# Eigenvalues and Eigenvectors

**Geometric Algorithms**

**Lecture 18**

CAS CS 132

# Practice Problem

*Suppose* $A$ *is a* $234 \times 300$ *matrix. What is the smallest possible value for* $\dim(\text{Nul}(A))$*? What is the largest possible value?*

*What is the smallest possible value for* $\text{rank}(A)$*? What is the largest possible value?*

# Answer

$A \in \mathbb{R}^{234 \times 300}$ ($m \times n$)

\# pivots in A = rank(A) + dim(Nul(A)) = 300

$\leq 234$

$66 \leq \dim(\text{Nul}(A)) \leq 300$

$0 \leq \text{rank}(A) \leq 234$

$\dim(\text{Nul}(A)) \geq 300 - 234 = 66$

consider $T(\vec{v}) = \vec{0}$ implemented by $\vec{0} \in \mathbb{R}^{234 \times 300}$

# Objectives

1. Motivate and introduce the fundamental notion of eigenvalues and eigenvectors
2. Determine how to verify eigenvalues and eigenvectors
3. Look at the subspace generated by eigenvectors
4. Apply the study of eigenvectors to dynamical linear systems

# Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

# Motivation

demo

# How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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# How can matrices transform vectors?*

In 2D and 3D we've seen:

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» projections
» shearing
» reflection
» scaling/stretching
» ... **Today's focus**

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We don't need a whole matrix to do scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

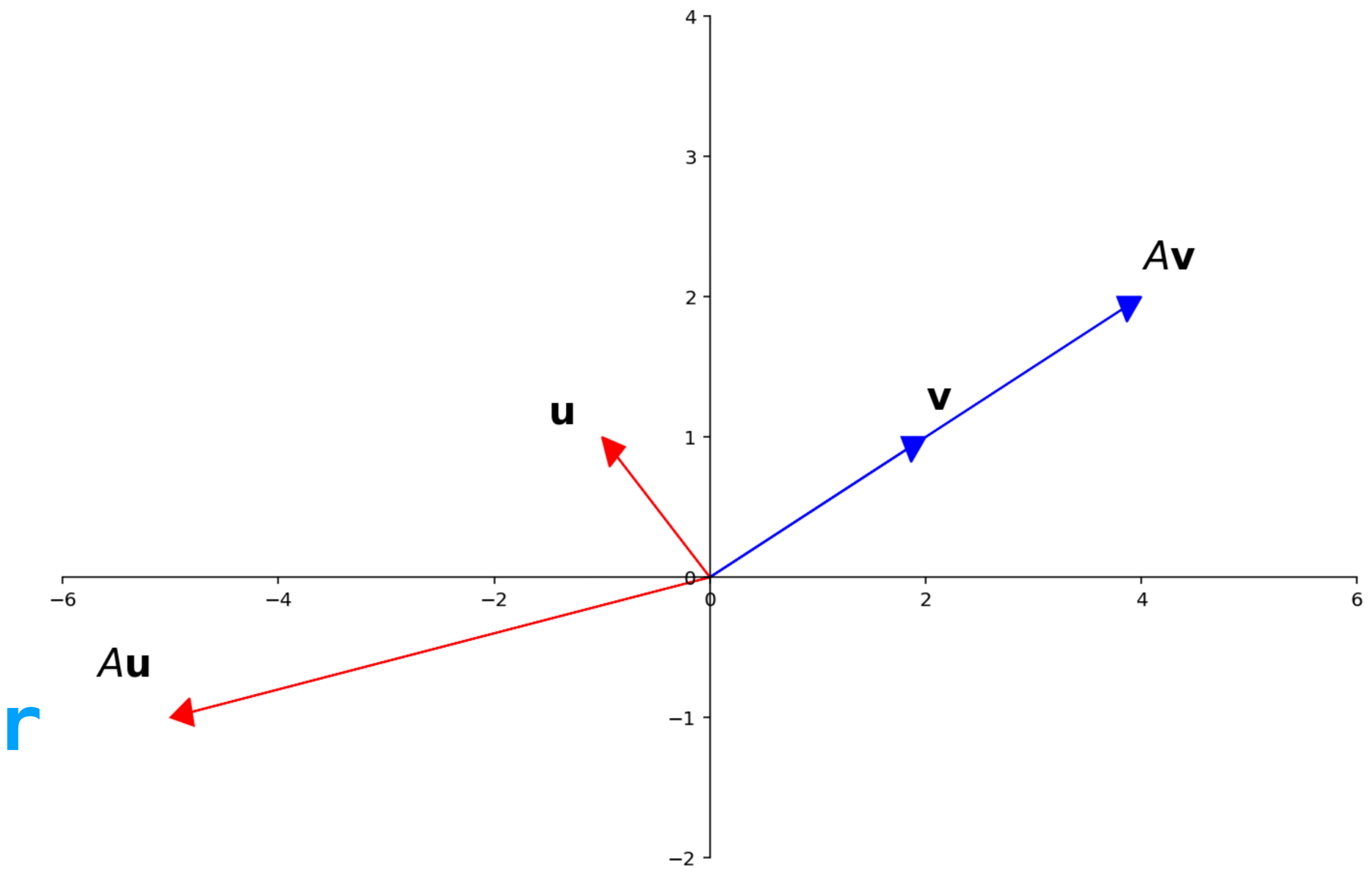
So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what $A$ does to $\mathbf{v}$.

# Eigenvectors (Informal)

eigenvalue

$$A\mathbf{v} = \lambda \mathbf{v}$$

eigenvector



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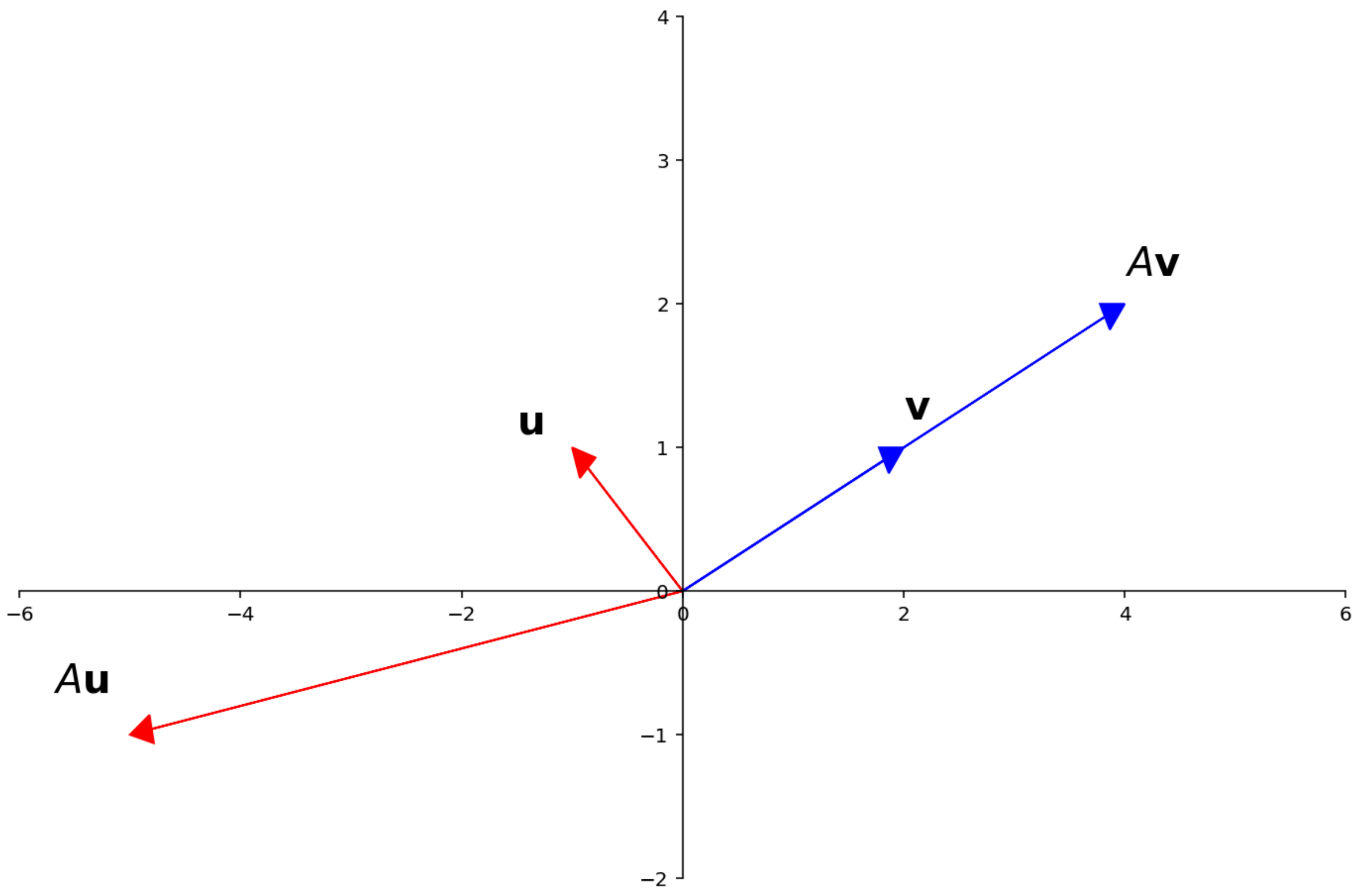


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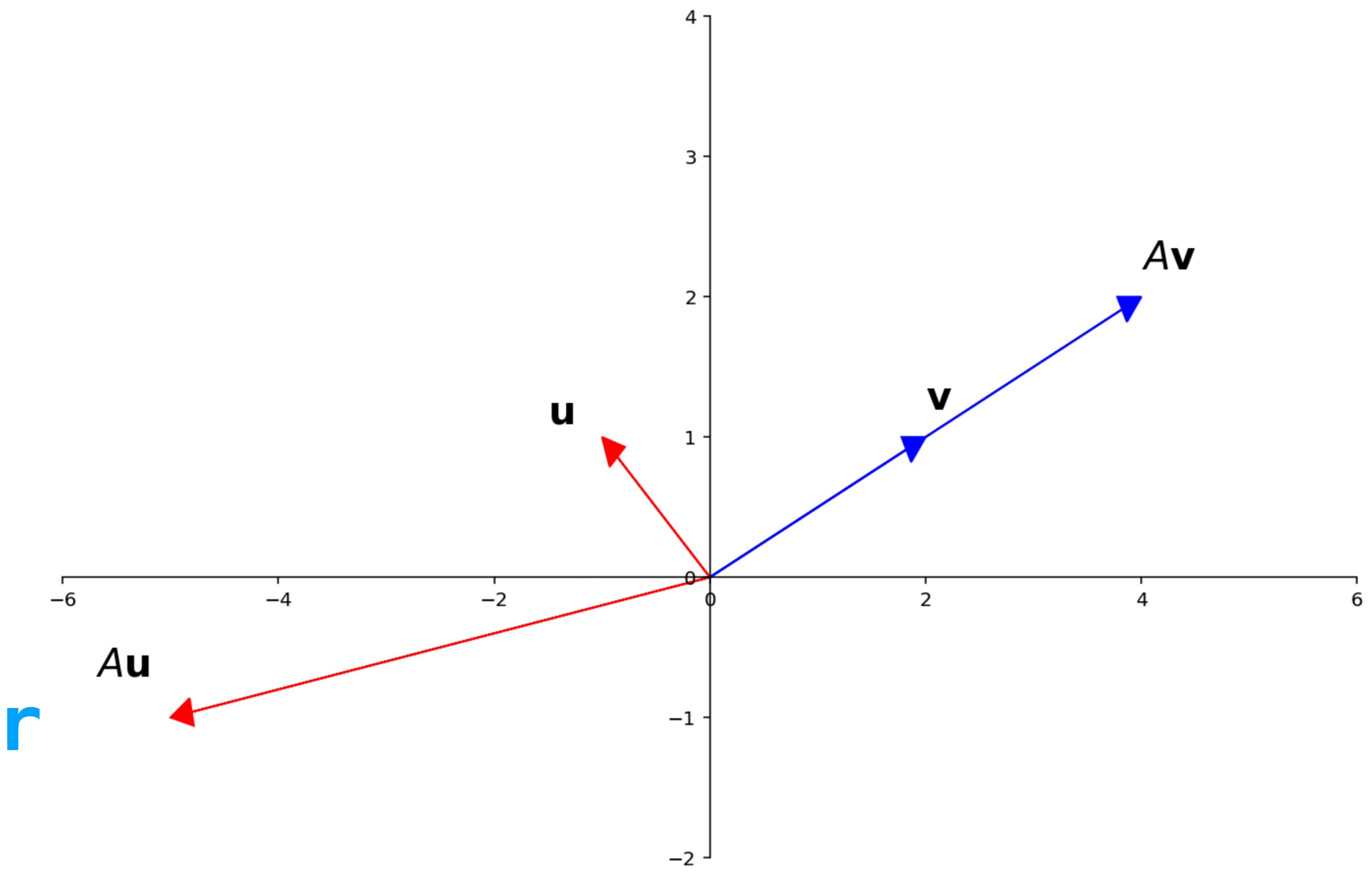



Eigenvectors of $A$ are stretched by $A$ without changing their direction.

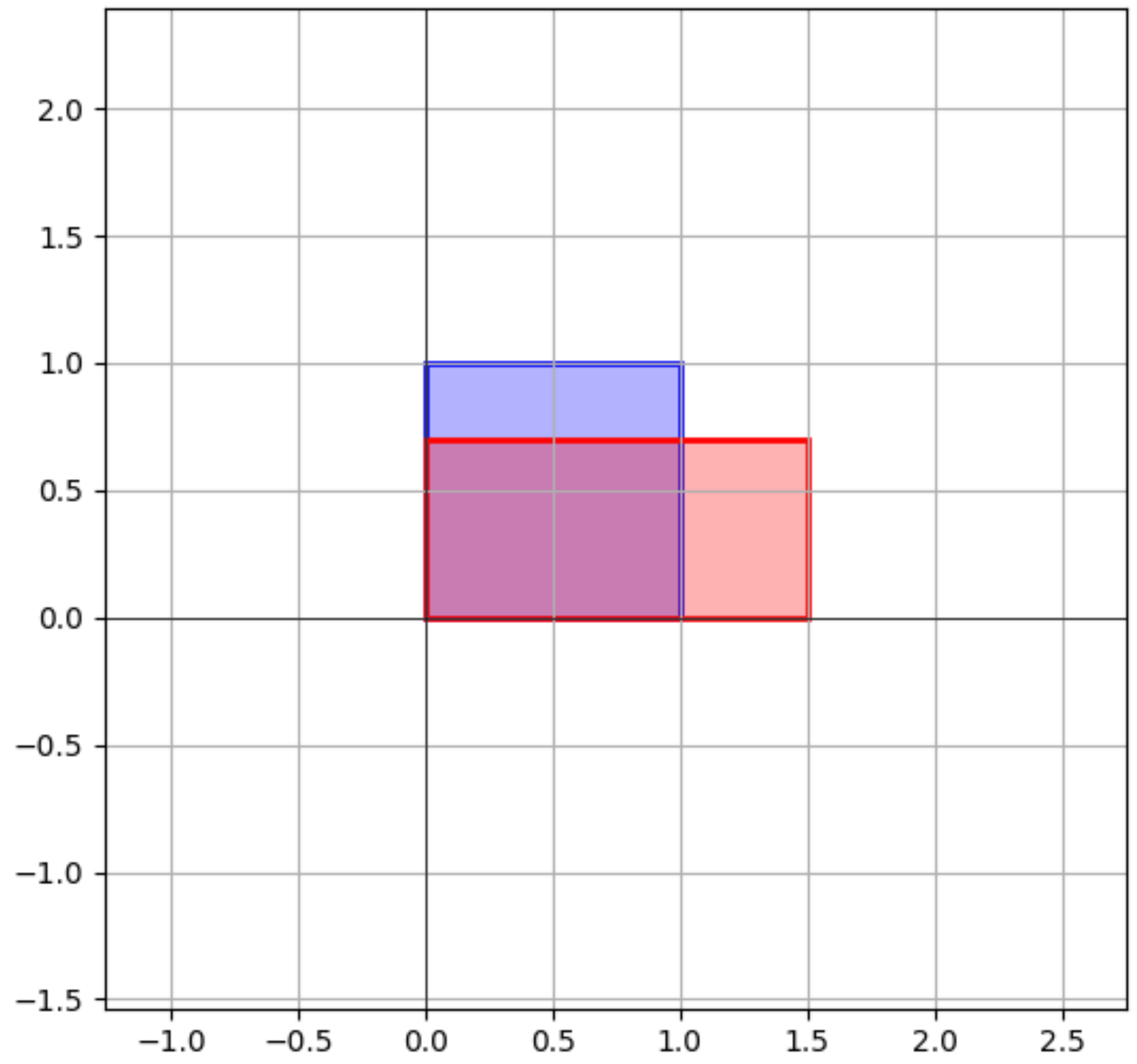
The amount they are stretched is called the **eigenvalue.**

# Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

*It transforms each entry individually and then combines them.*

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of* $A$. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

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$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

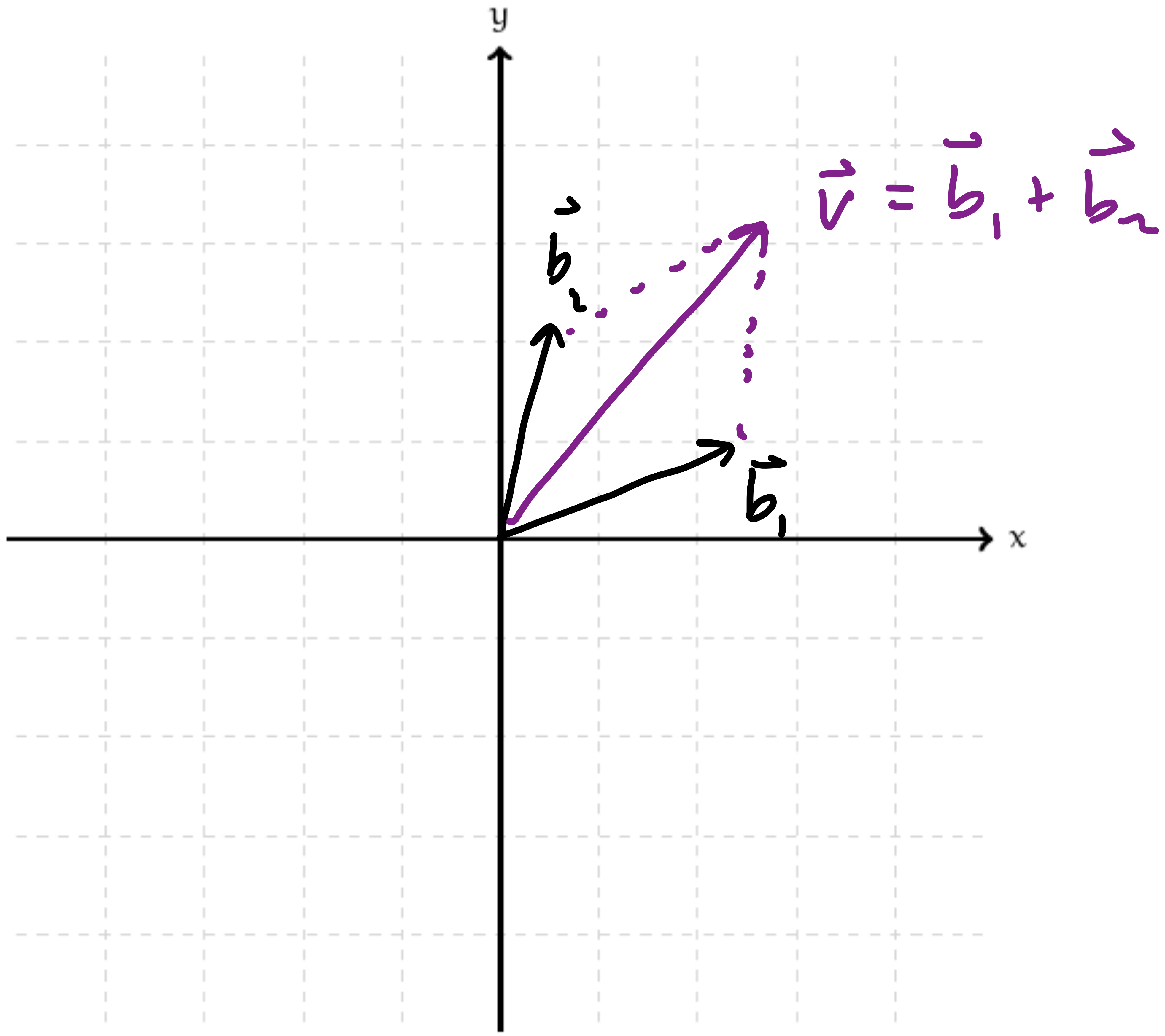
It's "easy to describe" how $A$ transforms $\mathbf{v}$.

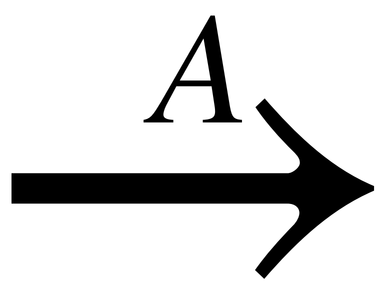
It *transforms each* "*component*" *individually and then combines them.*

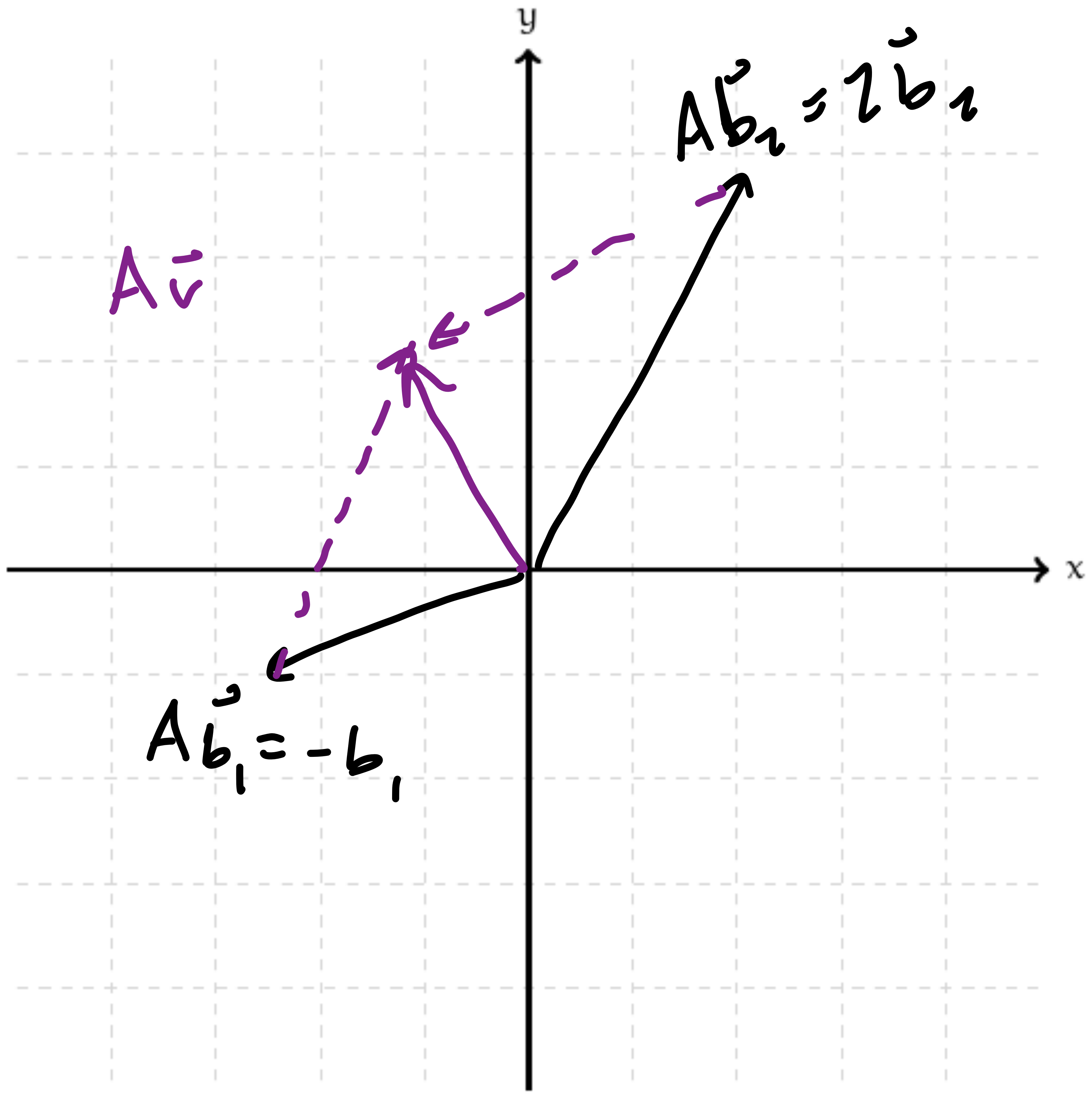
Verify: $A\vec{v} = A(2\vec{b}_1 - \vec{b}_2 - 5\vec{b}_3) = 2A\vec{b}_1 - A\vec{b}_2 - 5A\vec{b}_3$

$= 2\lambda_1\vec{b}_1 - \lambda_2\vec{b}_2 - 5\lambda_3\vec{b}_3$

# Eigenbases (Pictorially)

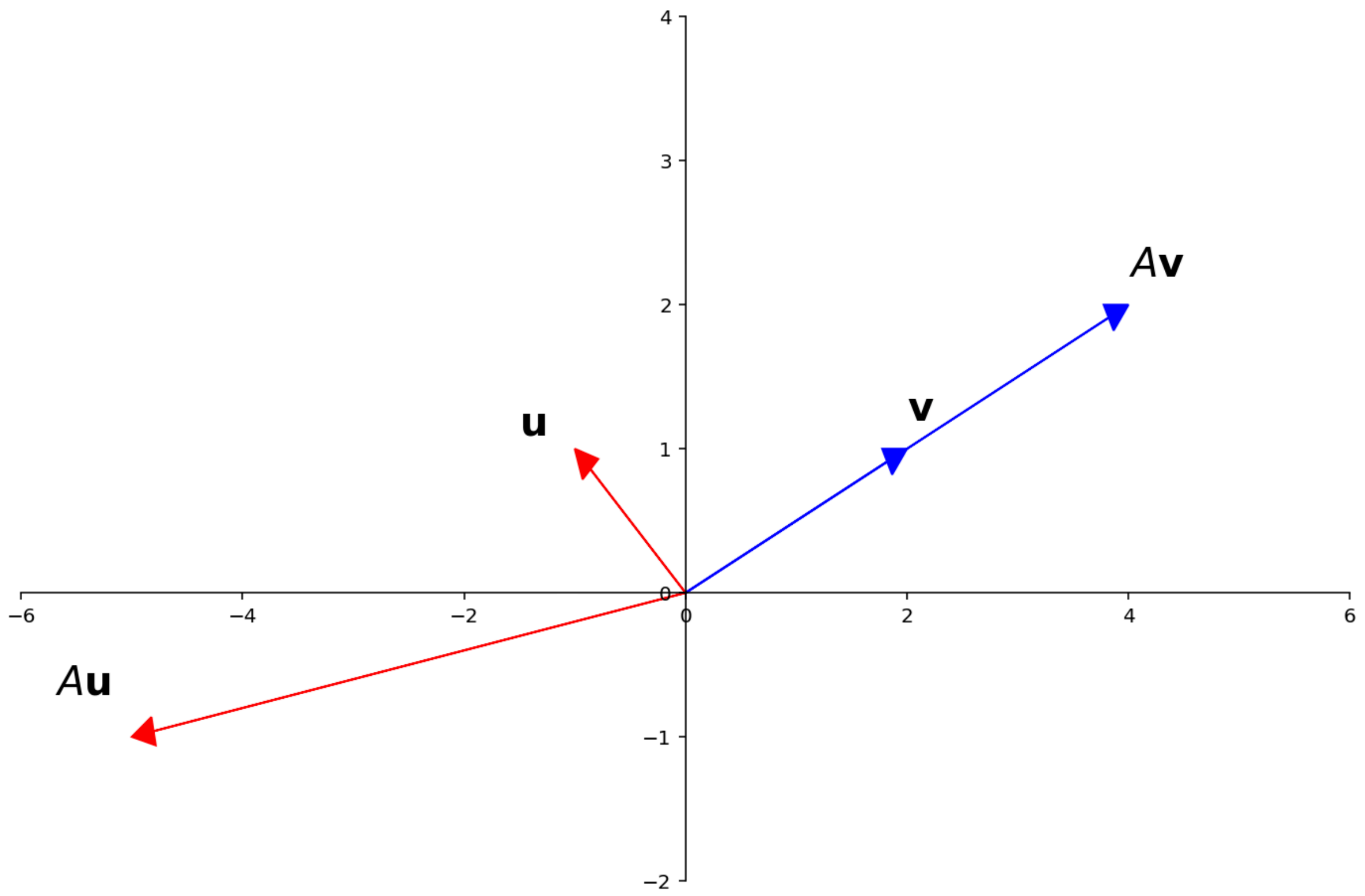


# Eigenvalues and Eigenvectors

# Formal Definition



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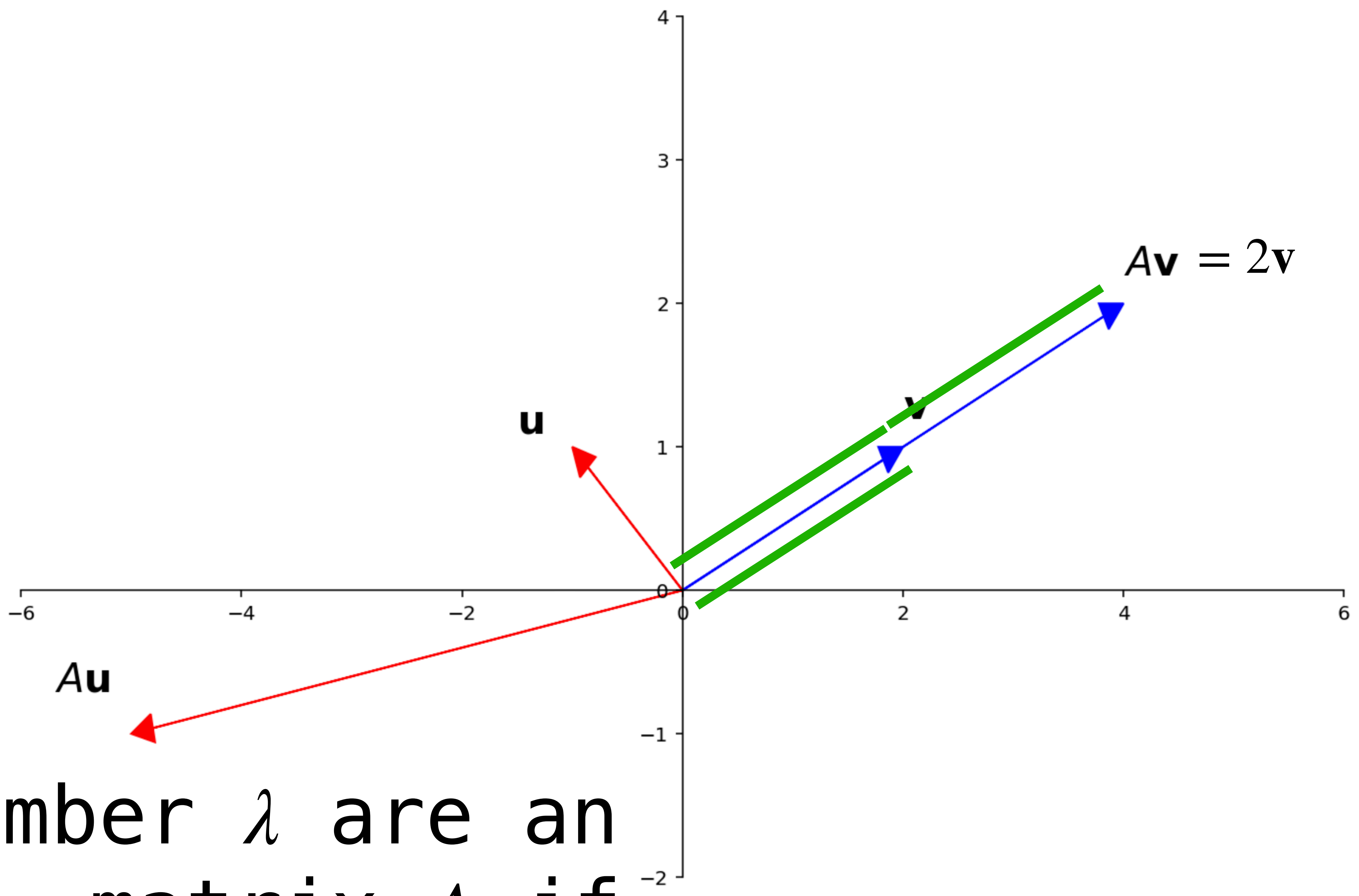


A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number $\lambda$ are an **eigenvector and eigenvalue** for a $n \times n$ matrix $A$ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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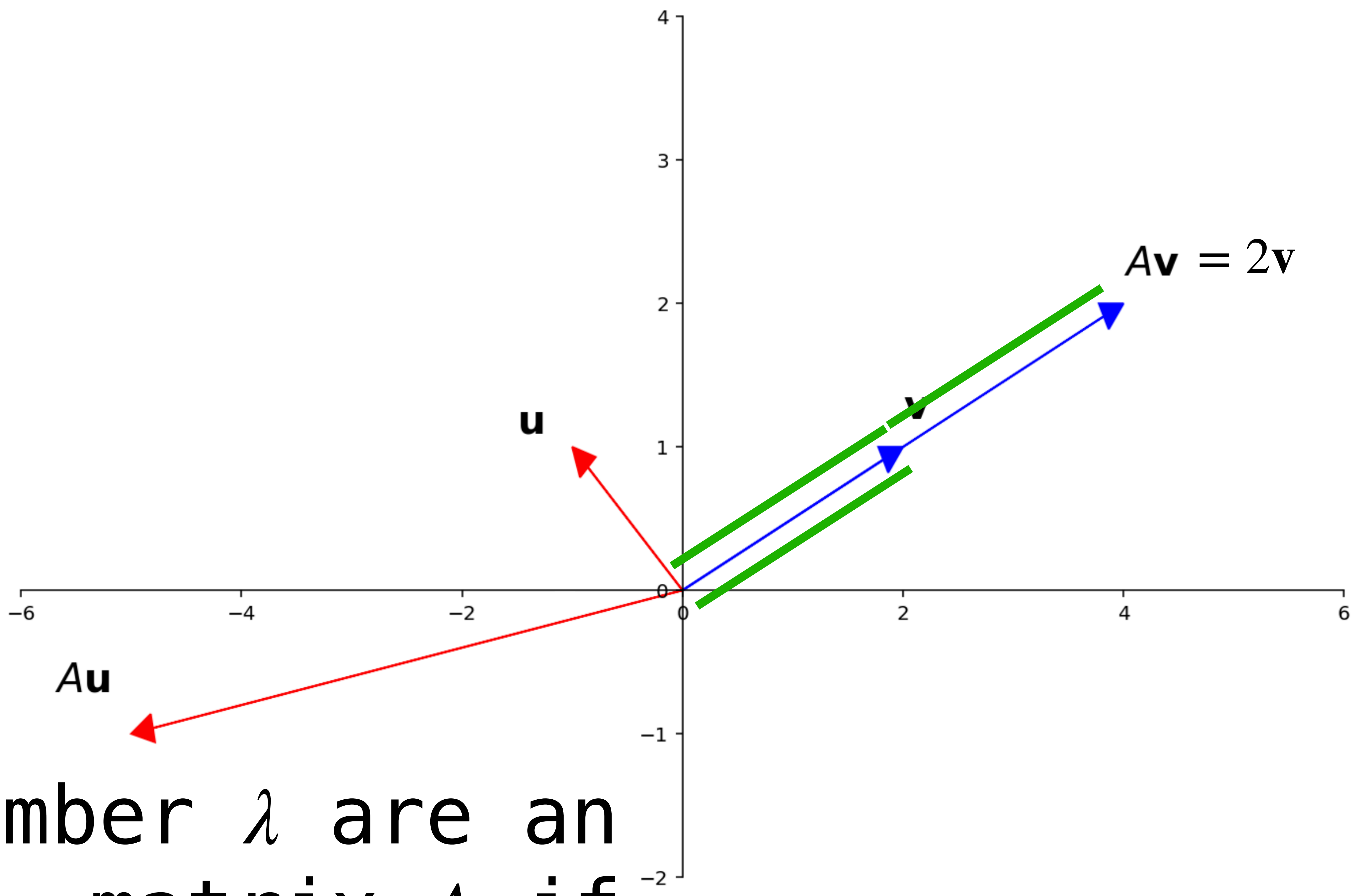


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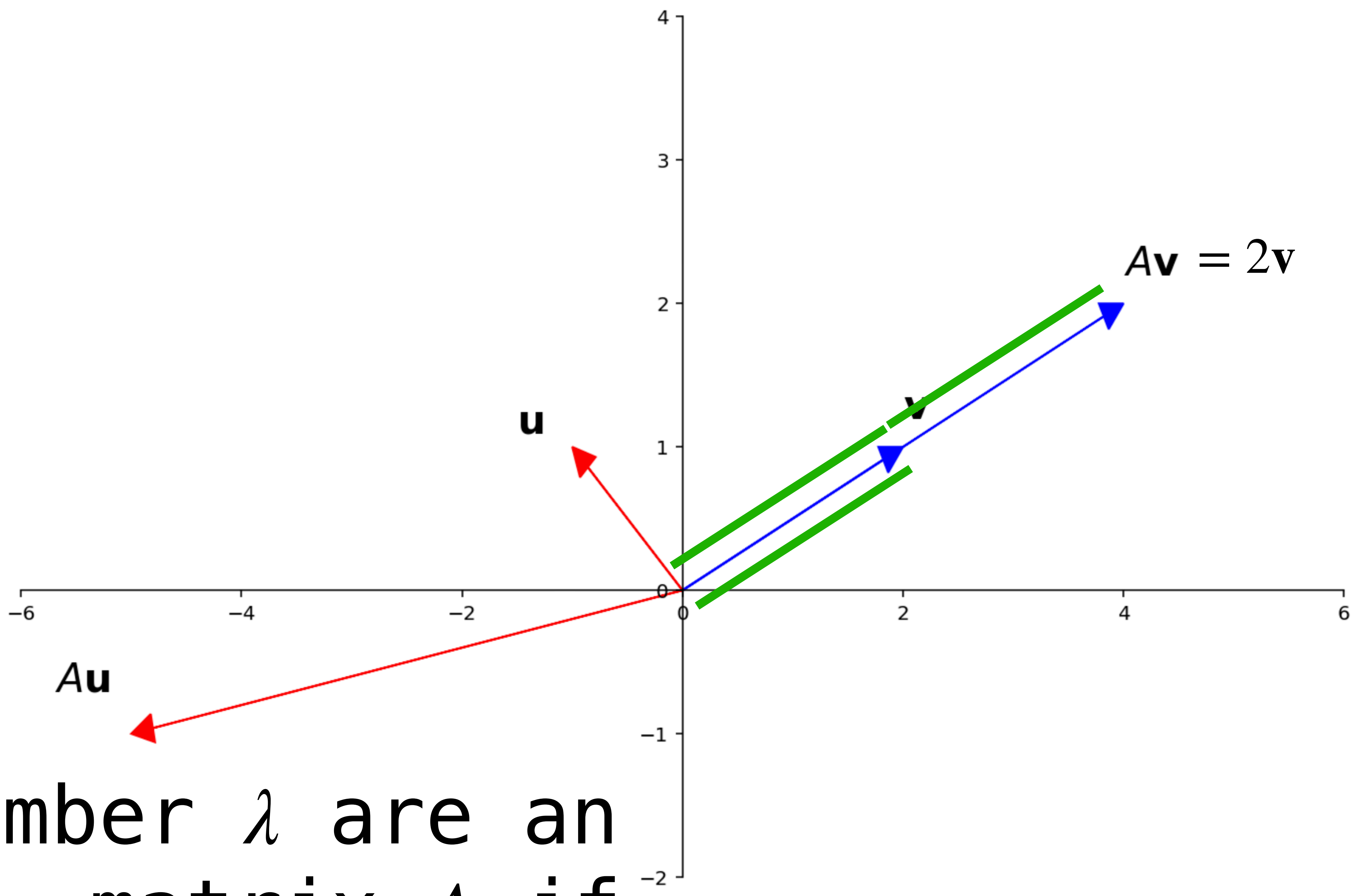


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*Note.* Eigenvectors <u>must</u> be nonzero, but it is possible for $0$ to be an eigenvalue.

# What if 0 is an eigenvalue?

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If $A$ has the eigenvalue $0$ with the eigenvector $\mathbf{v}$, then

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In other words,

- $\mathbf{v} \in \mathsf{Nul}(A)$
- $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

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**Theorem.** A $n \times n$ matrix is invertible if and only if it <u>does not</u> have $0$ as an eigenvalue.

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**Theorem.** A $n \times n$ matrix is invertible if and only if it <u>does not</u> have $0$ as an eigenvalue.

*To reiterate.* An eigenvalue $0$ is equivalent to

- » $A\mathbf{x} = \mathbf{0}$ has no nontrivial solutions
- » the columns of $A$ are linearly dependent
- » $\text{Col}(A) \neq \mathbb{R}^n$
- » ...

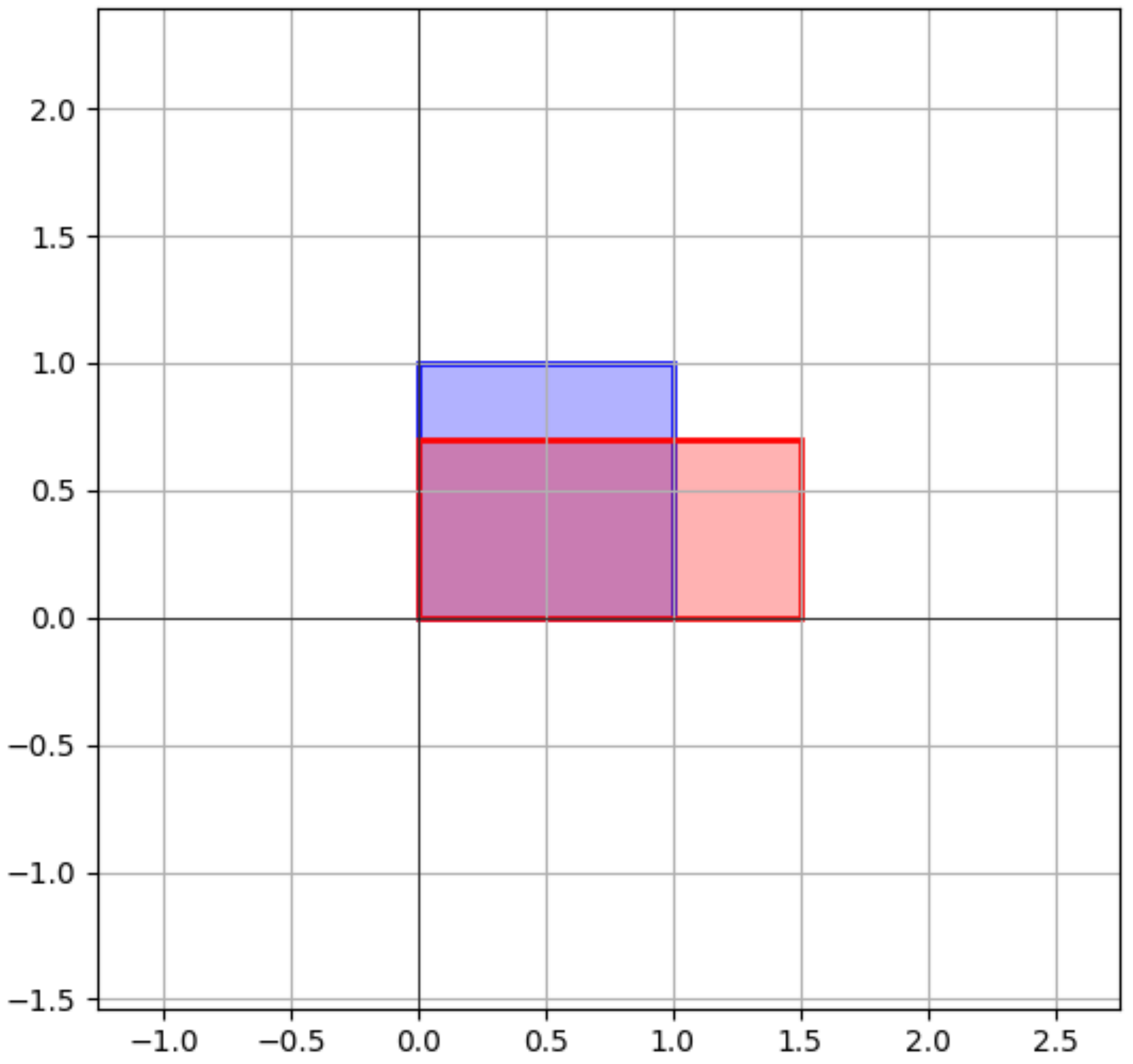
# Example: Unequal Scaling

Let's determine it's eigenvalues and eigenvectors:

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = 1.5 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda = 1.5$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix} = 0.7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



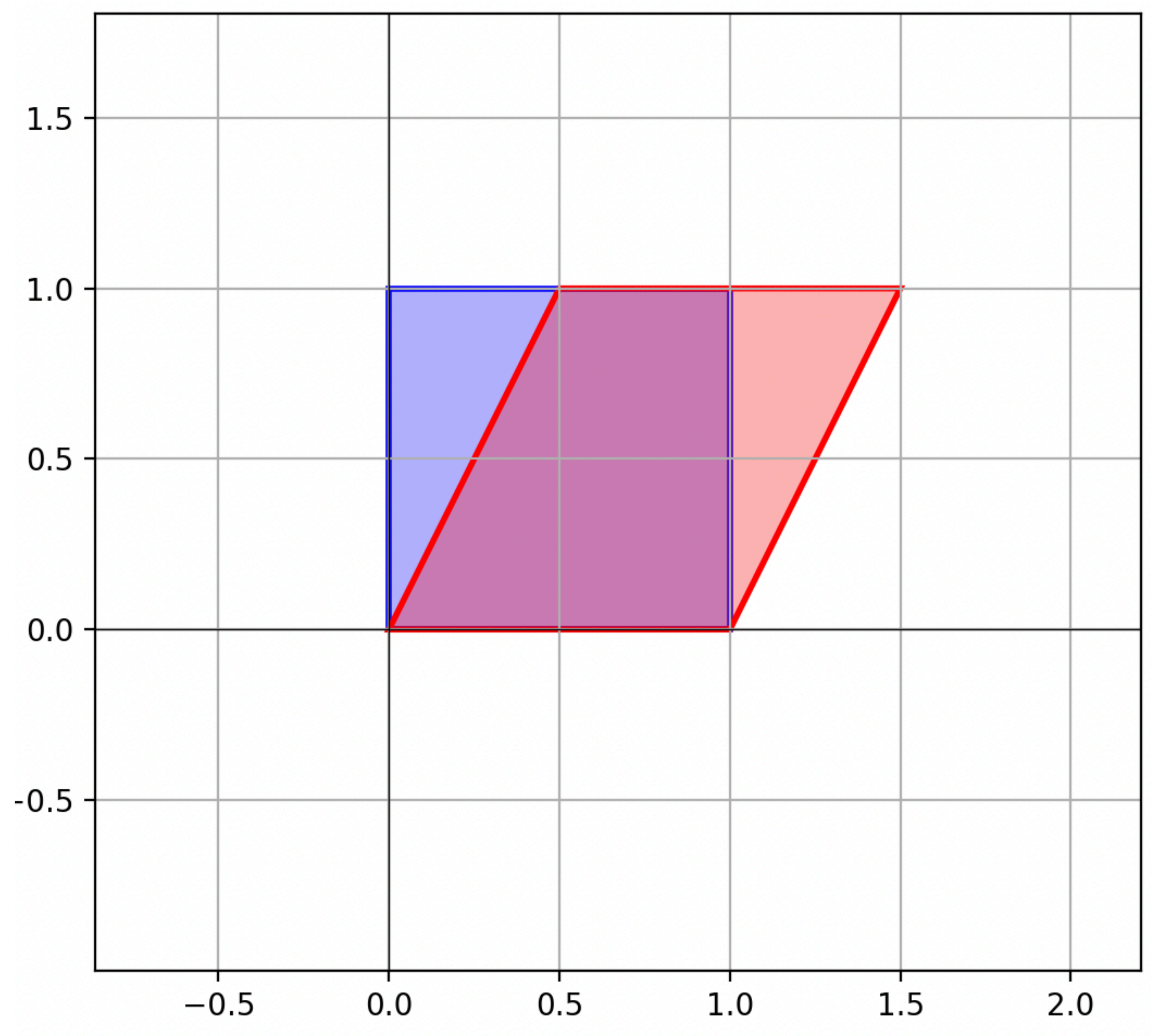

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

# Example: Shearing

Let's determine it's eigenvalues and eigenvectors:

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda = 1$



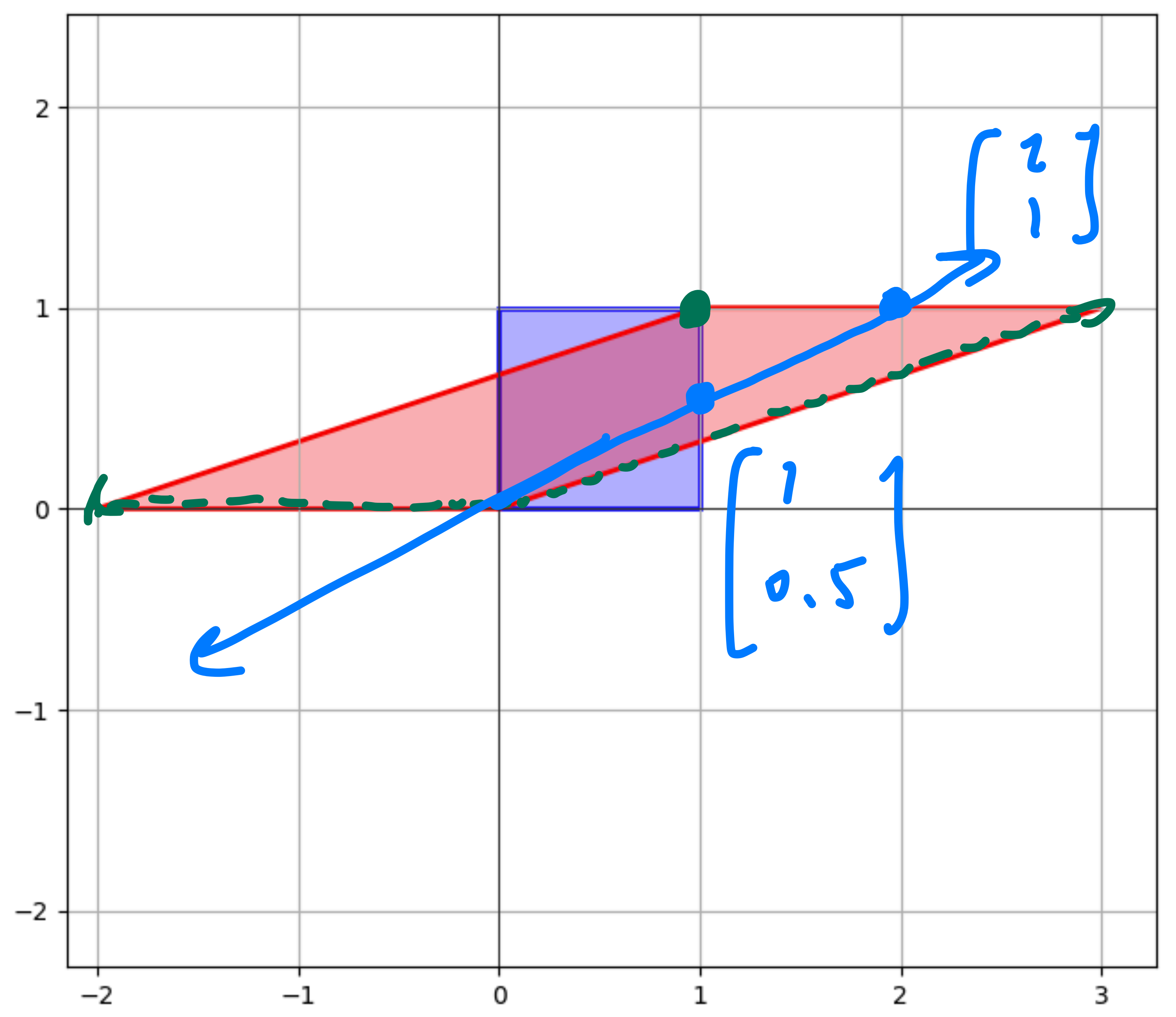

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

# Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \lambda = 2$$



# How do we verify eigenvalues and eigenvectors?

# Verifying Eigenvectors

# Verifying Eigenvectors

***Question.*** *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

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**Solution.** Easy. Work out the matrix-vector multiplication.

# Verifying Eigenvectors

$$\begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 - 30 \\ 30 - 10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\lambda = -4$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 - 12 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \quad X$$

# Verifying Eigenvalues

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This is harder...
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**Question.** Show that $7$ is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

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What vector do we check???

Before we go over how to do this...

# Verifying Eigenvalues (Warm Up)

**Question.** Verify that $1$ is an eigenvalue of

$$A := \begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

*Hint. Recall our discussion of Markov Chains.*

Solution: $A\vec{v} = \vec{v}$

eigenvectors of 1 are steady states

# Steady-States and Eigenvectors

Steady-state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue 1.

How did we find steady-state vectors?:

$$A\vec{r} = \vec{r} \qquad A\vec{r} - \vec{r} = \vec{0} \qquad A\vec{r} - I\vec{r} = \vec{0}$$

$$(A - I)\vec{r} = 0$$ } matrix equation

# Steady-States and Eigenvectors

$\mathbf{v}$ is a steady-state vector* $\equiv$ $\mathbf{v} \in \text{Nul}(A - I)$

*It must also be a probability vector

# Verifying Eigenvalues

This is harder...

**Question.** Show that $\lambda$ is an eigenvalue of $A$.

**Solution:** $A\vec{v} = \lambda\vec{v}$ $\quad A\vec{v} - \lambda\vec{v} = \vec{0}$ $\quad (A - \lambda I)\vec{v} = 0$

solve: $(A - \lambda I)\vec{x} = \vec{0}$

# Verifying Eigenvalues

$\mathbf{v}$ `is an eigenvector for` $\lambda \quad \equiv \quad \mathbf{v} \in \text{Nul}(A - \lambda I)$

solutions to
homogen. eq.

# Verifying Eigenvalues

This is harder...

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+12 \\ 10+4 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

**Question.** Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

**Solution:** solve: $(A - 7I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

$x_2$ is free

parametric: $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \checkmark$$

# Problem

*Verify that* $2$ *is an eigenvalue of* $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

# Answer

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore (A - 2I)\vec{x} = \vec{0}$ has nontrivial sol. and $\lambda = 2$ is and eigenvalue

$x_1 = 0.5x_2 - 3x_3$

$x_2$ is free

$x_3$ is free

$x_1 = -3$

$x_2 = 0$

$x_3 = 1$

$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector with $\lambda = 2$

How many eigenvectors can a matrix have?

# Linear Independence of Eigenvectors

**Theorem.*** If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

*So an $n \times n$ matrix can have at most $n$ eigenvalues.*

Why?: more than n eigenvalues ⟹

more than n L.I. vectors of $\mathbb{R}^n$

*We won't prove this.

# Eigenspace

**Fact.** The set of eigenvectors for a eigenvalue $\lambda$ of $A \in \mathbb{R}^{n\times n}$ form a subspace of $\mathbb{R}^n$.

Verify:

$\mathrm{Nul}(A - \lambda I)$

this is a subspace

closed under addition

$A\vec{v} = \lambda\vec{v}$

$A\vec{w} = \lambda\vec{w}$ $\Rightarrow$

$A(\vec{v} + \vec{w}) =$

$A\vec{v} + A\vec{w} =$

$\lambda\vec{v} + \lambda\vec{w} =$

$\lambda(\vec{v} + \vec{w})$

# Eigenspace

**Definition.** The set of eigenvectors for a eigenvalue $\lambda$ of $A$ is called the **eigenspace** of $A$ corresponding to $\lambda$.

It is the same as $\text{Nul}(A - \lambda I)$.

# How To: Basis of an Eigenspace

**Question.** Find a basis for the eigenspace of $A$ corresponding to $\lambda$.

**Solution.** Find a basis for $\mathrm{Nul}(A - \lambda I)$.

**We know how to do this.**

# Example

$$\begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -1 \\ -4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

*Determine a basis for the eigenspace corresponding to the eigenvalue* 1:

$$A - I = \begin{bmatrix} -3 & 0 & 3 \\ 1 & 0 & -1 \\ -4 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 1 x_3$

$x_2$ is free

$x_3$ is free

parametric:

$$x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis

# How do we find eigenvalues?

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**We'll cover this next time...**

# Eigenvalues of Triangular Matrices

**Theorem.** The eigenvalues of a triangular matrix are its entries along the diagonal.

Verify:

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$(A - \lambda I)$ must be in echelon form

and must have a free var.

# Example

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

*Determine the eigenvectors and values of the above matrix:*

$\lambda = 3, 0, 2$

$\lambda = 3$

$$\begin{bmatrix} 0 & 6 & -8 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -4 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1$ is free

$x_2 = 0 \quad x_3 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

# Linear Dynamical Systems

# Recall: Linear Dynamical Systems

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**Definition.** A **(discrete time) linear dynamical system** is described by a $n \times n$ matrix $A$. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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The possible states of the system are vectors in $\mathbb{R}^n$.

$A$ tells us how our system evolves over time.

Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after $i$ time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

# Recall: State Vectors

$$\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number $k$ time steps

This is also called a *recurrence relation* or a *linear difference function*

# Recall: State Vectors

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0)$$

$$\mathbf{v}_k = A^k\mathbf{v}_0$$

$$\mathbf{v}_5 = A\mathbf{v}_4 = A(AAAA\mathbf{v}_0)$$

$$\vdots$$

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It's defined in terms of $A$ itself, which doesn't tell us much about how the system behaves

*It's also difficult computationally because matrix multiplication is expensive*

# (Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain $A^k$ or previously defined terms

# (Closed-Form) Solutions

A **(closed–form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain $A^k$ or previously defined terms

In other word, it does not depend on $A^k$ and is not **recursive**

# Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed for the above linear dynamical system.

$$v_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \vec{v}_0 = \begin{bmatrix} 1 & k \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+k \\ 1 \end{bmatrix}$$

# Solutions with Eigenvectors as Initial States

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**<u>The Key Point.</u> This is still true of sums of eigenvectors.**

# Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for $A$ which span all of $\mathbb{R}^2$:

$$\vec{v} = \vec{b}_1 + \vec{b}_2$$

$$A^k \vec{v} = A^k(\vec{b}_1 + \vec{b}_2) = A^k \vec{b}_1 + A^k \vec{b}_2$$

$$= \lambda_1^k \vec{b}_1 + \lambda_2^k \vec{b}_2$$

# Eigenvectors and Growth in the Limit

**Theorem.** For a linear dynamical system $A$ with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of $A$ with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant $c_1$ (in other words, in the long term, the system grows exponentially in $\lambda_1$).

Verify:

asymptotically, long term behavior dominated by largest eigenvalue.

$f(x) \sim g(x)$

$\lim_{x \to \infty} \frac{f(x)}{f(x)} = 1$

# Eigenbases

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**Definition.** An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix $A$ is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of $A$.

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*We can represent vectors as **unique** linear combinations of eigenvectors.*

***Not all matrices have eigenbases.***

# Eigenbases and Growth in the Limit

**Theorem.** For a linear dynamical system $A$ with initial state $\mathbf{v}_0$, if $A$ has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of** $A$ **and** $\mathbf{b}_1$ **is its eigenvalue.**

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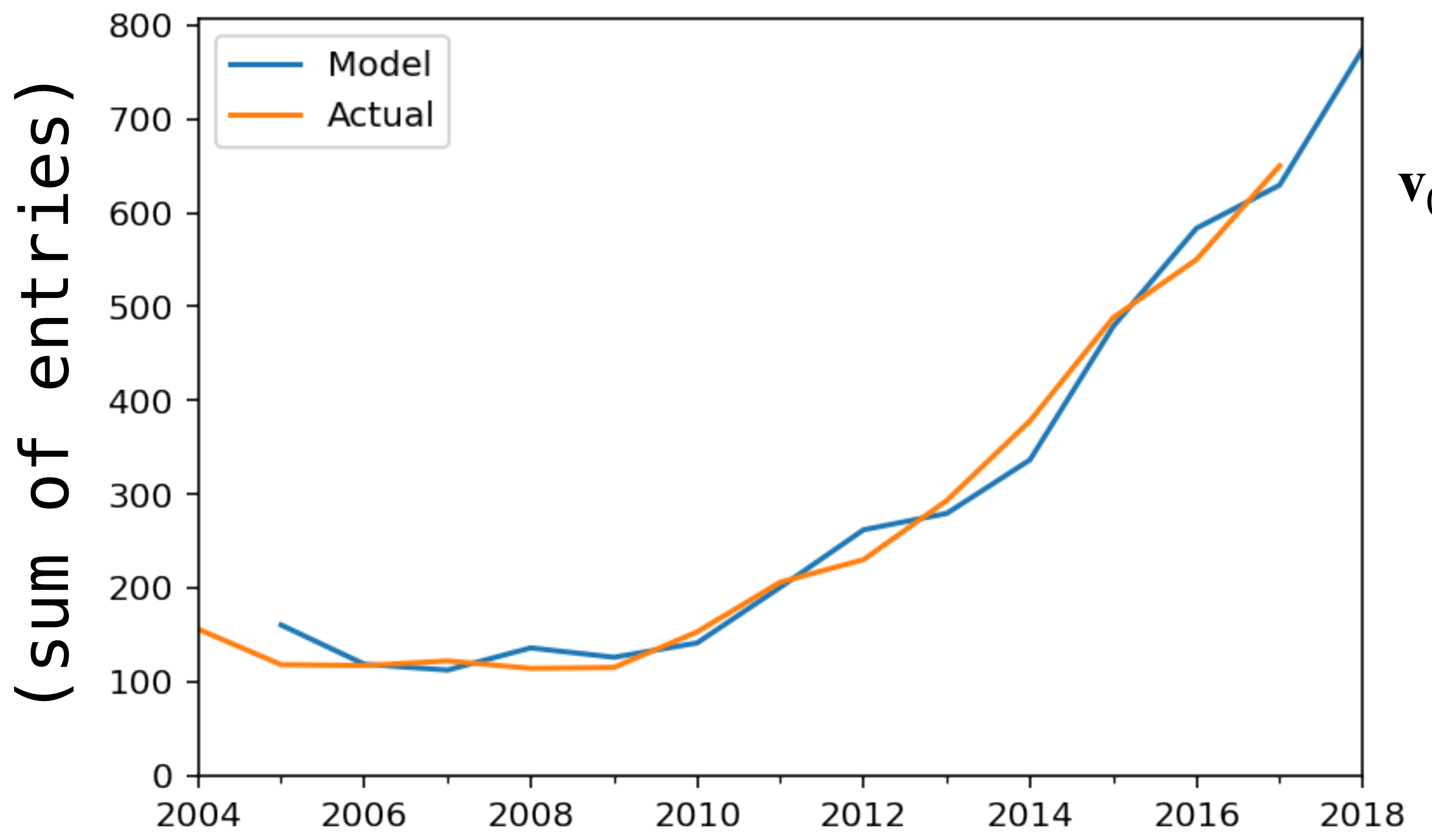
for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of $A$ and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

# Example: CS Major Growth

see the notes for more details




$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \text{\# of year 1 students enrolled in 2024} \\ \text{\# of year 2 students enrolled in 2024} \\ \text{\# of year 3 students enrolled in 2024} \\ \text{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

($A$ is determined by least squares)

**This is clearly exponential. If we want to "extract" the exponent, we need to look at the largest eigenvalue.**

# Another Example: Golden Ratio

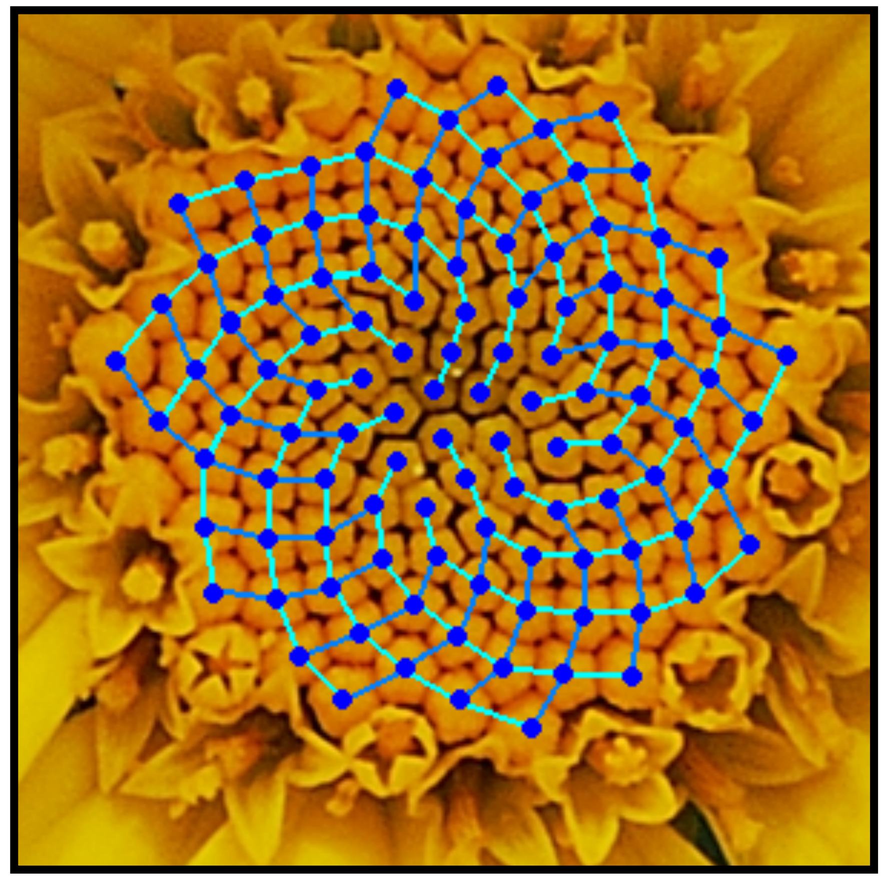
# A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the system given by the above matrix.

What does this matrix represent?:

# Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature.

https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG

# Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2}$$

The "long term behavior" is the Fibonacci sequence is defined by the golden ratio.

**This is the largest eigenvalue of** $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**