Subspaces

Geometric Algorithms
Lecture 16

CAS CS 132

Practice Problem

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 4 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1 + R_2 \\ R_3 \leftarrow 2R_3 \\ R_2 \leftrightarrow R_3}} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & 2 & 3 \end{bmatrix} = B$$

Consider the following pair of matrices A and B which are row equivalent. Find a matrix E such that $EA = B$.

Answer

$$R_1 \leftarrow R_1 + R_2$$
$$R_3 \leftarrow 2R_3$$
$$R_2, R_3 \leftarrow R_3, R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & 2 & 3 \end{bmatrix}$$

Objectives

1. Introduce the fundamental notions of **subspaces** and **bases**

2. Extend our intuitions about planes in $\mathbb{R}^3$ to subspaces in $\mathbb{R}^n$

3. Connected subspaces to matrices so that we can use the techniques we been honing in this course

Keywords

subspace
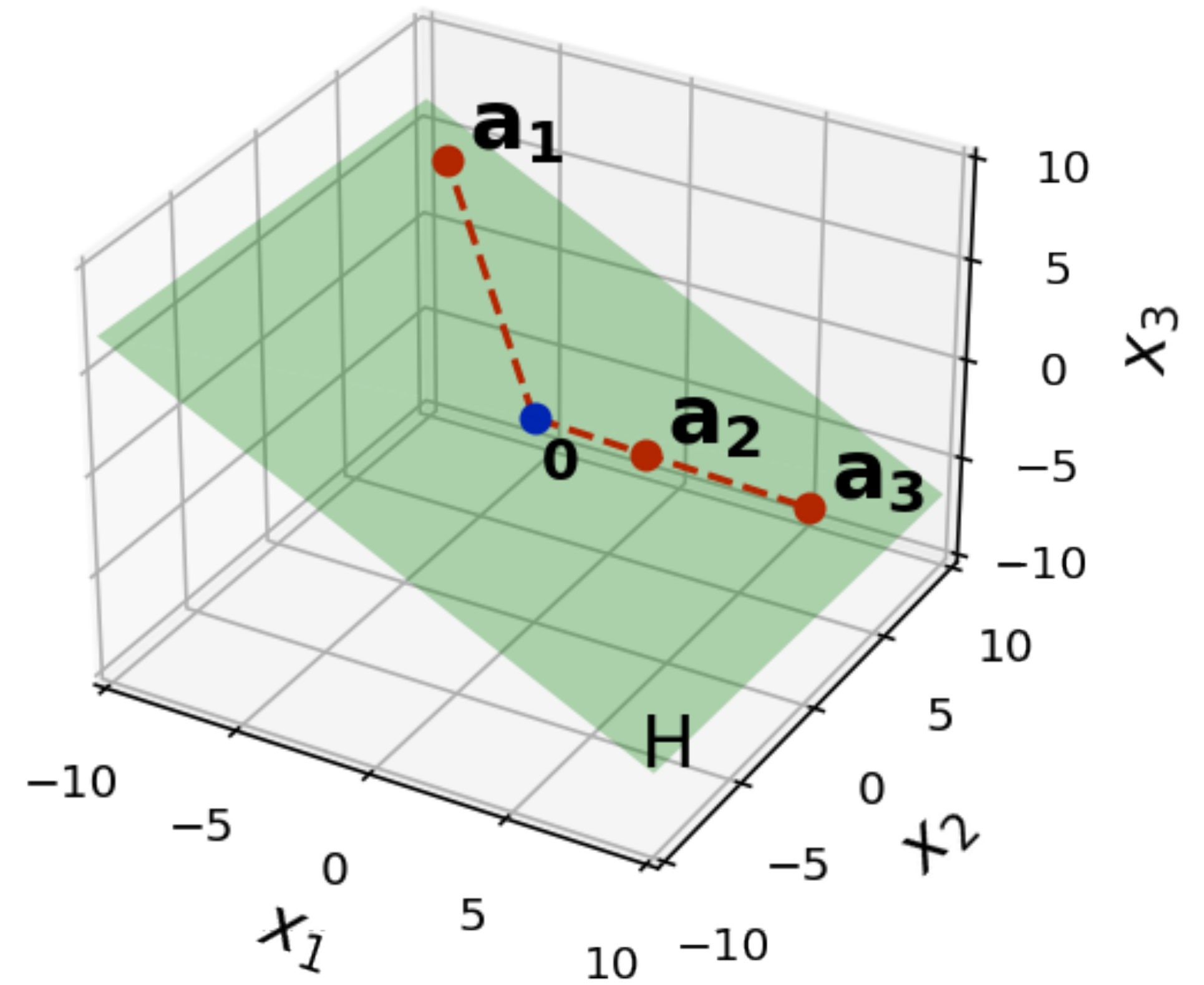
closed under addition

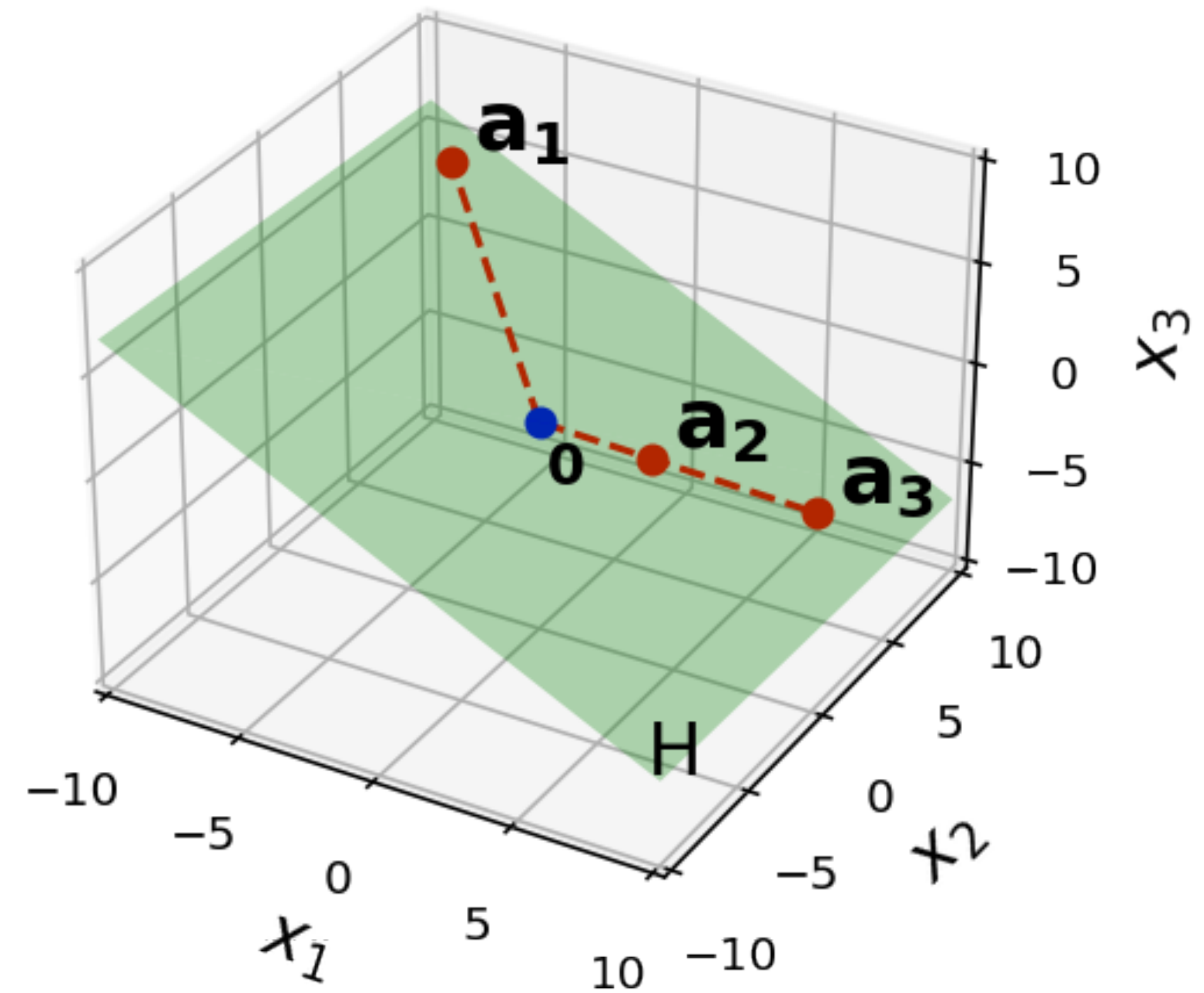closed under scaling

column space

null space

basis

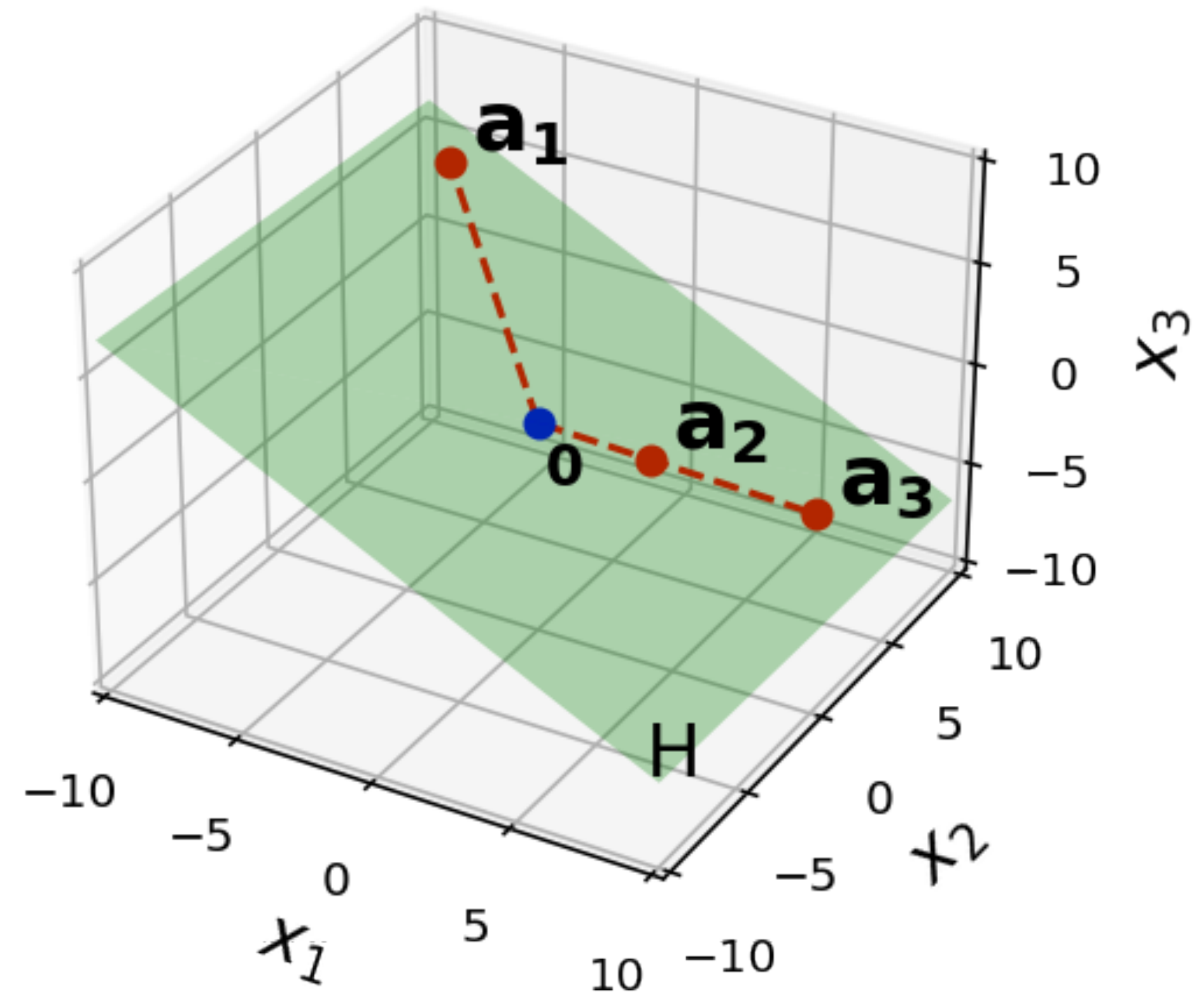# Subspaces

The Idea Behind Subspaces

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"sub" means "part of" or "below"

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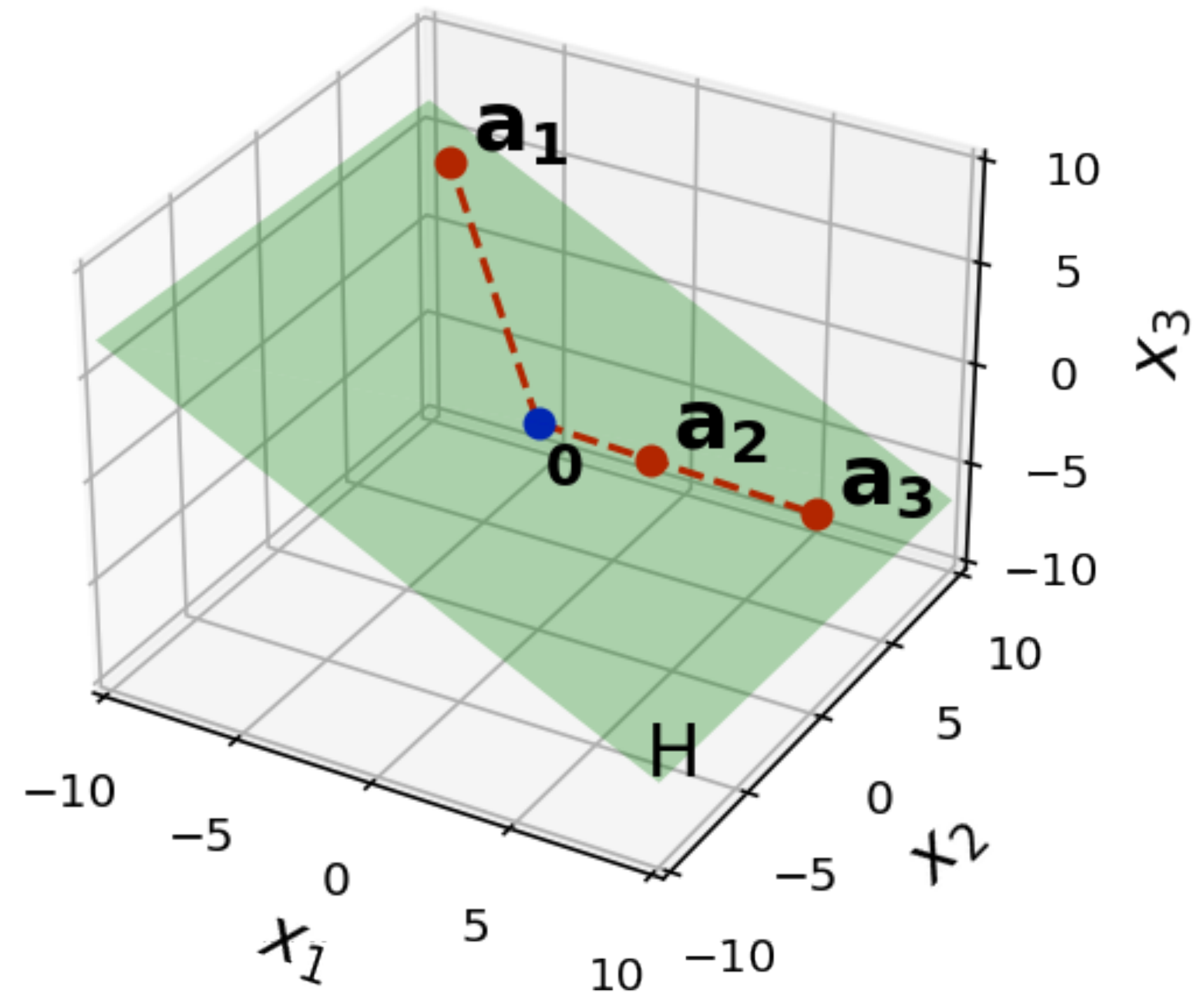
Subspaces *generalize* of this idea.

The Idea Behind **Subspaces**

"sub" means "part of" or "below"

A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$
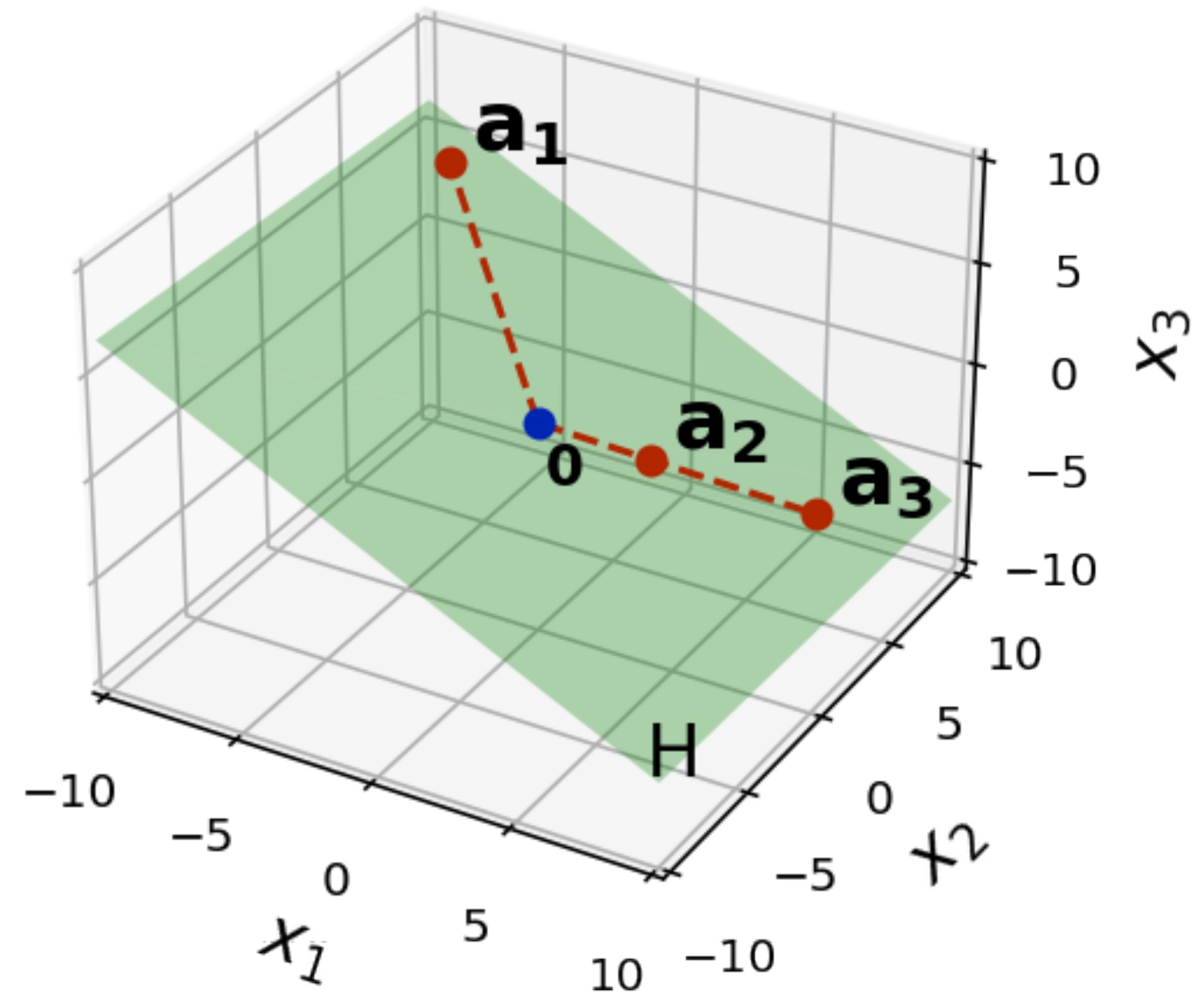
An Aside: Flatland, Relativity, Higher Dimensions

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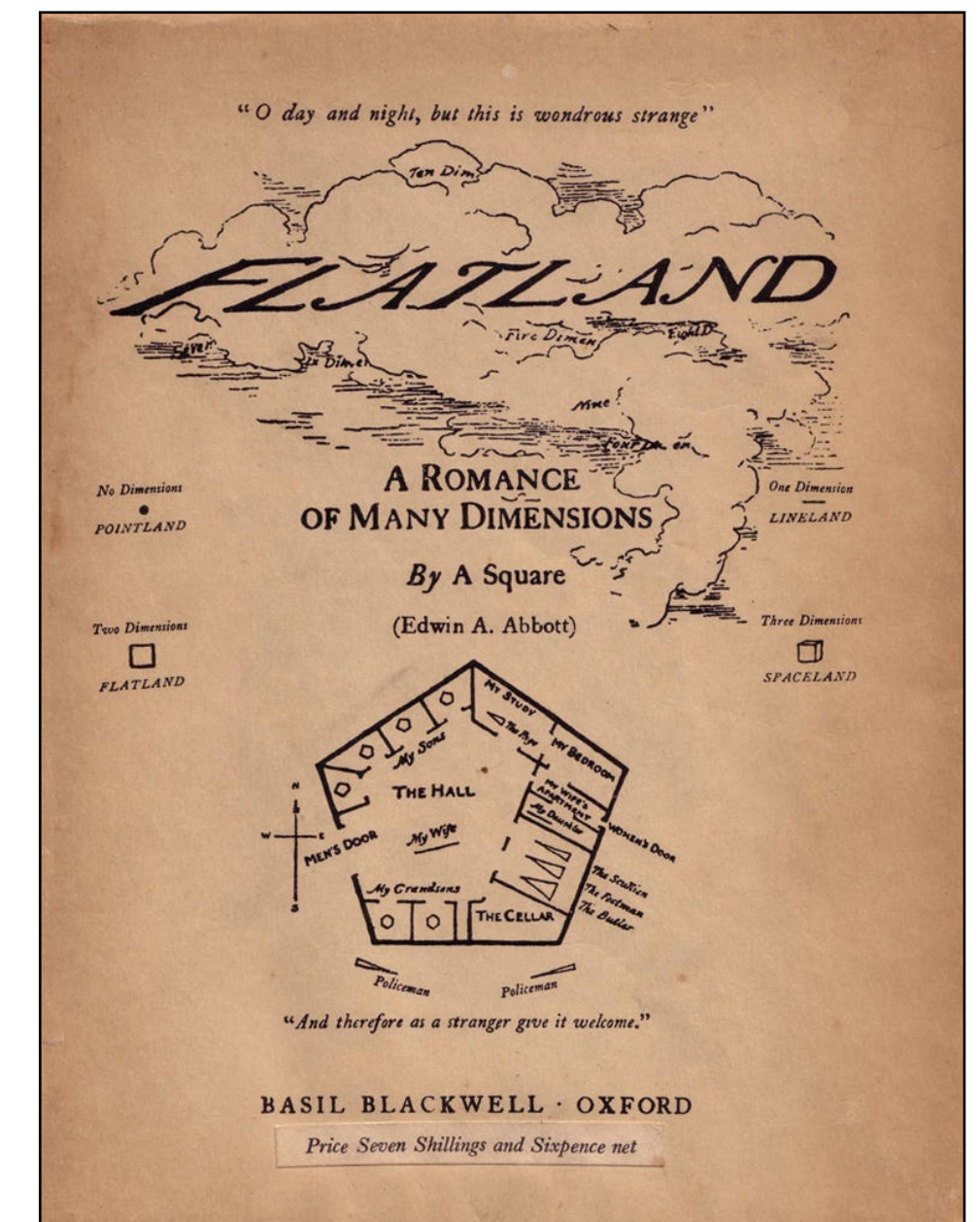
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The moral. We have to be careful regarding our intuitions about higher-dimensional subspaces.
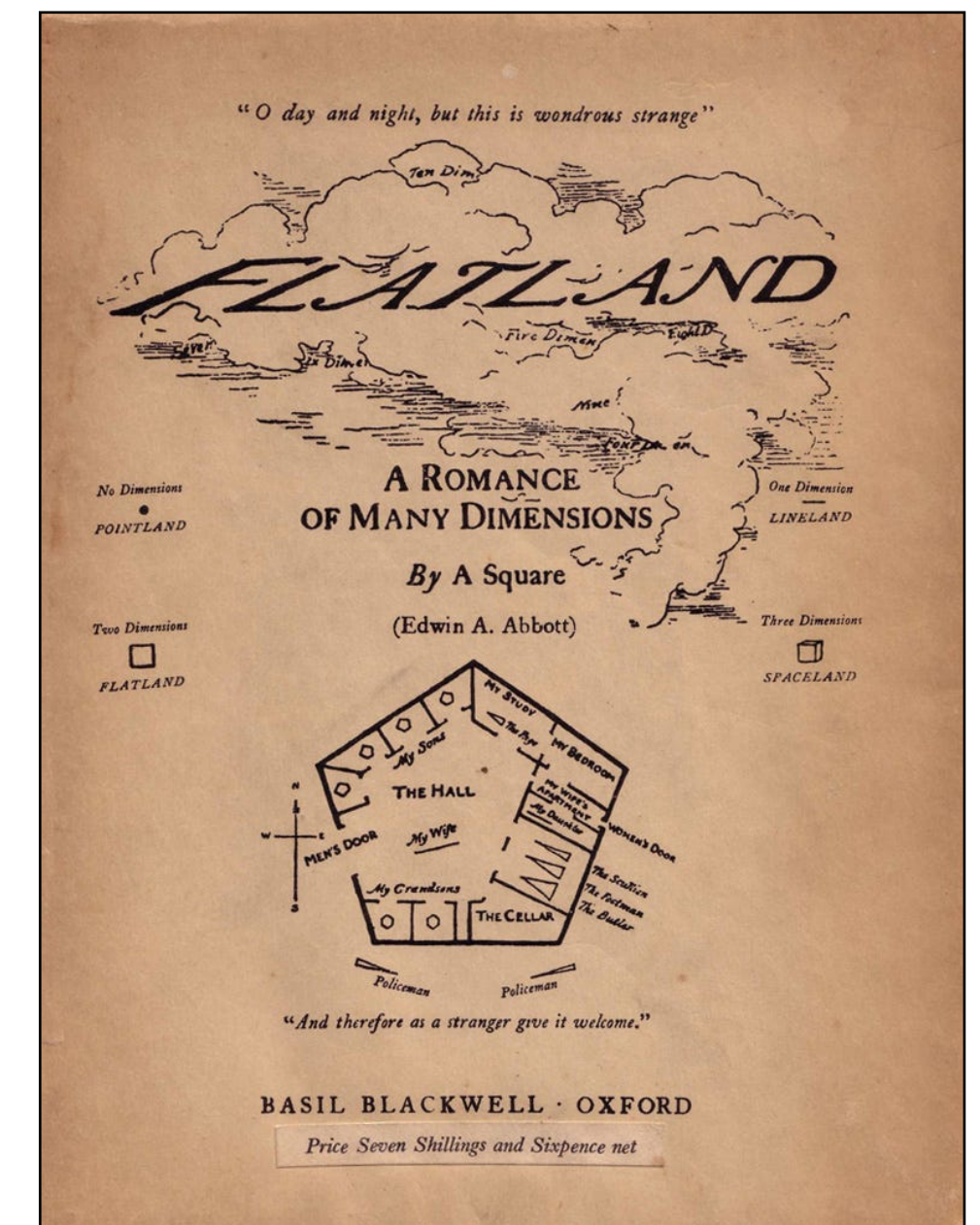
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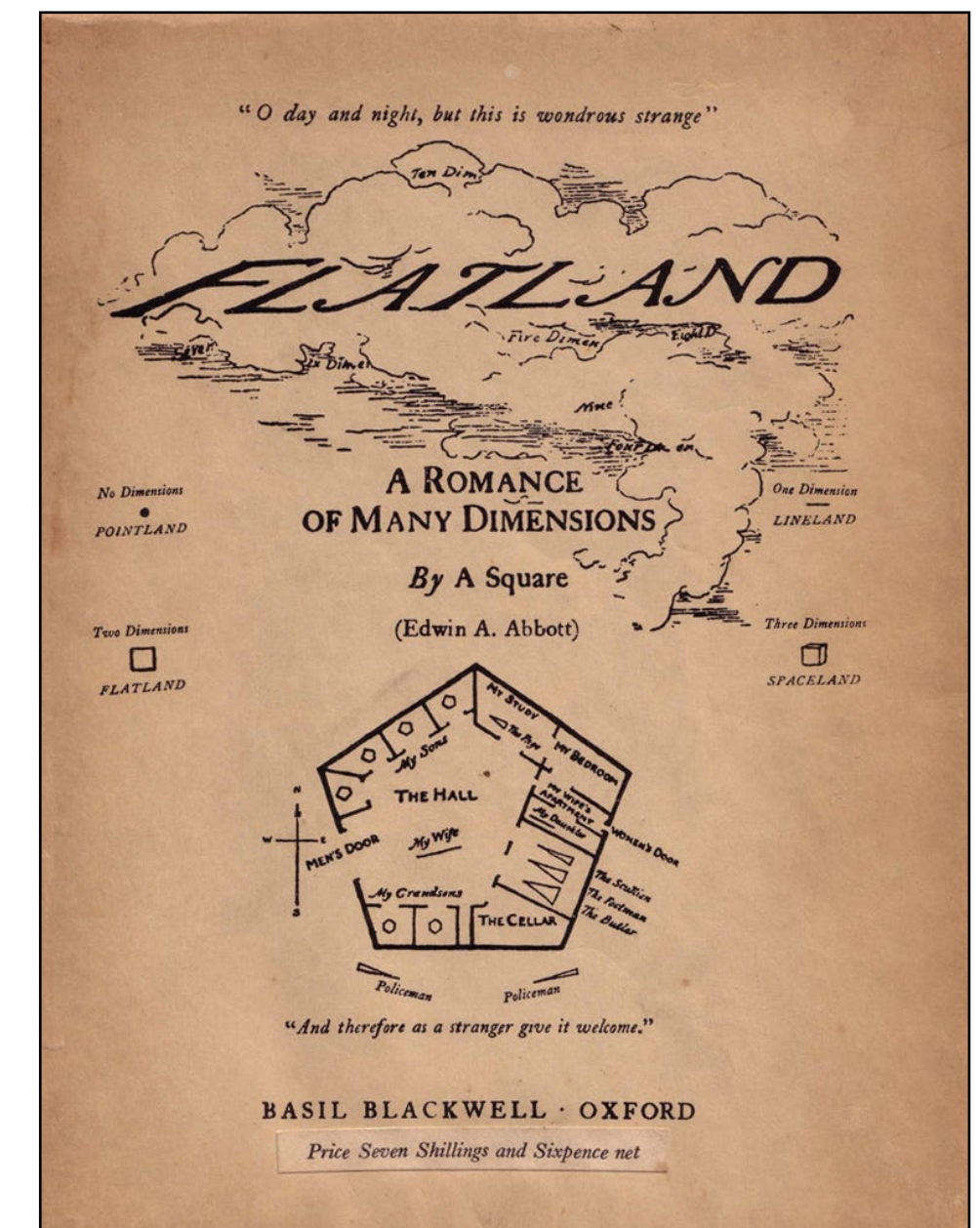
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A 3D subspace of $\mathbb{R}^7$ "looks like" 3D space from the inside, but from the outside it may be "tilted."

Projection of the 4D cube
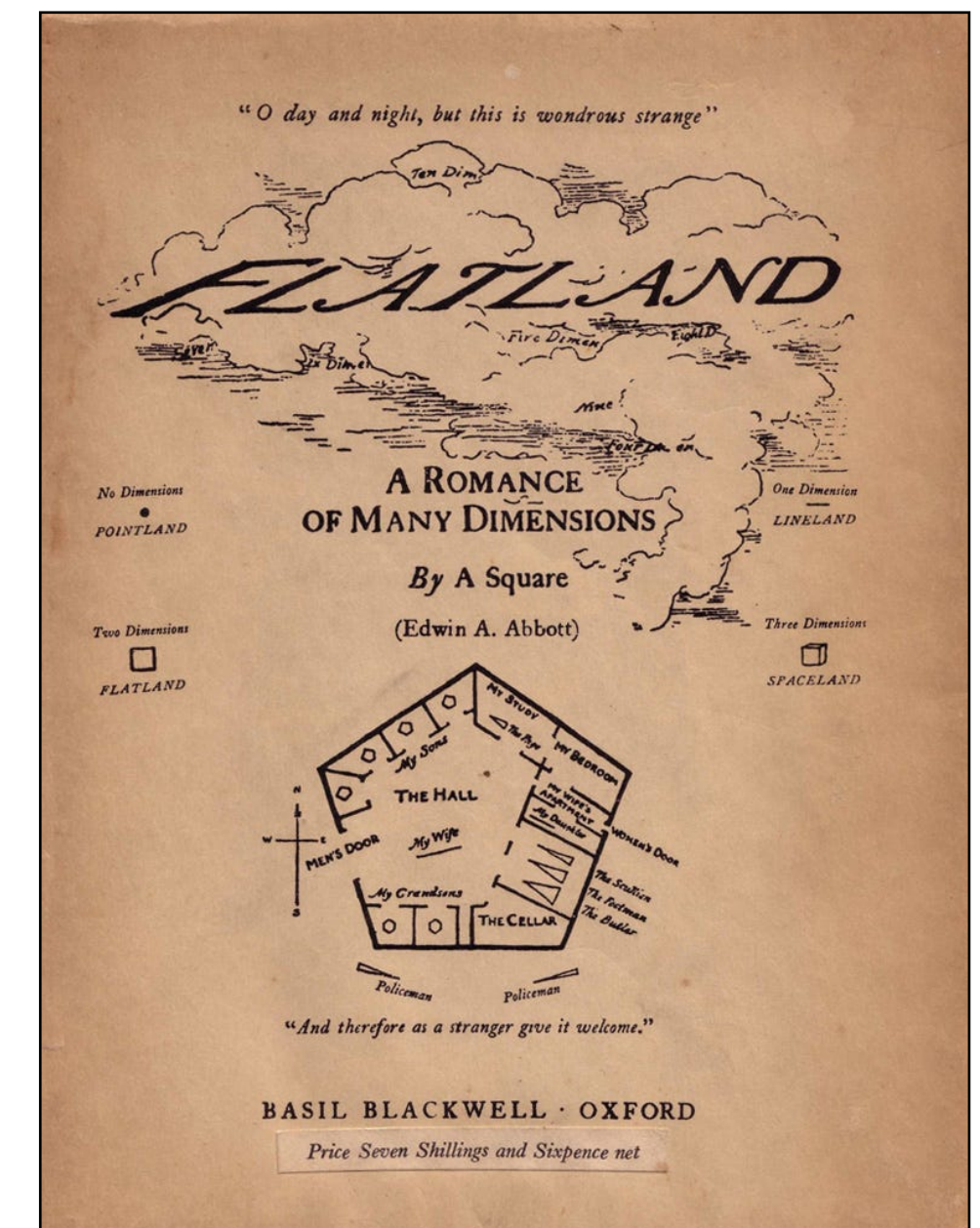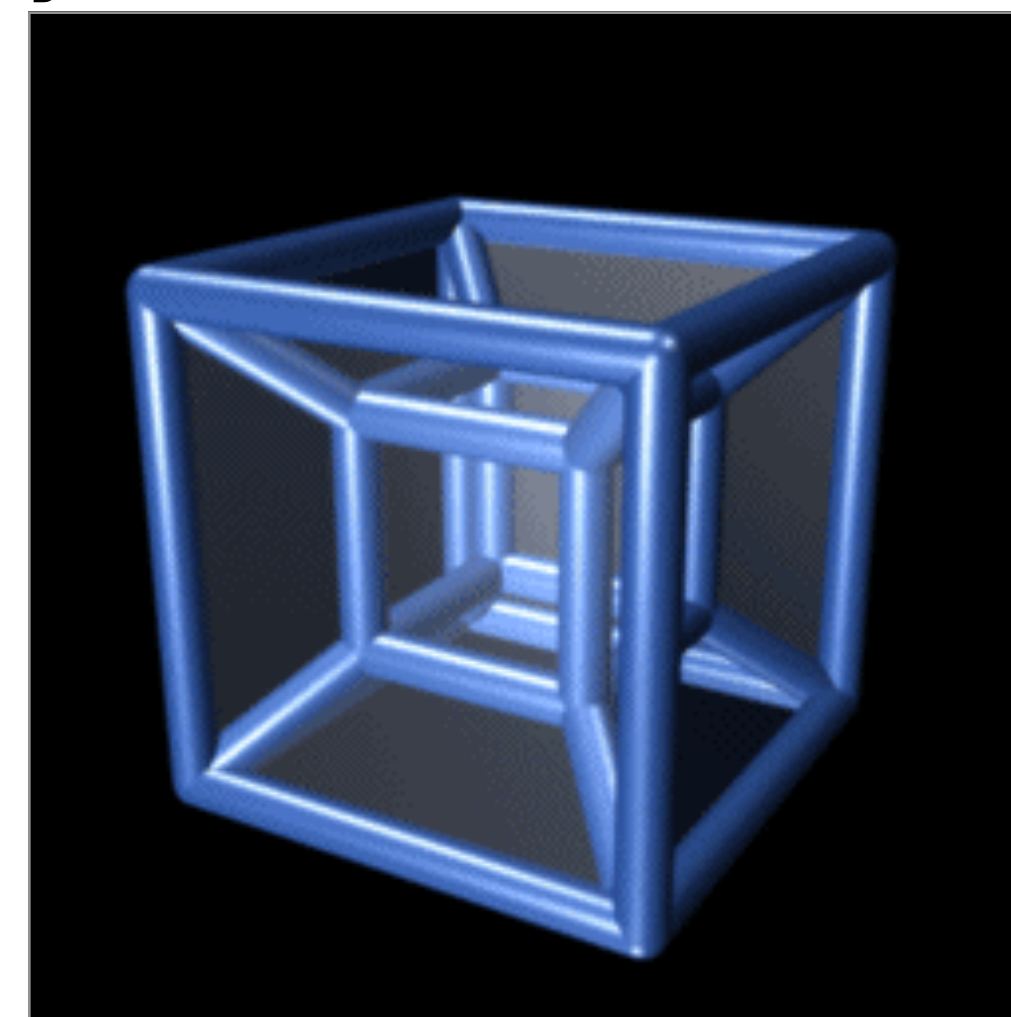
Subspace (Algebraic Definition)

Definition. A **subspace** of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $\mathbf{u} + \mathbf{v}$ is in H

2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H

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!! Subspaces must "live" somewhere !!

How to Think About this Definition

It's not possible to "leave" H by addition or scaling.

(recall this is also how we discussed spans)
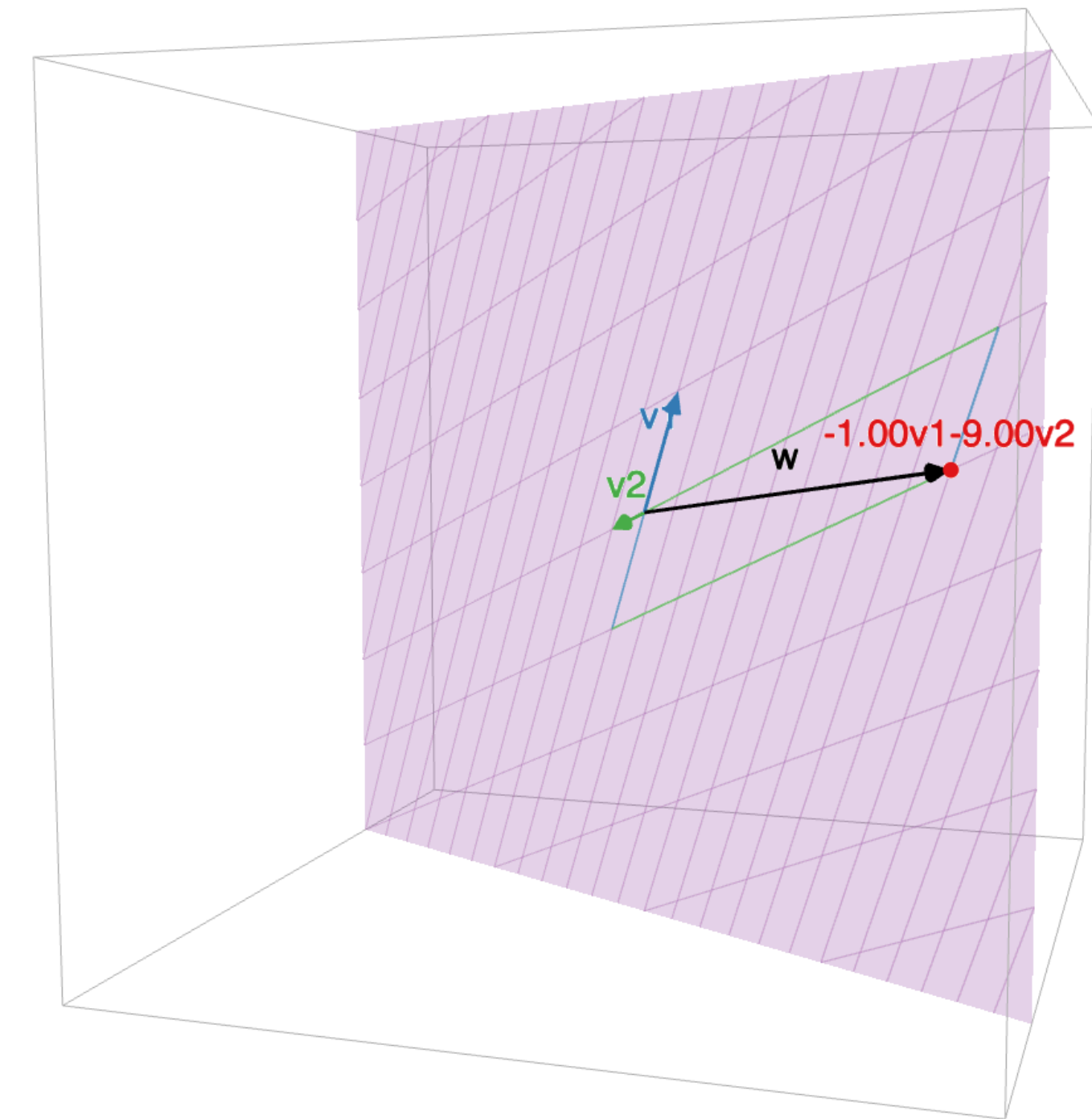
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Solution.

1. Show that if $\mathbf{u}$ and $\mathbf{v}$ are in H then so is $\mathbf{u}+\mathbf{v}$.

How To: Verifying Subspaces

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Solution.

1. Show that if $\mathbf{u}$ and $\mathbf{v}$ are in H then so is $\mathbf{u} + \mathbf{v}$.

2. Show that if $\mathbf{u}$ is in H then so is $c\mathbf{u}$ for any scalar c.

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Find $\mathbf{u}$ and $\mathbf{v}$ in H such that $\mathbf{u} + \mathbf{v}$ is not in H.

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Find $\mathbf{u}$ and $\mathbf{v}$ in H such that $\mathbf{u} + \mathbf{v}$ is not in H.

OR

Find $\mathbf{u}$ in H such that $c\mathbf{u}$ is not in H for *some* scalar c.

Subspaces must include the origin

Fact. For any subspace H of $\mathbb{R}^n$, the zero vector is in H. In set notation: $\mathbf{0} \in H$

Verify:

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Find $\mathbf{u}$ in H such that $c\mathbf{u}$ is not in H for *some* scalar c.

OR

Show that $\mathbf{0}$ is not in H.

Example: The Origin

Fact. The set $\{\mathbf{0}_n\}$ is a subspace of $\mathbb{R}^n$

Verify:

Example: $\mathbb{R}^n$

Fact. The set $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ (in other words, $\mathbb{R}^n$ is a subspace of itself).

Example: Spans

Fact. For any set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$, the set $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of $\mathbb{R}^n$.

Verify:

Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin

2. lines (through the origin)

3. planes (through the origin)

4. All of $\mathbb{R}^3$
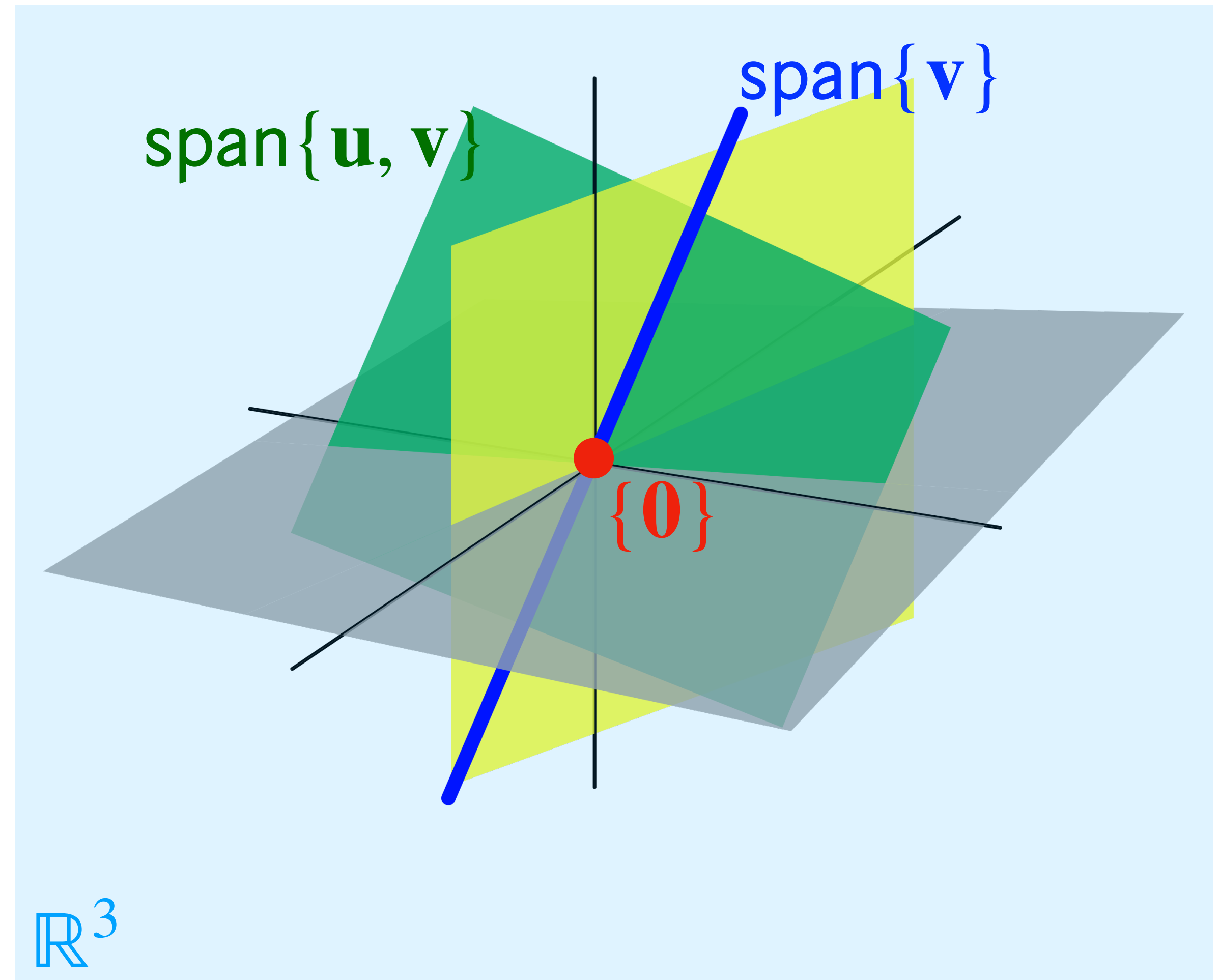
Non-Example: Bounded Sets

Fact. The set $\{(x,y) : x \geq 3\}$ is *not* a subspace of $\mathbb{R}^2$.

Verify:
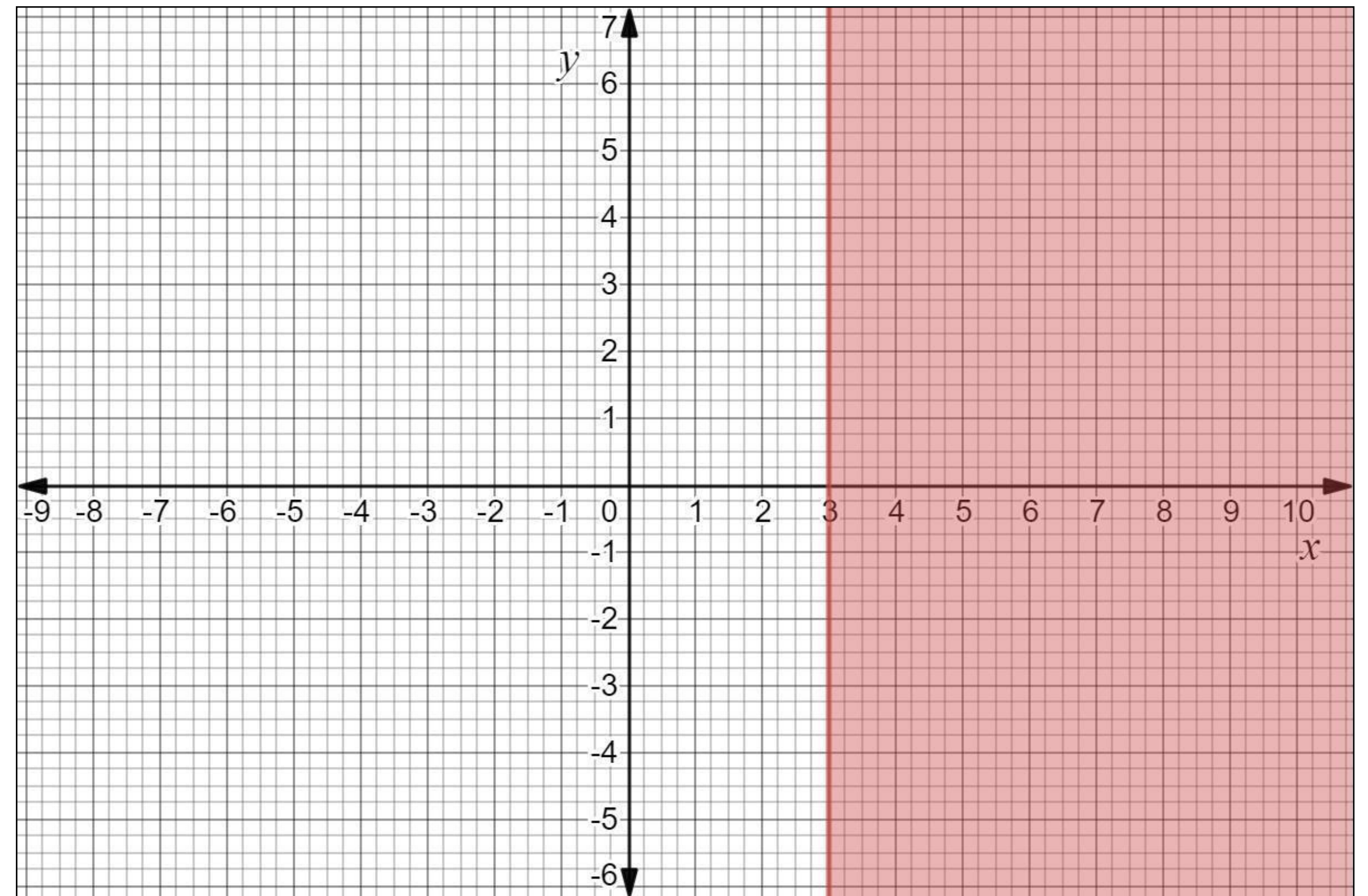
Question

1. *Show that the unit sphere* $\{(x, y, z) : x^2 + y^2 + x^2 = 1\}$ *is <u>not</u> a subspace of* $\mathbb{R}^3$.

2. *Show that the range of a linear transformation* $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$.

Answer (1)

Answer (2)

How To: Subspaces and Span

Question. Show that v lies in the subspace generated by $\mathbf{u}_1,\ldots,\mathbf{u}_k$.

Solution. Show that v is in $\mathrm{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$.

We will start using "subspace generated by" and "span of" interchangeably.

Subspaces and Matrices

The Connection between Subspaces and Matrices

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» collections of vectors
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Today we'll look at:

» column space
» null space

Column Space

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The column space of a matrix is the span of its columns.

The column space of a matrix is the <u>range</u> of the linear transformation it implements.

Subspace of What?

$$m \left| \begin{array}{c} \underset{n}{\overline{}} \\ \begin{bmatrix} | & | & \dots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \dots & | & | \end{bmatrix} \end{array} \right.$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\mathrm{Col}(A)$

is a subspace of

$\mathbb{R}^m$

Examples

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 5 \\ 3 & -3 & 6 \end{bmatrix}$$

Col(A) is all of $\mathbb{R}^3$

Col(B) is just span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

Null Space

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The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A.

Subspace of What?

$$\overset{n \text{ columns}}{\underset{\underset{m \times n}{} \quad \underset{n \times 1}{}}{\overset{m}{\underset{\text{rows}}{}} \;\; A \, \mathbf{v}}} = \underset{m \times 1}{\mathbf{0}}$$

$\mathbf{v}$ **is a vector in** $\mathbb{R}^n$

$\mathsf{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

The Null Space is a Subspace

Fact. For any $m \times n$ matrix A, the set $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Verify:

Examples

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 5 \\ 3 & -3 & 6 \end{bmatrix}$$
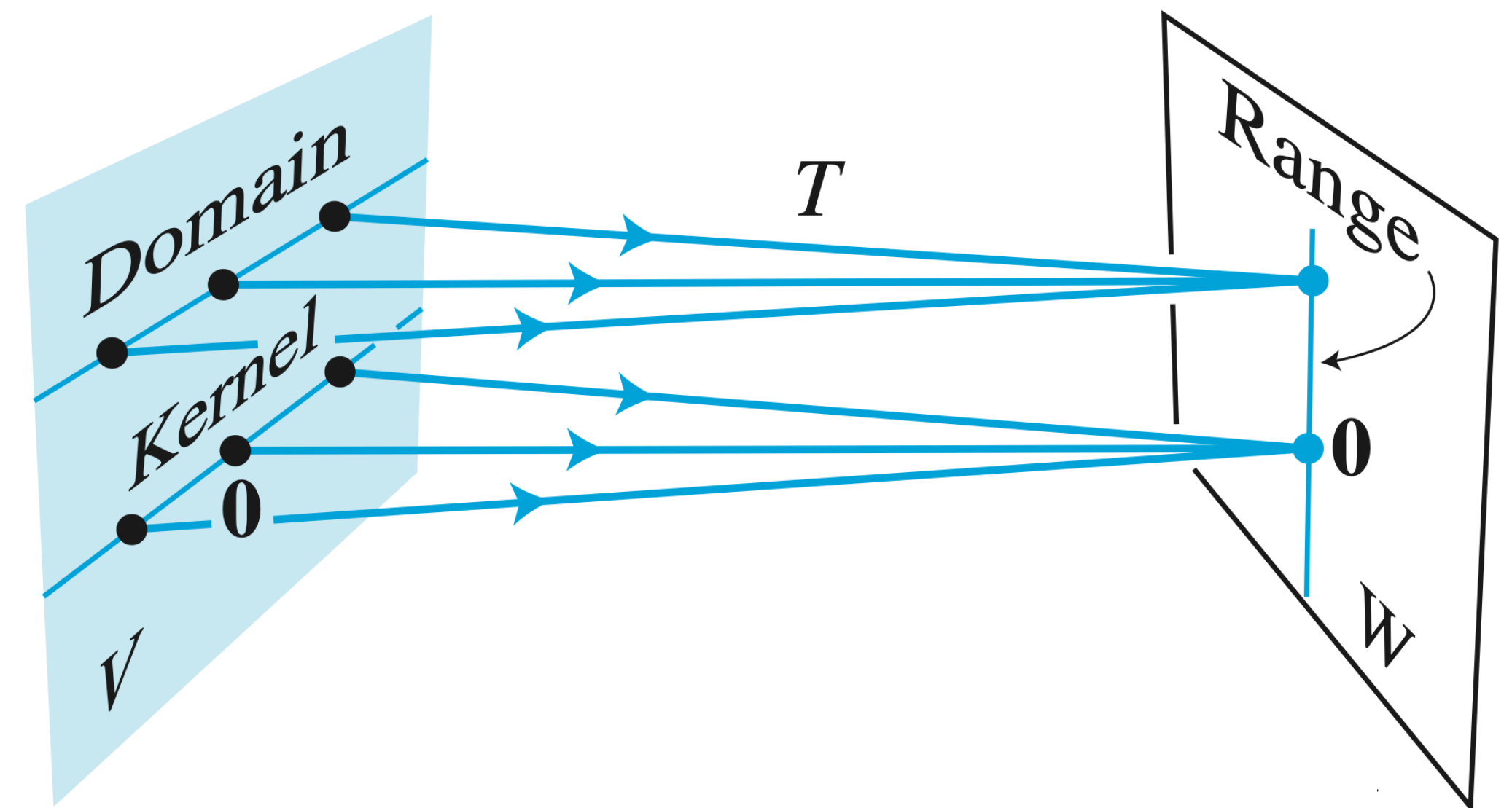
$\text{Nul}(A) = \{\mathbf{0}\}$ 　　　　 Verify:

$\text{Nul}(B) = \text{span}\{[1\ \ 1\ \ 0]^T\}$

Linear Transformations Perspective

If A implements the linear transformation T then:

» Col(A) is the same as ran(T), where vectors are "sent" by T

» Nul(A) is the set of vectors "zeroed out" by T, which is sometimes called the **kernel** of T.

Comparing Column Space and Null Space

The column space and the null space live can live in entirely different spaces.

The point. They are not easily comparable

Contrast Between Nul A and Col A for an m x n Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

(just for reference)

Bases

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A basis is a "minimal" choice of these vectors.

A basis is a "compact representation" of a subspace.

Recall: Standard Basis

Definition. The n**-dimensional standard basis vectors** (or standard coordinate vectors) are the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ where

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ n-1 \\ n \end{matrix}$$

Recall: Standard Basis

Definition (Alternative). The n-dimensional standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix.

$$I = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]$$

What was interesting about the standard basis?
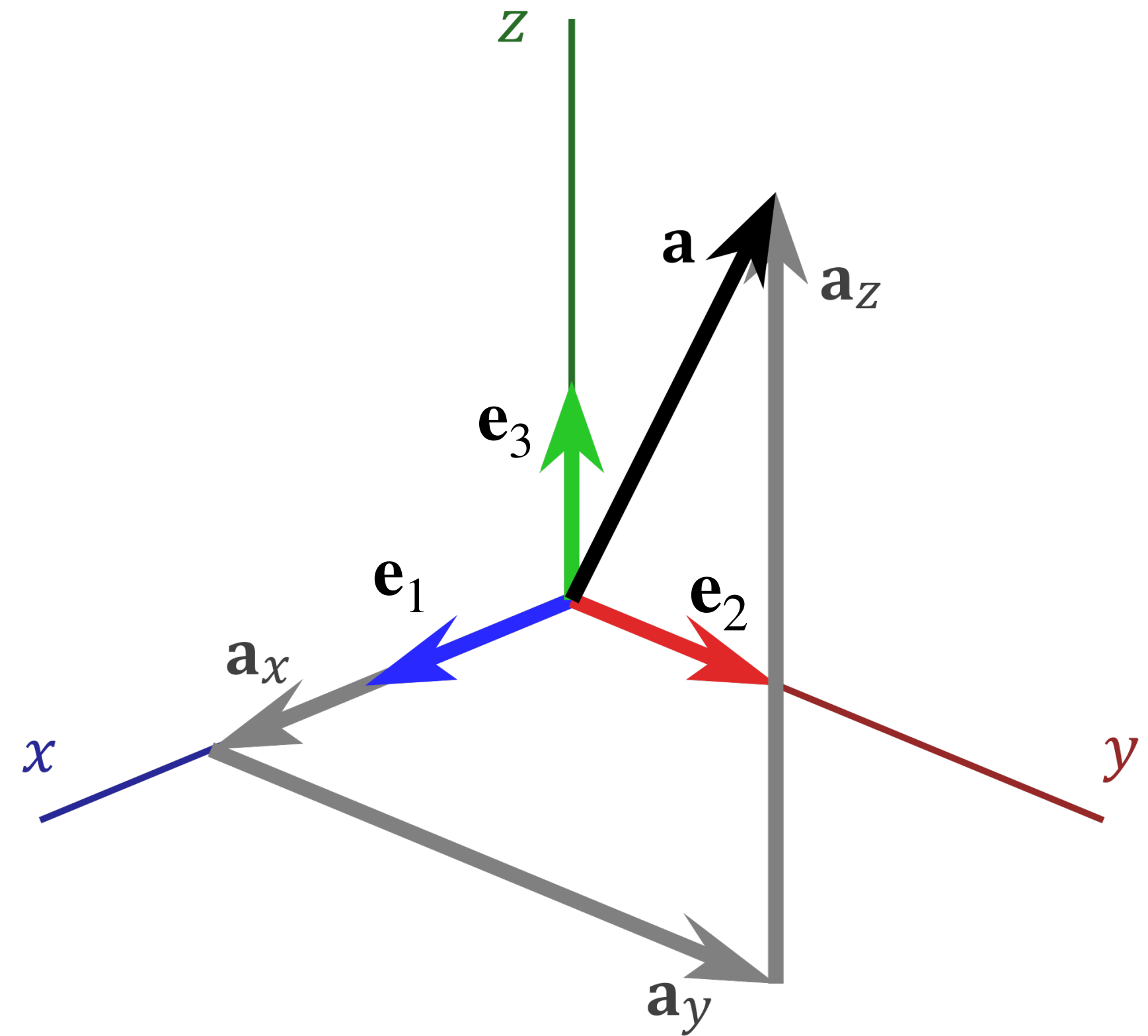
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The n standard basis vectors
in $\mathbb{R}^n$:

» are linearly independent
» span all of $\mathbb{R}^n$
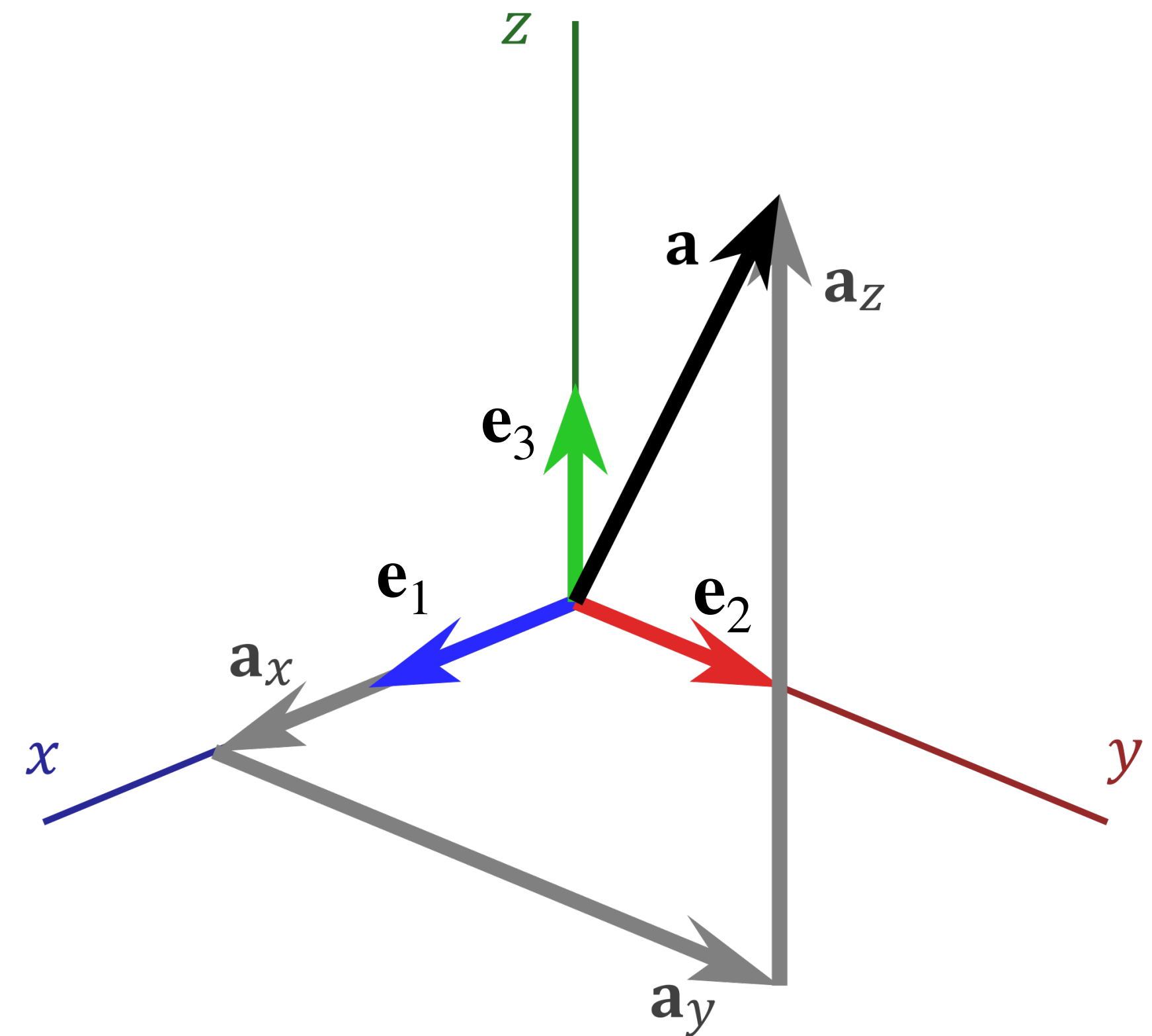
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Their span is as "large" as possible while the set of vectors generating the span is as "small" as possible.
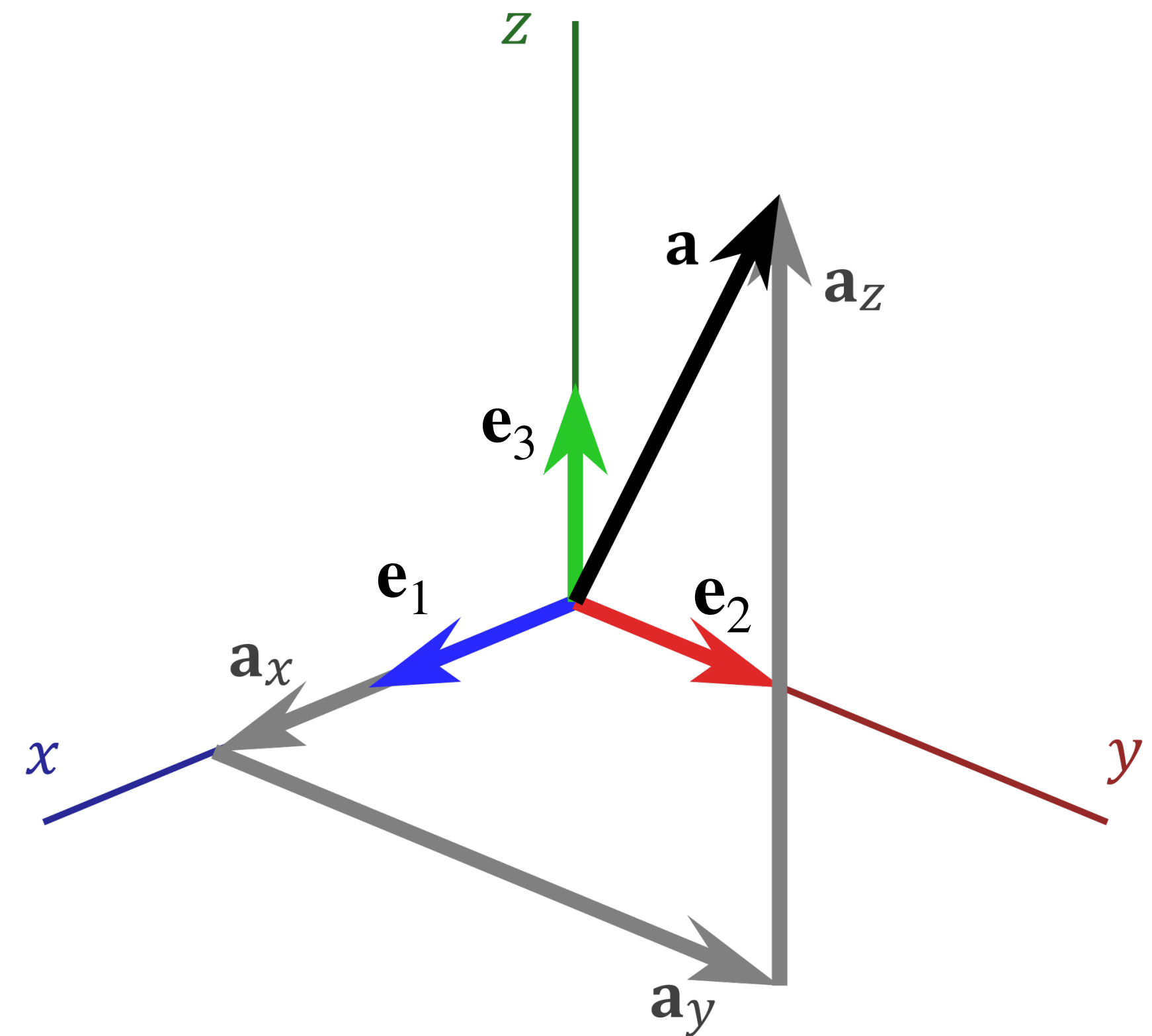
Basis

Basis

Definition. A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$).

A basis is a *minimal* set of vectors which spans all of H.

Example: Standard basis

The standard basis is a basis of $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Column vectors are just weights for a linear combination of the standard basis

Example: Column Space of Invertible Matrices

Fact. The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

Verify:

Example: Subsets of Spanning Sets

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Theorem. If the vectors $\mathbf{v}_1$, $\mathbf{v}_2,\ldots,\mathbf{v}_k$ span a subspace H then a subset of them form a basis of H.

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We can *remove* vectors from a spanning set until we get a basis.

How do we do this?

As usual, by connecting back to matrices.

Question

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Is this set of vectors a basis for $\mathbb{R}^3$?

Answer

Solving tip. A set of vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$ if the standard basis is in their span.

Bases of Column Space and Null Space

The Goal of this Last Section

Determine how to find <u>bases</u> for the **column space** and the **null space** of a given matrix.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for Nul(A).

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Question. Given a $m \times n$ matrix A find a basis for Nul(A).

The idea. Describe the solutions of $A\mathbf{x} = 0$ as linear combination of vectors

Example

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above reduced echelon form.

Let's write down a general form solution for A:

Parametric Solutions

We can think of our general form solution as a (linear) transformation.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

$\equiv$

"given values for x_2, x_3, and x_4, I can give you a solution"

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$$\equiv \qquad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Parametric Solutions

We can think of our general form solution as a (linear) transformation. !! this transformation is only linear !!
!! in the case of homogeneous equations !!

$$x_1 = 2x_2 + x_4 - 3x_5$$

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x_4 is free

x_5 is free

$$\equiv \qquad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Example

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Let's find the matrix implementing this linear transformation:

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its <u>columns</u>.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its <u>columns</u>.

The columns of this matrix <u>span</u> $\text{Nul}(A)$.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix are linearly independent.

Verify:

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix <u>span</u> Nul(A).

The columns of this matrix are linearly independent.

The columns of this matrix form a basis for Nul(A).

Example

Alternatively, we can think of writing a general form solution so that it is a linear combination of vectors with <u>free variables as weights</u>:

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for Nul(A).

Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.

2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.

3. The resulting vectors form a basis for Nul(A).

An Observation

The *number* of vectors in the basis we found is the same as the number of <u>free variables</u> in a general form solution.

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = (-2)x_4 + 2x_5$

x_4 is free

x_5 is free

$$\equiv \qquad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Practice Problem

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above RREF. Determine a basis for $\text{Col}(A)$

Answer

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

onto column space...

How To: Finding a basis for the column space

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\text{Col}(A)$.

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We already know the columns of A span $\text{Col}(A)$.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathrm{Col}(A)$.

We already know the columns of A span $\mathrm{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\mathrm{Col}(A)$.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\mathrm{Col}(A)$.

We already know the columns of A span $\mathrm{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\mathrm{Col}(A)$.

Which vectors should we choose?

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4 \ \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \blacksquare \quad \mathbf{a}_3 \quad \blacksquare \quad \blacksquare] \sim \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare & \blacksquare \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \quad \mathbf{a}_3]$ has 2 pivots.

Column Space and Reduced Echelon form

$$\begin{bmatrix} \mathbf{a}_1 & \blacksquare & \mathbf{a}_3 & \blacksquare & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare & \blacksquare \\ 0 & \blacksquare & 1 & \blacksquare & \blacksquare \\ 0 & \blacksquare & 0 & \blacksquare & \blacksquare \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \quad \mathbf{a}_3]$ has 2 pivots.

So the pivot columns are <u>linearly independent</u>.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$\begin{bmatrix} \overset{2}{\mathbf{a}_1} & \overset{1}{\mathbf{a}_2} & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \sim \begin{bmatrix} \overset{2}{1} & \overset{1}{-2} & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

Column Space and Reduced Echelon form

$$\begin{bmatrix} {}^{\color{cyan}2}\mathbf{a}_1 & {}^{\color{red}1}\mathbf{a}_2 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \sim {}^{\color{cyan}2}\begin{bmatrix} 1 & {}^{\color{red}1}-2 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $\mathbf{2a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4 \ \ \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4 \ \ \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> Col(A).

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The takeaway. The pivot columns of A form a basis for $\text{Col}(A)$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The takeaway. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

!! IMPORTANT !!
Choose the columns <u>of A</u>.

($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\mathrm{Col}(A)$)

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for Col(A).

Solution.

1. Find the pivot columns in an echelon form of A.

2. The associated columns <u>in A</u> form a basis for Col(A).

General Tip

A lot of information can be gleaned from the (reduced) echelon form of a matrix.

You shouldn't do reductions without thinking, but when you're stuck, reduce and maybe you can find a solution in that matrix.

Question

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A.

Answer

Summary

Subspaces define "tilted versions" of $\mathbb{R}^k$ in $\mathbb{R}^n$ (where $k \le n$).

Bases are compact representation of subspaces as minimal spanning sets.

Matrices have useful associated subspaces like the column space and null space.