Computer Graphics

Geometric Algorithms

Lecture 15

CAS CS 132

Practice Problem

$$\begin{bmatrix} 2 & 1 & 3 \\ -2 & 0 & -4 \\ 6 & 3 & 9 \end{bmatrix}$$

`Find the LU decomposition of the above matrix.`

Answer

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 0 & -4 \\ 6 & 3 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ -2^{+2} & 0^{+1} & -4^{+3} \\ 6^{-6} & 3^{-3} & 9^{-9} \end{bmatrix} \xrightarrow[E_3:\ R_3 \leftarrow R_3 - 0R_2]{E_1:\ R_2 \leftarrow R_2 + R_1 \quad (1) \qquad E_2:\ R_3 \leftarrow R_3 - 3R_1 \quad (-3)} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$A = LU$

$$E_1^{-1} E_2^{-1} E_3^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Objectives

1. Look at linear algebraic methods in graphics
2. Briefly discuss Homework 8

Keywords

elementary matrices

LU factorization

wireframe objects

homogeneous coordinates

translation

perspective projections

Recap: Solving Systems using the LU Factorization

Connecting back to Matrix Equations

$$A\mathbf{x} = \mathbf{b}$$

Question. Solve the above matrix equation (in other words, find a general form solution).

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What does the LU factorization give us?

Connecting back to Matrix Equations

$$(LU)\mathbf{x} = \mathbf{b}$$

Question. Solve the above matrix equation (in other words, find a general form solution).

Substitute LU for A

Connecting back to Matrix Equations

$$L(U\mathbf{x}) = \mathbf{b}$$

Question. Solve the above matrix equation (in other words, find a general form solution).

Rearrange matrix-vector multiplications

Connecting back to Matrix Equations

$$U\mathbf{x} = L^{-1}\mathbf{b}$$

Question. Solve the above matrix equation (in other words, find a general form solution).

Multiply by L^{-1} on both sides

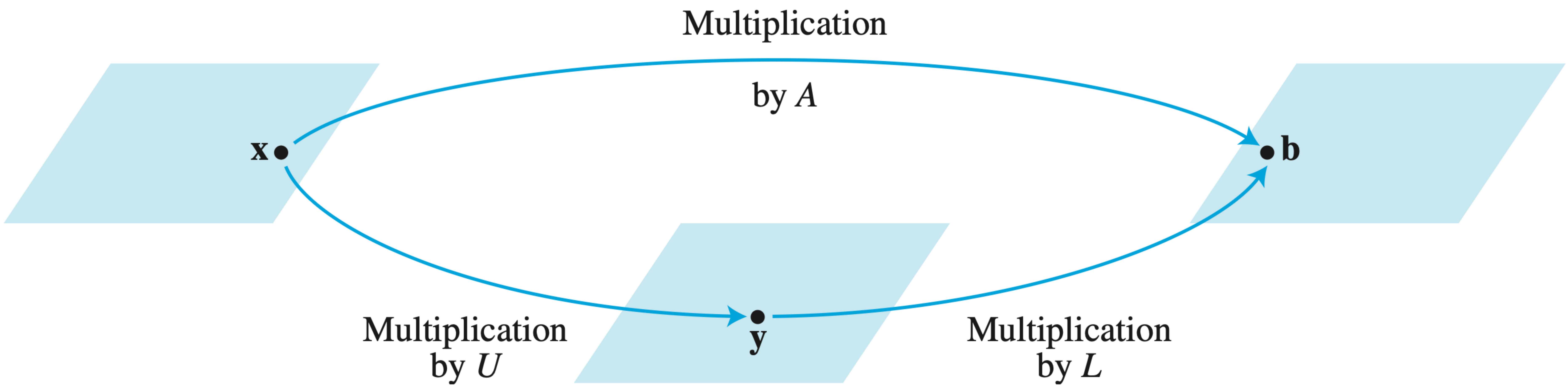
Connecting back to Matrix Equations

$$U\mathbf{x} = L^{-1}\mathbf{b}$$

Question. Solve the above matrix equation (in other words, find a general form solution).

A solution to $A\mathbf{x} = \mathbf{b}$ is the same as a solution to $U\mathbf{x} = L^{-1}\mathbf{b}$

Solving systems with the LU (Pictorially)



$$U\mathbf{x} = \mathbf{y} = L^{-1}\mathbf{b}$$

If A maps $\mathbf{x}$ to $\mathbf{b}$, then U maps $\mathbf{x}$ to some vector $\mathbf{y}$ which is mapped to $\mathbf{b}$ by L.

FLOPS for $L\mathbf{x} = \mathbf{b}$

L is a **lower triangular** matrix. The system can be solved in $\sim n^2$ FLOPS by <u>forward</u> substitution.

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = b_1$$
$$x_2 = b_2 - a_{21}x_1$$
$$x_3 = b_3 - a_{31}x_1 - a_{32}x_2$$

FLOPS for $U\mathbf{x} = \mathbf{v}$

U is in *echelon form*. We only need to perform back substitution, which can be done in $\sim n^2$ FLOPS.

$$\begin{bmatrix} \blacksquare & * & * & * & * & | \\ 0 & \blacksquare & * & * & * & \mathbf{v} \\ 0 & 0 & 0 & \blacksquare & * & \\ 0 & 0 & 0 & 0 & 0 & | \end{bmatrix} \xrightarrow{\text{back substitution}} \begin{bmatrix} 1 & 0 & * & 0 & * & | \\ 0 & 1 & * & 0 & * & \mathbf{w} \\ 0 & 0 & 0 & 1 & * & \\ 0 & 0 & 0 & 0 & 0 & | \end{bmatrix}$$

FLOP Comparison

	Preprocessing	Solving
Gaussian Elimination	0	$\sim \frac{2}{3}n^3$
Matrix Inversion	$\sim 2n^3$	$\sim 2n^2$
LU Factorization	$\sim \frac{2}{3}n^3$	$\sim 2n^2$

Graphics

Disclaimer

I am not an expert in this field.

Motivation (or Pretty Pictures)

Graphics doesn't need much motivation.

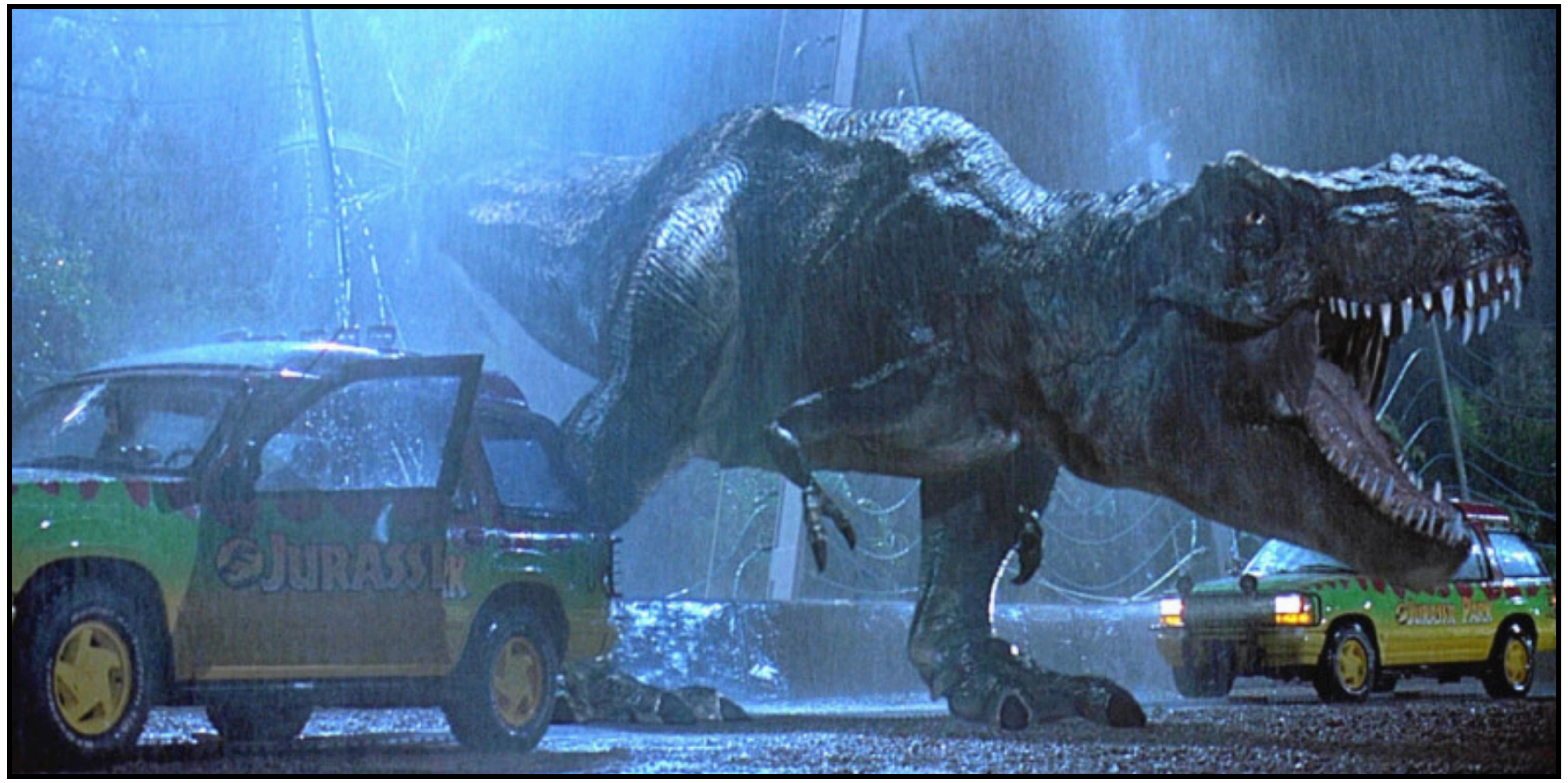
We spend so much time interacting graphics in one form or another.

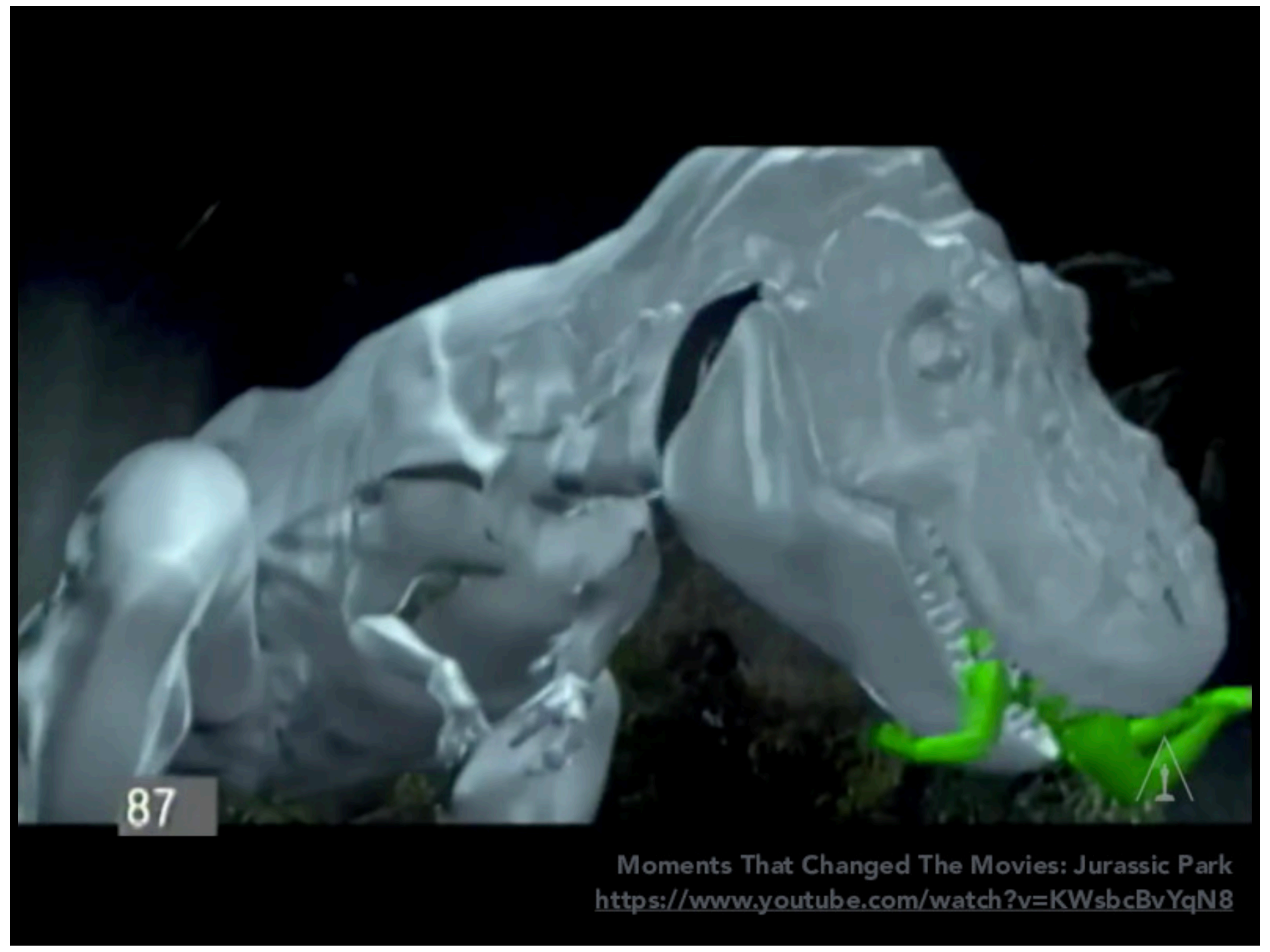
But in case you haven't thought too much about it, some examples...

source: CS184 Lecture Slides, UC Berkeley, Ng Ren

Movies

Jurassic Park (1993)





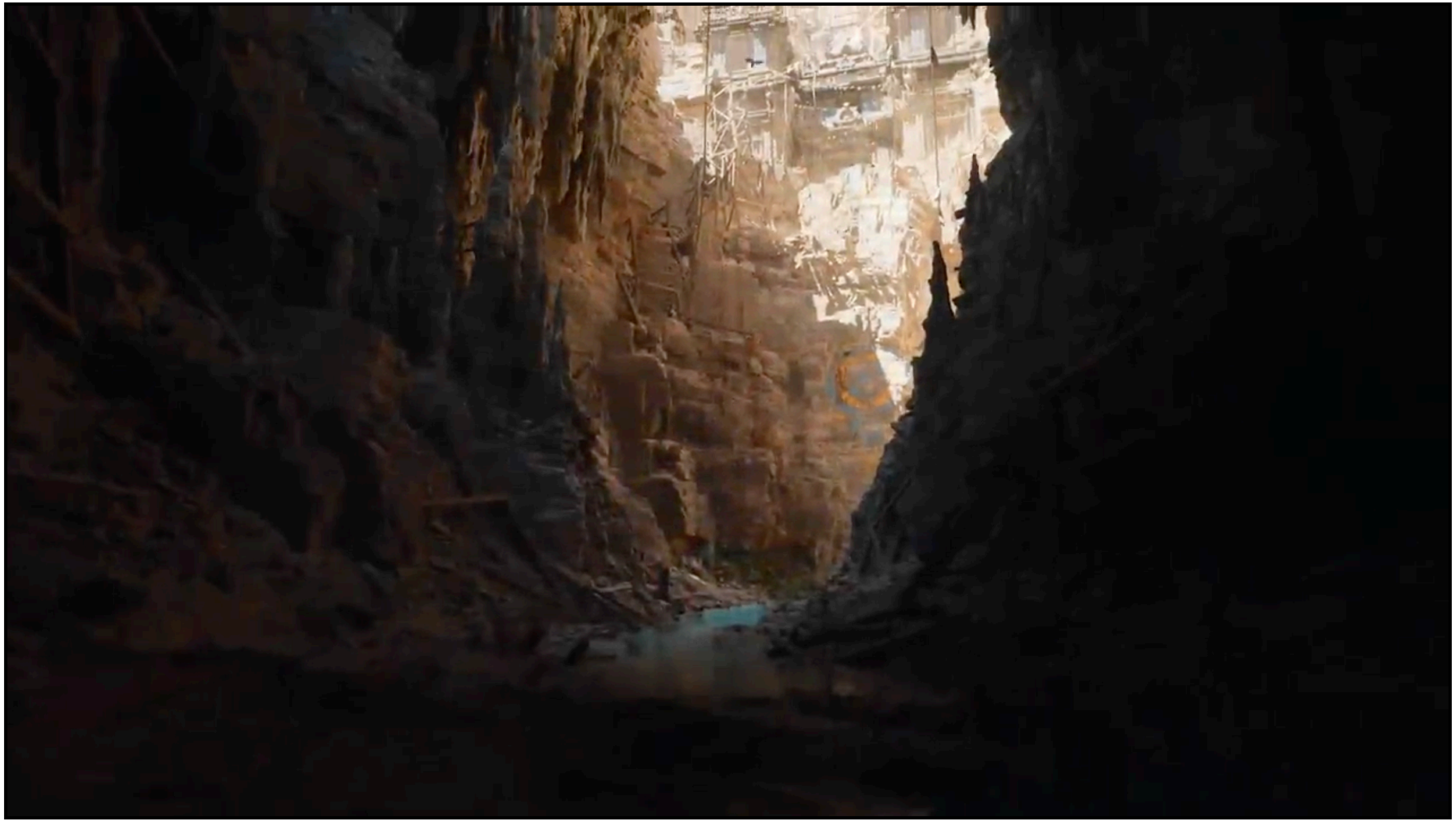
Alice in Wonderland (2010)

Motion Capture

Two Towers (2002)

Video Games

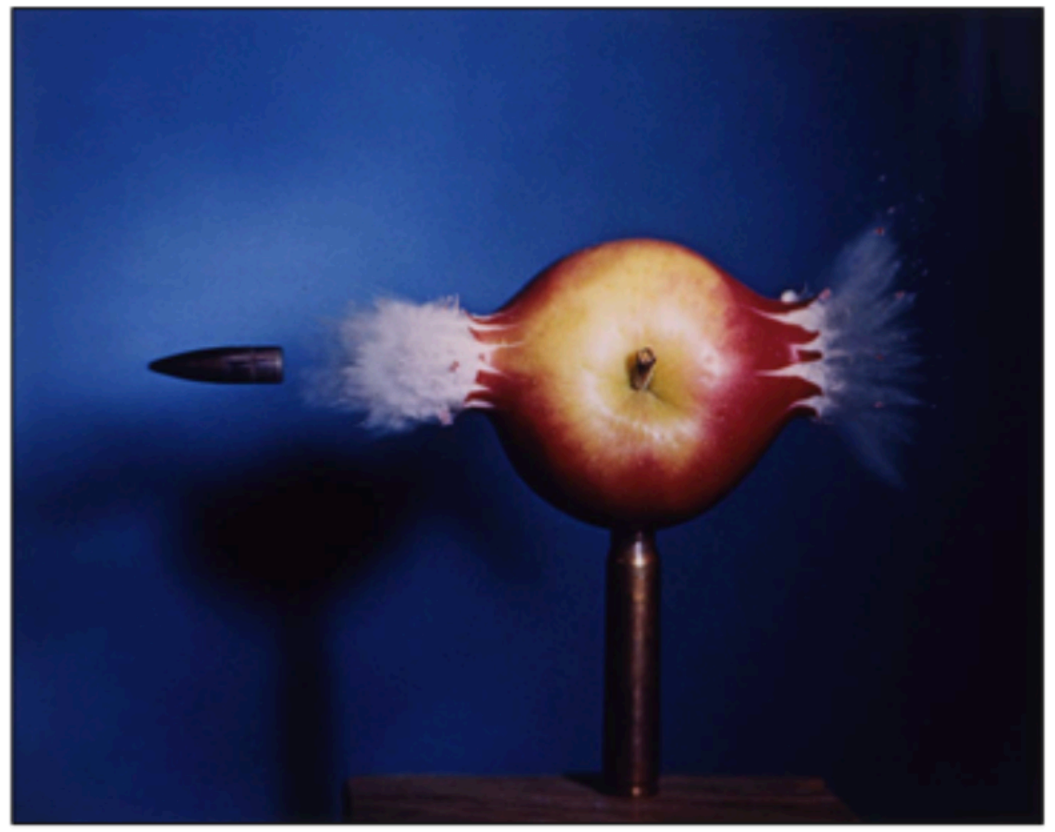
Unreal Engine 5 (2020)



Scientific Visualization

First image of a black hole (2022)

Photography



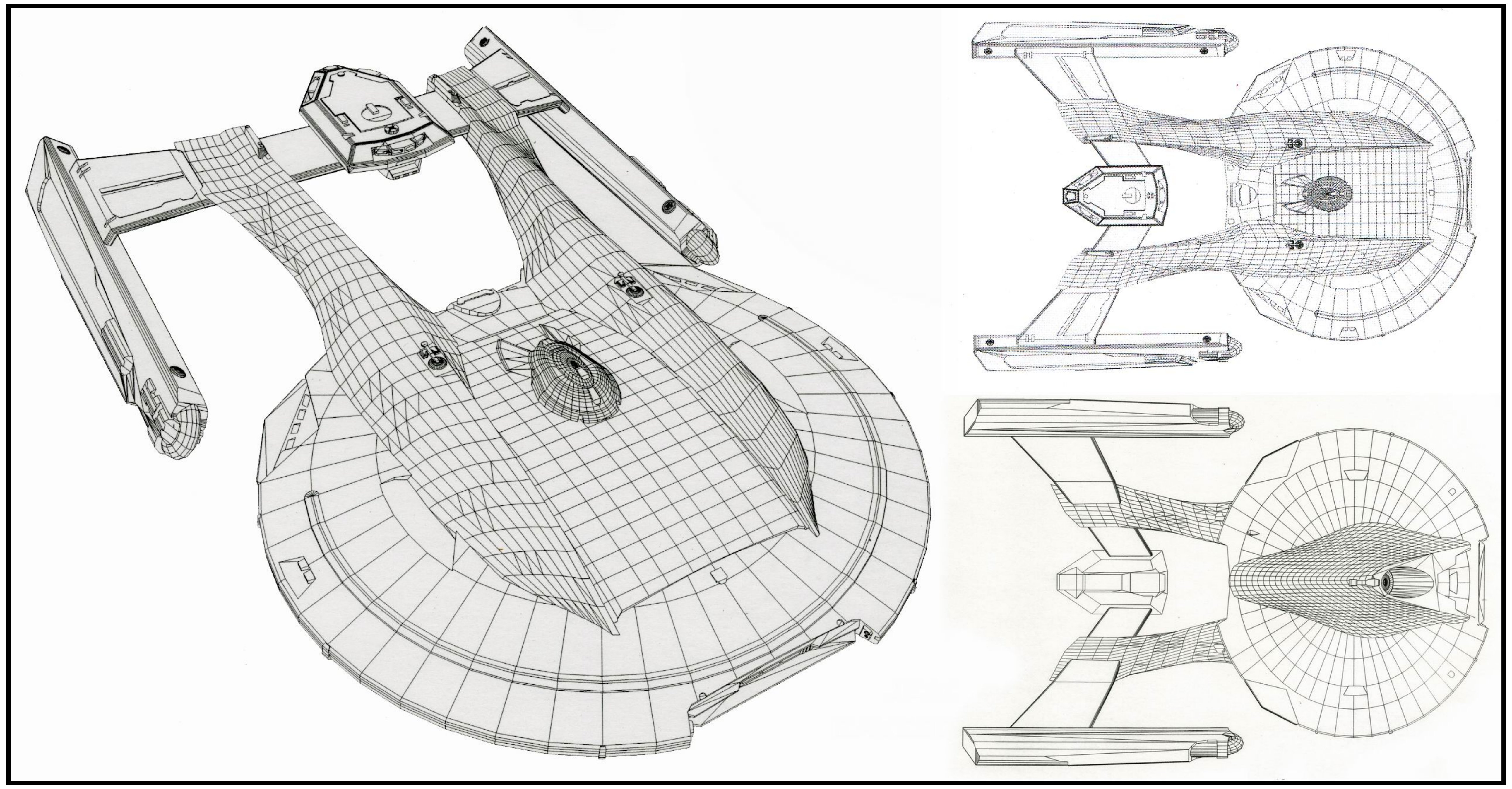
NASA | Walter Iooss | Steve McCurry
Harold Edgerton | NASA | National Geographic

Graphics and Linear Algebra

3D Graphics

There are many facets of computer graphics, but we will be focusing on one problem today:

Manipulating and Transforming 3D objects and rendering them on a screen.



3D Graphics Pipeline

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Today

Wire Frames

A **wire frame** is representation of a surface as a collection of polygons and line segments.

Transformations on line segments and polygons are **linear**.

https://commons.wikimedia.org/wiki/File:Wireframe_Render_of_Digital_Clothing_Bathingrobe_3D_Model.jpg

Transformations

We've seen many 2D transformations

» Reflections
» Expansion
» Shearing
» Projection

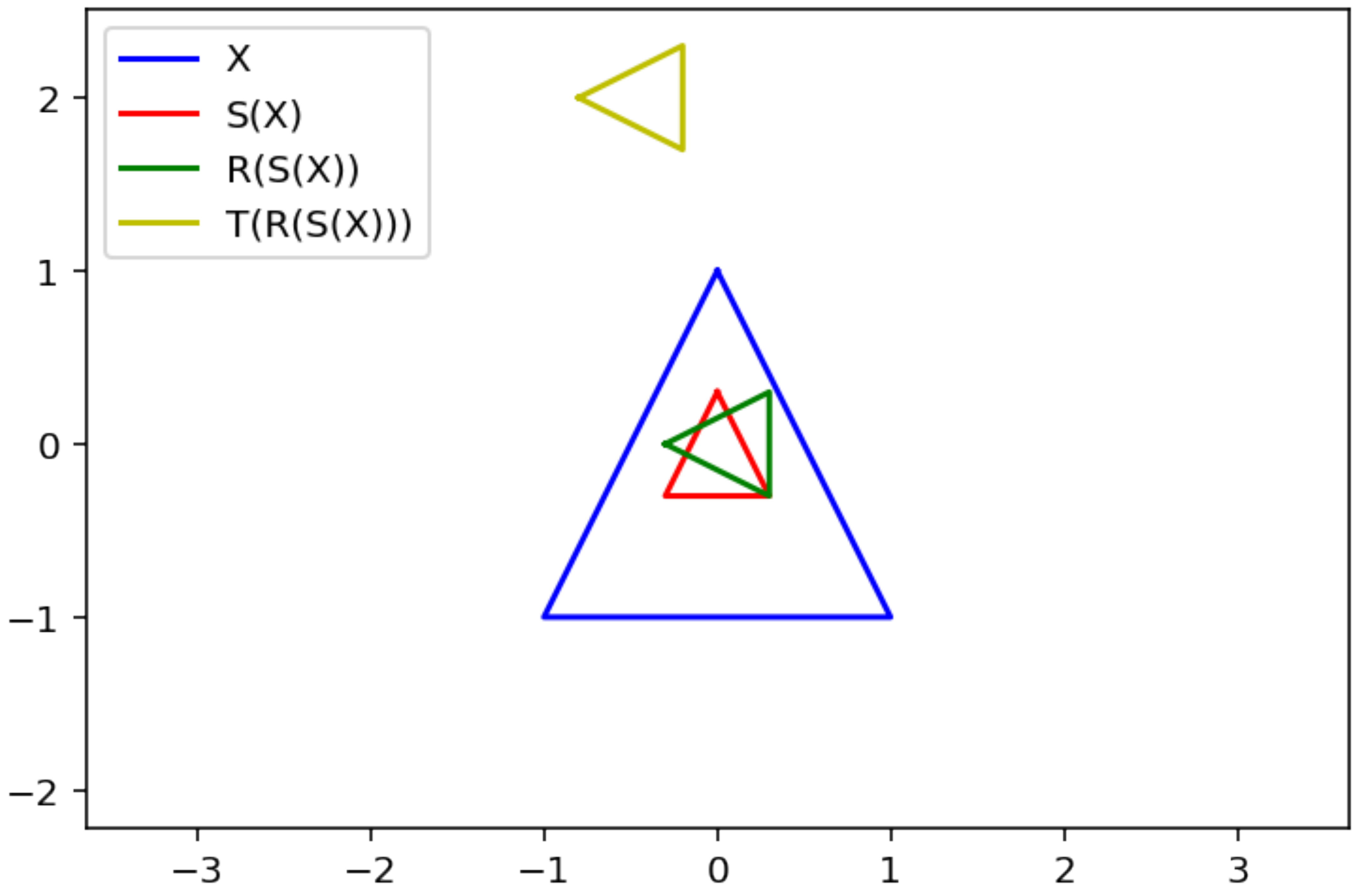
We've seen some 3D transformations

» Rotations
» Projections

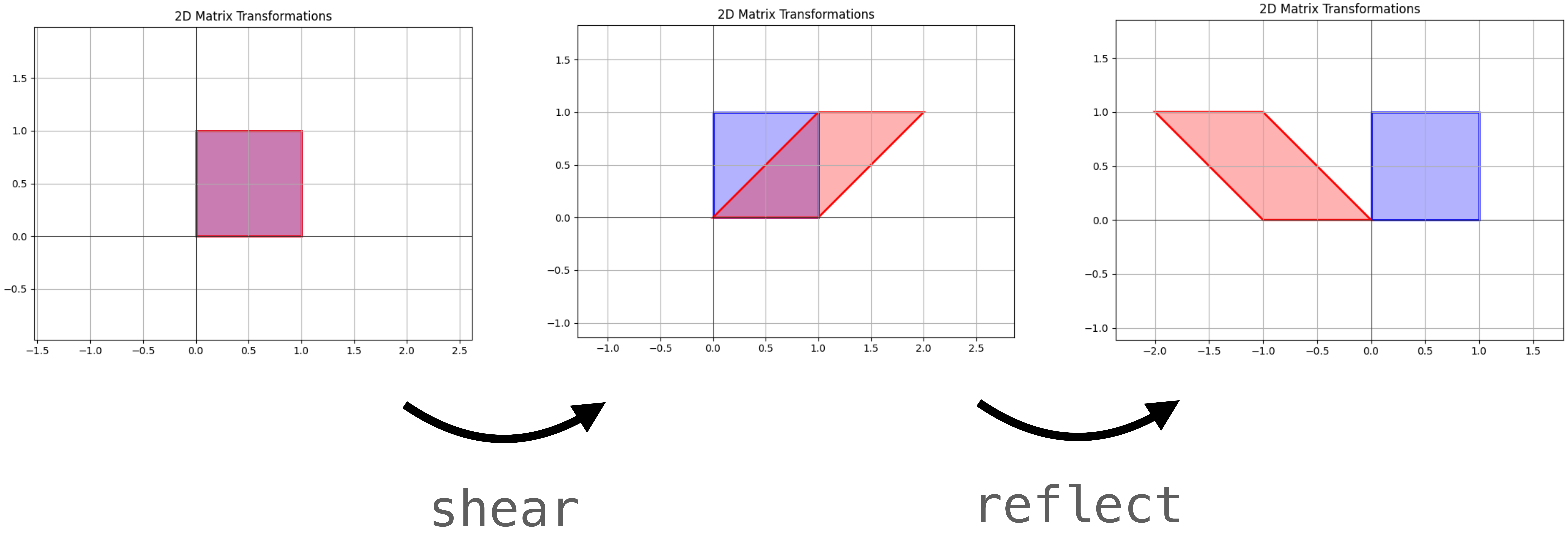
Composing Transformations

Recall. Multiplying matrices **composes** their associated transformations.

So complex graphical transformations can be combined into a single matrix.



Shearing and Reflecting (Geometrically)



More Transformations

What we're adding today:

» More on rotations
» translations
» perspective projections

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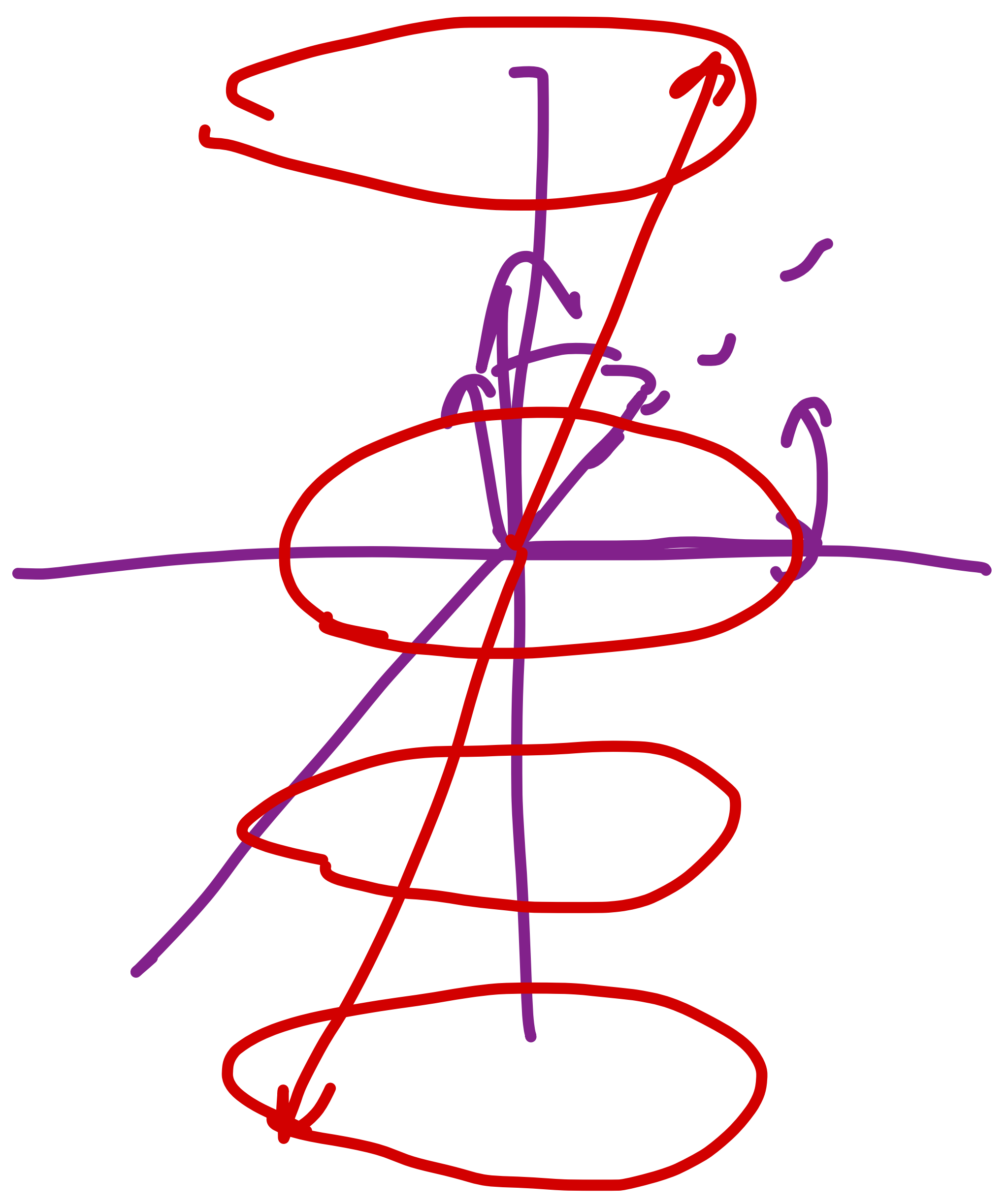
These aren't linear, but they are incredibly important so we have to address them.

3D Rotation Matrices

$$R_x^\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y^\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_z^\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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These are the matrices for counterclockwise rotation around x, y, and z axes.

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Fact. Any rotation can be done by some matrix of the form

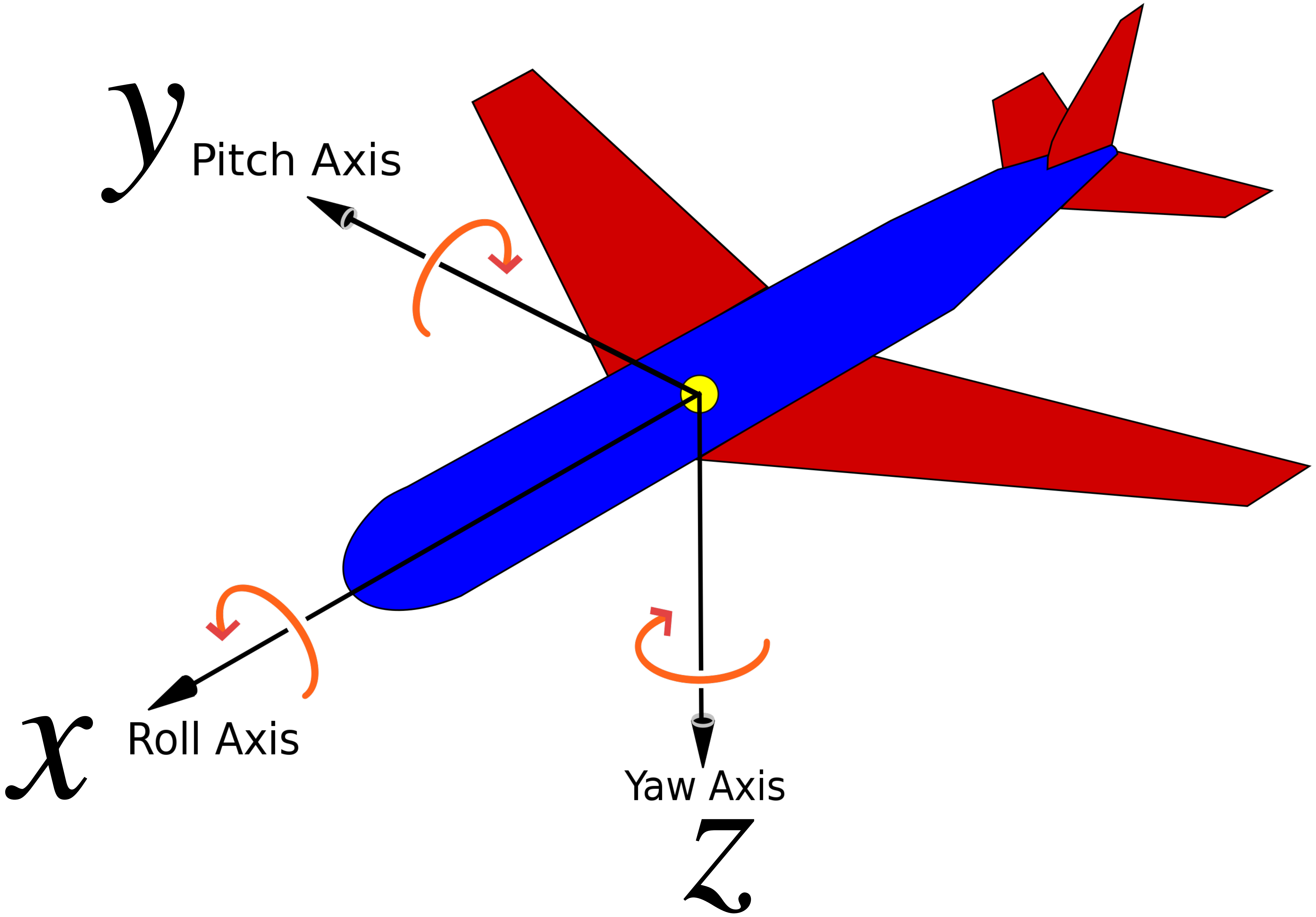
$$R_z^\theta R_y^\gamma R_x^\eta$$

Roll, Pitch and Yaw

roll changes the side-to-side tilt

pitch changes the up-down tilt

yaw changes direction



https://commons.wikimedia.org/wiki/File:Yaw_Axis_Corrected.svg

General Rotations

$$R_z^{\theta} R_y^{\gamma} R_x^{\eta}$$

yaw pitch roll

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Exactly what rotation you get is not obvious (this a hard problem in control theory).

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Remember. !!Matrix multiplication does not commute!!

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yaw pitch roll

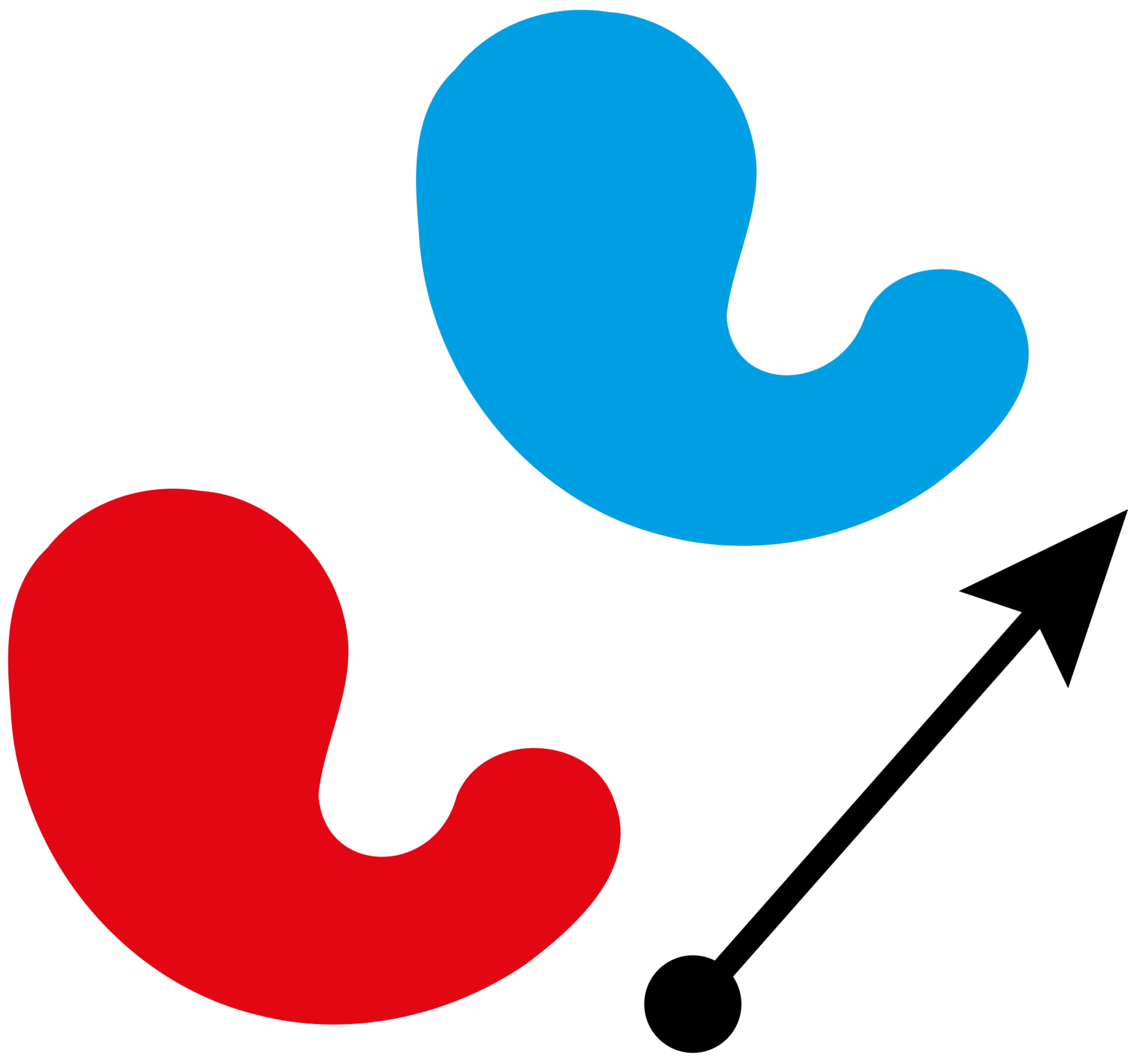
Exactly what rotation you get is not obvious (this a hard problem in control theory).

Remember. !!Matrix multiplication does not commute!!

So changing η above doesn't just rotate the object around the x–axis (that axis might be tilted along the pitch axis, for example).

demo

Translation

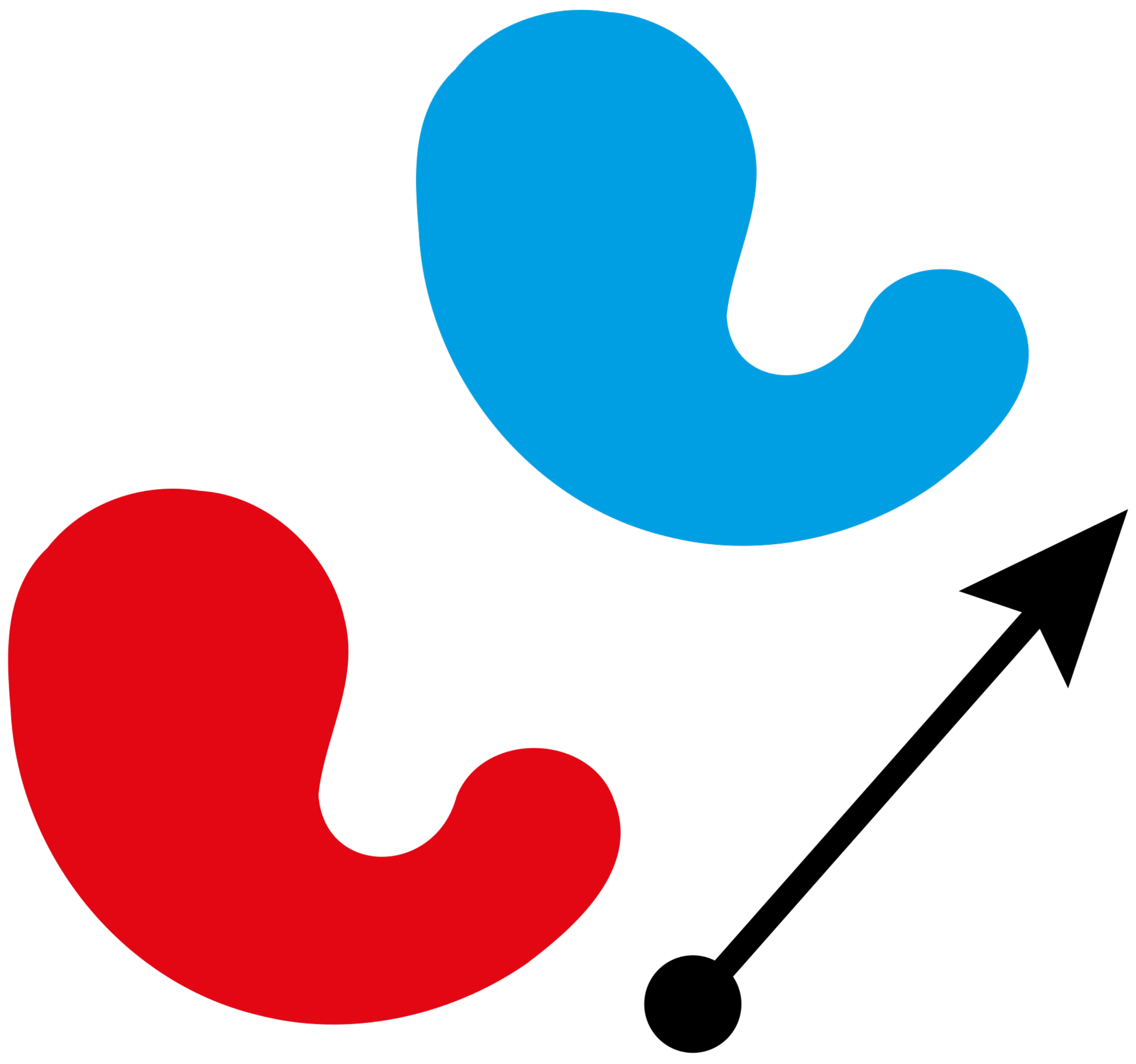


In 2D

https://commons.wikimedia.org/wiki/File:Traslazione_OK.svg

Translation

Given a vector $\mathbf{t}$ a **translation** is the transformation



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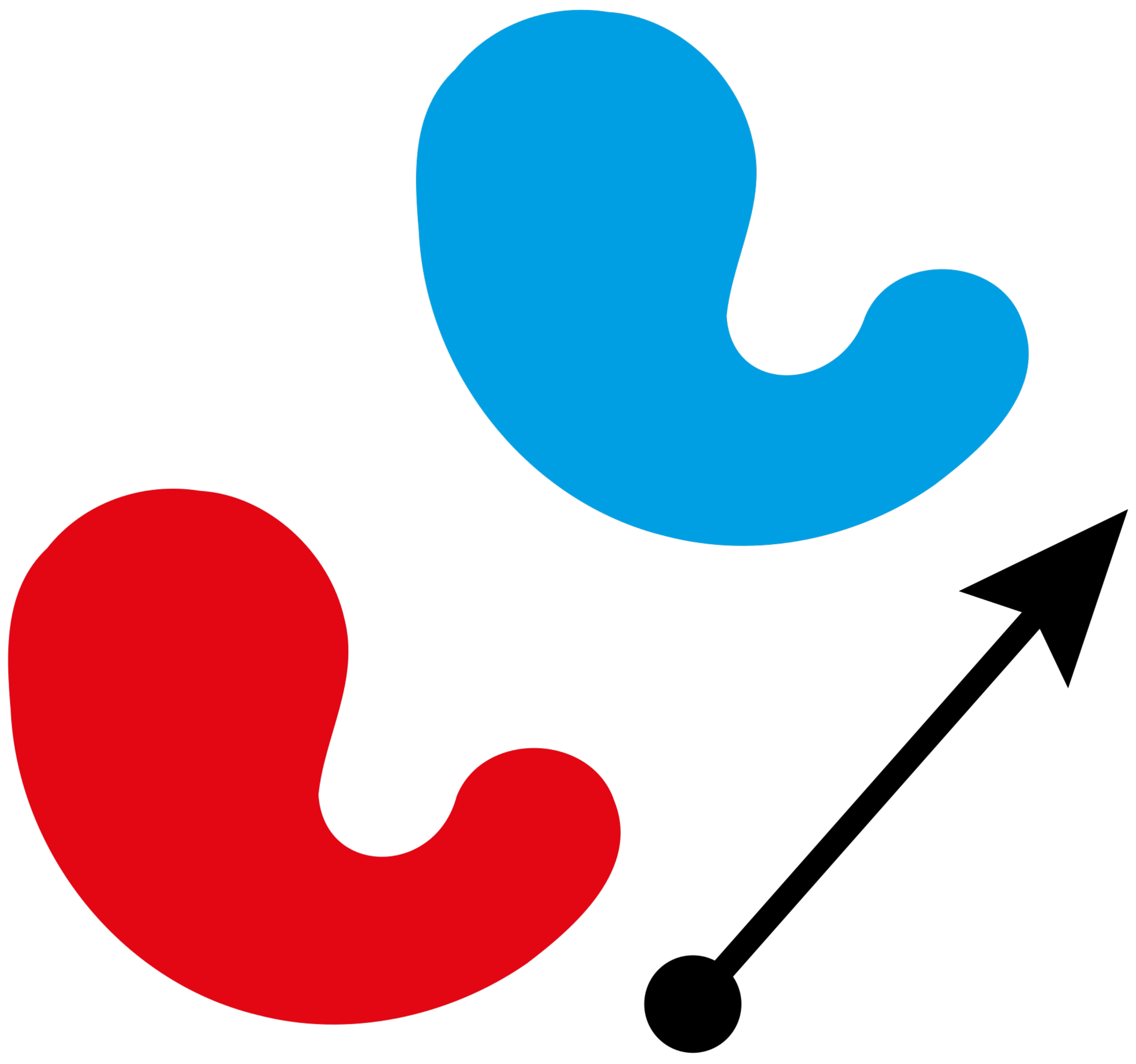
$$T(\mathbf{x}) = \mathbf{x} + \mathbf{t}$$



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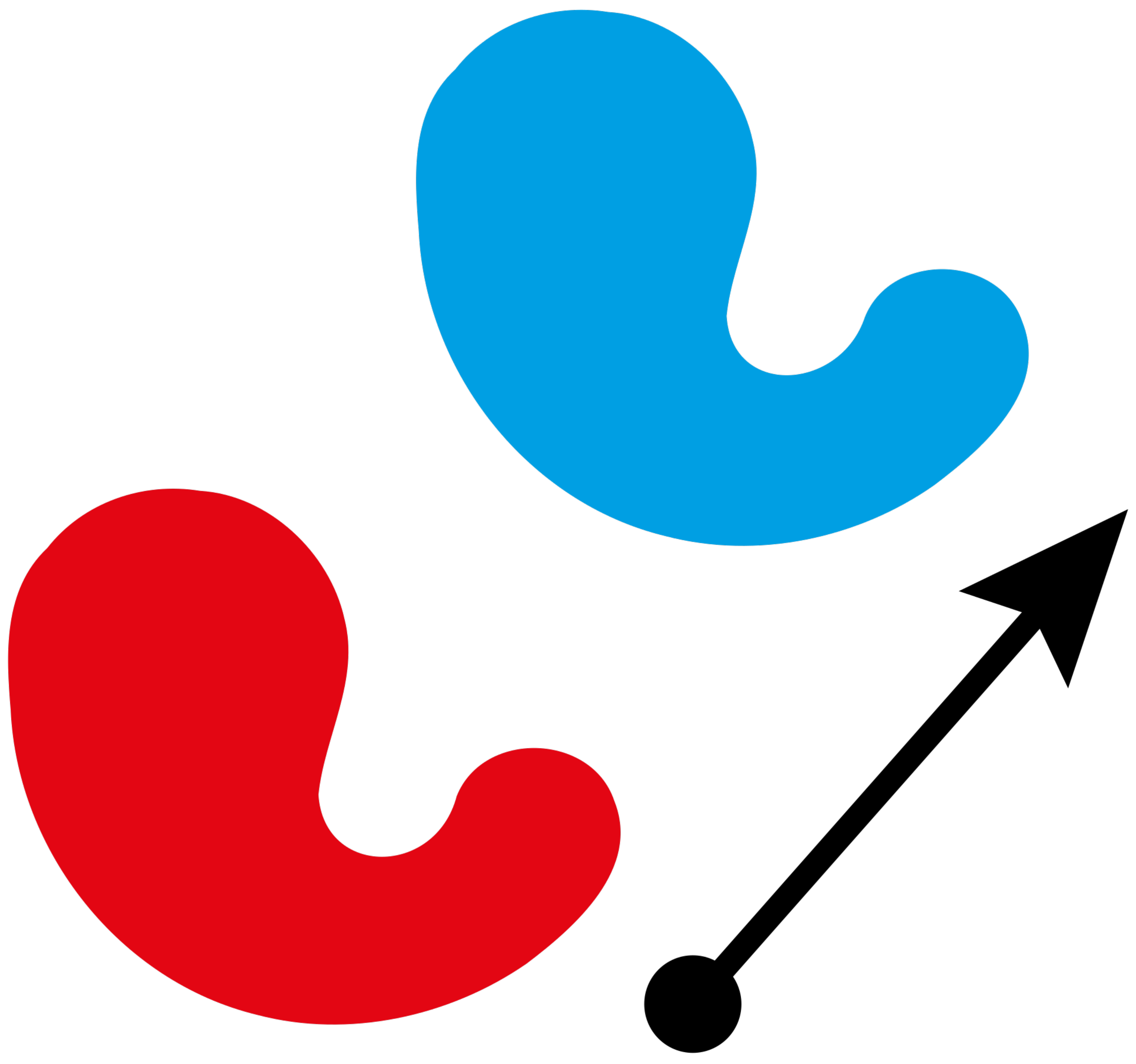
As we've seen, **translation is not linear:**



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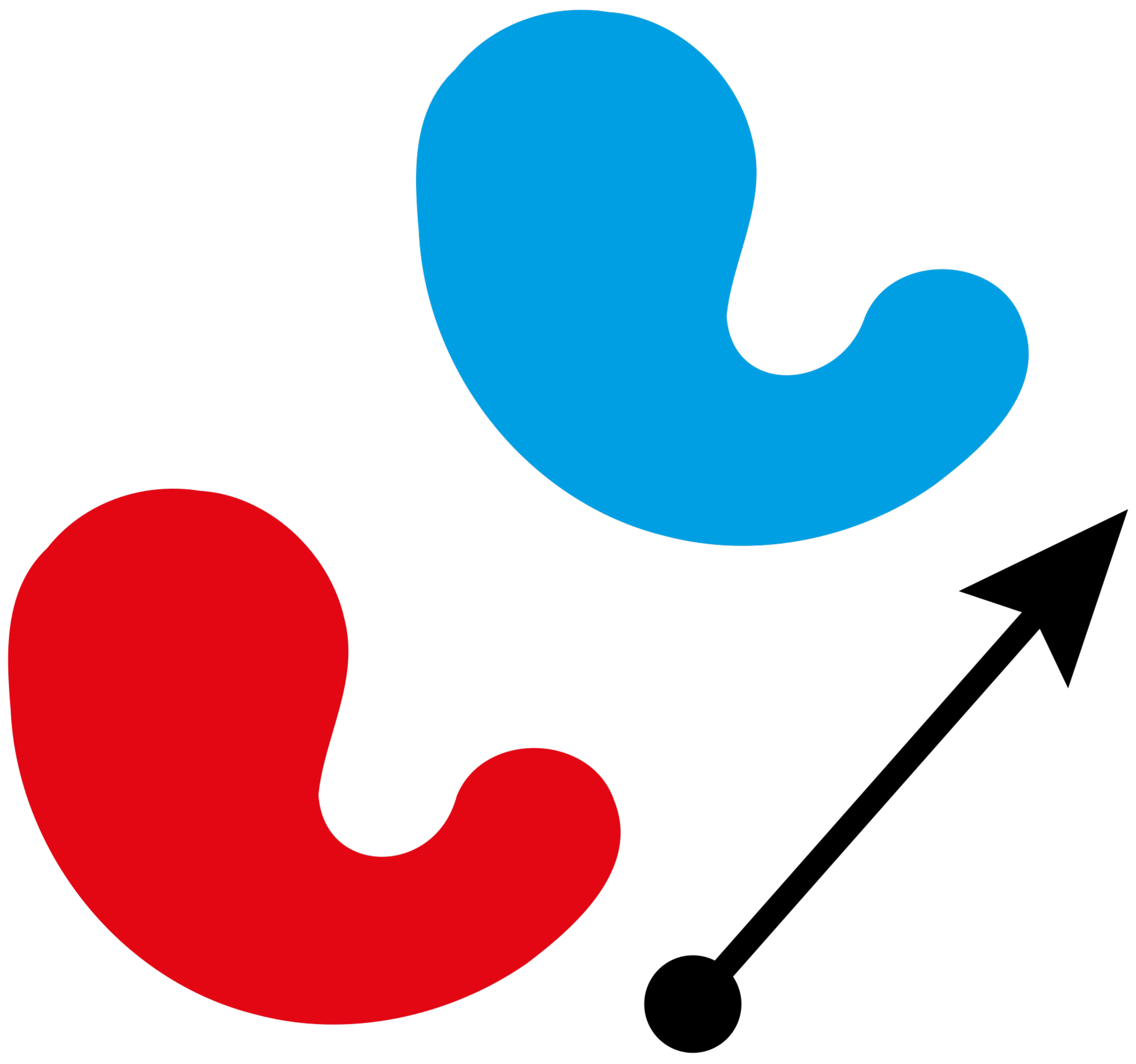

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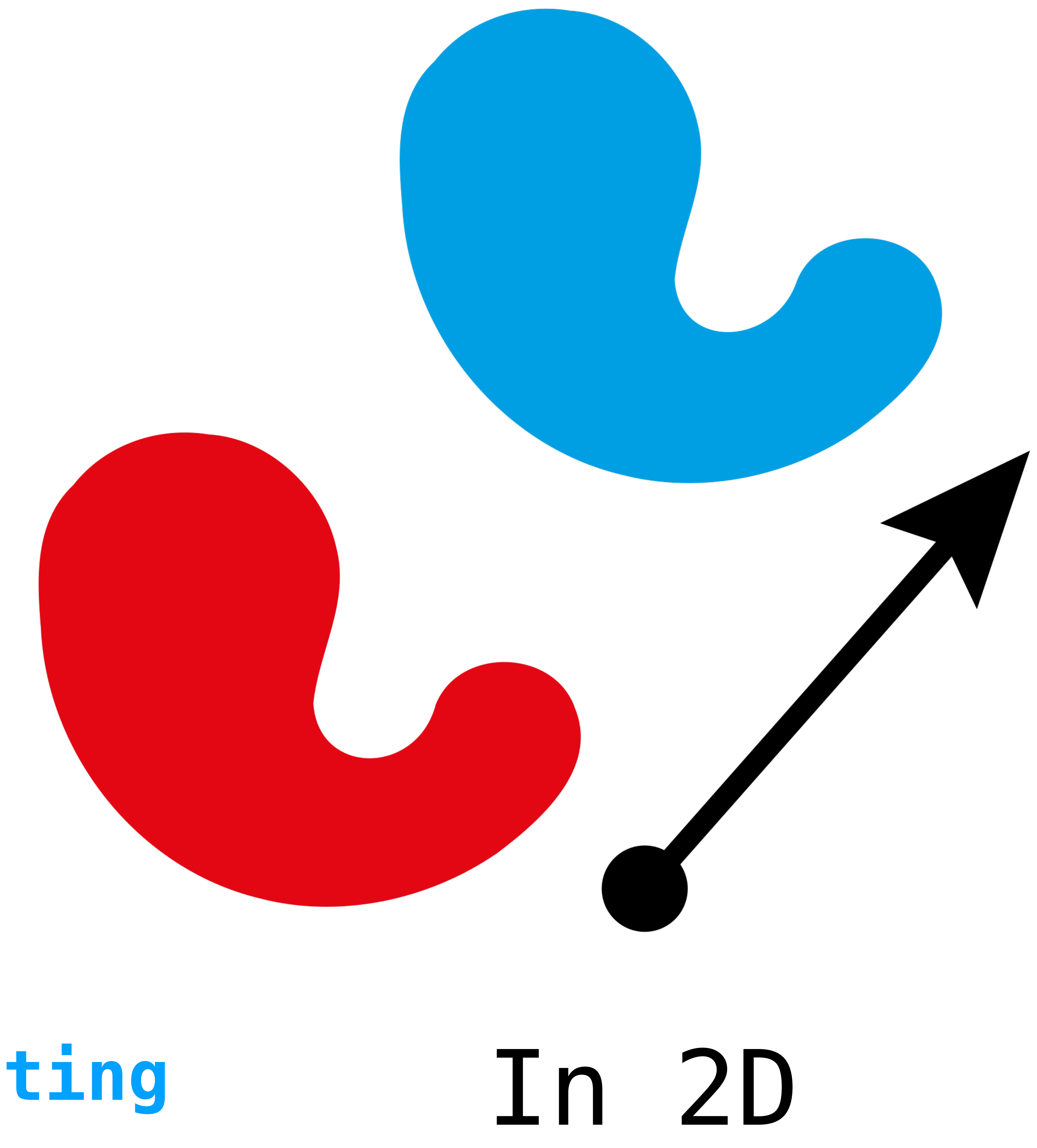
For this to be interesting $\mathbf{t}$ will be nonzero



https://commons.wikimedia.org/wiki/File:Traslazione_OK.svg

Translation (3D)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$$

Observation. This would be linear if we had another variable.

Translation (3D)

$$\begin{bmatrix} x \\ y \\ z \\ q \end{bmatrix} \mapsto \begin{bmatrix} x + aq \\ y + bq \\ z + cq \\ q \end{bmatrix}$$

Observation. This would be linear if we had another variable.

Translation (3D)

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}a\\b\\c\\1\end{bmatrix}$$

Observation. This would be linear if we had another variable.

Translation (3D)

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observation. This would be linear if we had another variable.

So if we are willing to keep around an extra entry, we can do translation **linearly.**

Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

For initializing to homogeneous coordinates, we set this to 1

Cartesian to homogeneous

The **homogeneous coordinate** for vector in $\mathbb{R}^3$ is the same except "sheared" into the 4th dimension.

We use the extra entry to perform simple nonlinear transformations in a linear setting.

Translation (3D)

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x+a \\ x+b \\ x+c \\ 1 \end{bmatrix}$$

Definition. The **3D translation matrix** for homogeneous coordinates which translates by $(a, b, c)^T$ is the following.

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+2 \\ y+2 \\ z+2 \\ 1 \end{bmatrix}$$

Matrix Transformations for Homogeneous Coordinates

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \longrightarrow \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Now all our transformations need to be 4×4 matrices.

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But it's easy make 3×3 matrices work for homogeneous coordinates.

Matrix Transformations for Homogeneous Coordinates

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Now all our transformations need to be 4×4 matrices.

But it's easy make 3×3 matrices work for homogeneous coordinates.

If a transformation is linear, it doesn't need the extra coordinate.

Example: Homogeneous Rotation

Rotating counterclockwise about the x–axis in homogeneous coordinates is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projections

Vanishing Points

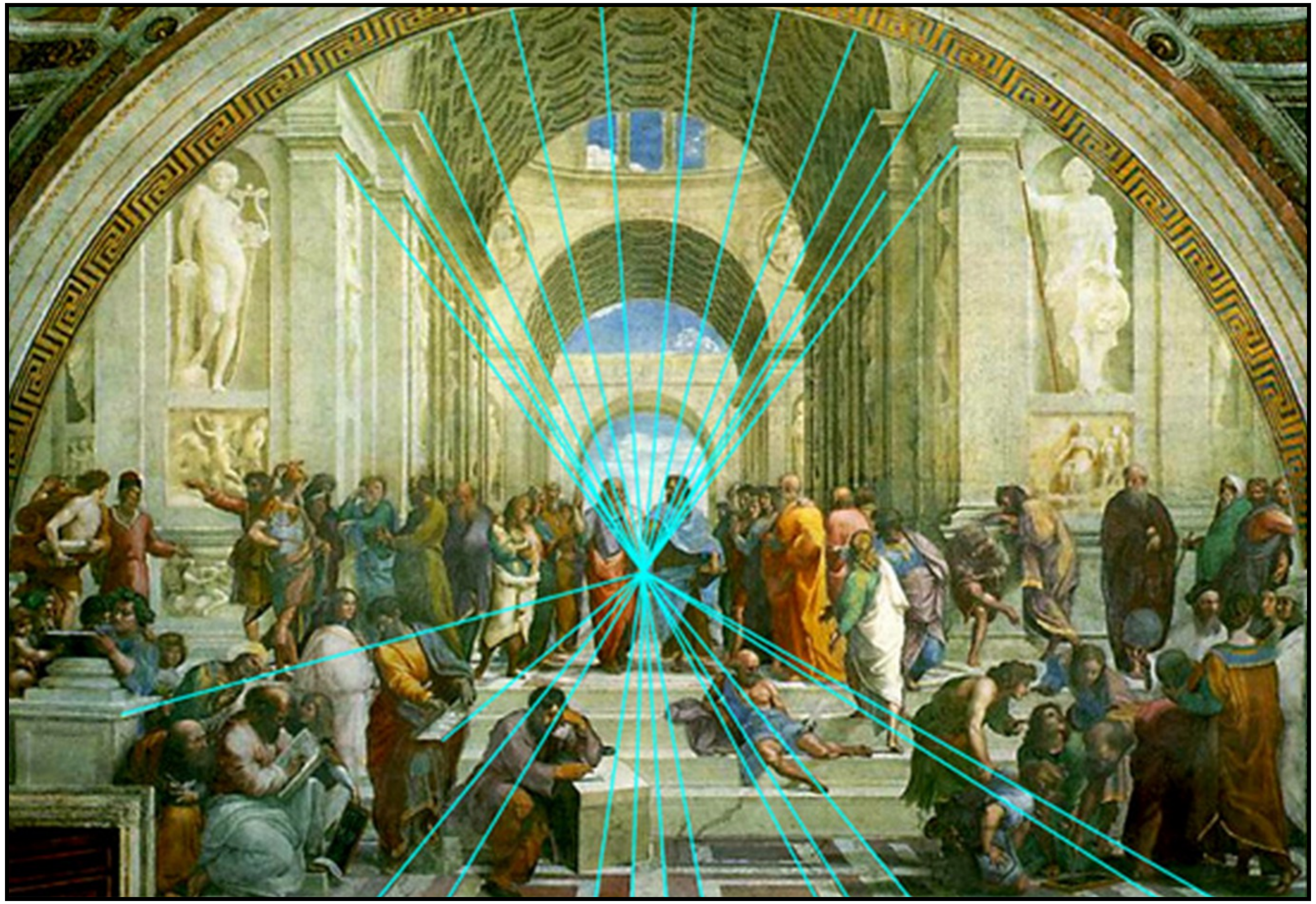
Parallel lines in space don't necessarily look parallel at a distance, they angle towards a point in the distance.

This is a side effect of **perspective projection**.

https://commons.wikimedia.org/wiki/File:Railroad-Tracks-Perspective.jpg

Vanishing Point

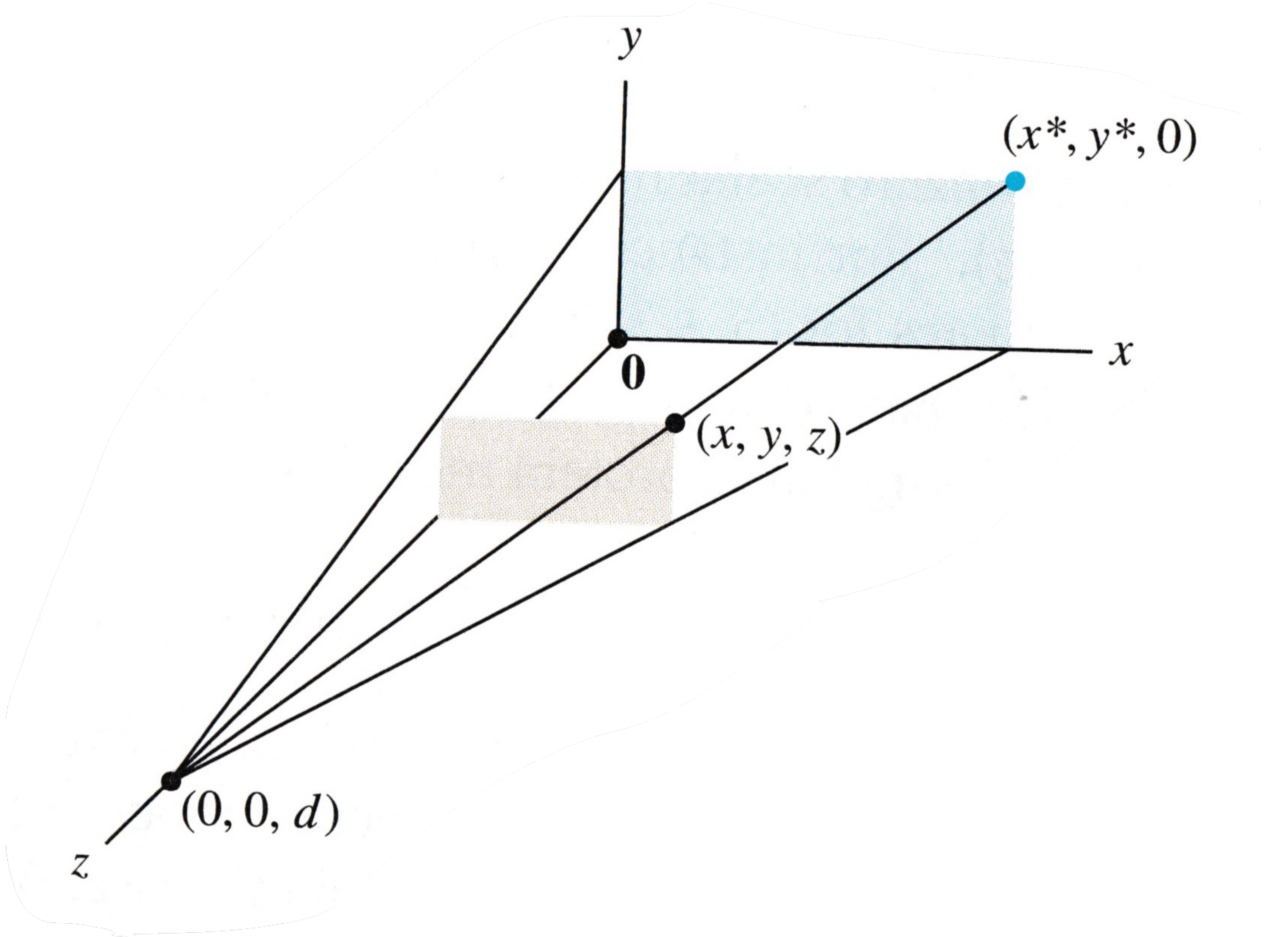
The School of Athens (~1510)



Computing Perspective

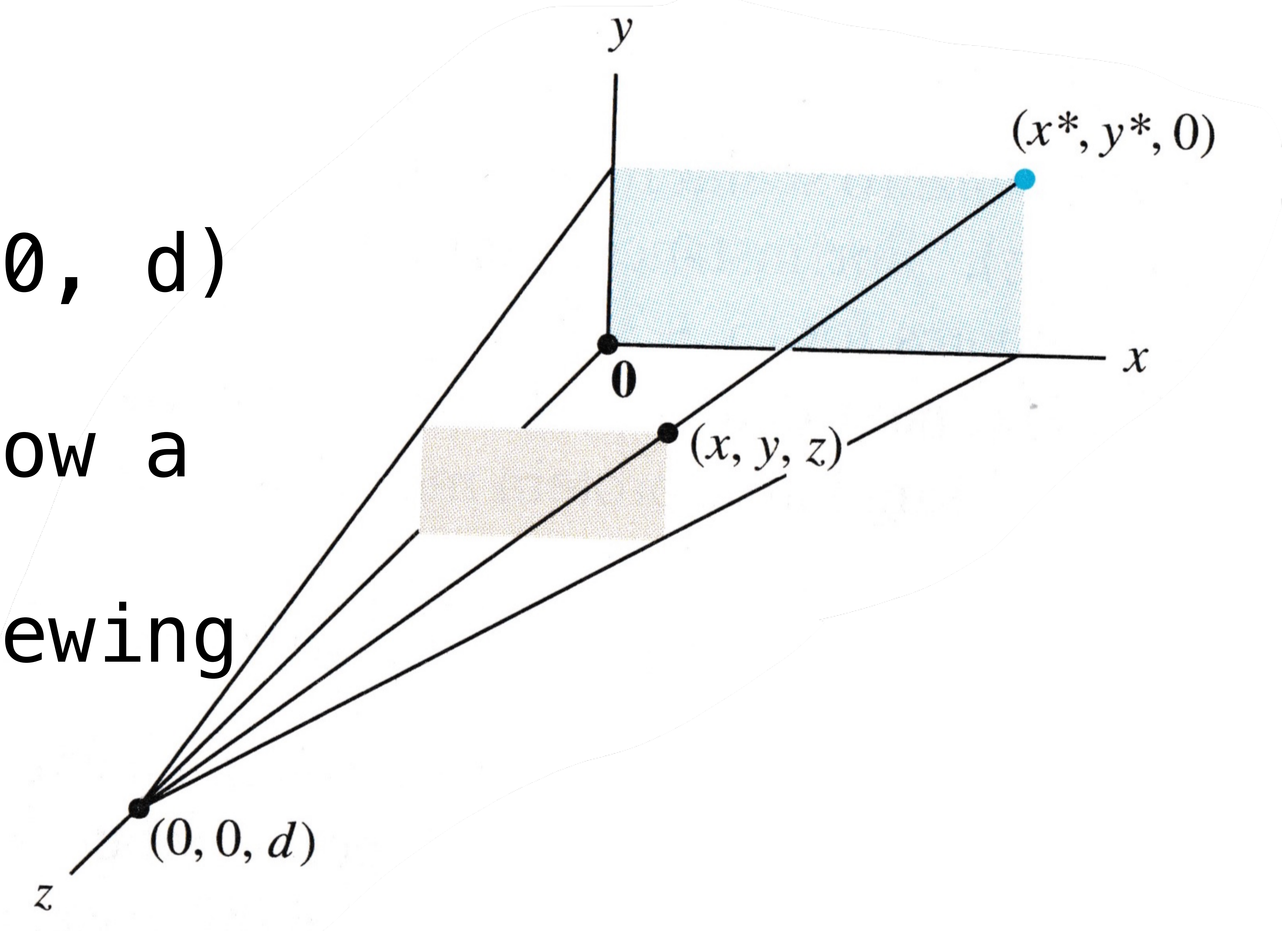
Light enters our eyes (or camera) at a single point from all directions.

Closer things "appear bigger" in our field of vision.

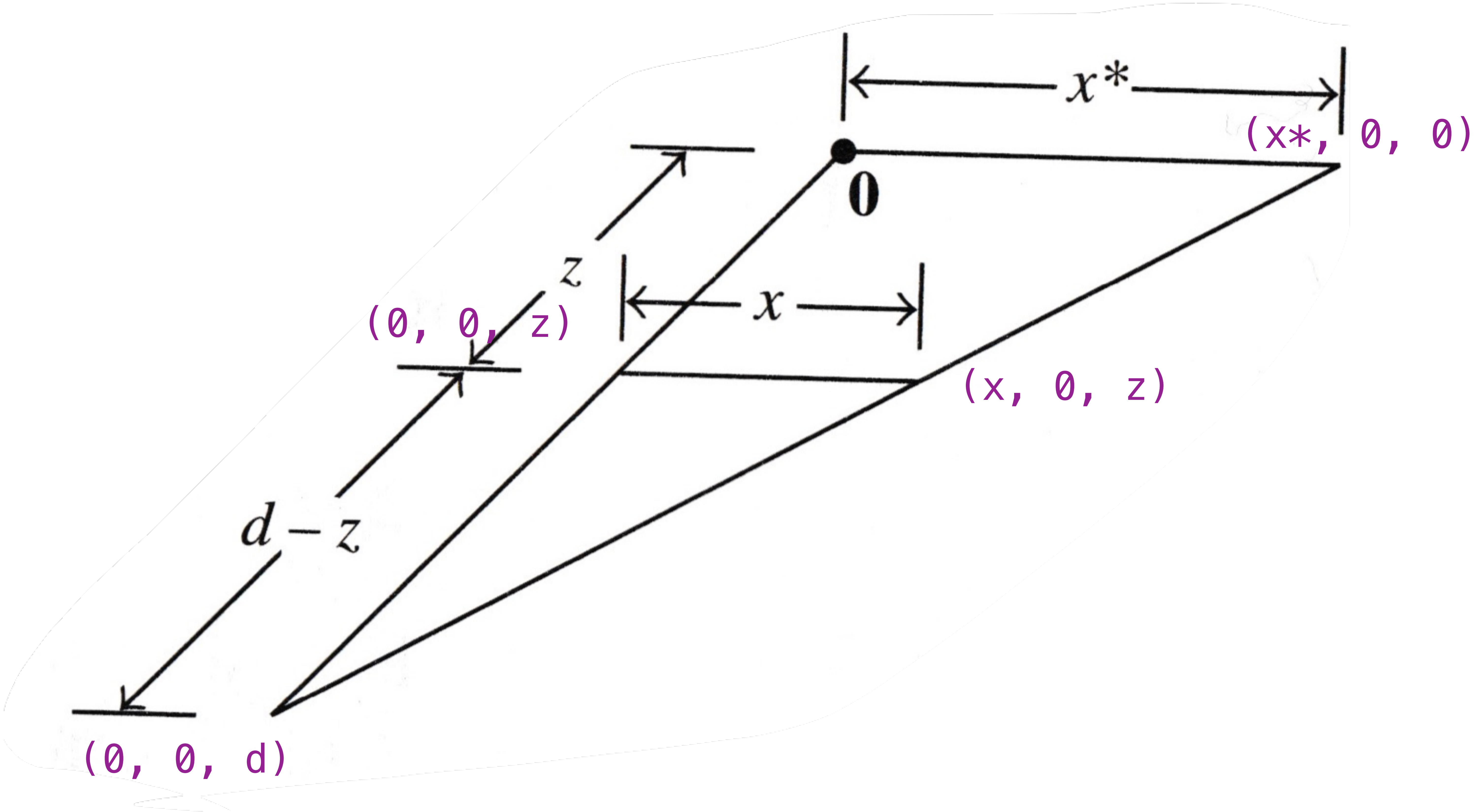


Computing Perspective

Problem. Given a **viewing position** (0, 0, d) and a **viewing plane** (xy–axis) determine how a point (x, y, z) is *projected* onto the viewing plane.

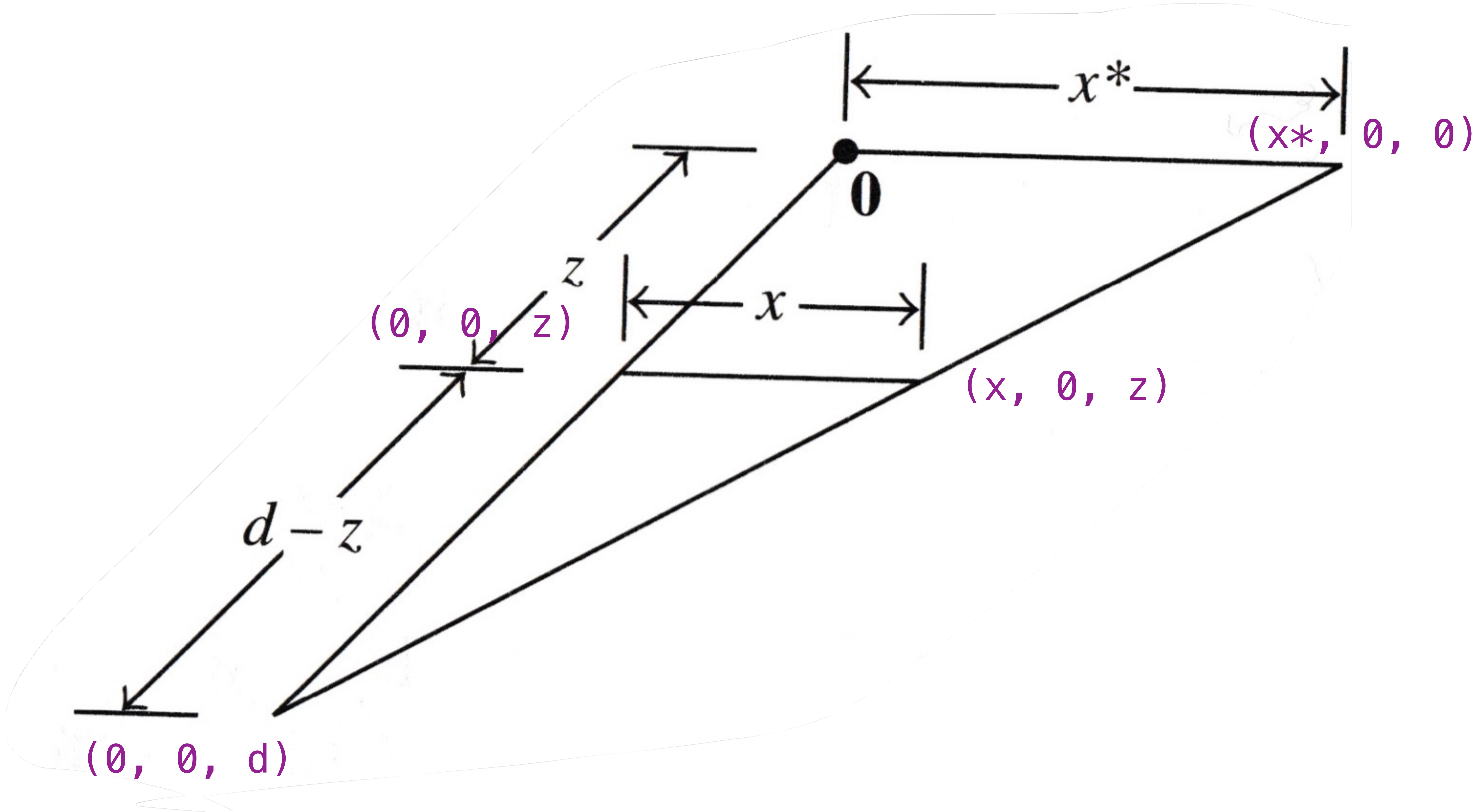


Similar Triangles



Similar Triangles

Similar triangles are triangles with the same angles (in the same order).



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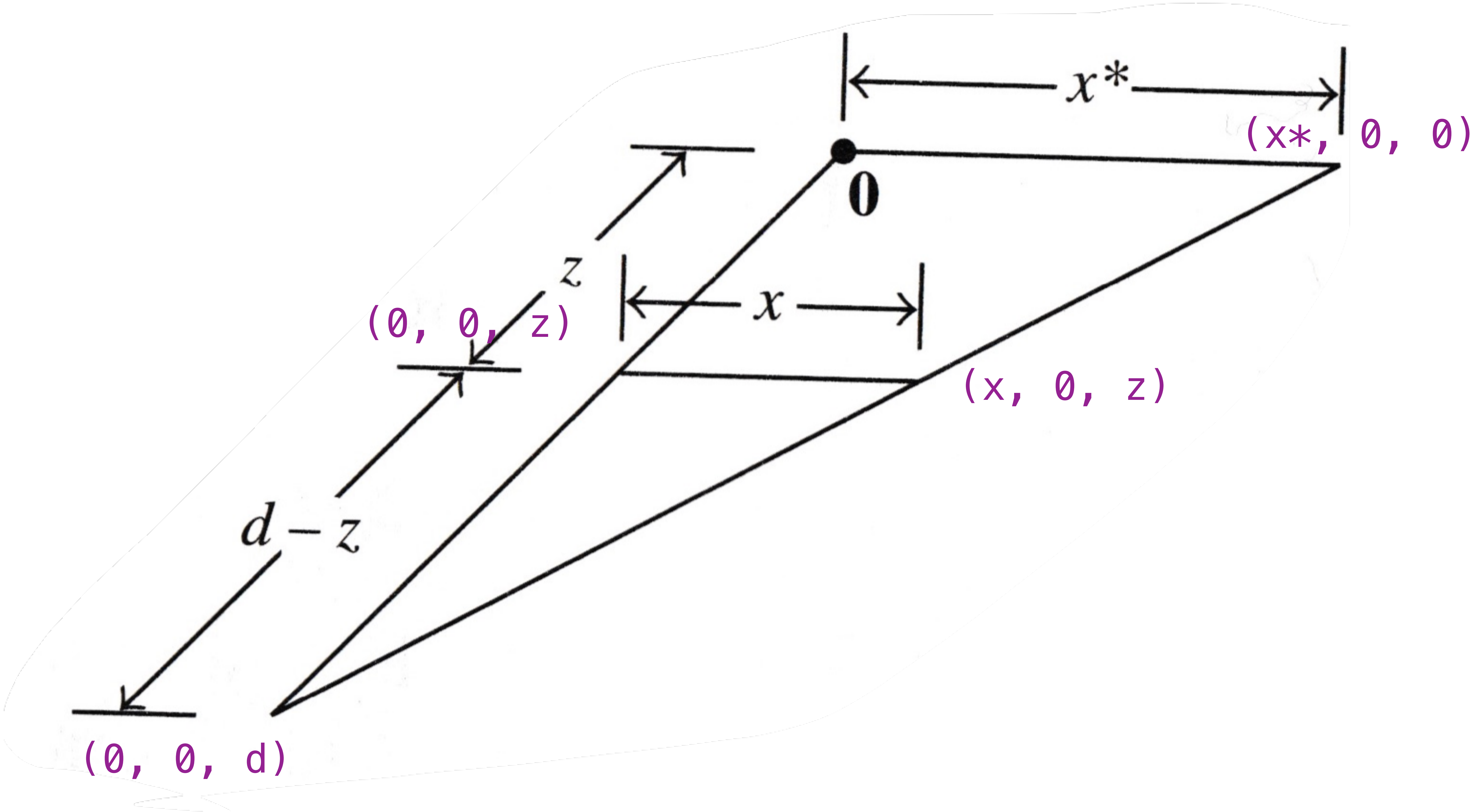
Similar triangles preserve side ratios.



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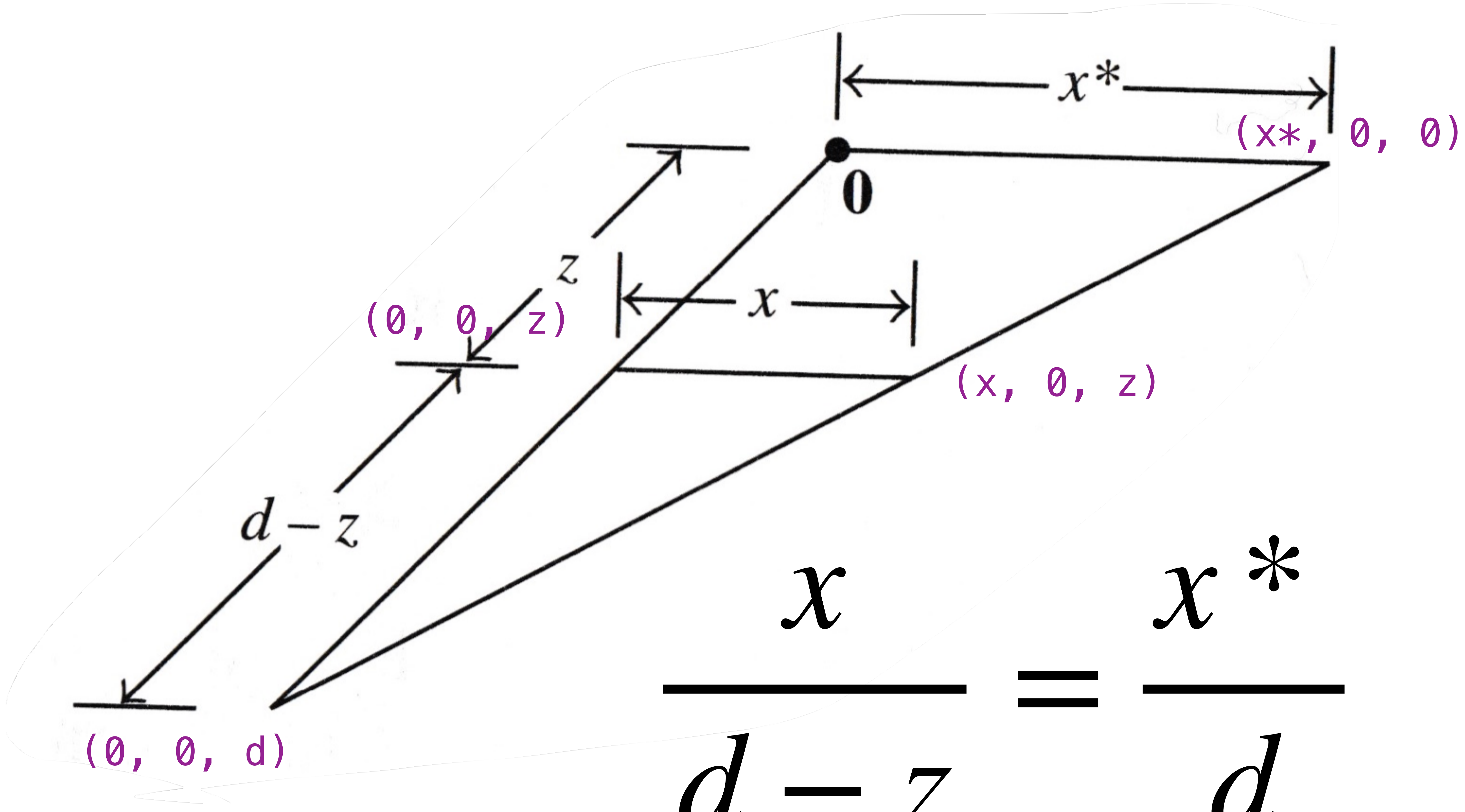
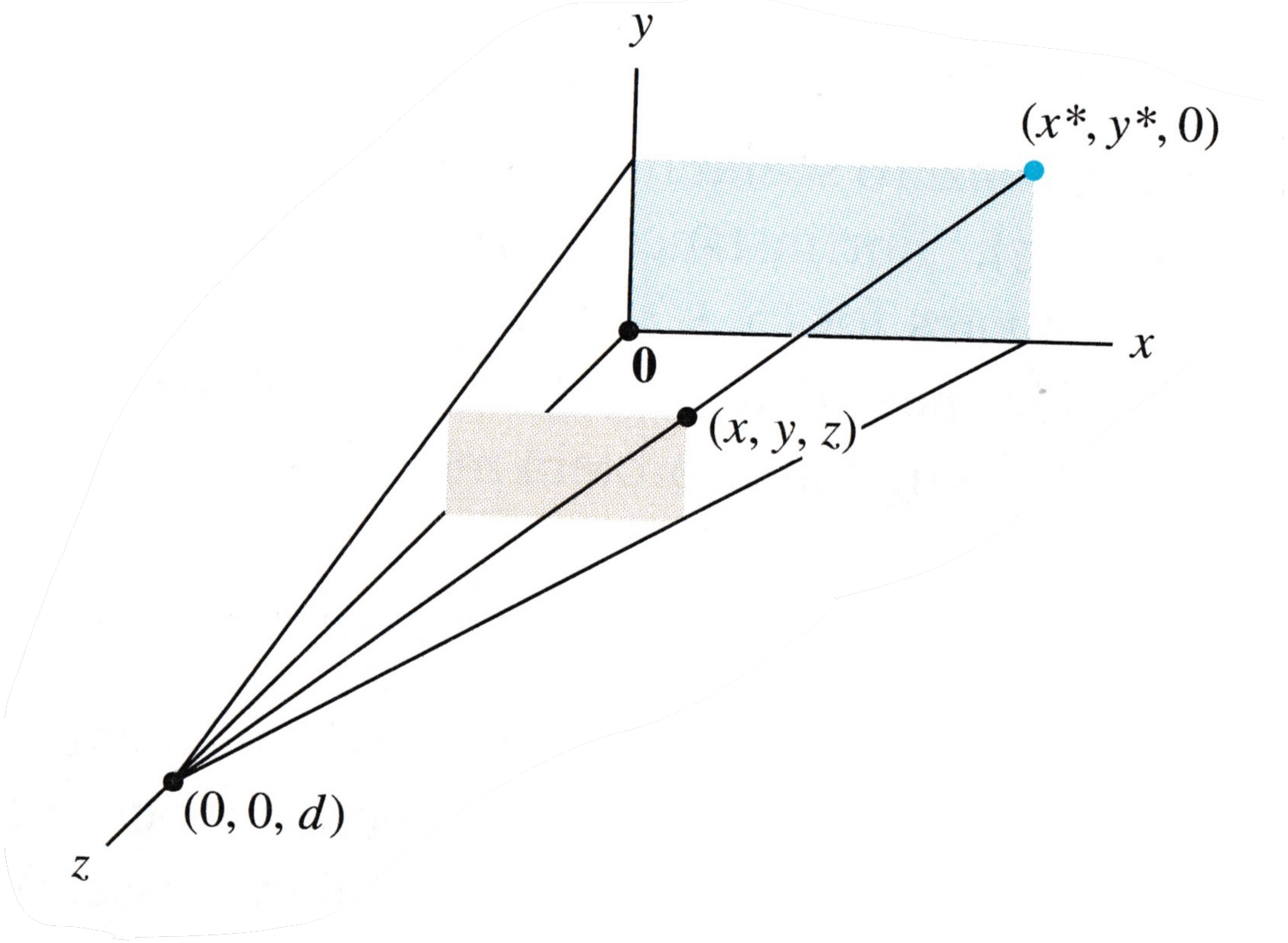


$$\frac{x}{d-z} = \frac{x^*}{d}$$

The Transformation

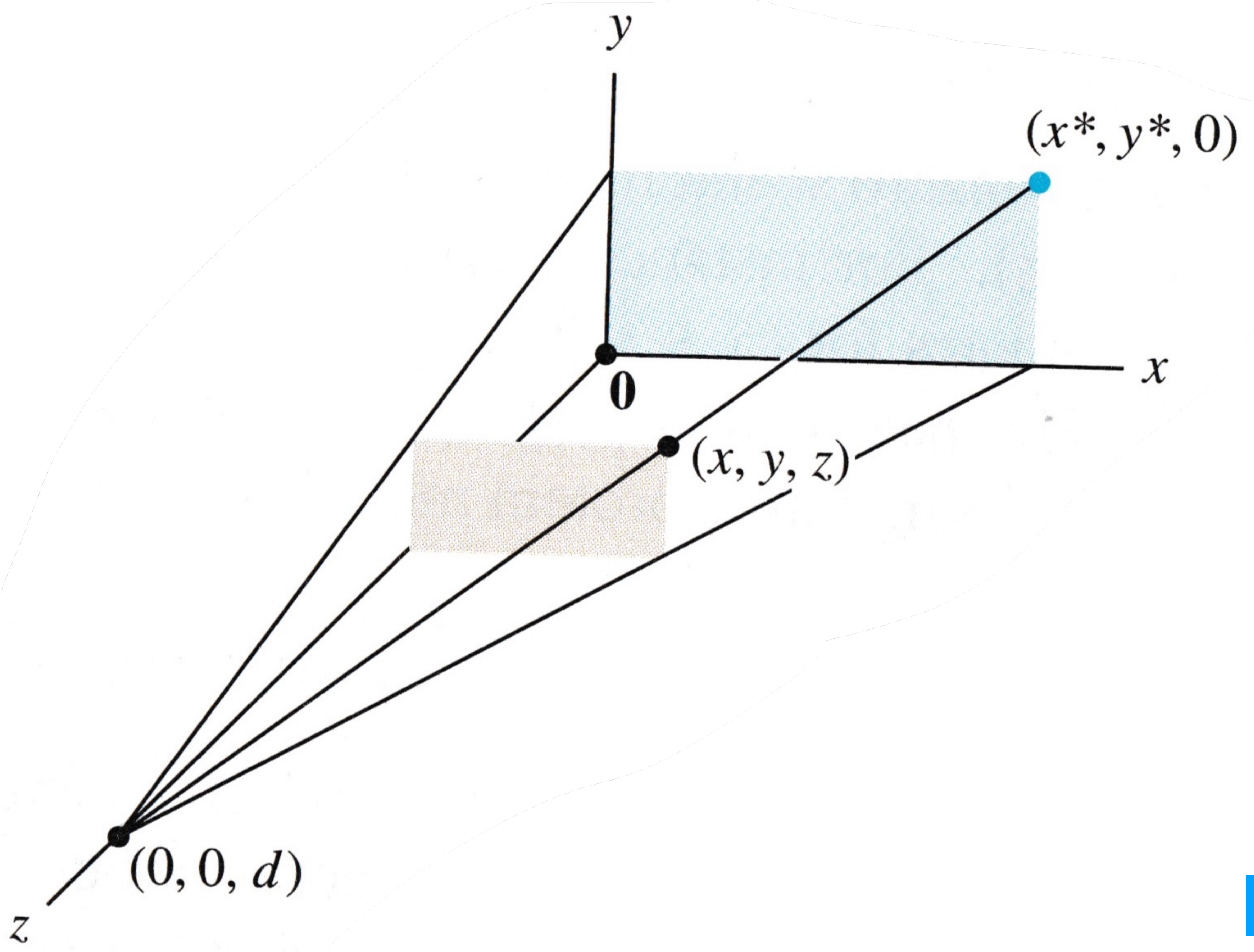


not differential

$$x^* = \frac{dx}{d - z} = \frac{x}{1 - z/d}$$

$$y^* = \frac{dy}{d - z} = \frac{y}{1 - z/d}$$

The Transformation



$$x^* = \frac{dx}{d - z} = \frac{x}{1 - z/d}$$

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Not linear, But we will homogeneous coordinates to address this

A Trick with Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \mapsto \begin{bmatrix} x/h \\ y/h \\ z/h \end{bmatrix}$$

homogeneous to Cartesian

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homogeneous to Cartesian

We can compute perspective using homogeneous coordinates if we allow the extra entry to **vary.**

A Trick with Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \mapsto \begin{bmatrix} x/h \\ y/h \\ z/h \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x/1 \\ y/1 \\ z/1 \end{bmatrix}$$

homogeneous to Cartesian

We can compute perspective using homogeneous coordinates if we allow the extra entry to **vary.**

When we convert back to normal coordinates, we divide by the extra entry (this is consistent with before).

Perspective Projection

point on the screen → $\begin{bmatrix} x/(1-z/d) \\ y/(1-z/d) \end{bmatrix}$

Definition. The **perspective projection** (and matrix) is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix} \mapsto \begin{bmatrix} x/(1-z/d) \\ y/(1-z/d) \\ 0 \end{bmatrix}$$

When we convert back to usual coordinates, we divide by $1 - z/d$ as desired.

always zero

Homework 8

The Rough Outline

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5. Draw the resulting image on the screen.

demo

A Couple Words of Warning

Check your system now. Make sure you can run matplotlib (in particular matplotlib widgets).

Post on piazza if there seems to be a platform dependent issue.