Dimension and Rank

Geometric Algorithms
Lecture 17

Practice Problem

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Consider the subspace H generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.
Show that $\mathbf{v}_3$ and $\mathbf{v}_4$ form a basis for H.

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Hint. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the span of $\mathbf{v}_3$ and $\mathbf{v}_4$

Objectives

1. Discuss the coordinate systems.

2. Introduce the fundamental notion of <u>dimension</u>, which quantifies how "large" a space is

3. Relate the dimension of the column space and the null space of a matrix

Keywords

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basis

column space

null space

coordinate system

change of basis

dimension

rank

rank theorem

invertible matrix theorem (extended)
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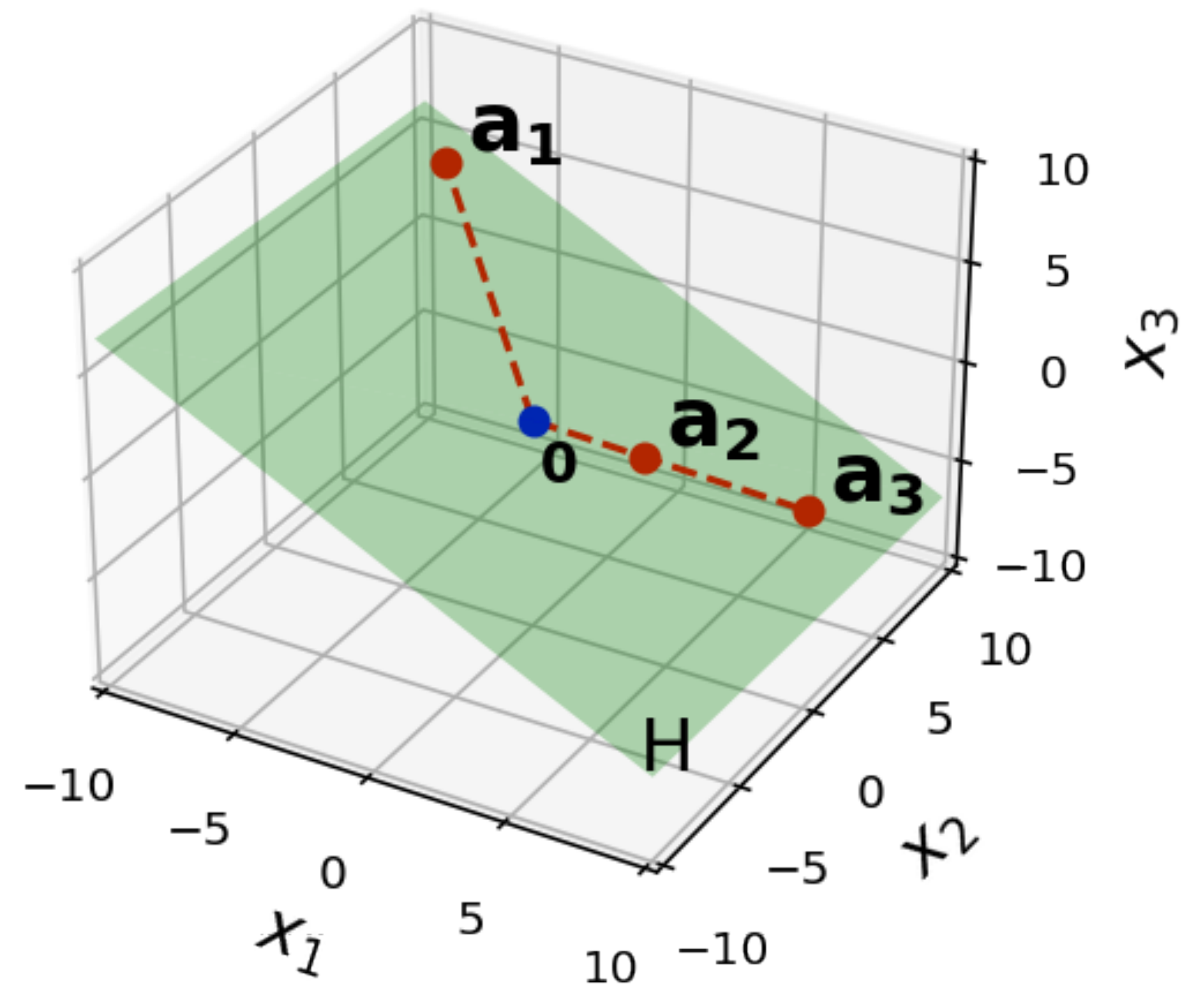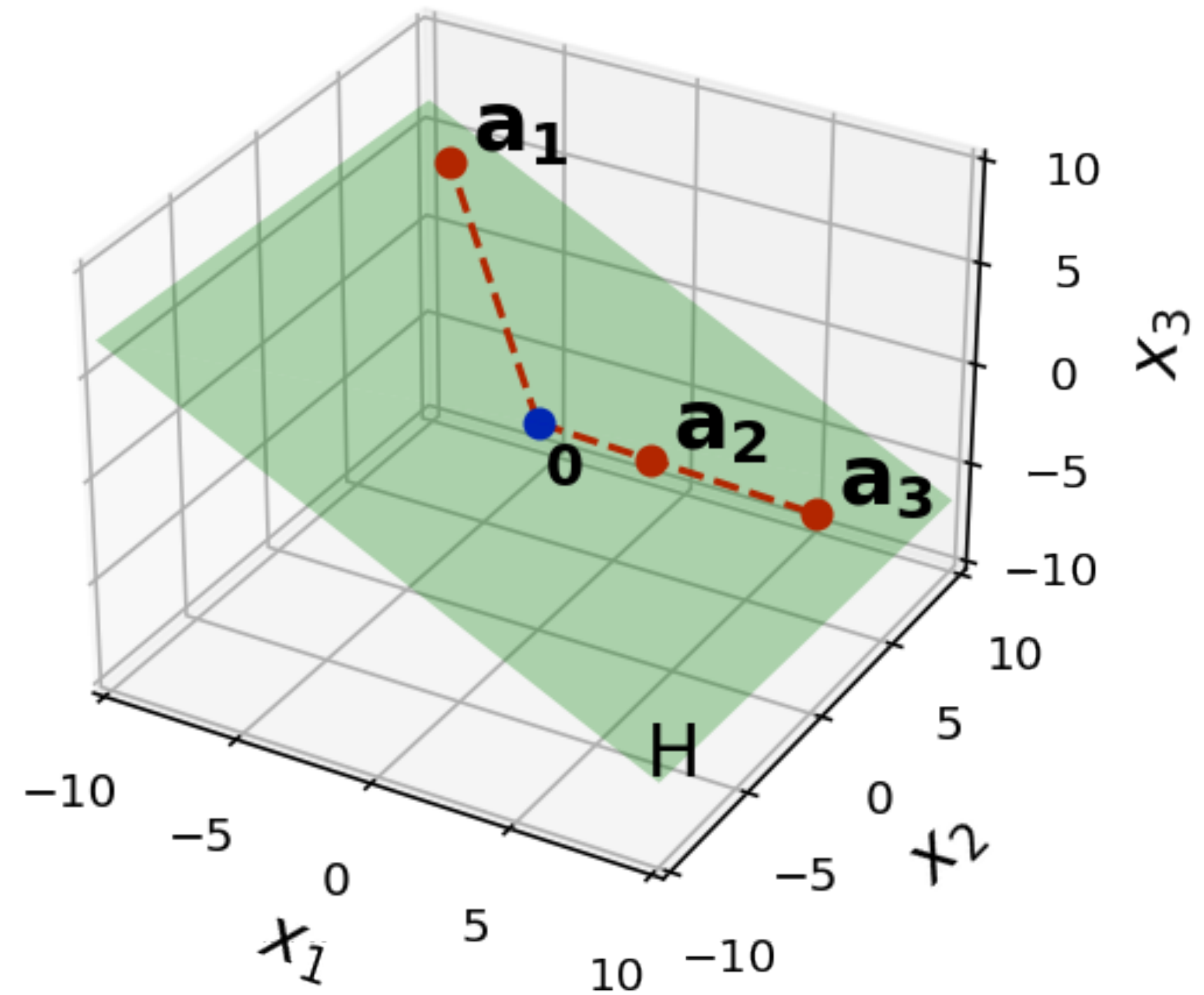
Recap

Recall: The Idea Behind Subspaces

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Subspaces *generalize* of this idea.
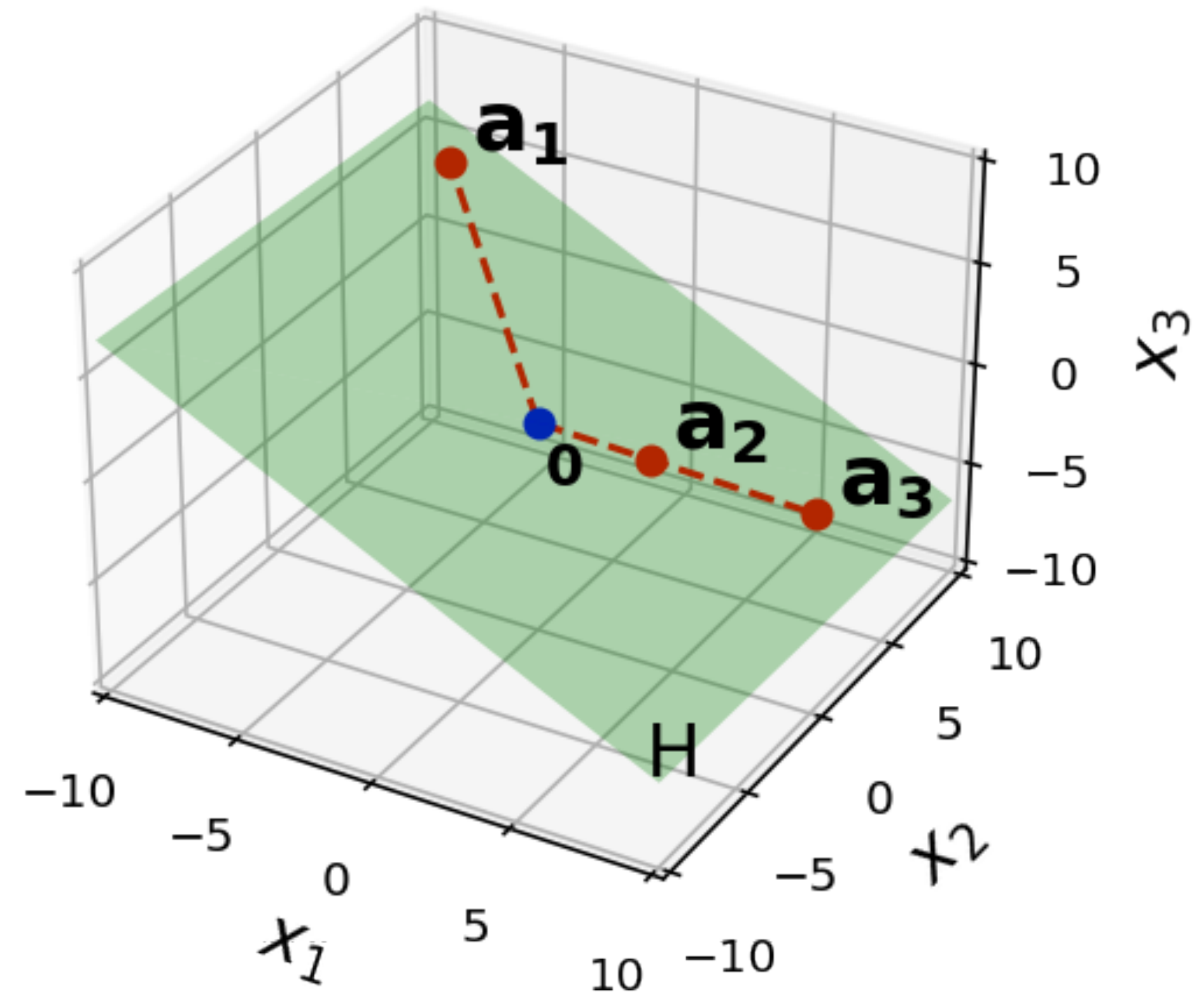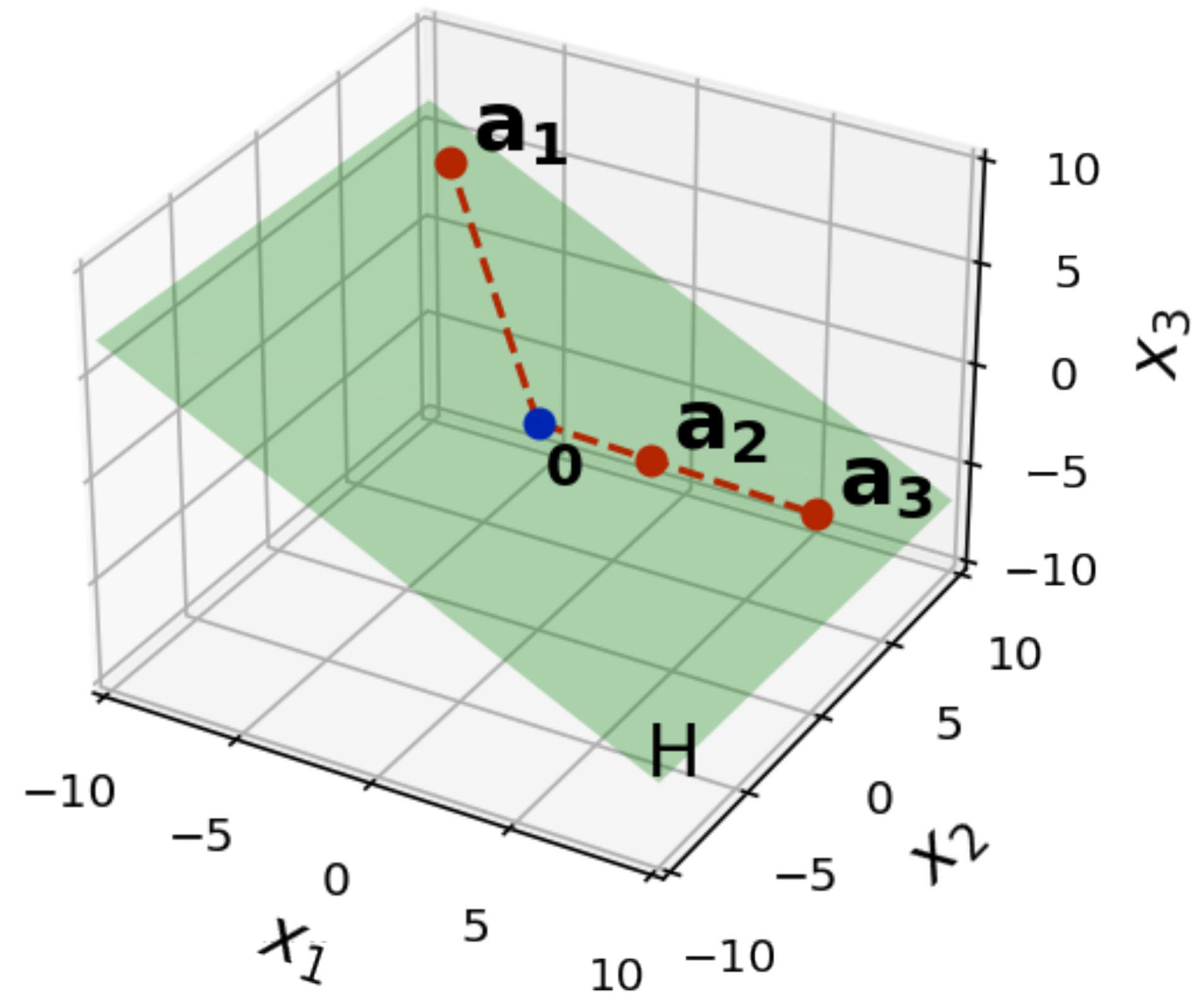
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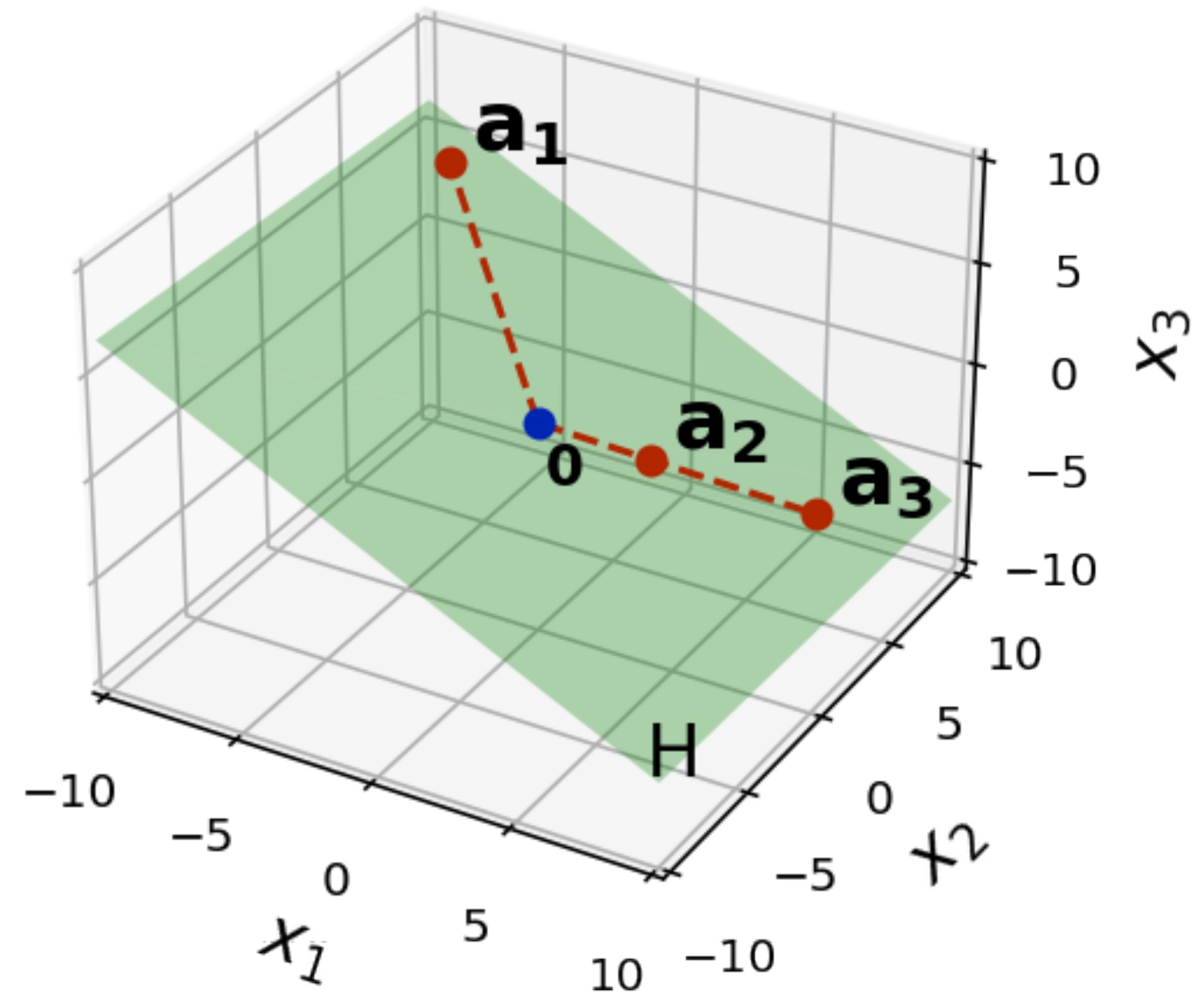
Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$

Recall: Subspace (Algebraic Definition)

Definition. A **subspace** of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $\mathbf{u}+\mathbf{v}$ is in H

2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H

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        H **is closed under scaling**

!! Subspaces must "live" somewhere !!

Column Space

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The column space of a matrix is the <u>range</u> of the linear transformation it implements.

Subspace of What?

$$n$$

$$m \left| \begin{bmatrix} | & | & \dots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \dots & | & | \end{bmatrix} \right.$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\text{Col}(A)$

is a subspace of

$\mathbb{R}^m$

Null Space

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Definition. The **null space** of a matrix A, written $\text{Nul}(A)$ or $\text{Nul}\,A$, is the set of all solutions to the homogenous equation

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The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A.

Subspace of What?

$$\overset{n \text{ columns}}{\underset{m \times n}{A}} \underset{n \times 1}{\mathbf{v}} = \underset{m \times 1}{\mathbf{0}}$$

m rows

$\mathbf{v}$ **is a vector in** $\mathbb{R}^n$

$\text{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

Recall: Basis

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Definition. A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$).

A basis is a *minimal* set of vectors which spans all of H.

Recall: Parametric Solutions

We can think of our general form solution as a (linear) transformation.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

$$\equiv$$

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Recall: Parametric Solutions

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<u>(linear) transformation</u>. **!! this transformation is only linear !!**
!! in the case of homogeneous equations !!

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Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

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Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.

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1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.

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1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.

2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.

3. The resulting vectors form a basis for Nul(A).

Recall: Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The idea. What if we cover up the non-pivot columns?

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$$[\mathbf{a}_1 \ \blacksquare \ \mathbf{a}_3 \ \blacksquare \ \blacksquare] \sim \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare & \blacksquare \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

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Then we see $[\mathbf{a}_1 \ \mathbf{a}_3]$ has 2 pivots.

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So the pivot columns are <u>linearly independent</u>.

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$$\begin{bmatrix} \overset{\textcolor{blue}{2}}{\mathbf{a}_1} & \overset{\textcolor{red}{1}}{\mathbf{a}_2} & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \sim \begin{bmatrix} \overset{\textcolor{blue}{2}}{1} & \overset{\textcolor{red}{1}}{-2} & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

Recall: Column Space and Reduced Echelon form

$$\begin{bmatrix} \overset{2}{\mathbf{a}_1} & \overset{1}{\mathbf{a}_2} & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \sim \begin{bmatrix} \overset{2}{1} & \overset{1}{-2} & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1.$

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> Col(A).

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The takeaway. The pivot columns of A form a basis for $\text{Col}(A)$.

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!! IMPORTANT !!

Choose the columns <u>of A</u>.

($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\mathrm{Col}(A)$)

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1. Find the pivot columns in an echelon form of A.

2. The associated columns <u>in A</u> form a basis for $\mathrm{Col}(A)$.

Example

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A.

Answer

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
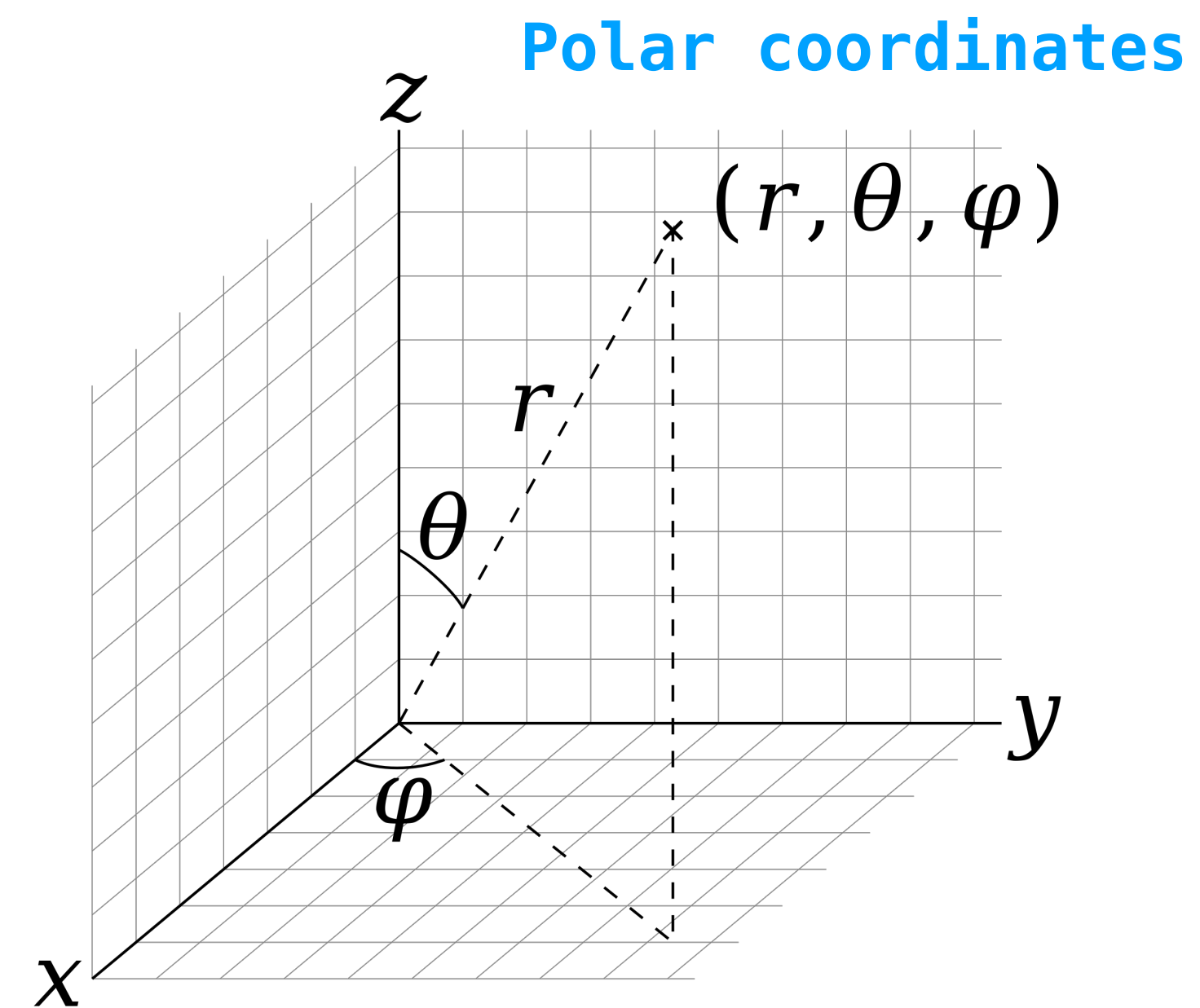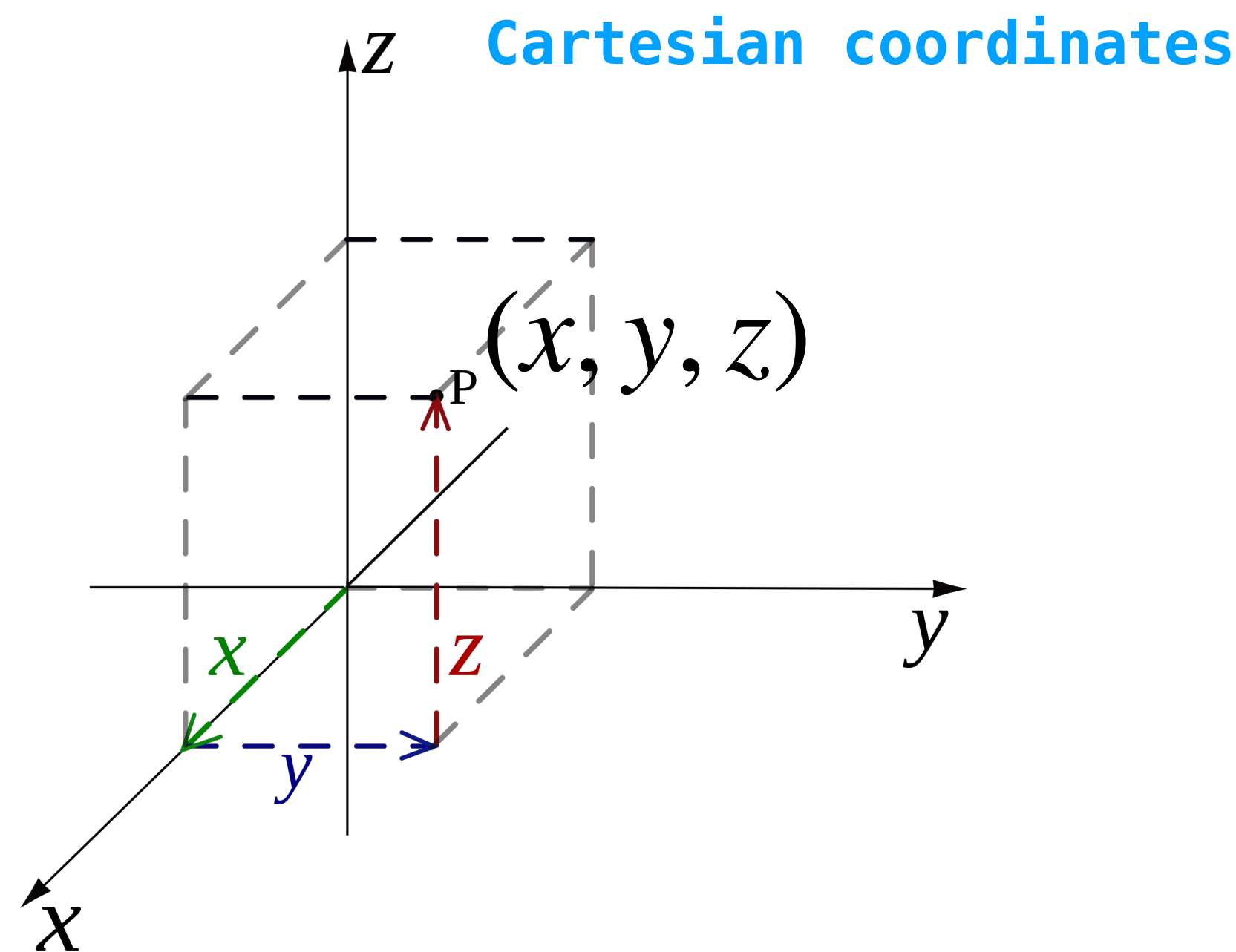
moving on...

Coordinate Systems

At a High Level

A coordinate system is a way of representing <u>positions</u> in terms of a <u>sequence of numbers</u>.

Examples.

Cartesian coordinates

(x, y, z)

Polar coordinates

(r, θ, φ)

Question (Conceptual)*

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This question is non-sensical.

It's just a sequence of numbers. We need to be *told* if it should be interpreted in the **polar** coordinate system or the **Cartesian** coordinate system.

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Verify:

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Every basis provides a way to write down *coordinates* of a vector.

And every time we write down a vector, we are **assuming a coordinate system**.

what do we mean by this?

A Thought Experiment

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Then one day, you get tired of talking about "abstract" vectors, you want to work with *numbers.*

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Because we've learned everything up to now, we know that there is a basis $\mathbf{b}_1$, $\mathbf{b}_2$,...., $\mathbf{b}_n$ for the space $\mathbb{R}^n$.

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + ... + (-0.1)\mathbf{b}_n \qquad\qquad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \\ -0.1 \end{bmatrix}$$

and then choose those weights as a representation of v as a sequence of numbers

But wait...

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This *depends* on the choice of basis.

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2, \ldots, \mathbf{c}_n$ then we would get some other representation.

$$\mathbf{v} = (-10)\mathbf{c}_1 + (4.3)\mathbf{c}_2 + \ldots + 0\mathbf{c}_n = \begin{bmatrix} -10 \\ 4.3 \\ \vdots \\ 0 \end{bmatrix}$$

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Every basis defined a different coordinate system

Standard Basis

The standard basis defines the Cartesian coordinate system for $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$
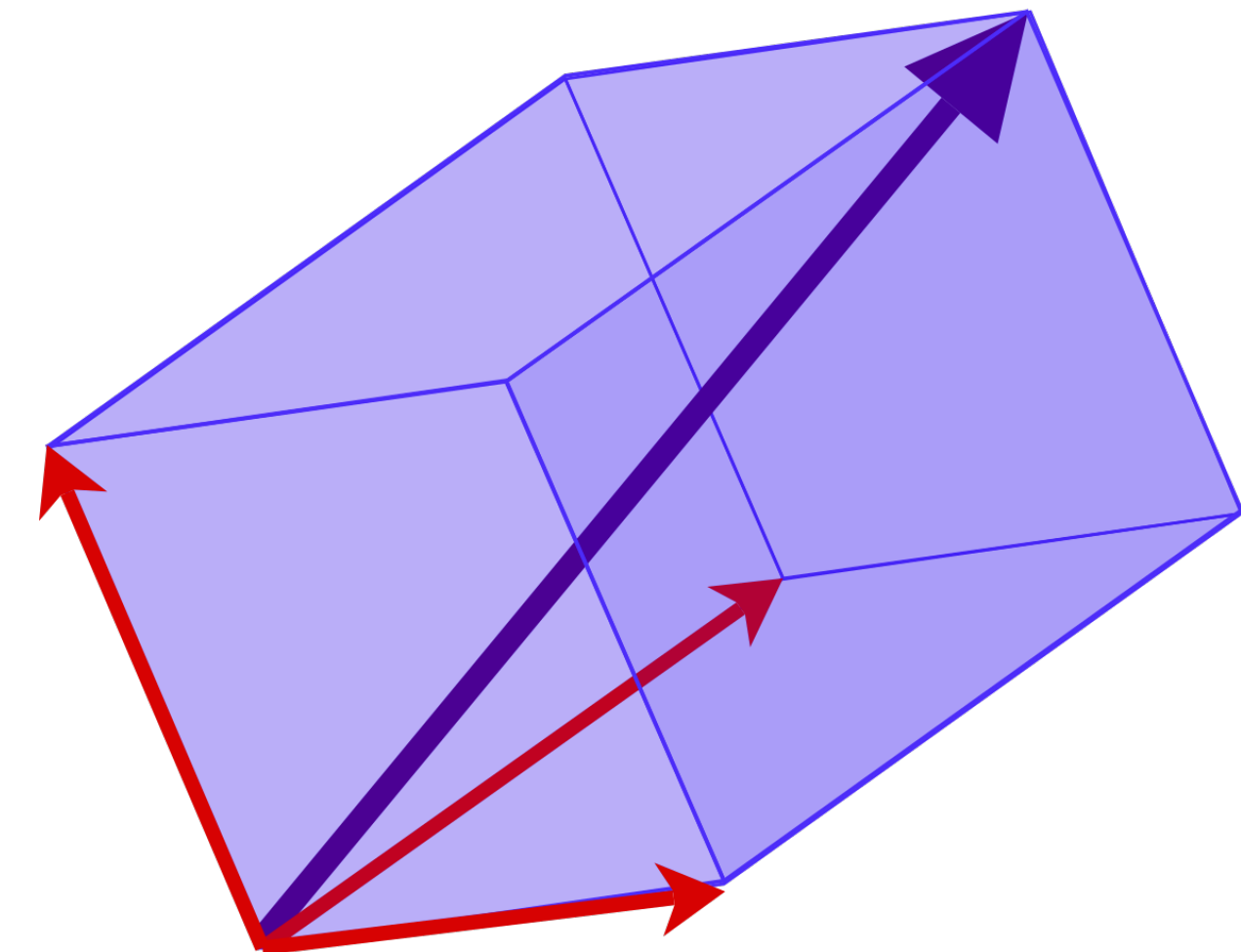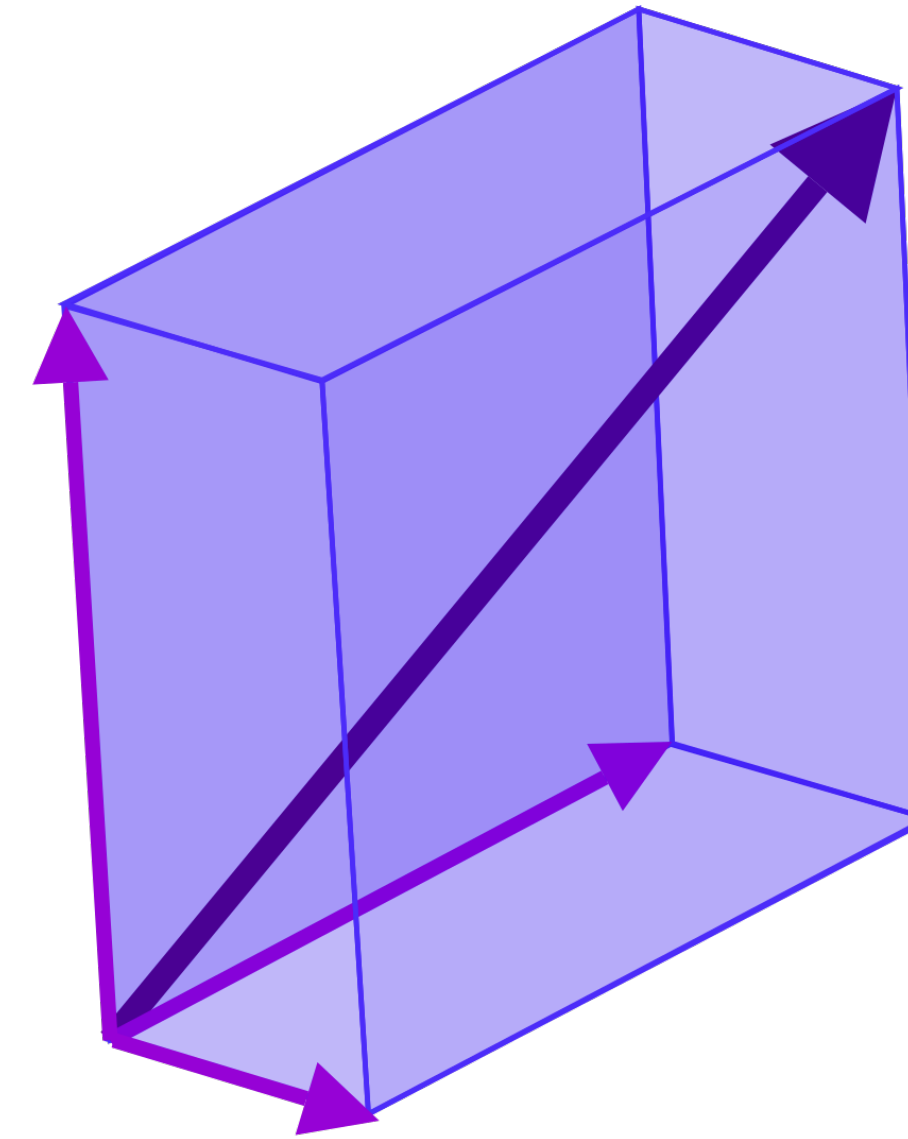
Column vectors are just weights for a linear combination of the standard basis

but we can also use
different coordinate systems

How to think about this

Changing the coordinate system "warps space".

The question is: how do we represent a vector $\mathbf{v}$ in the warped space if we wanted it to "be in the same place"?

Coordinate Vectors

Coordinate Vectors

Let $\mathbf{v}$ be a vector in a subspace H of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis of H where

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_k\mathbf{b}_k$$

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Definition. The **coordinate vector of $\mathbf{v}$ relative to** $\mathscr{B}$ is

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

Coordinate Vectors and the Standard Basis

When we write down a vector $\mathbf{v}$ in $\mathbb{R}^n$, we're really writing down a coordinate vector **relative to the standard basis** $\mathscr{E}$.

$$[\mathbf{v}]_{\mathscr{E}} = \mathbf{v}$$

How do we find coordinate vectors?

For an arbitrary basis $\mathscr{B}$, to determine $[\mathbf{v}]_{\mathscr{B}}$, we need to find weights $a_1, \ldots, a_k$ such that
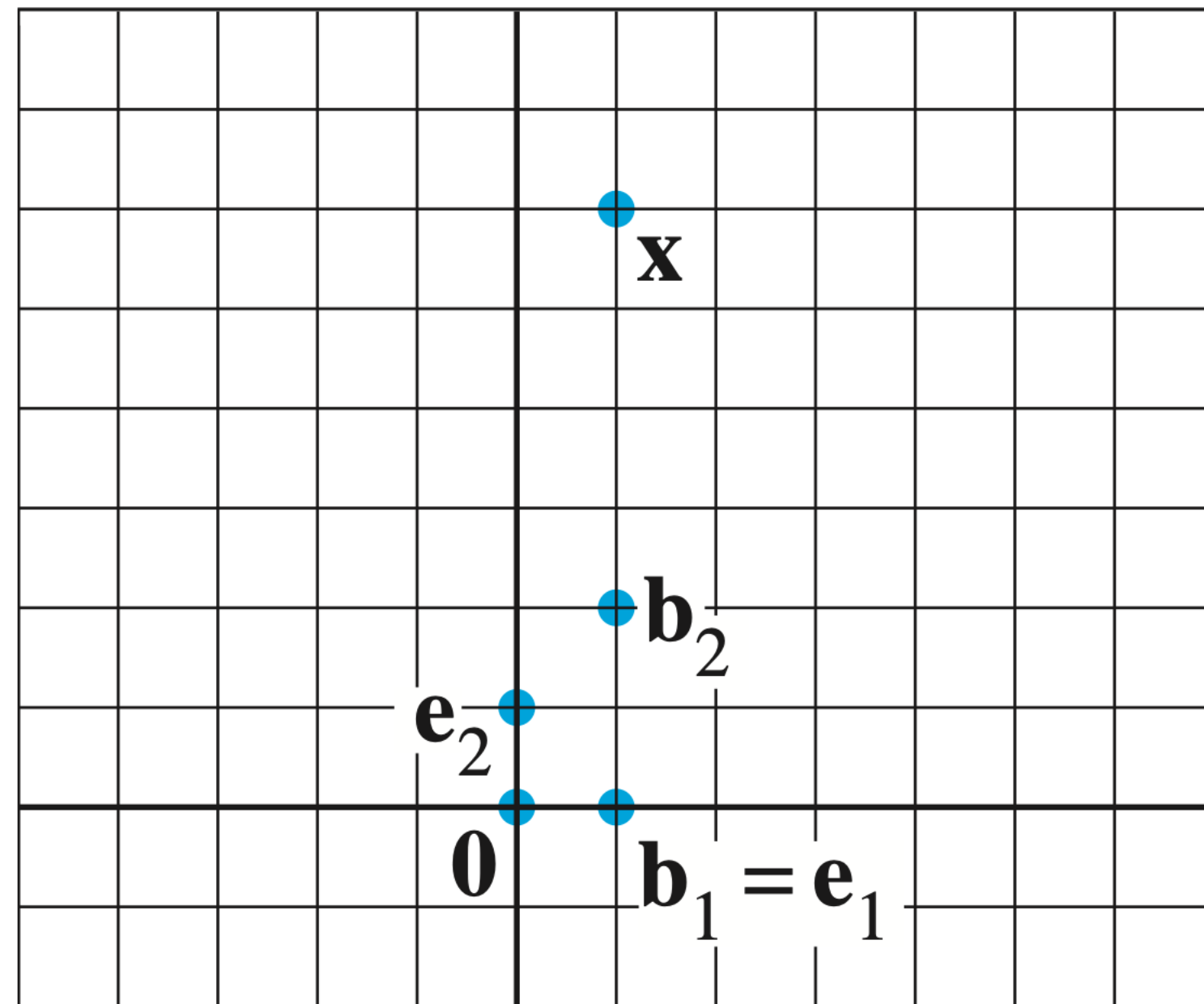
$$a_1\mathbf{b}_1 + \ldots + a_k\mathbf{b}_k = \mathbf{v}$$
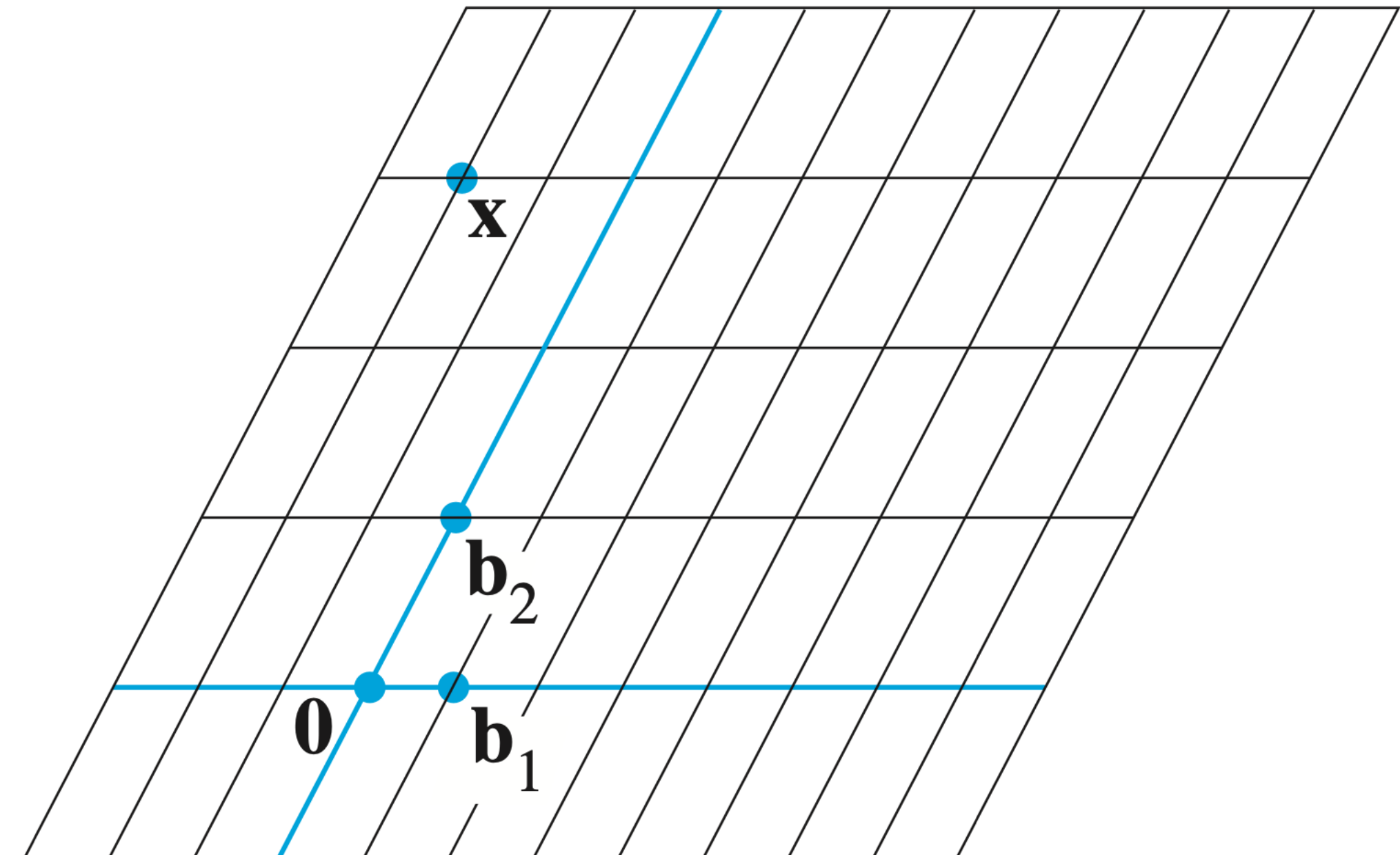
This is just <u>solving a vector equation</u>.

Example: 2D Case

Write the coordinate vector for $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ relative to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

Example: 2D Case (Geometrically)

FIGURE 1 Standard graph paper.

FIGURE 2 $\mathcal{B}$-graph paper.

$\mathcal{B}$ defines a "different grid for our graph paper"

How To: Coordinate Vectors

Question. Find the coordinate vector for $\mathbf{v}$ in the subspace H relative to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$.

Solution. Solve the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_k\mathbf{b}_k = \mathbf{v}$$
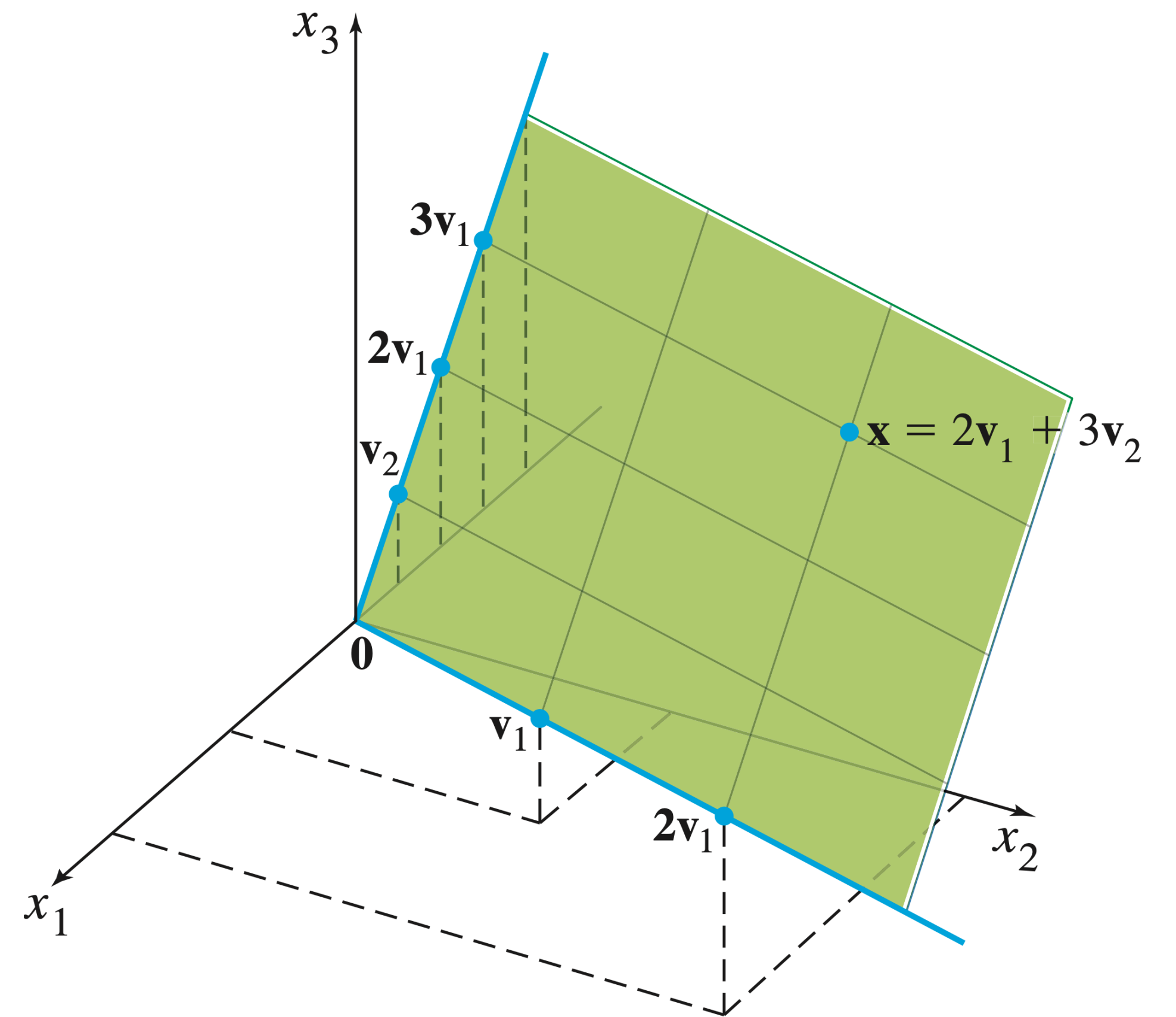
A solution $(a_1, \ldots, a_k)$ means

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

Example: 3D Case

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

Find the coordinate vector for $\mathbf{u}$ relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of a subspace H (of $\mathbb{R}^3$):
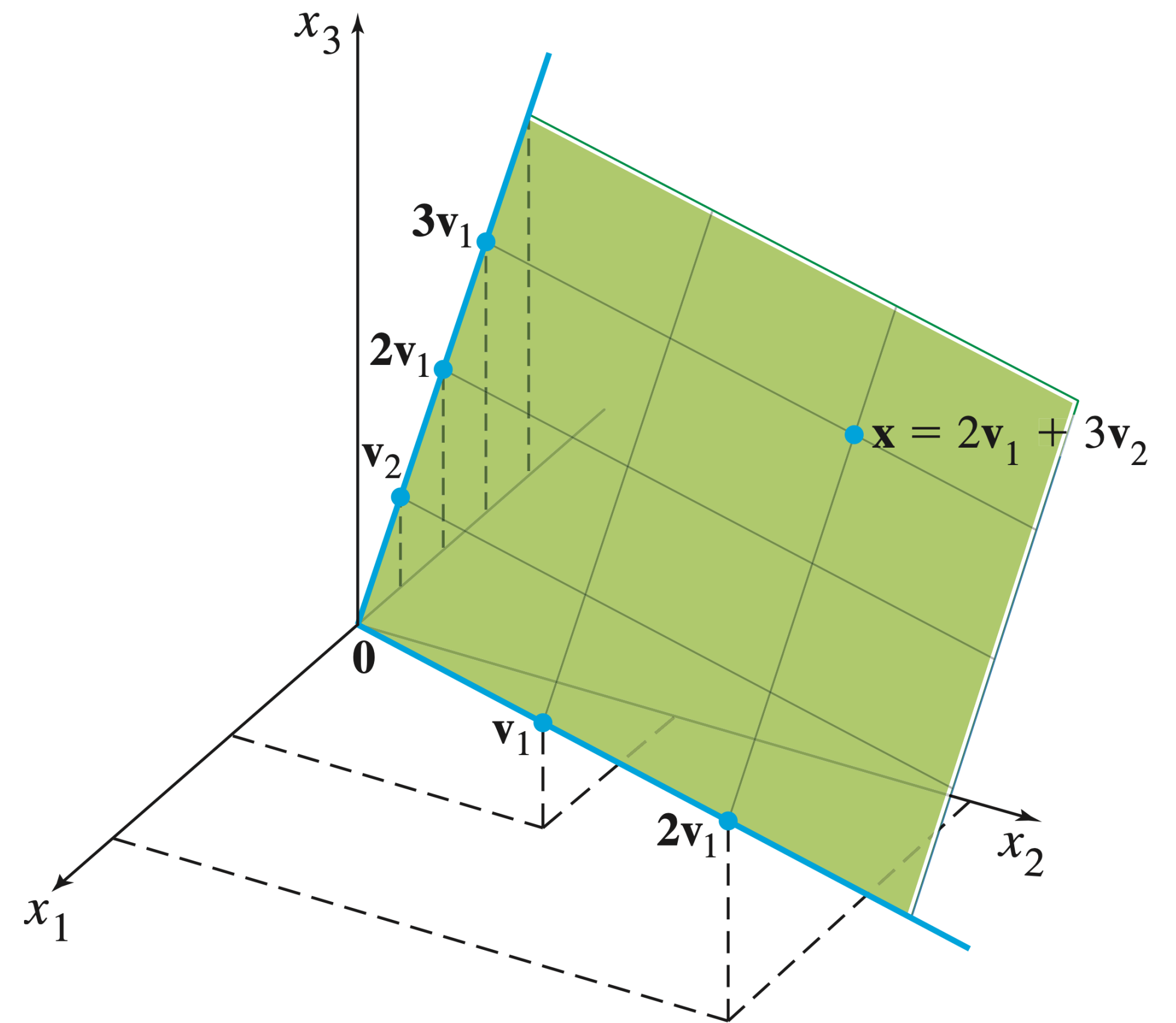
An Aside: Coordinates and one-to-one correspondences

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In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a <u>one-to-one correspondence</u> from H to $\mathbb{R}^2$. This is also sometimes called an **isomorphism**.

An Aside: Coordinates and one-to-one correspondences

In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a <u>one-to-one correspondence</u> from H to $\mathbb{R}^2$. This is also sometimes called an **isomorphism**.

Isomorphic things "look and behave the same up to simple transformations."

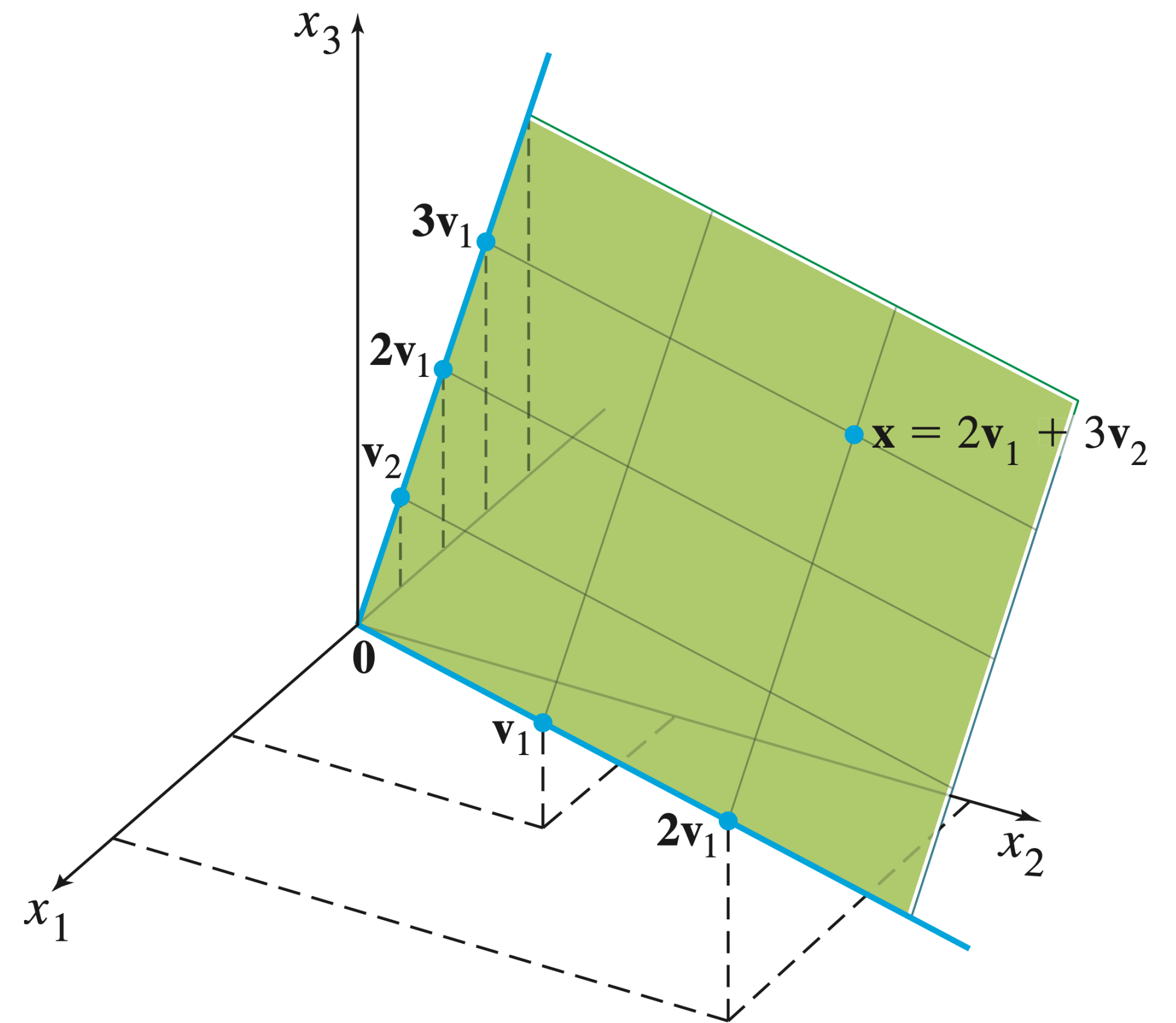# An Aside: Coordinates and one-to-one correspondences

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An Aside: Coordinates and one-to-one correspondences

In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a <u>one–to–one correspondence</u> from H to $\mathbb{R}^2$. This is also sometimes called an **isomorphism**.

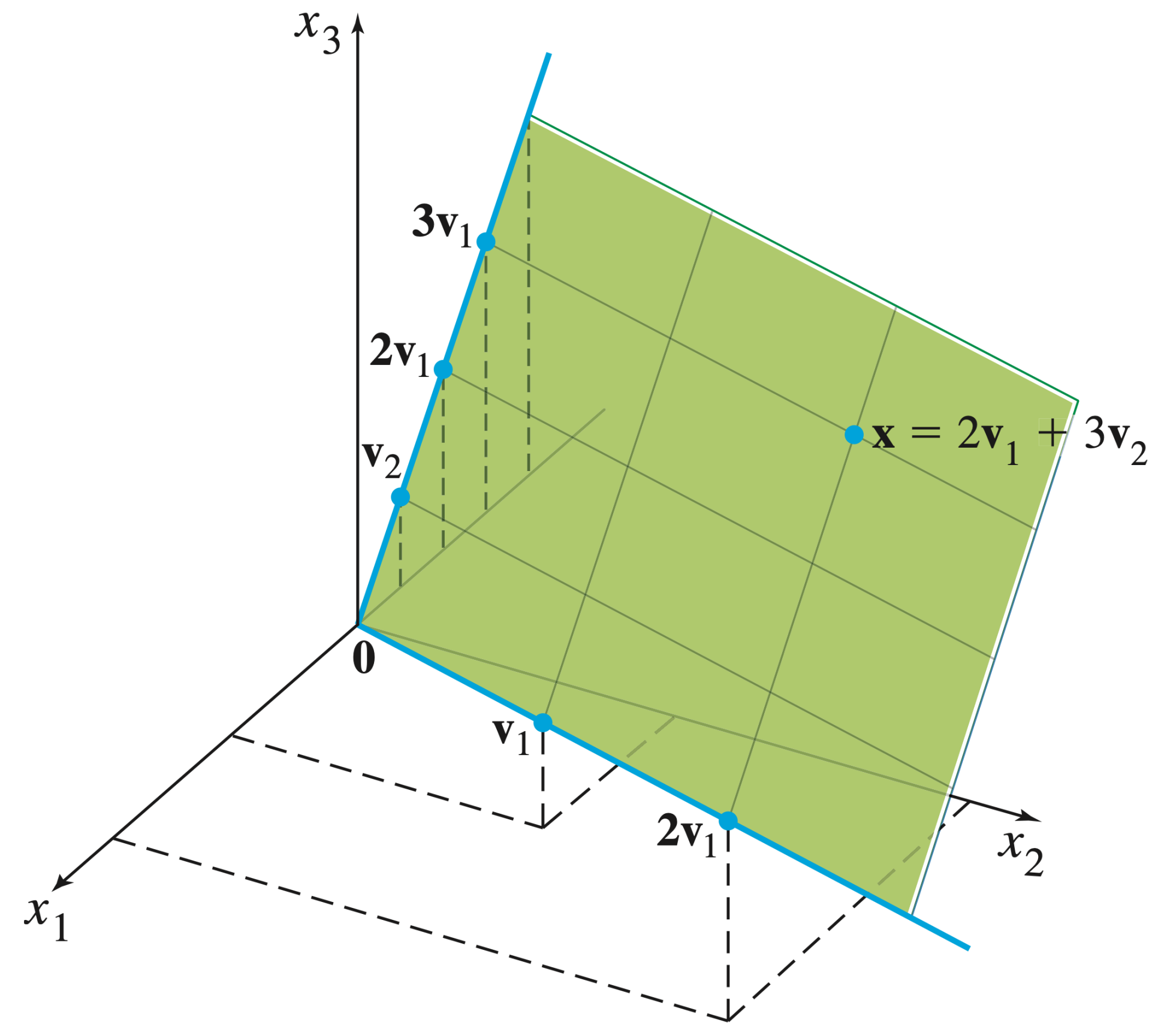Isomorphic things "look and behave the same up to simple transformations."

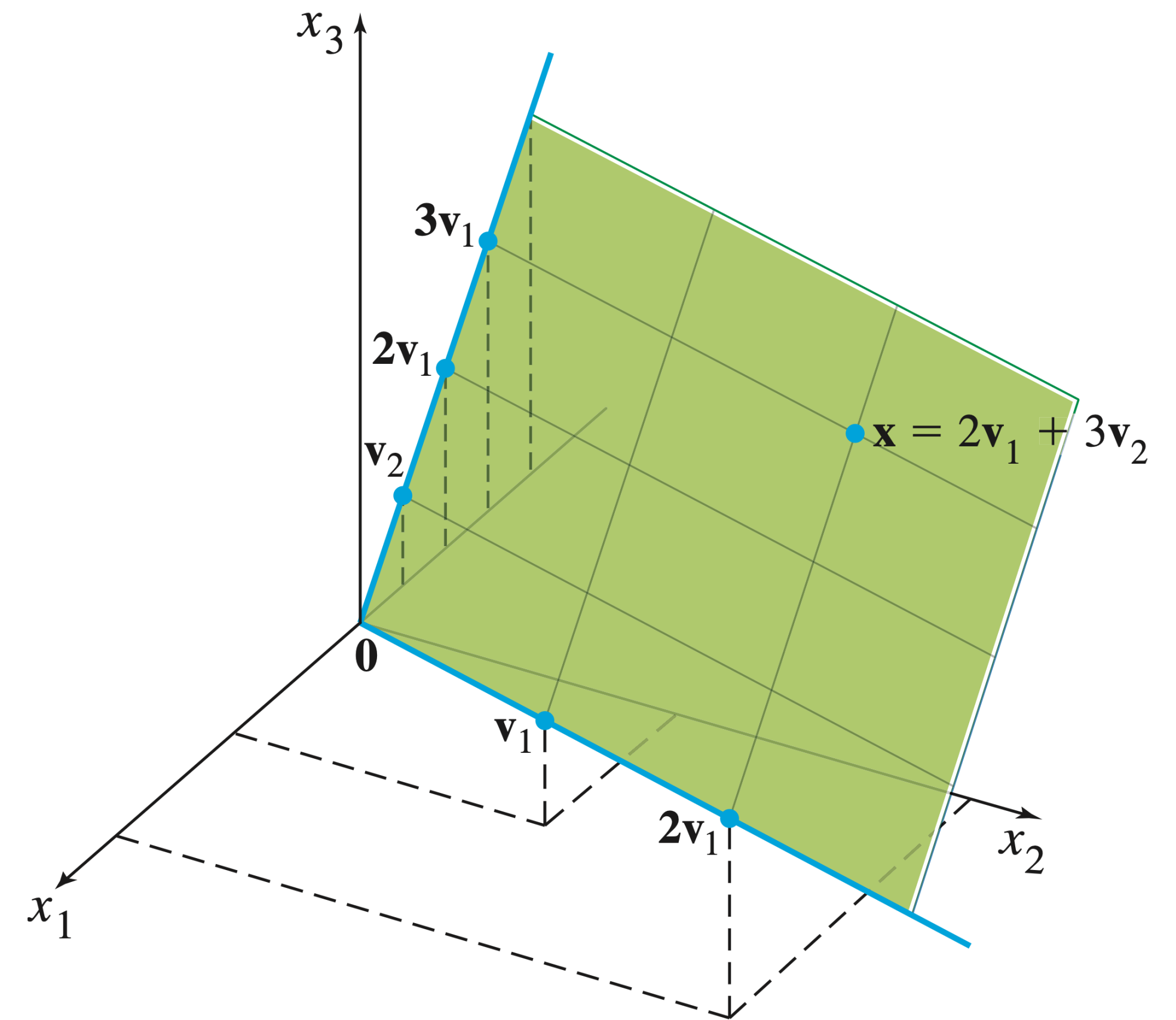So $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to $\mathbb{R}^2$.

This is a formal way of saying that $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a "copy of $\mathbb{R}^2$."

Question

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Suppose $[\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, *where* $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. *Find* $\mathbf{u}$.

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad [\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Dimension and Rank

The Idea of Dimension

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Theorem. Every basis of a subspace H has <u>exactly the same number of vectors</u>.

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This number is a measure of how "large" H is.

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<u>This should confirm our intuitions:</u>

» a plane (through the origin) is a 2D subspace

» a line (through the origin) is a 1D subspace

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin

2. lines (through the origin)

3. planes (through the origin)

4. All of $\mathbb{R}^3$

$\mathrm{span}\{\mathbf{u}, \mathbf{v}\}$

$\mathrm{span}\{\mathbf{v}\}$

$\{\mathbf{0}\}$

$\mathbb{R}^3 = \mathrm{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
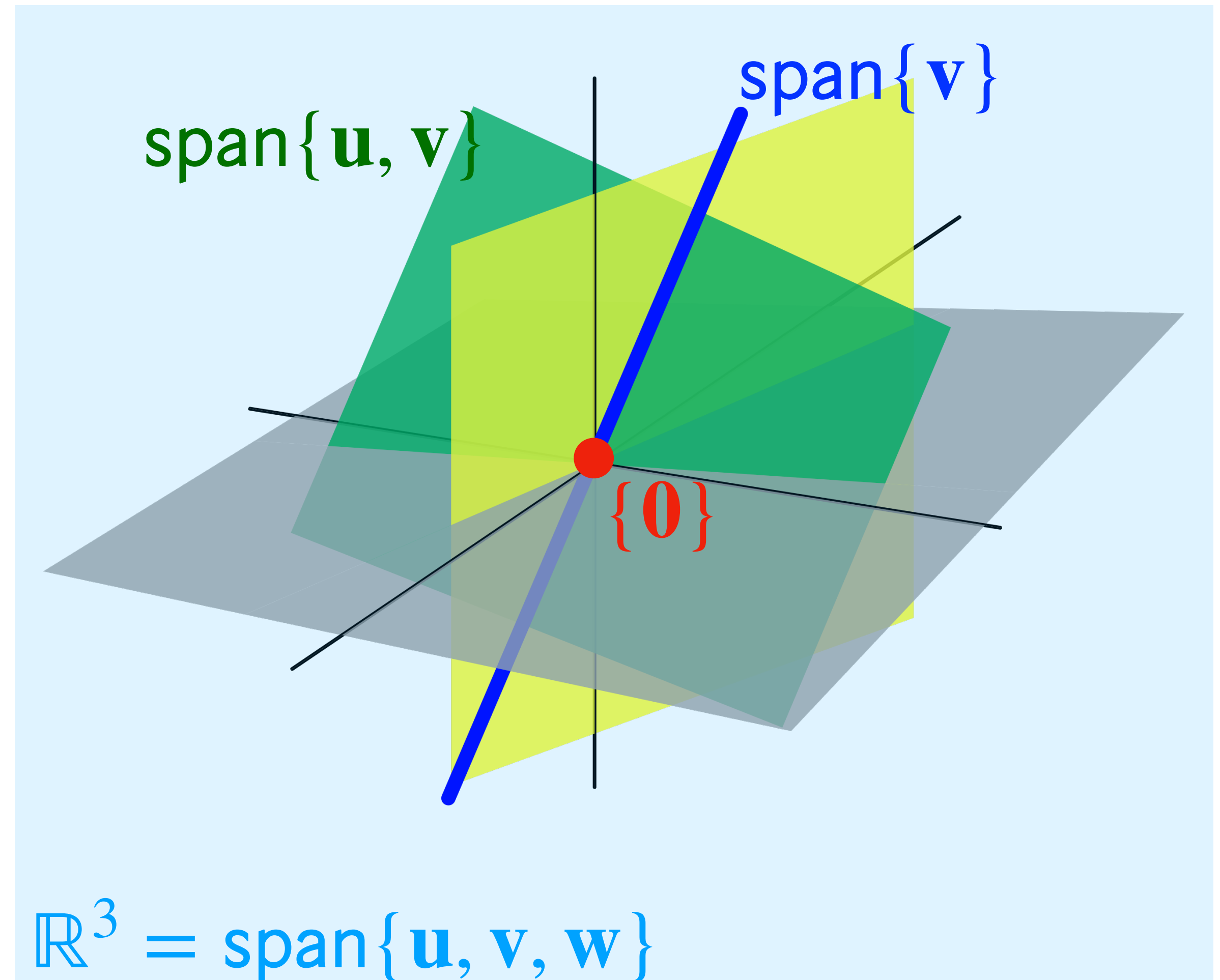
Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. 0-dimensional subspace

2. 1-dimensional subspaces

3. 2-dimensional subspaces

4. 3-dimensional subspace

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$\mathbb{R}^3 = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
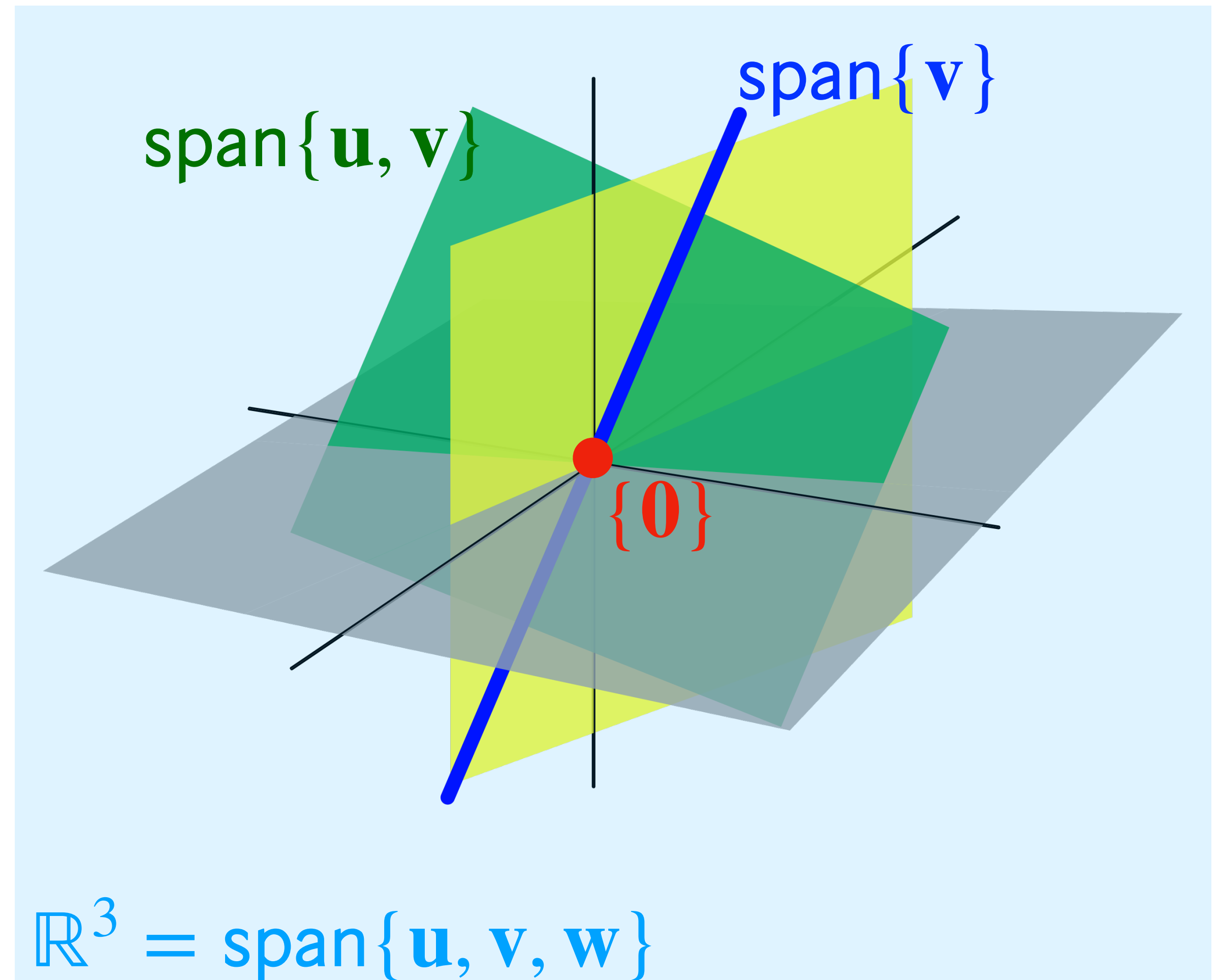
How does this connect to null space and column space?

Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of <u>free variables</u> in a general form solution.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

$$\equiv$$

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Dimension of the Null Space

The **dimension of** $\text{Nul}(A)$ is the number of <u>free variables</u> in a general form solution to $A\mathbf{x} = \mathbf{0}.$

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Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of <u>basic variable</u> or equivalently the number of <u>pivot columns</u>.

$$[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4 \ \ \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Dimension of the Column Space

The **dimension of** $\text{Col}(A)$ is the number of <u>basic variable</u> in our solution, or equivalently the number of <u>pivot columns</u> of A.

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Rank

Definition. The **rank** of a matrix A, written rank(A) or rank A, is the dimension of Col(A).

This is just terminology.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A,

$$\text{rank}(A) + \text{dim}(\text{Nul}(A)) = n$$

Verify:

This is incredibly important.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A,

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Example. If a "line's worth of stuff" is pulled into the null space (and mapped to $\mathbf{0}$) then

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For a $m \times n$ matrix A, its columns space $\mathrm{Col}(A)$ *could* have n dimensions.

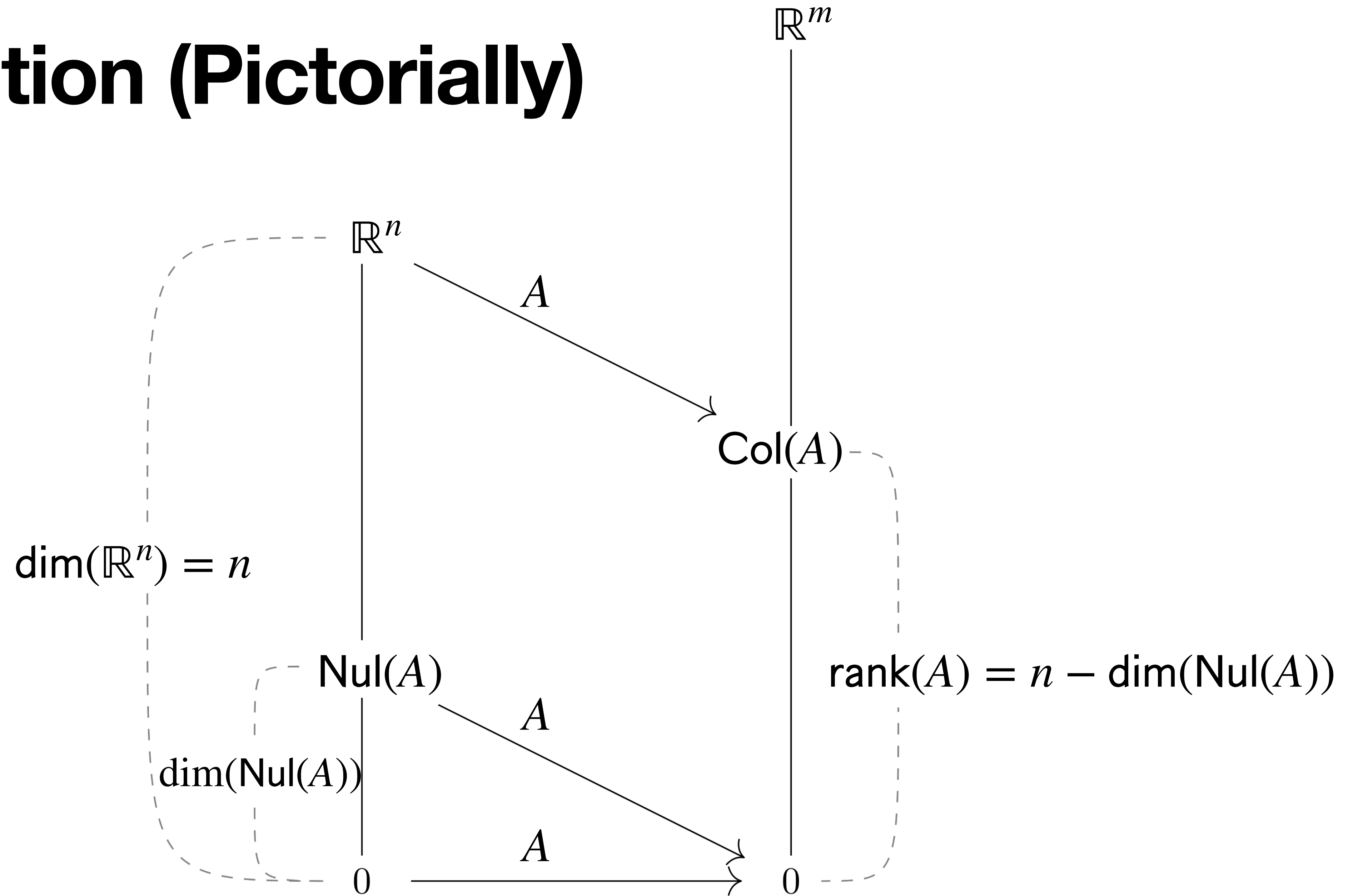In this case: $\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n + 0 = n$

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Example. If a "line's worth of stuff" is pulled into the null space (and mapped to $\mathbf{0}$) then

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = (n-1) + 1 = n$$

The null space "takes away" some of the dimensions of the column space.

The Intuition (Pictorially)

$$\mathbb{R}^m$$

$$\mathbb{R}^n$$

$$A$$

$$\text{Col}(A)$$

$$\dim(\mathbb{R}^n) = n$$

$$\text{Nul}(A)$$

$$\text{rank}(A) = n - \dim(\text{Nul}(A))$$

$$A$$

$$\dim(\text{Nul}(A))$$

$$A$$

$$0$$

$$A$$

$$0$$

Question (Conceptual)

Let A be a 5×7 *matrix such that* $\dim(\operatorname{Nul}(A)) = 3$*.*
What is the dimension of $\operatorname{Col}(A)$*?*

Answer: 4

Extending the IMT

Theorem. For an $n \times n$ invertible matrix A, the following are logically equivalent (they must all by true or all by false.

- » $\text{Col}(A) = \mathbb{R}^n$
- » $\dim(\text{Col}(A)) = n$
- » $\text{rank}(A) = n$
- » $\text{Nul}(A) = \{\mathbf{0}\}$
- » $\dim(\text{Nul}(A)) = 0$

Summary

We can find bases for the column space and null space by looking at the reduced echelon form of a matrix.

Column vectors are written in terms of a coordinate system, which we can change.

Dimension is a measure of how large a space is.