Subspaces

Geometric Algorithms
Lecture 16

CAS CS 132

Practice Problem

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 4 & 0 & 2 \end{bmatrix} \xrightarrow[]{\substack{R_1 \leftarrow R_1 + R_2 \\ R_3 \leftarrow 2R_3 \\ R_2 \leftrightarrow R_3}} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & 2 & 3 \end{bmatrix} = B$$

Consider the following pair of matrices A and B which are row equivalent. Find a matrix E such that $EA = B$.

Answer

$$\begin{aligned} R_1 &\leftarrow R_1 + R_2 \\ R_3 &\leftarrow 2R_3 \\ R_2, R_3 &\leftarrow R_3, R_2 \end{aligned} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\longrightarrow \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}}$$

Objectives

1. Introduce the fundamental notions of **subspaces** and **bases**
2. Extend our intuitions about planes in $\mathbb{R}^3$ to subspaces in $\mathbb{R}^n$
3. Connected subspaces to matrices so that we can use the techniques we been honing in this course

Keywords

subspace

closed under addition

closed under scaling

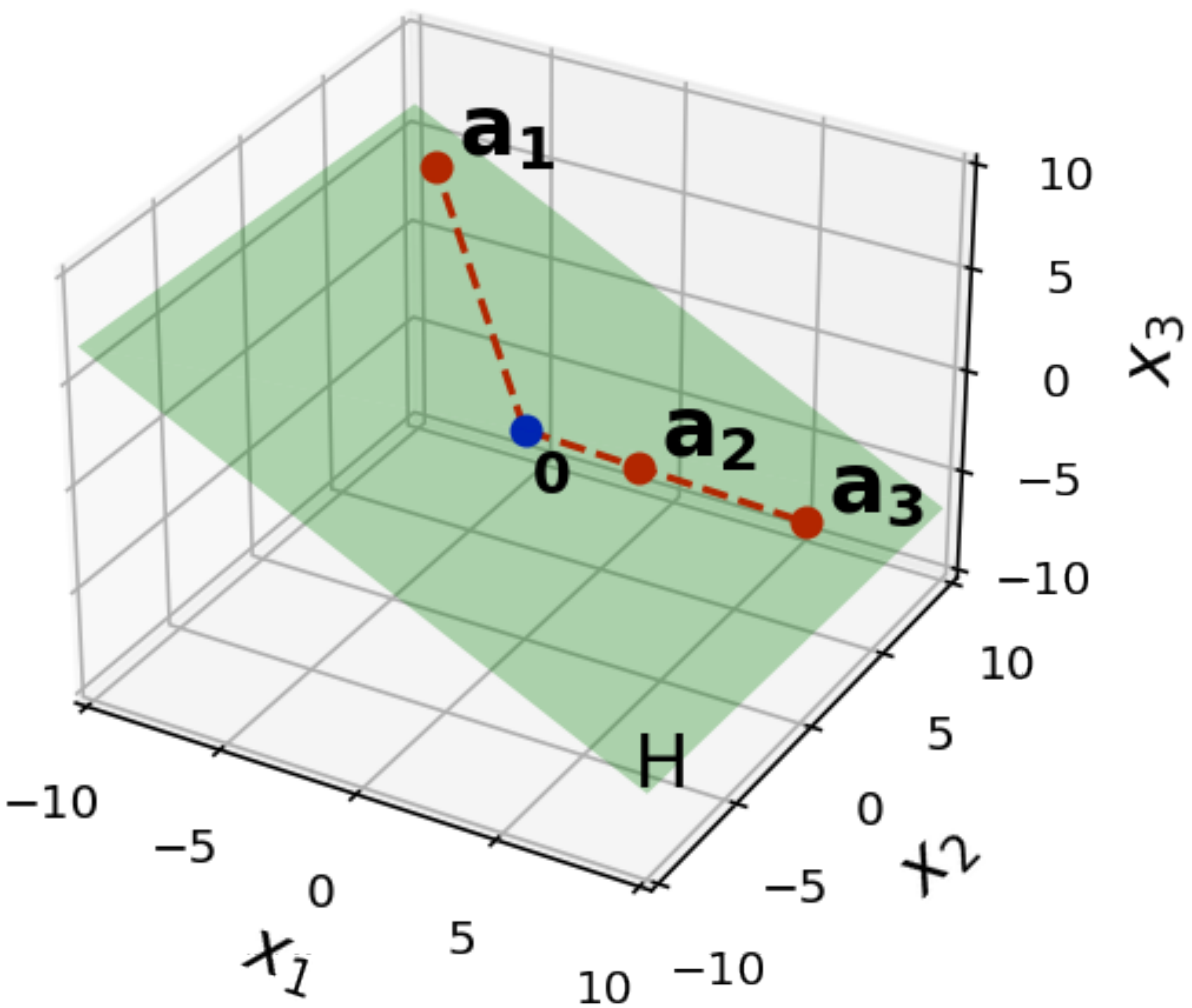
column space

null space

basis

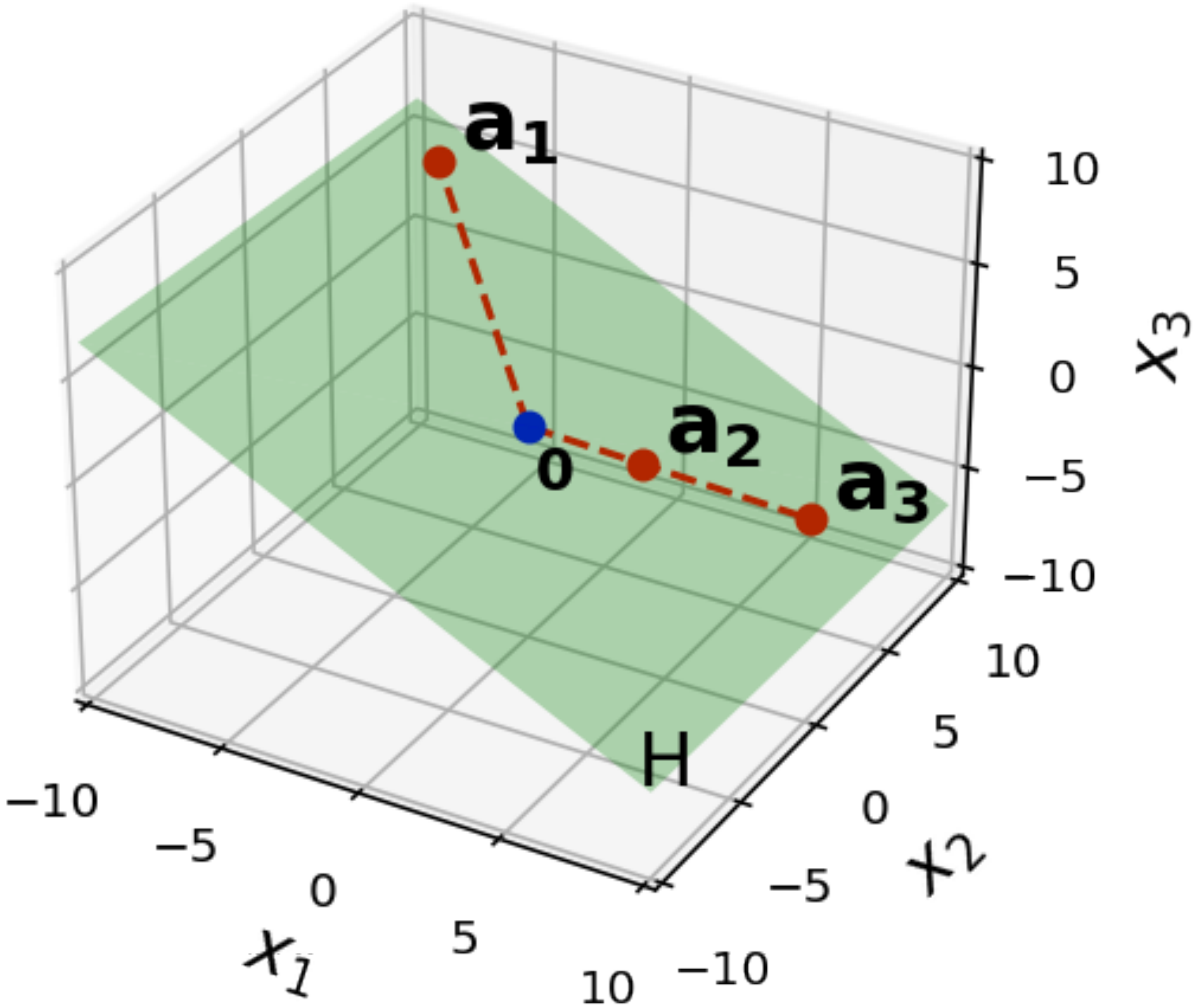
Subspaces

The Idea Behind Subspaces



The Idea Behind Subspaces

"sub" means "part of" or "below"



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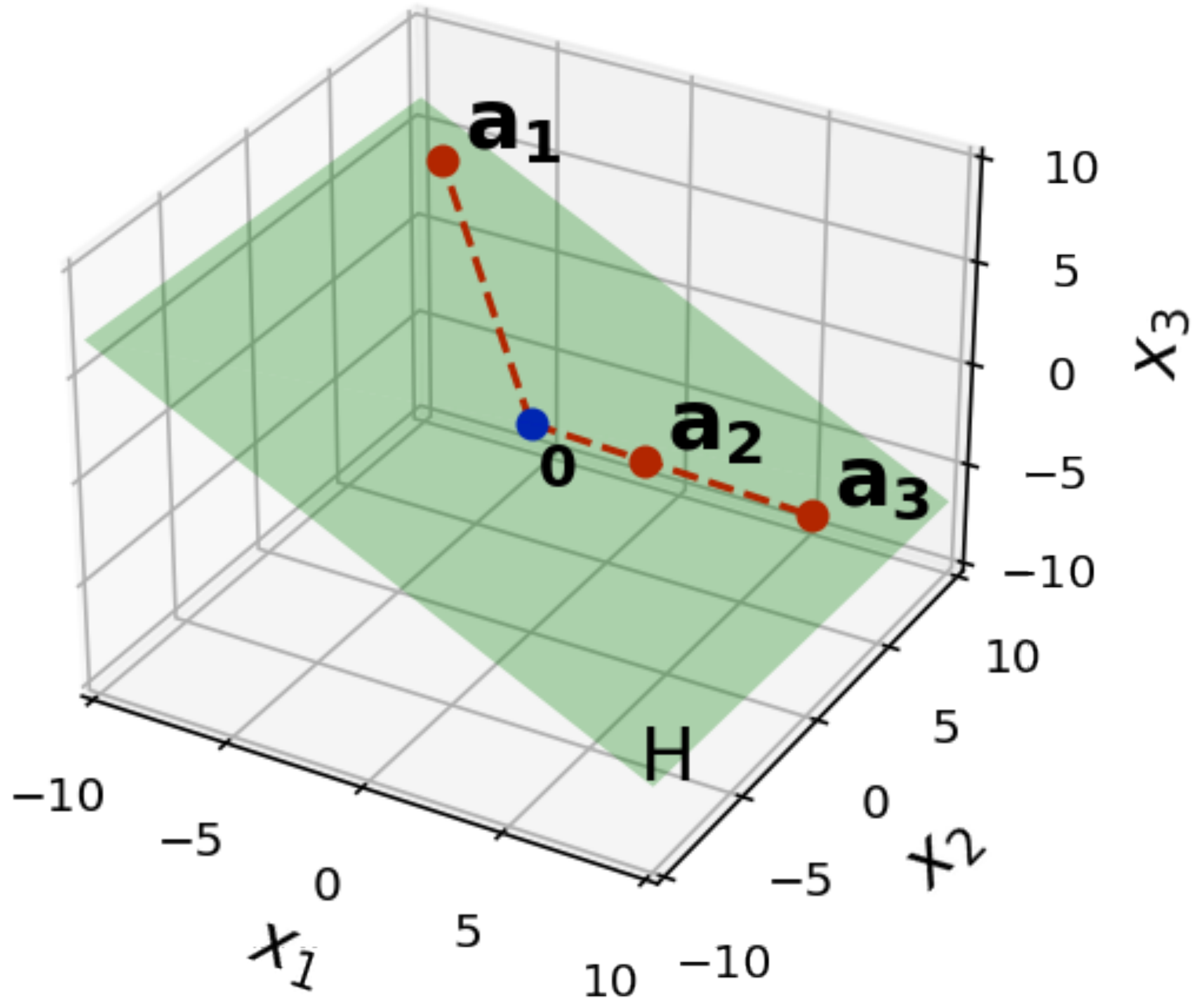

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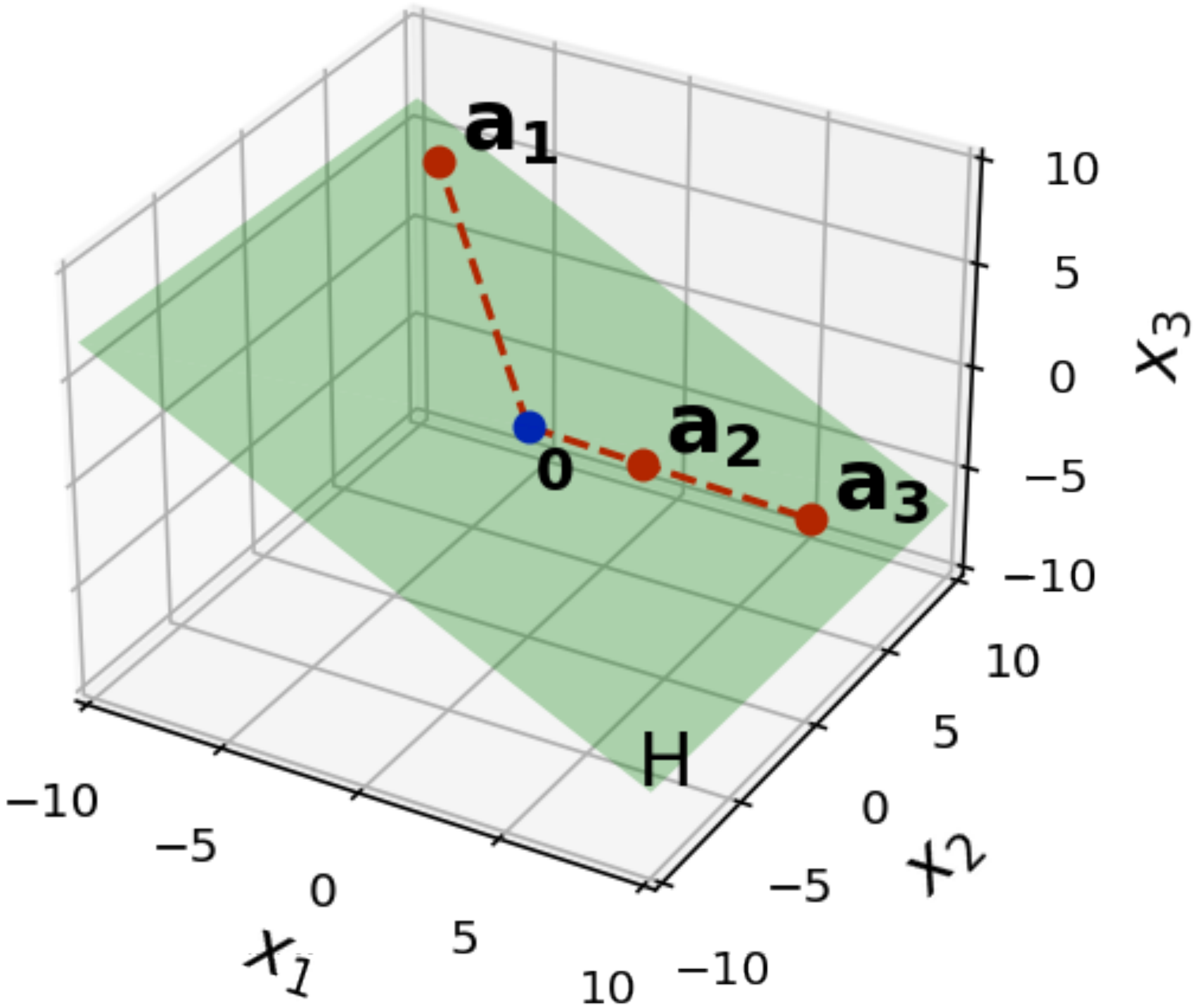
Subspaces *generalize* of this idea.



The Idea Behind Subspaces

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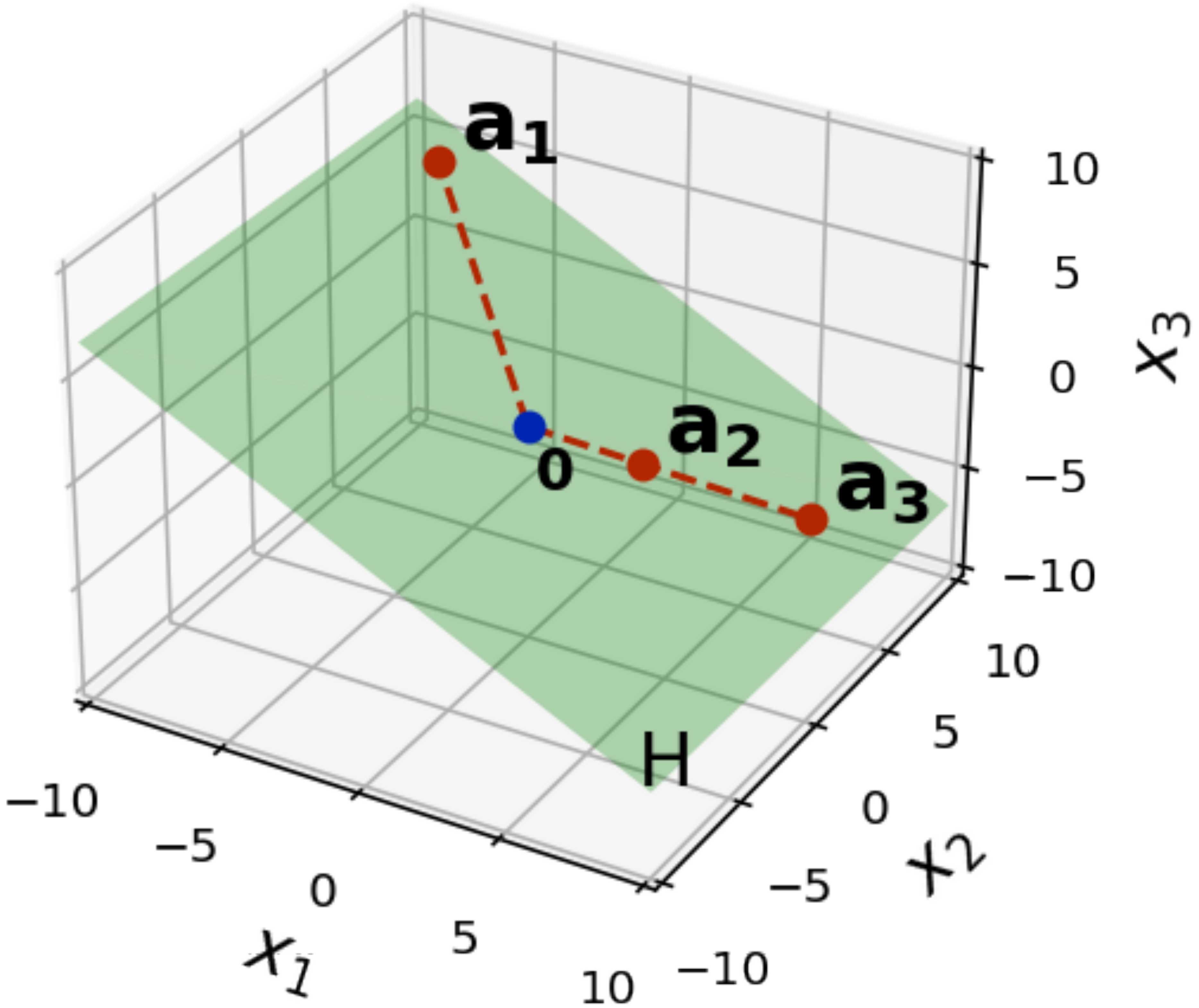
A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$



An Aside: Flatland, Relativity, Higher Dimensions

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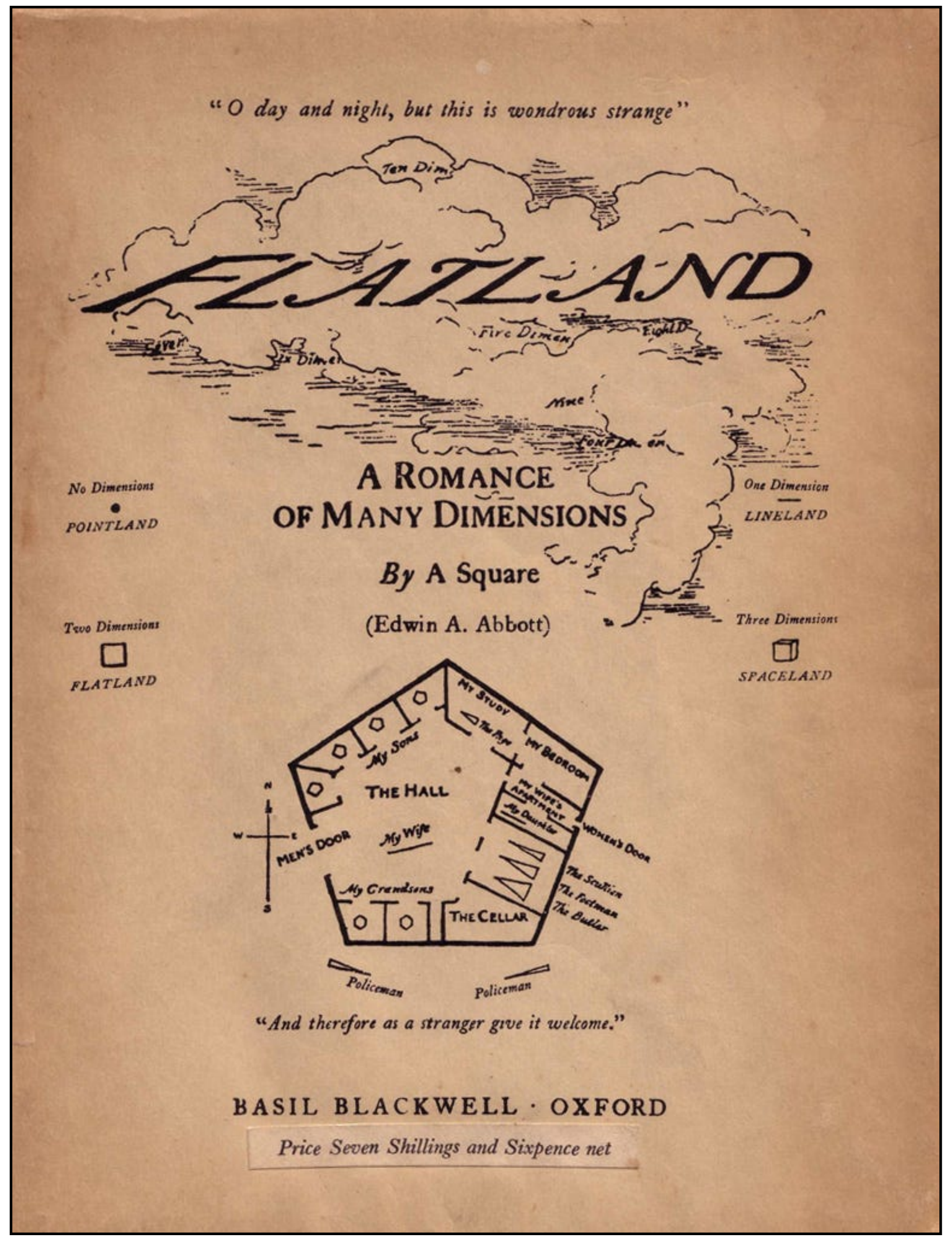
You would never "know" if that plane was sitting in some 3D space, and you'd never know if it was tilted.

Flatland by Edwin A. Abbott



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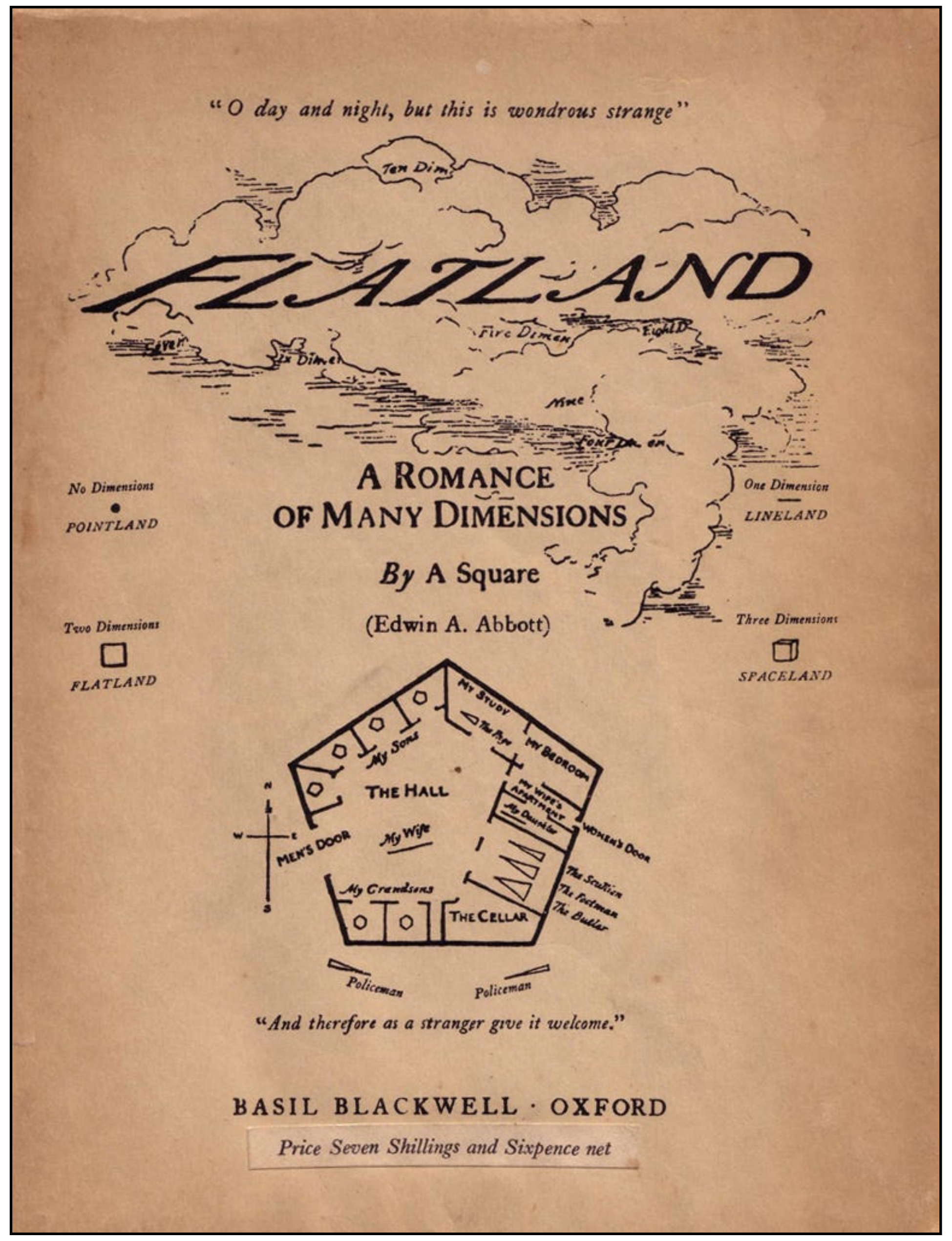

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The moral. We have to be careful regarding our intuitions about higher-dimensional subspaces.

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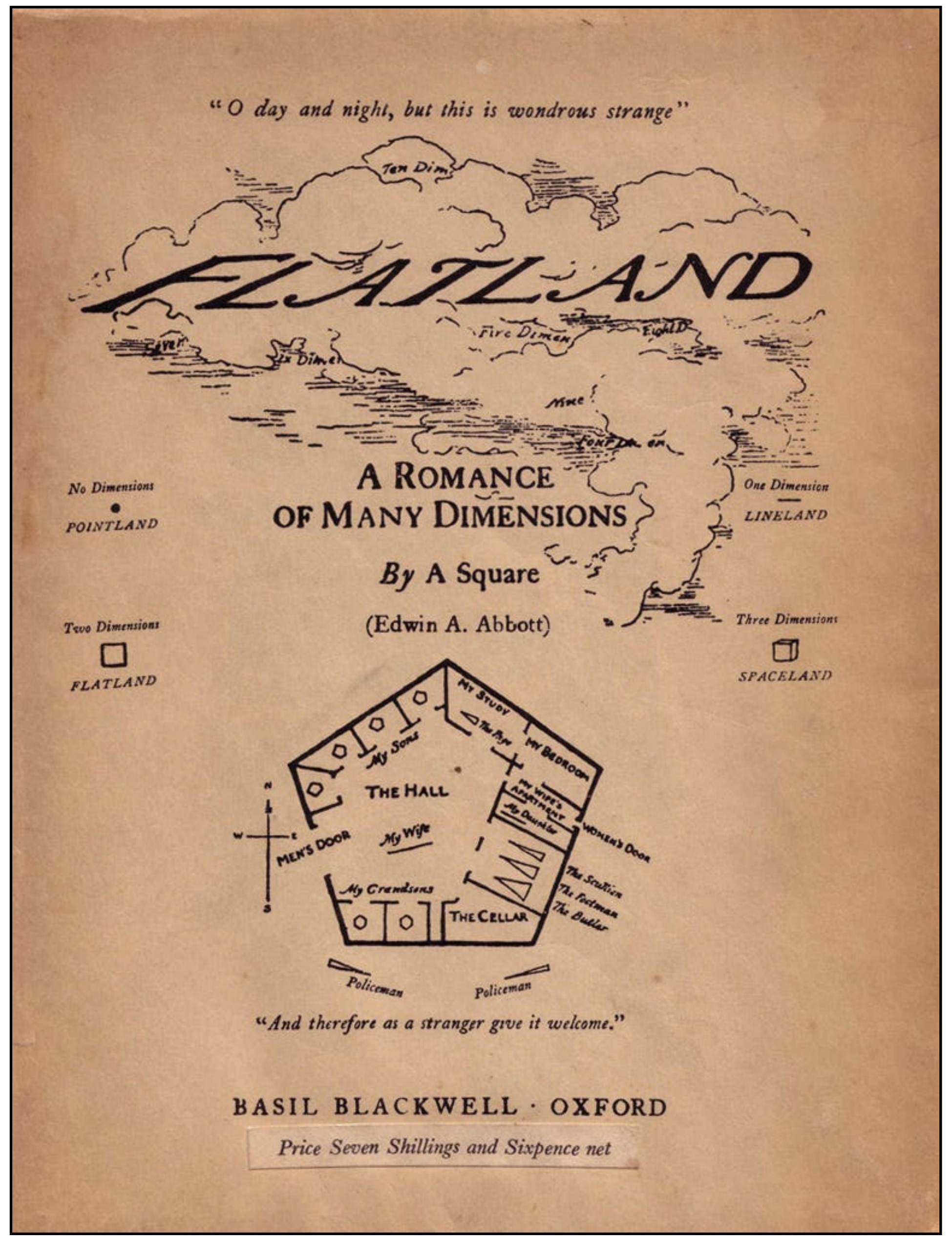

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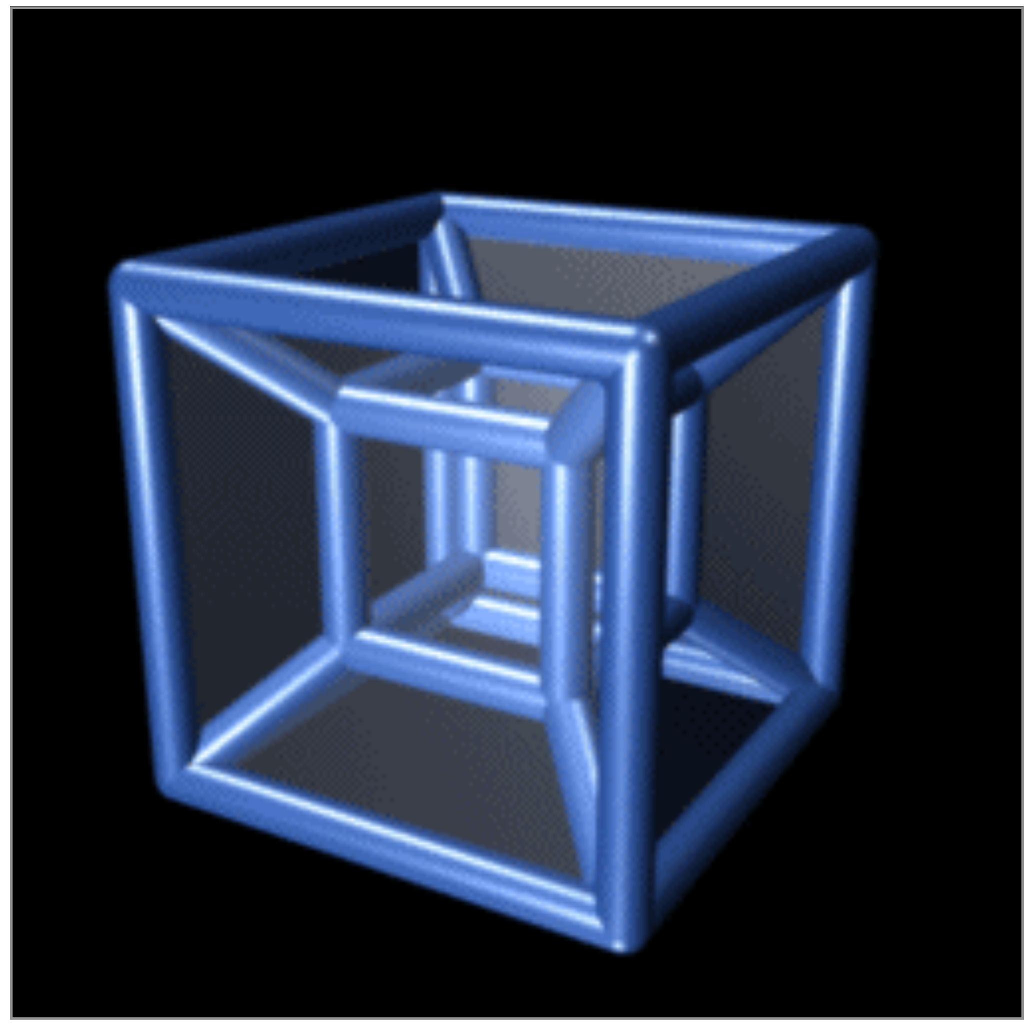
The moral. We have to be careful regarding our intuitions about higher-dimensional subspaces.

A 3D subspace of $\mathbb{R}^7$ "looks like" 3D space from the inside, but from the outside it may be "tilted."

Projection of the 4D cube



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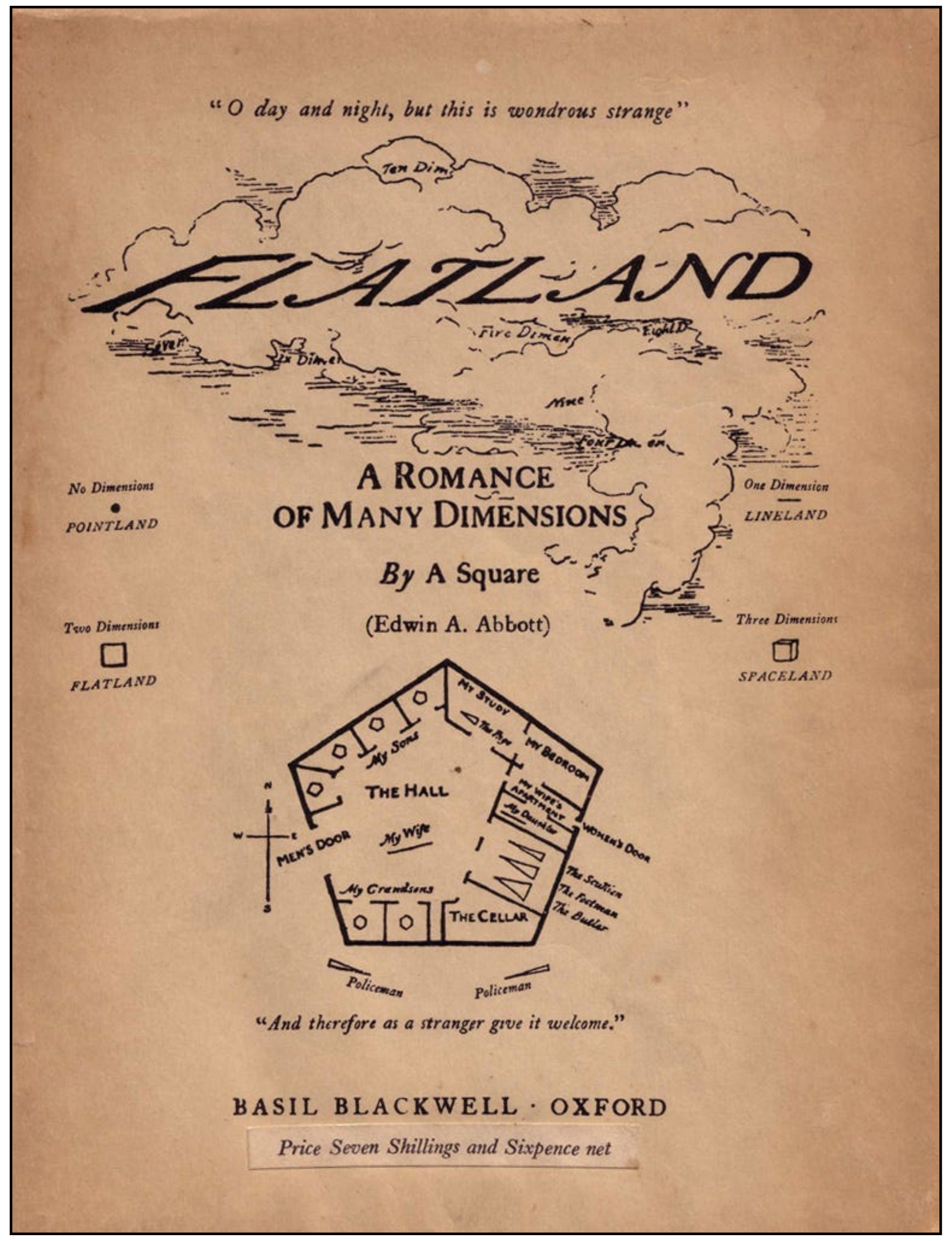

Subspace (Algebraic Definition)

Definition. A **subspace** of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $\mathbf{u}+\mathbf{v}$ is in H
2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H

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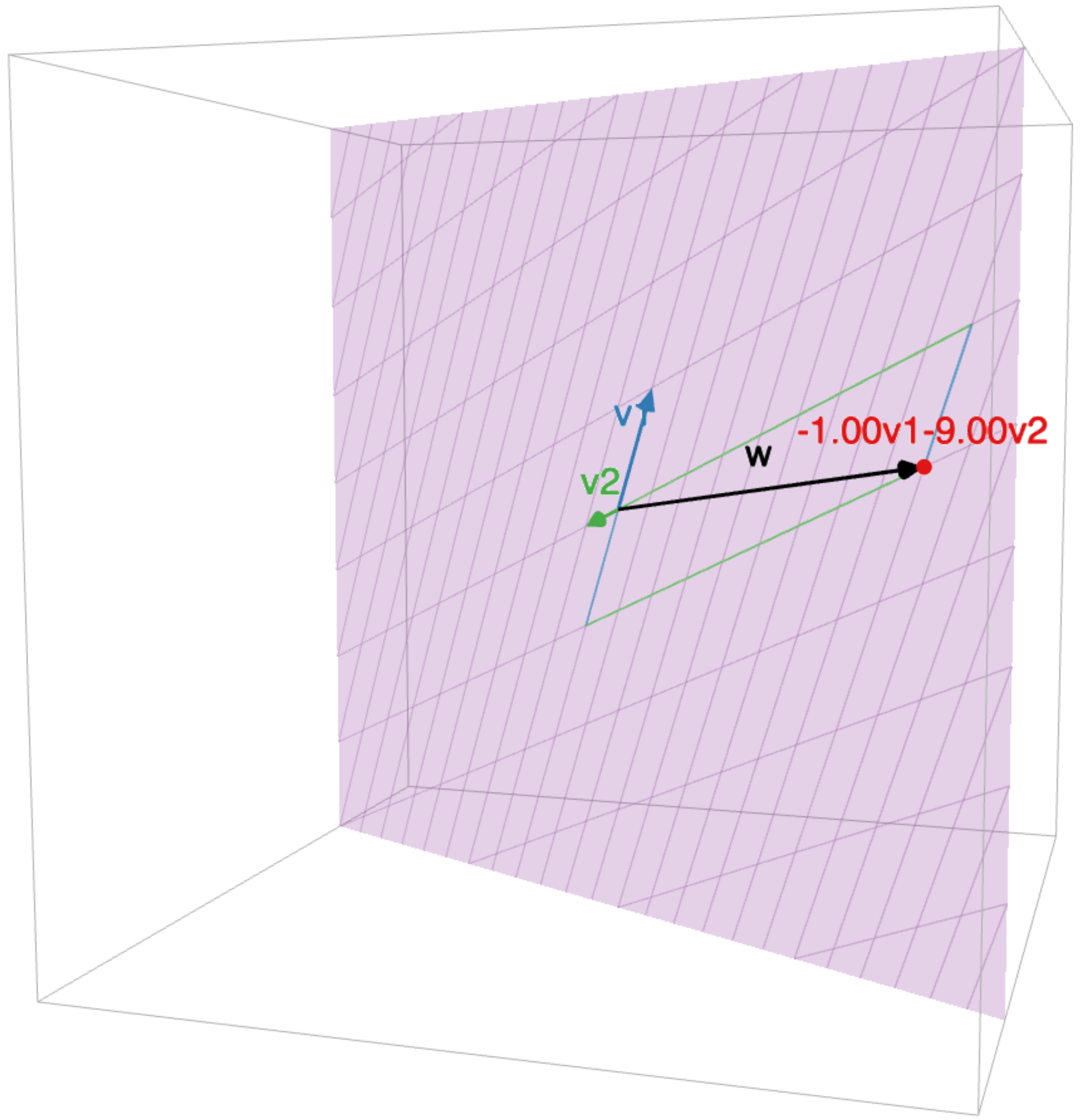
!! Subspaces must "live" somewhere !!

How to Think About this Definition

It's not possible to "leave" H by addition or scaling.

lin. comb.

(recall this is also how we discussed spans)



https://textbooks.math.gatech.edu/ila/spans.html

How To: Verifying Subspaces

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Question. Verify that H is a subspace of $\mathbb{R}^n$.

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1. Show that if $\mathbf{u}$ and $\mathbf{v}$ are in H then so is $\mathbf{u} + \mathbf{v}$.

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1. Show that if $\mathbf{u}$ and $\mathbf{v}$ are in H then so is $\mathbf{u} + \mathbf{v}$.
2. Show that if $\mathbf{u}$ is in H then so is $c\mathbf{u}$ for any scalar c.

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Solution.

Find $\mathbf{u}$ and $\mathbf{v}$ in H such that $\mathbf{u} + \mathbf{v}$ is not in H.

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Find $\mathbf{u}$ and $\mathbf{v}$ in H such that $\mathbf{u} + \mathbf{v}$ is not in H.

OR

Find $\mathbf{u}$ in H such that $c\mathbf{u}$ is not in H for *some* scalar c.

Subspaces must include the origin

Fact. For any subspace H of $\mathbb{R}^n$, the zero vector is in H. In set notation: $\mathbf{0} \in H$

Verify:

$$\vec{r} \in H$$

$\downarrow$ closure under scaling

$$\underset{\vec{0}}{\overset{\|}{0\vec{r}}} \in H$$

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Find $\mathbf{u}$ in H such that $c\mathbf{u}$ is not in H for *some* scalar c.

OR

<u>Show that $\mathbf{0}$ is not in H.</u>

Example: The Origin

$$\overset{\displaystyle S}{\overset{\shortparallel}{}}$$

Fact. The set $\{\mathbf{0}_n\}$ is a subspace of $\mathbb{R}^n$

Verify: ~~addition:~~ $u \in S$ $\quad v \in S$ $\quad$ so $\quad u = \vec{0} \quad v = \vec{0}$

$$u + v = \vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\} = S \checkmark$$

~~scaling:~~ $u \in S \quad u = 0 \qquad cu = c\vec{0} = \vec{0} \in S \checkmark$

Example: $\mathbb{R}^n$

Fact. The set $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ (in other words, $\mathbb{R}^n$ is a subspace of itself).

add: $u+v \in \mathbb{R}^n$ given $u \in \mathbb{R}^n$ $v \in \mathbb{R}^n$

scale: $cu \in \mathbb{R}$ given $u \in \mathbb{R}^n$

Example: Spans

Fact. For any set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of $\mathbb{R}^n$, the set $\underbrace{\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}}_{S}$ is a subspace of $\mathbb{R}^n$.

Verify: add: $u \in S \quad w \in S$ then

$$u = \sum_{i=1}^{k} \alpha_i \vec{v}_i$$

$$w = \sum_{i=1}^{k} \gamma_i \vec{v}_i$$

$$u + w = \sum_{i=1}^{k} \alpha_i \vec{v}_i + \sum_{i=1}^{k} \gamma_i \vec{v}_i$$

$$= \sum_{i=1}^{k} (\alpha_i + \gamma_i) \vec{v}_i \Big\} \text{ lin comb. } \vec{v}_1 \ldots \vec{v}_k$$

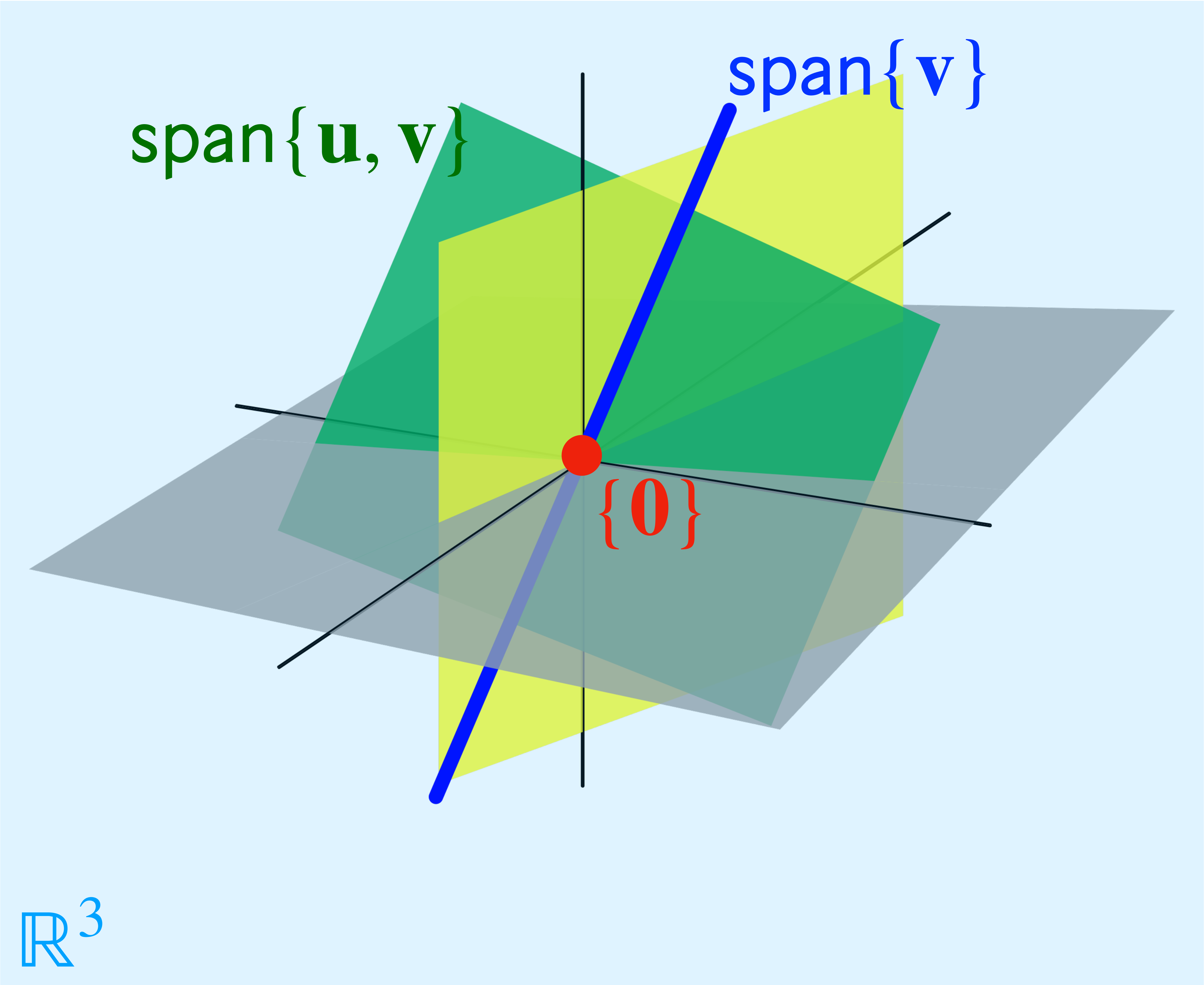
$$\in S$$

scaling: exercise

Subspace in $\mathbb{R}^{\cancel{3}\,n}$ (Geometrically)

There are only ~~4~~ n+1 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of $\mathbb{R}^3$



https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

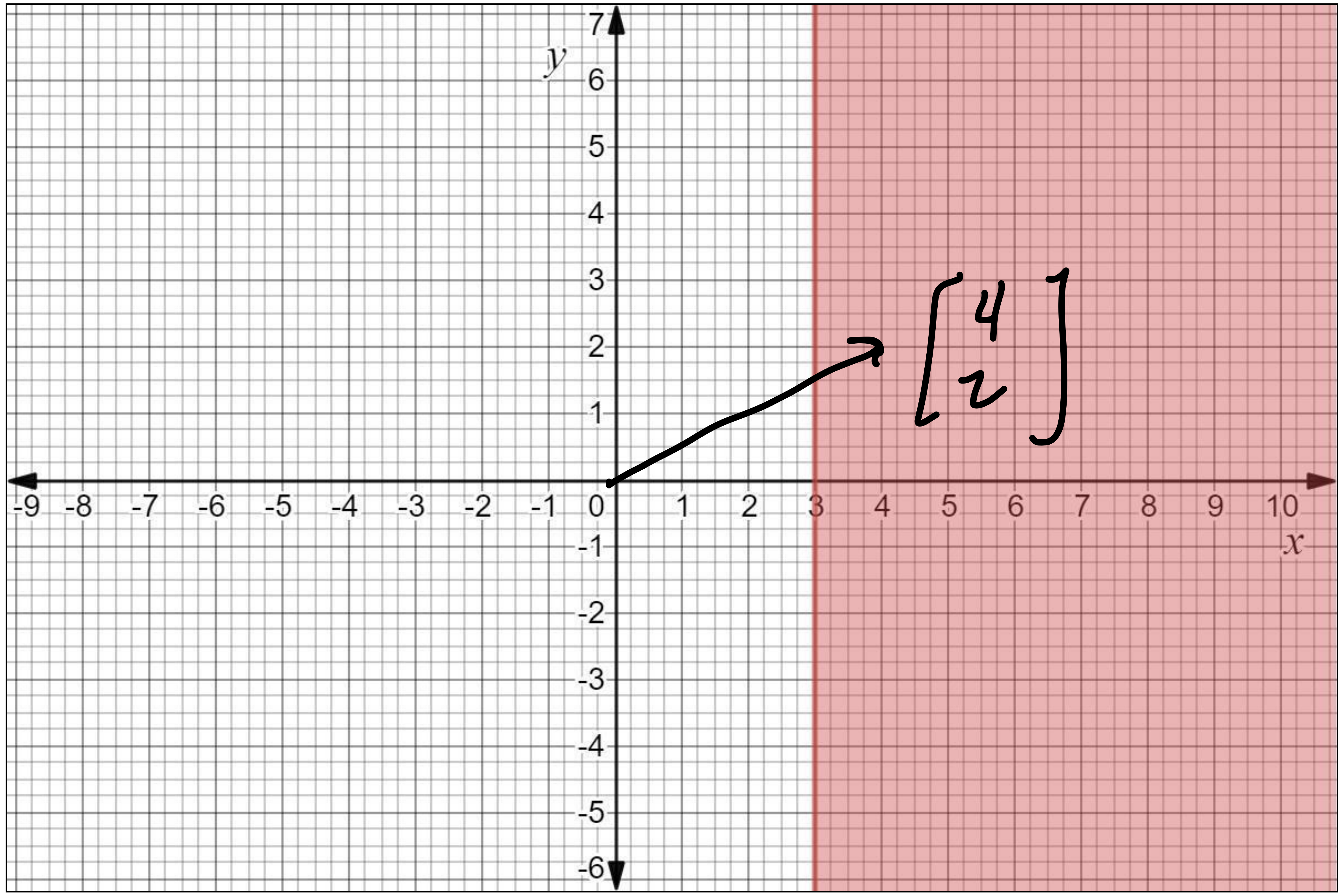
Non-Example: Bounded Sets

$x+z \geq 3+3 \geq 3$

$\begin{bmatrix} x \\ y \end{bmatrix} \in S$ $\quad x \geq 3$ $\qquad \begin{bmatrix} z \\ w \end{bmatrix} \in S$ $\quad z \geq 3$ $\qquad \begin{bmatrix} x+z \\ y+w \end{bmatrix} \overset{?}{\in} S$

$S =$

Fact. The set $\{(x, y) : x \geq 3\}$ is *not* a subspace of $\mathbb{R}^2$.



Verify:

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \in S$

$0\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S}$

Question

1. *Show that the unit sphere* $\{(x, y, z) : x^2 + y^2 + \overset{z}{\cancel{x}}{}^2 = 1\}$ *is <u>not</u> a subspace of* $\mathbb{R}^3$.

2. *Show that the range of a linear transformation* $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$.

Answer (1)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$$

$$0^2 + 0^2 + 0^2 = 0 \neq 1$$

Answer (2)

add: $u, v \in \operatorname{ran}(T)$ then there are

$w, q \in \operatorname{dom}(T)$ $\quad T(w) = u, \; T(q) = v$

$$u + v = T(w) + T(q) = T(\vec{w} + \vec{q})$$

so $\quad u + v \in \operatorname{ran}(T)$ ✓

scaling: exercise

How To: Subspaces and Span

Question. Show that $\mathbf{v}$ lies in the subspace generated by $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

Solution. Show that $\mathbf{v}$ is in $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

We will start using "subspace generated by" and "span of" interchangeably.

Subspaces and Matrices

The Connection between Subspaces and Matrices

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Since matrices can be viewed as...

» collections of vectors
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`Since matrices can be viewed as...`

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- `implementing linear transformations`

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The Connection between Subspaces and Matrices

Since matrices can be viewed as...

» collections of vectors
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...they have many associated subspaces.

Today we'll look at:

» column space
» null space

Column Space

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The column space of a matrix is the span of its columns.

The column space of a matrix is the range of the linear transformation it implements.

Subspace of What?

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \cdots & | & | \end{bmatrix} \end{array}$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\text{Col}(A)$

is a subspace of

$\mathbb{R}^m$

Examples

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{array}{ccc} 1 & & 3 \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 5 \\ 3 & -3 & 6 \end{bmatrix}} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Col(A) is all of $\mathbb{R}^3$

Col(B) is just span $\left\{ \begin{array}{c} 1 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array}, \begin{array}{c} 3 \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \end{array} \right\}$

Null Space

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$$A\mathbf{x} = \mathbf{0}$$

The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A.

$$\{ \vec{v} : A\vec{v} = \vec{0} \}$$

Subspace of What?

$$\begin{array}{r|l} & n \text{ columns} \\ m \text{ rows} & A\mathbf{v} = \mathbf{0} \end{array}$$

$m \times n$ $\quad n \times 1 \quad$ $m \times 1$

$\mathbf{v}$ **is a vector in** $\mathbb{R}^n$

$\text{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

The Null Space is a Subspace

Fact. For any $m \times n$ matrix A, the set $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

addition

Verify: $u, v \in \mathrm{Nul}(A)$, $\quad A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$ ✓

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

scaling : exercise

Examples

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 5 \\ 3 & -3 & 6 \end{bmatrix}$$

$\mathrm{Nul}(A) = \{\mathbf{0}\}$ IMT

$\mathrm{Nul}(B) = \mathrm{span}\{[1\ 1\ 0]^T\}$

Verify: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in \mathrm{Nul}(B)$

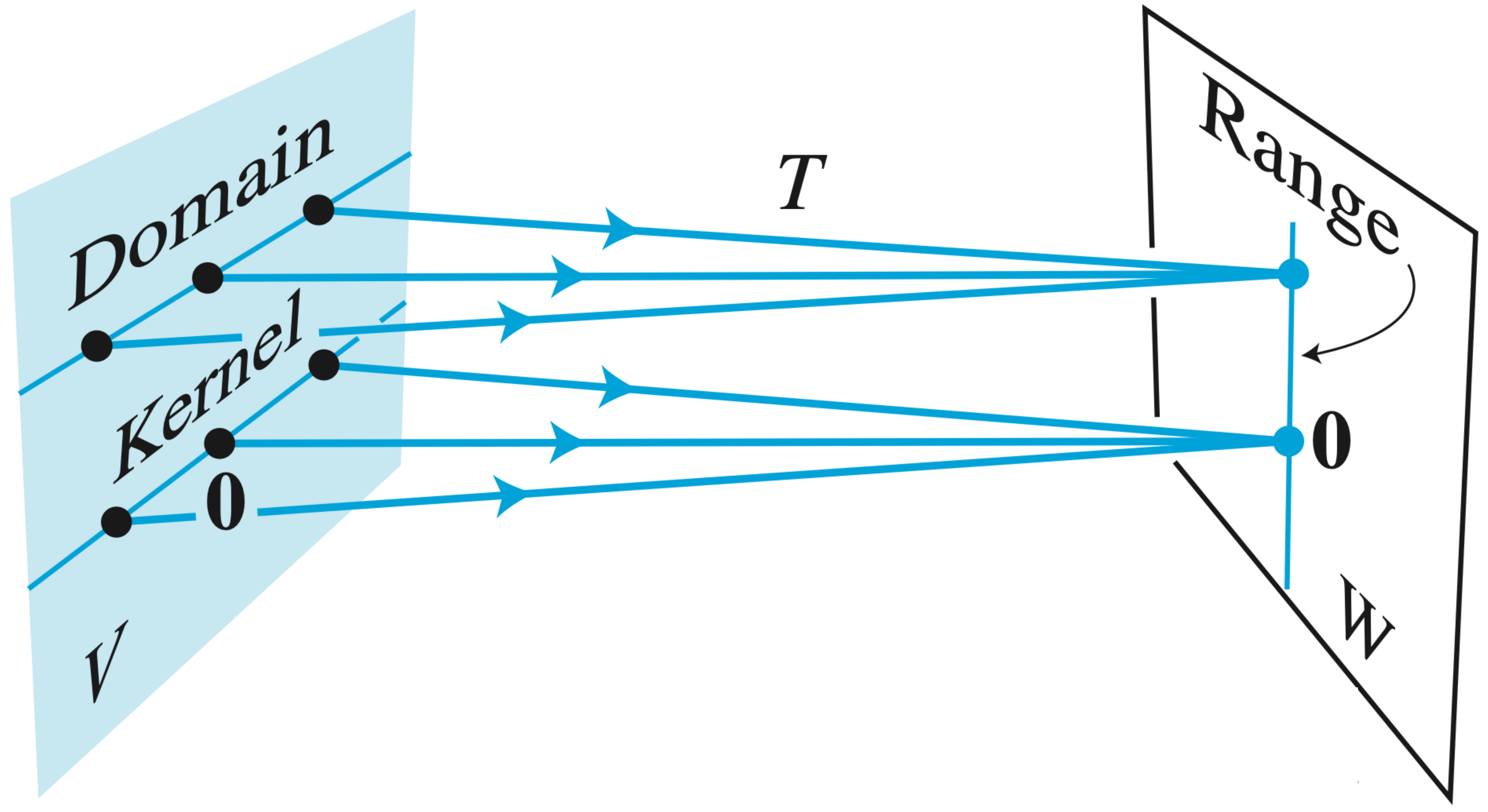
$B\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Linear Transformations Perspective

If A implements the linear transformation T then:

» $\text{Col}(A)$ is the same as $\text{ran}(T)$, where vectors are "sent" by T

» $\text{Nul}(A)$ is the set of vectors "zeroed out" by T, which is sometimes called the **kernel** of T.



Linear Algebra and its Applications (Lay, Lay, McDonald)

Comparing Column Space and Null Space

The column space and the null space live can live in entirely different spaces.

The point. They are not easily comparable

Contrast Between Nul A and Col A for an m x n Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\, A \quad \mathbf{0} \,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\, A \quad \mathbf{v} \,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

(just for reference)

Linear Algebra and its Applications (Lay, Lay, McDonald)

Bases

The idea behind a basis

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We've already said spans are subspaces, but the converse true too.

Every subspace is the span of a collection of vectors.

A basis is a "minimal" choice of these vectors.

A basis is a "compact representation" of a subspace.

Recall: Standard Basis

Definition. The n***-dimensional standard basis vectors*** (or standard coordinate vectors) are the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ where

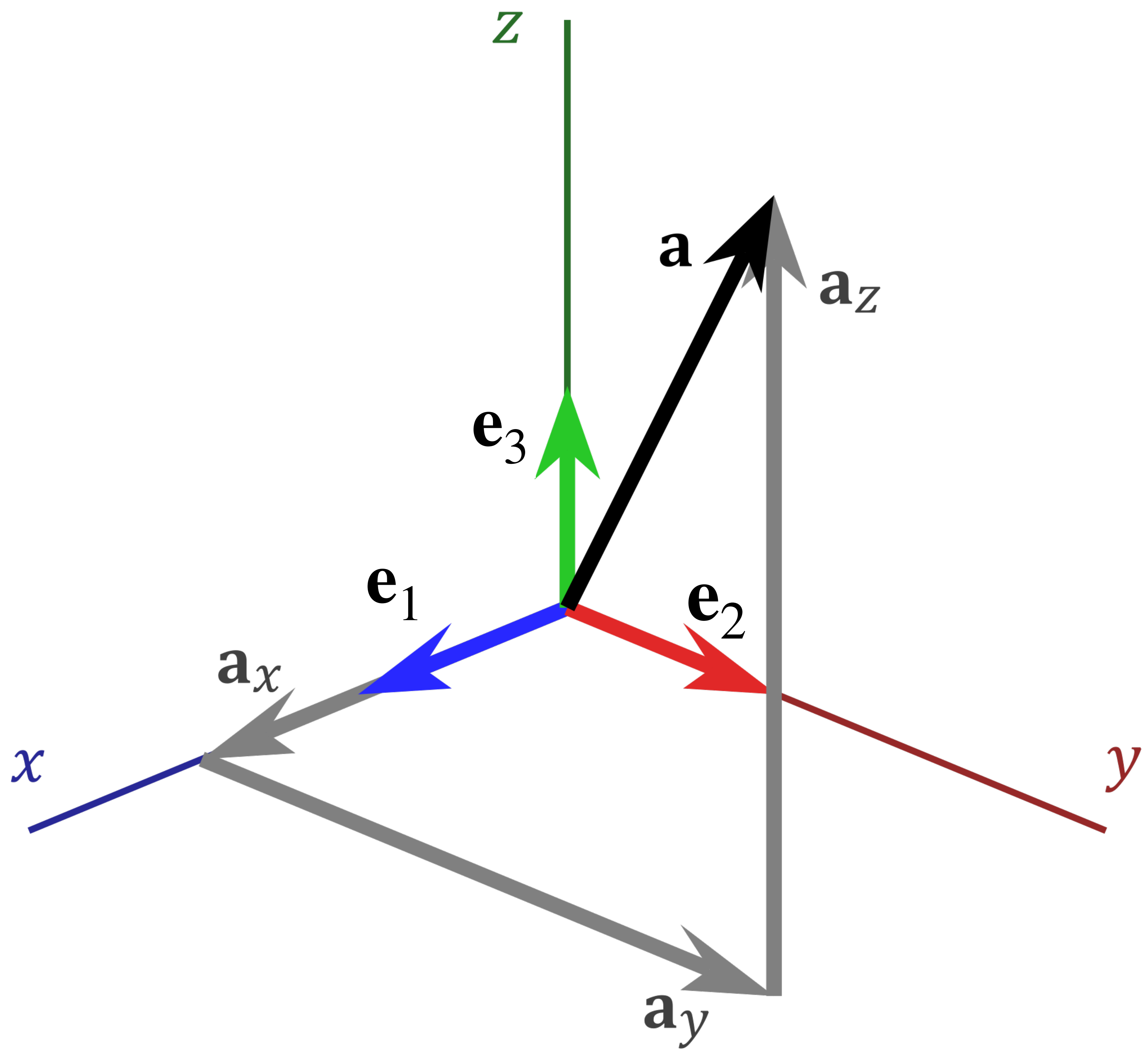
$$\mathbf{e}_i = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ n-1 \\ n \end{array}$$

Recall: Standard Basis

Definition (Alternative). The n–dimensional standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix.

$$I = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]$$

What was interesting about the standard basis?



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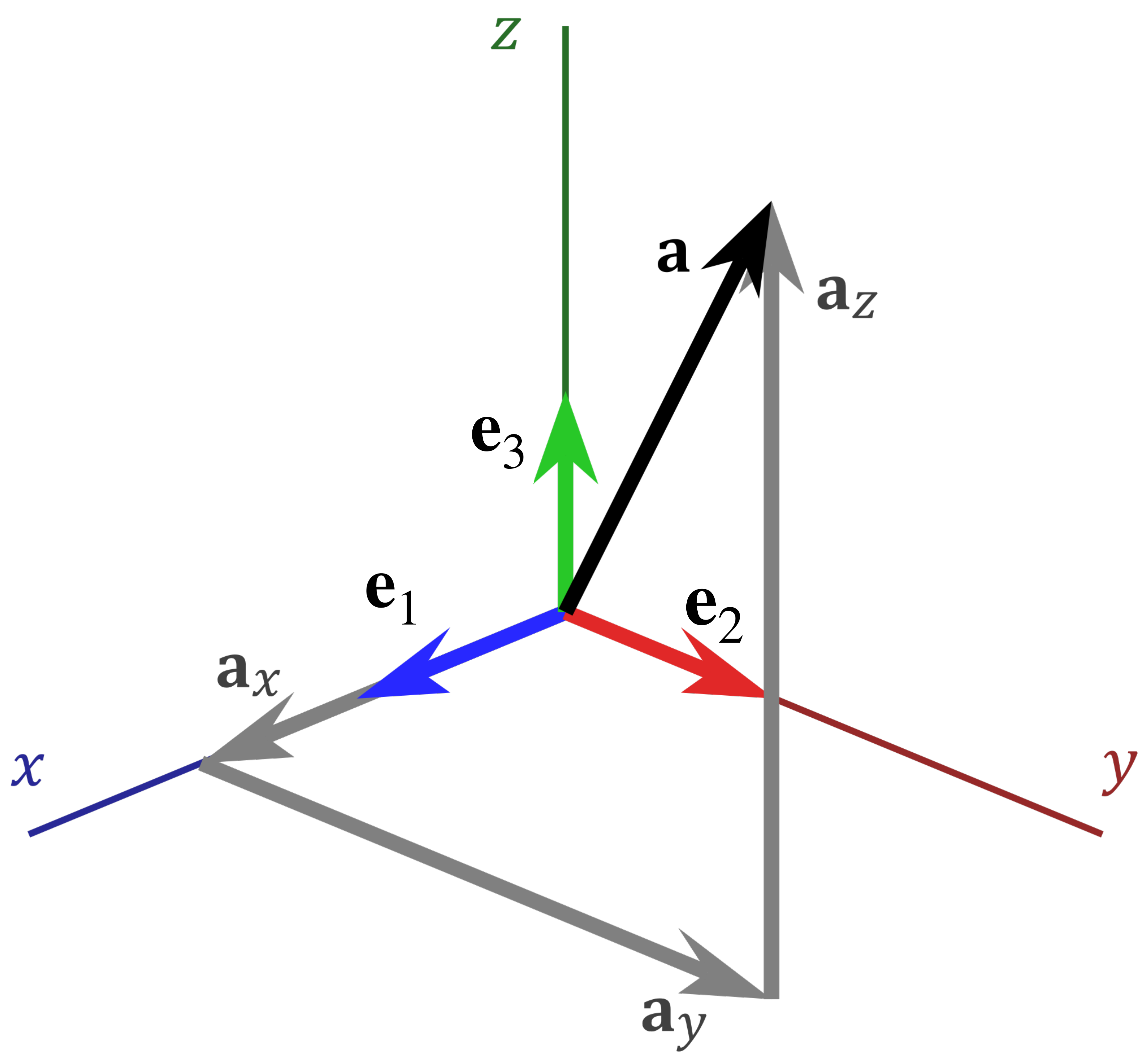
- » are linearly independent
- » span all of $\mathbb{R}^n$



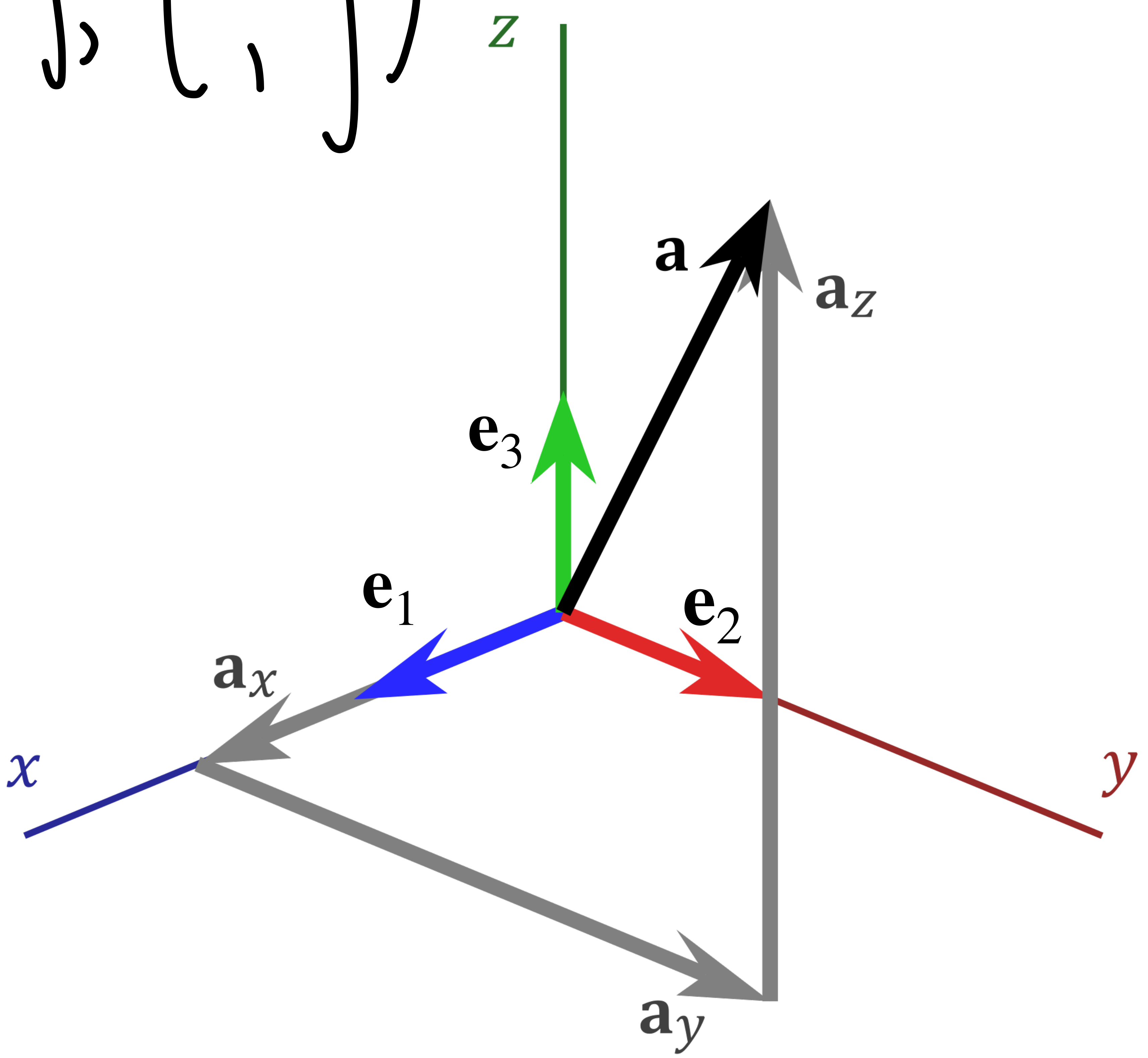
What was interesting about the standard basis?

$\begin{bmatrix}0\\1\\0\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

The n standard basis vectors in $\mathbb{R}^n$:

- » are linearly independent
- » span all of $\mathbb{R}^n$

Their span is as "large" as possible while the set of vectors generating the span is as "small" as possible.



Basis

Basis

Definition. A **basis** for a subspace H of $\mathbb{R}^n$ is a $\boxed{\text{linearly independent}}$ set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$).

A basis is a *minimal* set of vectors which spans all of H.

$\{v_1, \ldots, v_{k+1}\}$ Lin. Dep.

$v_i \in \mathrm{span}\{v_1, v_2, \ldots v_{i-1}, v_{i+1}, \ldots, v_k\}$

Example: Standard basis

The standard basis is a basis of $\mathbb{R}^n$.

$$\begin{bmatrix} \boxed{v_1} \\ \boxed{v_2} \\ \vdots \\ v_n \end{bmatrix} = \boxed{v_1}\mathbf{e}_1 + \boxed{v_2}\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Column vectors are just weights for a linear combination of the standard basis

Example: Column Space of Invertible Matrices

Fact. The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

Verify: IMT says col. L.I. and span $\mathbb{R}^n$

Example: Subsets of Spanning Sets

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Theorem. If the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \dots, \mathbf{v}_k$ span a subspace H then a subset of them form a basis of H.

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We can *remove* vectors from a spanning set until we get a basis.

Example: Subsets of Spanning Sets

Theorem. If the vectors $\mathbf{v}_1$, $\mathbf{v}_2$,...,$\mathbf{v}_k$ span a subspace H then a subset of them form a basis of H.

We can *remove* vectors from a spanning set until we get a basis.

How do we do this?

Example: Subsets of Spanning Sets

Theorem. If the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_k$ span a subspace H then a subset of them form a basis of H.

We can *remove* vectors from a spanning set until we get a basis.

How do we do this?

As usual, by connecting back to matrices.

Question

$$\left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}} \right\}$$

Is this set of vectors a basis for $\mathbb{R}^3$?

$$v_1 + v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{v_2 + 3e_3}{-2}$$

$$\frac{v_1 + v_3}{3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Answer

Solving tip. A set of vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$ if the standard basis is in their span.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & -2 \\ 3 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bases of Column Space and Null Space

The Goal of this Last Section

Determine how to find <u>bases</u> for the **column space** and the **null space** of a given matrix.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

$A\vec{x} = \vec{0}$

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

The idea. Describe the solutions of $A\mathbf{x} = 0$ as linear combination of vectors

Example

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above reduced echelon form.

Let's write down a general form solution for A:

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = -2x_4 + 2x_5$

x_4 is free

x_5 is free

Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

$$
\begin{aligned}
&x_1 = 2x_2 + x_4 - 3x_5 \\
&x_2 \text{ is free} \\
&x_3 = (-2)x_4 + 2x_5 \\
&x_4 \text{ is free} \\
&x_5 \text{ is free}
\end{aligned}
\quad \equiv
$$

"given values for x_2, x_3, and x_4, I can give you a solution"

Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

$$
\begin{aligned}
x_1 &= 2x_2 + x_4 - 3x_5 \\
\boxed{x_2} &\ \text{is free} \\
x_3 &= (-2)x_4 + 2x_5 \\
\boxed{x_4} &\ \text{is free} \\
\boxed{x_5} &\ \text{is free}
\end{aligned}
\quad \equiv \quad
\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}
$$

Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

!! this transformation is only linear !!
!! in the case of homogeneous equations !!

$$
\begin{aligned}
x_1 &= 2x_2 + x_4 - 3x_5 \\
x_2 &\text{ is free} \\
x_3 &= (-2)x_4 + 2x_5 \\
x_4 &\text{ is free} \\
x_5 &\text{ is free}
\end{aligned}
\quad \equiv \quad
\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}
$$

Example

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Let's find the matrix implementing this linear transformation:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its columns.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its <u>columns</u>.

The columns of this matrix <u>span</u> $\mathrm{Nul}(A)$.

Example

$$\begin{array}{cc} \begin{array}{c} \\ 1 \\ \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array}$$

The columns of this matrix are linearly independent.

Verify: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$

1 $x_1 + 0 + 0 = 0$

2 $0 + x_2 + 0 = 0$

3 $0 + 0 + x_3 = 0$

$$\boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}}$$

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix <u>span</u> $\text{Nul}(A)$.

The columns of this matrix are linearly independent.

The columns of this matrix form a basis for $\text{Nul}(A)$.

Example

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = (-2)x_4 + 2x_5$

x_4 is free

x_5 is free

Alternatively, we can think of writing a general form solution so that it is a linear combination of vectors with <u>free variables as weights</u>:

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.
3. The resulting vectors form a basis for $\mathrm{Nul}(A)$.

An Observation

The *number* of vectors in the basis we found is the same as the number of <u>free variables</u> in a general form solution.

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = (-2)x_4 + 2x_5$

x_4 is free

x_5 is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Practice Problem

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above RREF. Determine a basis for $\mathrm{Col}(A)$

Answer

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

onto column space...

How To: Finding a basis for the column space

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathrm{Col}(A)$.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\text{Col}(A)$.

We already know the columns of A span $\text{Col}(A)$.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathsf{Col}(A)$.

We already know the columns of A span $\mathsf{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\mathsf{Col}(A)$.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathsf{Col}(A)$.

We already know the columns of A span $\mathsf{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\mathsf{Col}(A)$.

Which vectors should we choose?

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Column Space and Reduced Echelon form

$$\begin{bmatrix}\mathbf{a}_1 & \text{[redacted]} & \mathbf{a}_3 & \text{[redacted]} & \text{[redacted]}\end{bmatrix} \sim \begin{bmatrix} 1 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]} \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_3\end{bmatrix}$ has 2 pivots.

Column Space and Reduced Echelon form

$$\begin{bmatrix} \mathbf{a}_1 & \text{[redacted]} & \mathbf{a}_3 & \text{[redacted]} \end{bmatrix} \sim \begin{bmatrix} 1 & \text{[redacted]} & 0 & \text{[redacted]} \\ 0 & & 1 & \\ 0 & & 0 & \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \ \ \mathbf{a}_3]$ has 2 pivots. $\begin{bmatrix} \vec{a}_1 & \vec{a}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

So the pivot columns are <u>linearly independent</u>.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$\begin{array}{cc} 2 & 1 \\ \mathbf{a}_1 & \mathbf{a}_2 \end{array}$$ [redacted] $$\sim \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$ [redacted]

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

$$[a_1\ a_2] \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$\begin{array}{cc} 2 & 1 \\ \end{array}$$
$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \text{[redacted]} \end{bmatrix} \sim \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix} 1 & -2 & \text{[redacted]} \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix} \end{array}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \end{array}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

$-v_1 + 2v_3 = v_4 \Rightarrow -a_1 + 2a_3 = a_4$

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $\mathbf{2a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> $\text{Col}(A)$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The takeaway. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The takeaway. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

!! IMPORTANT !!

Choose the columns of A.

($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\mathrm{Col}(A)$)

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathrm{Col}(A)$.

Solution.

1. Find the pivot columns in an echelon form of A.
2. The associated columns <u>in A</u> form a basis for $\mathrm{Col}(A)$.

General Tip

A lot of information can be gleaned from the (reduced) echelon form of a matrix.

You shouldn't do reductions without thinking, but when you're stuck, reduce and maybe you can find a solution in that matrix.

Question

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A.

Answer

Summary

Subspaces define "tilted versions" of $\mathbb{R}^k$ in $\mathbb{R}^n$ (where $k \leq n$).

Bases are compact representation of subspaces as minimal spanning sets.

Matrices have useful associated subspaces like the column space and null space.