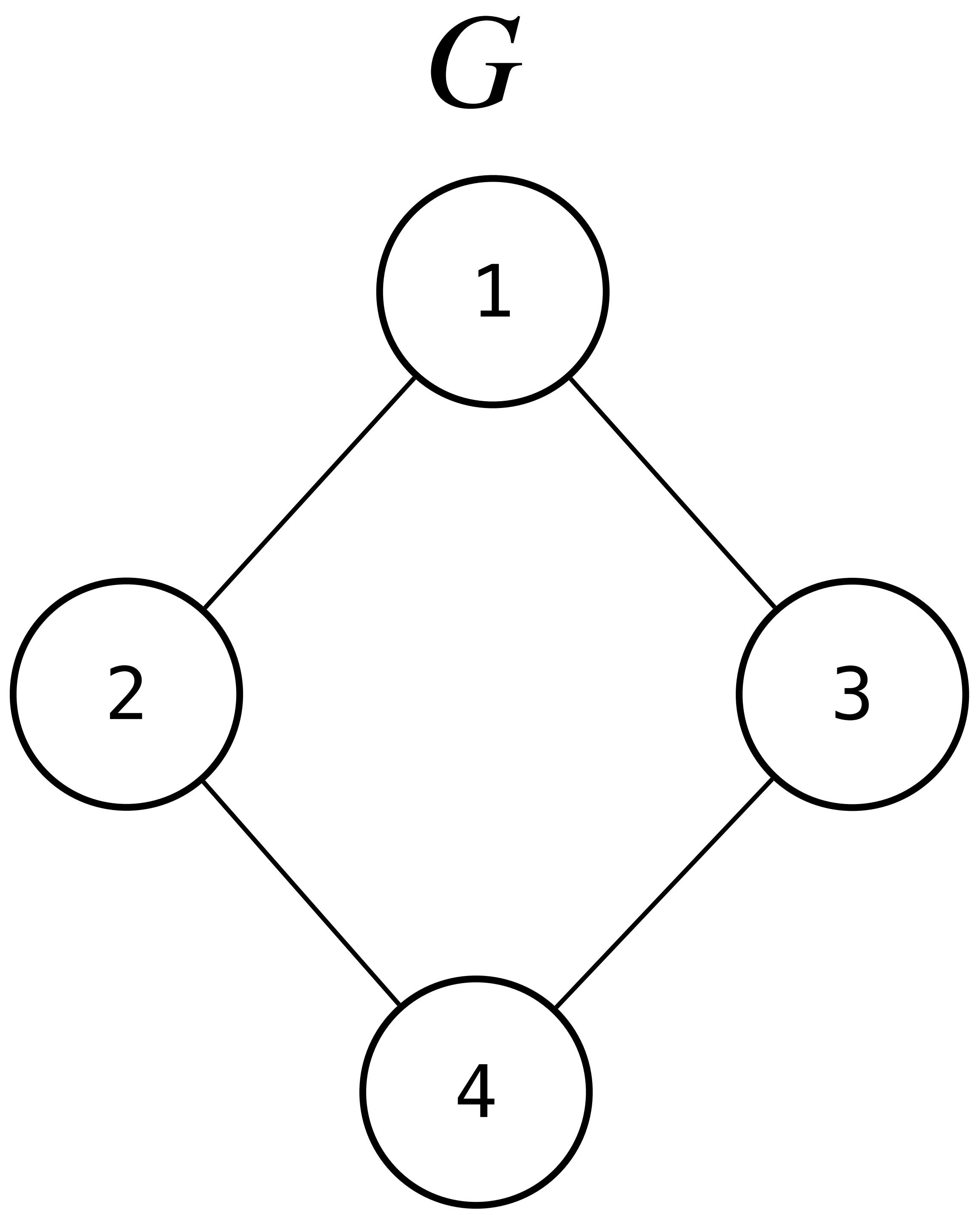
- $\left((A_G + I)^k\right)_{ij}$ is nonzero if and only if there is a path of length at most k from v_i to v_j.

Example



Example

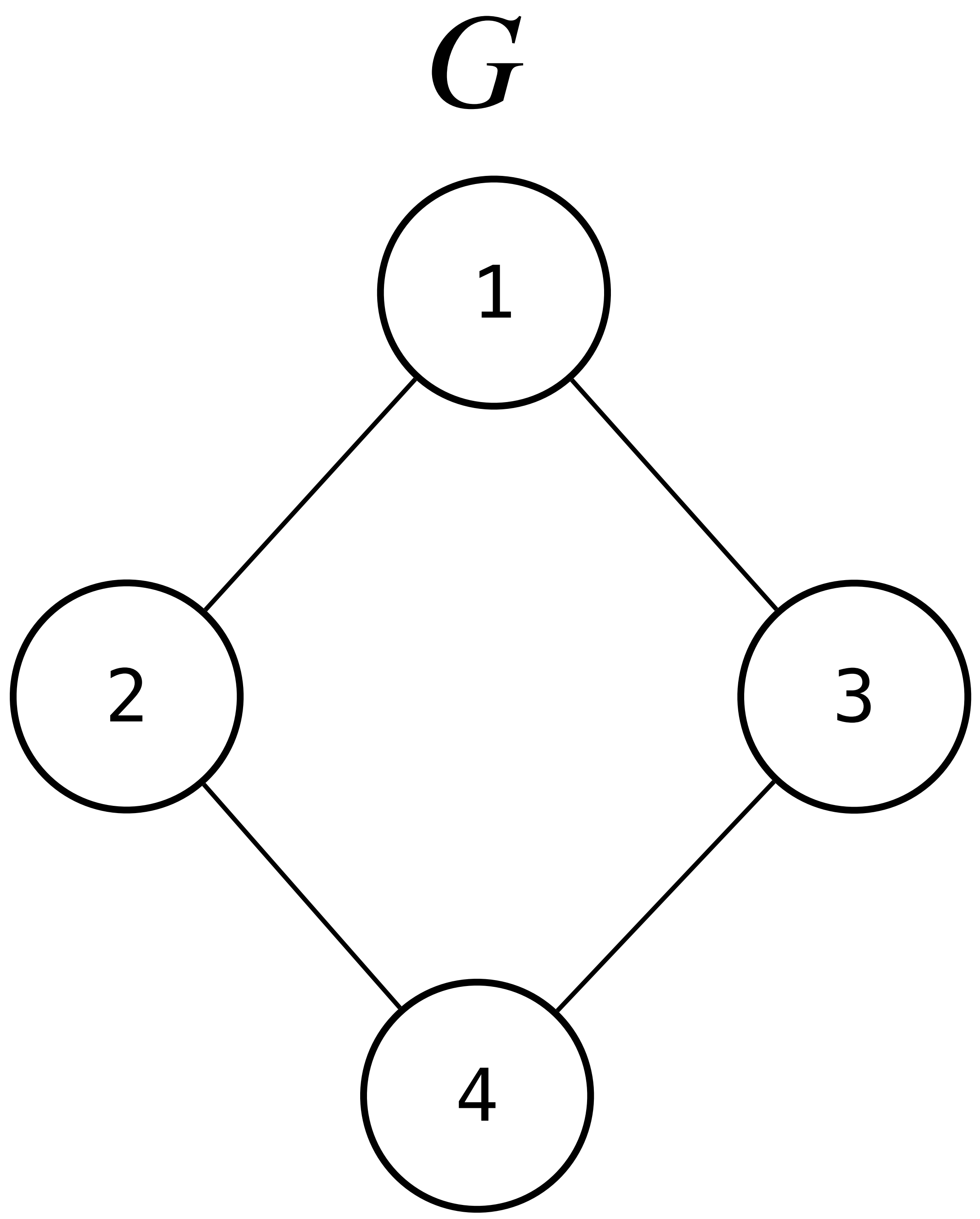
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \texttt{(adjacency matrix for } G\texttt{) + } I$$



Example

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \texttt{(adjacency matrix for } G\texttt{) + } I$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$



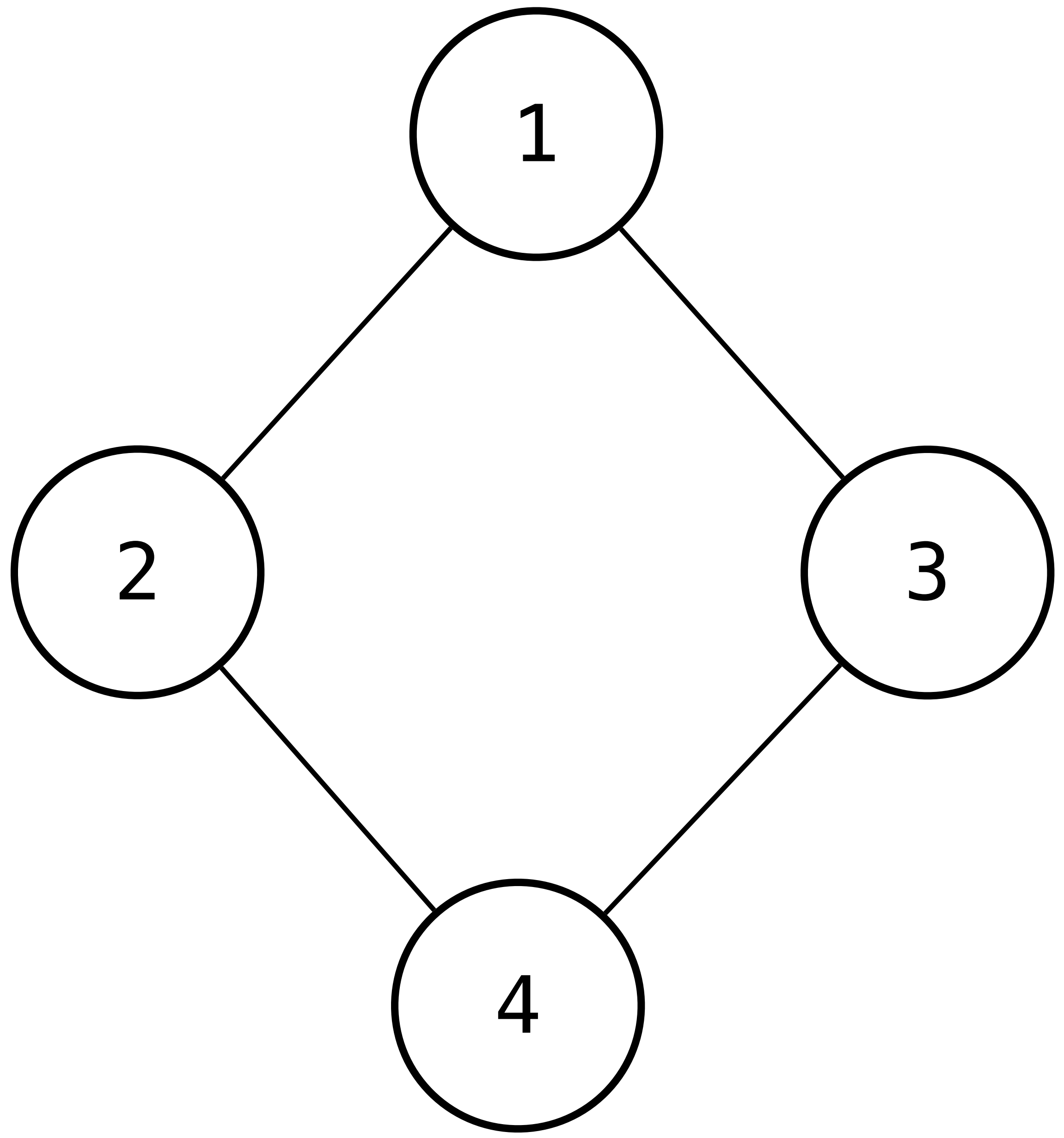
Example

$$\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix} = \texttt{(adjacency matrix for } G\texttt{) + } I$$

$$\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix}^2 = \begin{bmatrix}3&2&2&2\\2&3&2&2\\2&2&3&2\\2&2&2&3\end{bmatrix}$$

$$\begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix}^3 = \begin{bmatrix}3&2&2&2\\2&3&2&2\\2&2&3&2\\2&2&2&3\end{bmatrix}\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix} = \begin{bmatrix}6&6&6&7\\6&6&7&6\\6&7&6&6\\7&6&6&6\end{bmatrix}$$

G



How To: Reachability

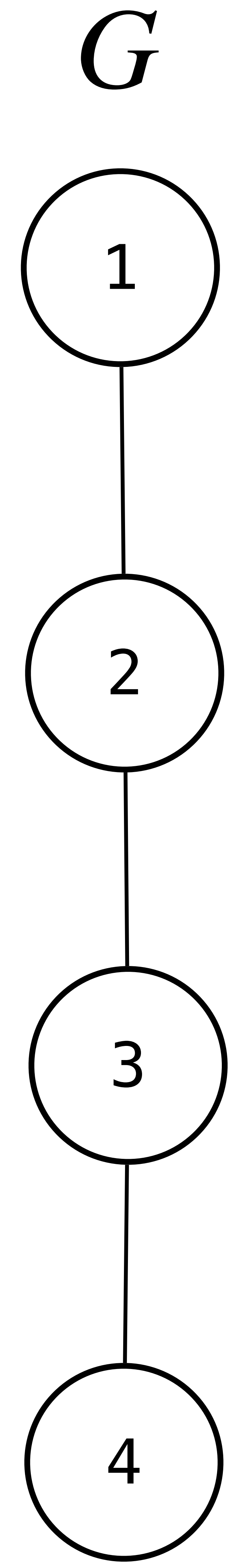
Question: Given a simple graph G determine how many nodes, v_i can reach in at least k steps.

Answer: Find $(A_G + I)^k$ and count the number of nonzero elements in column i.

(This could be useful for homework 6.)

Question

Determine the $(A_G + I)^2$ *and* $(A_G + I)^3$ *and interpret the results.*



Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.

Adjacency matrices are linear algebraic representations of graphs.