Eigenvalues and Eigenvectors

Geometric Algorithms

Lecture 18

CAS CS 132

Practice Problem

Suppose A *is a* 234×300 *matrix. What is the smallest possible value for* $\dim(\mathrm{Nul}(A))$*? What is the largest possible value?*

What is the smallest possible value for $\mathrm{rank}(A)$*? What is the largest possible value?*

Answer

Objectives

1. Motivate and introduce the fundamental notion of eigenvalues and eigenvectors
2. Determine how to verify eigenvalues and eigenvectors
3. Look at the subspace generated by eigenvectors
4. Apply the study of eigenvectors to dynamical linear systems

Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

Motivation

demo

How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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How can matrices transform vectors?*

In 2D and 3D we've seen:

- » rotations
- » projections
- » shearing
- » reflection
- » scaling/stretching
- » ... **Today's focus**

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What's special about scaling?

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$$\mathbf{x} \mapsto c\mathbf{x}$$

What's special about scaling?

We don't need a whole matrix to do scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

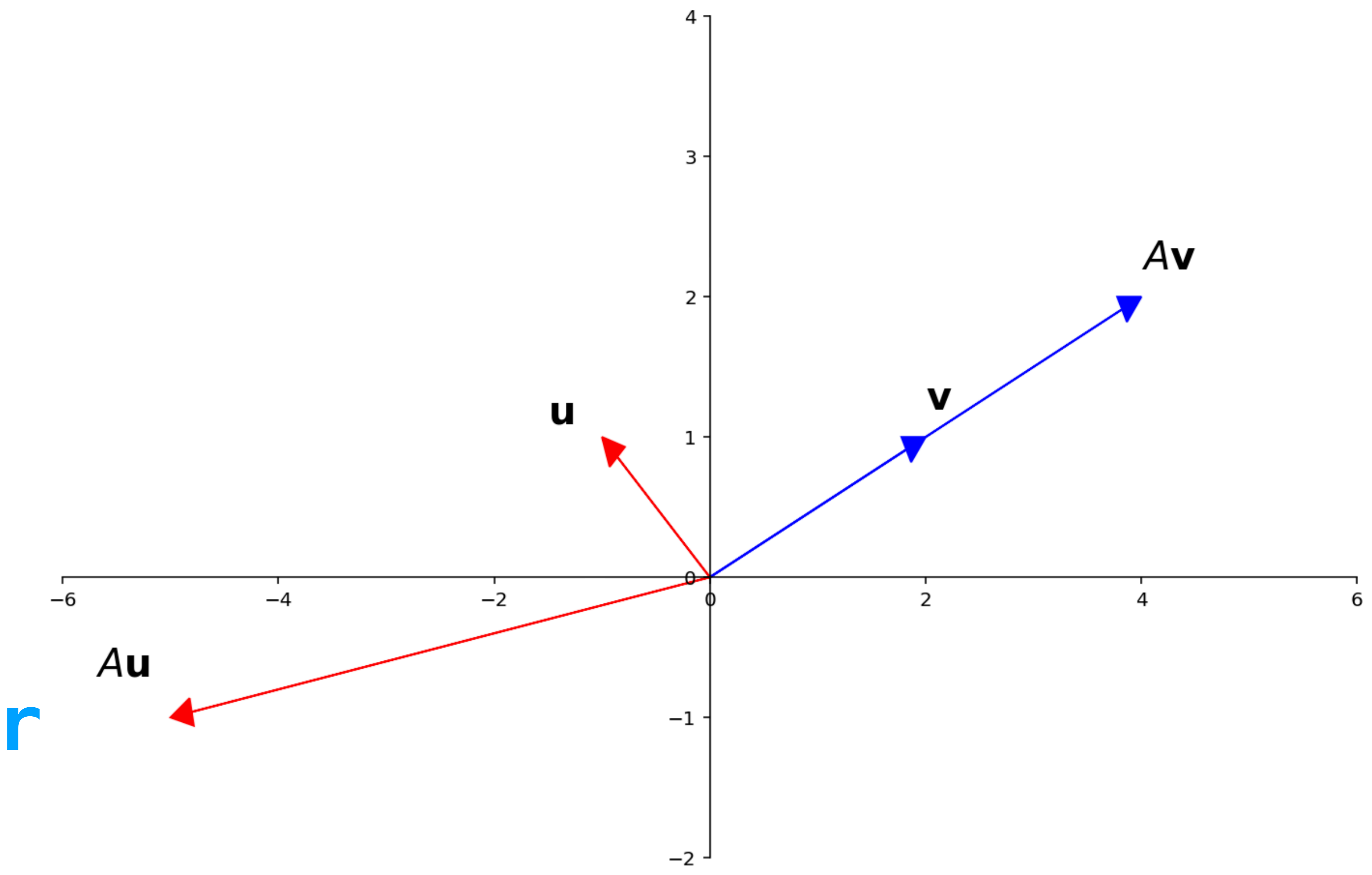
So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what A does to $\mathbf{v}$.

Eigenvectors (Informal)

eigenvalue

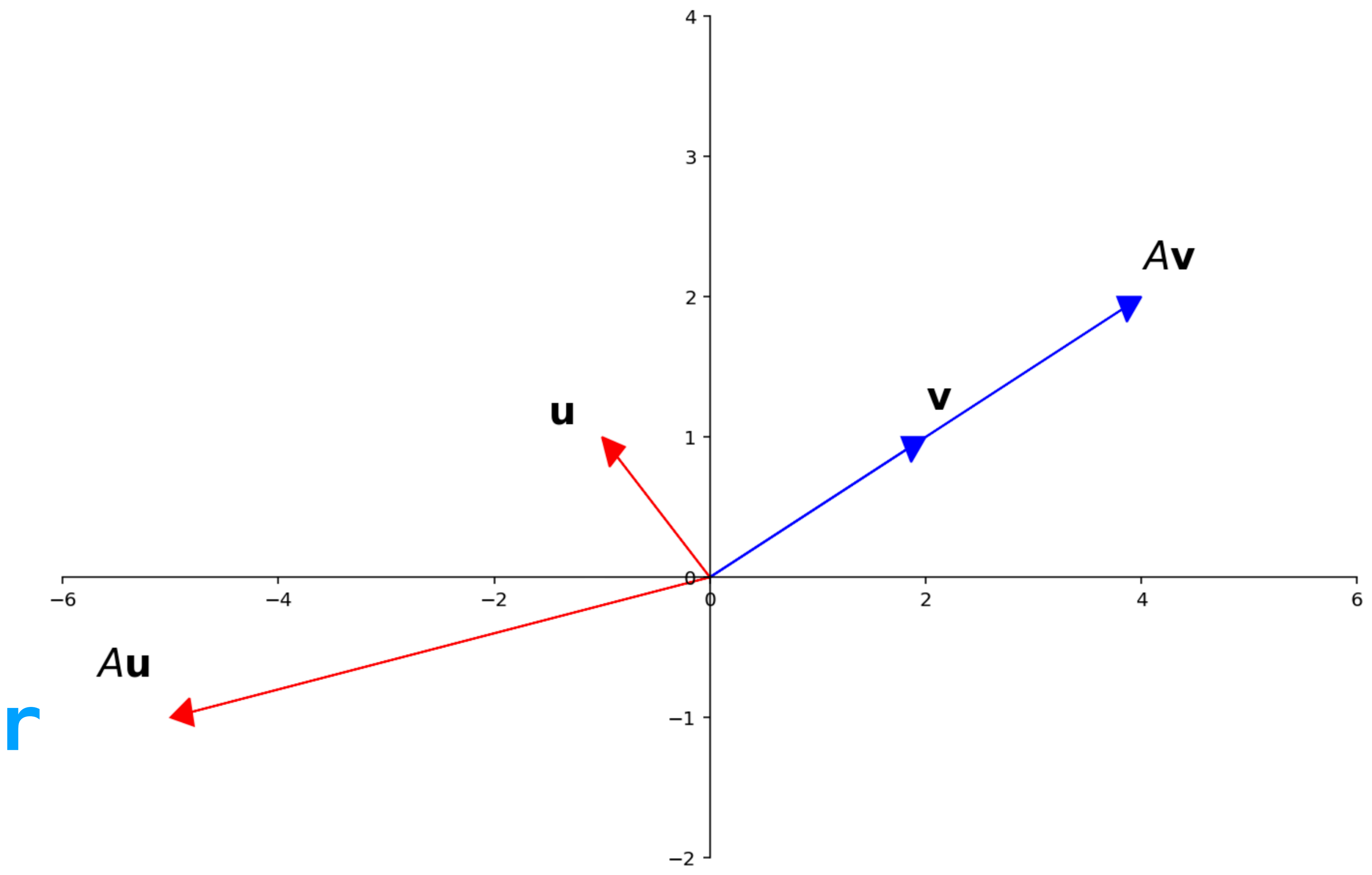
$$A\mathbf{v} = \lambda\mathbf{v}$$

eigenvector



Eigenvectors (Informal)

$$\underbrace{\lambda}_{\text{eigenvalue}}\qquad A\mathbf{v} = \lambda \underbrace{\mathbf{v}}_{\text{eigenvector}}$$



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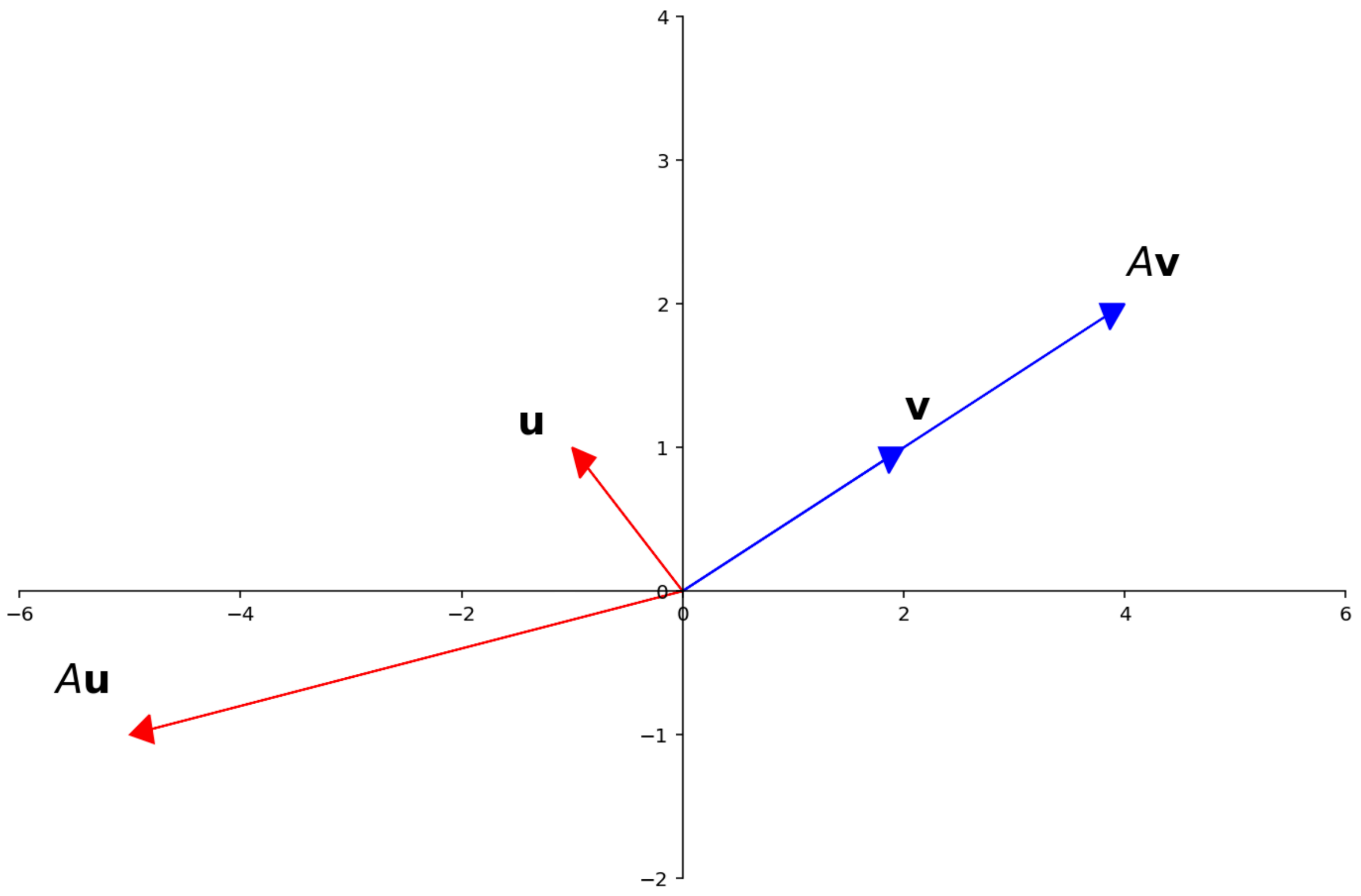


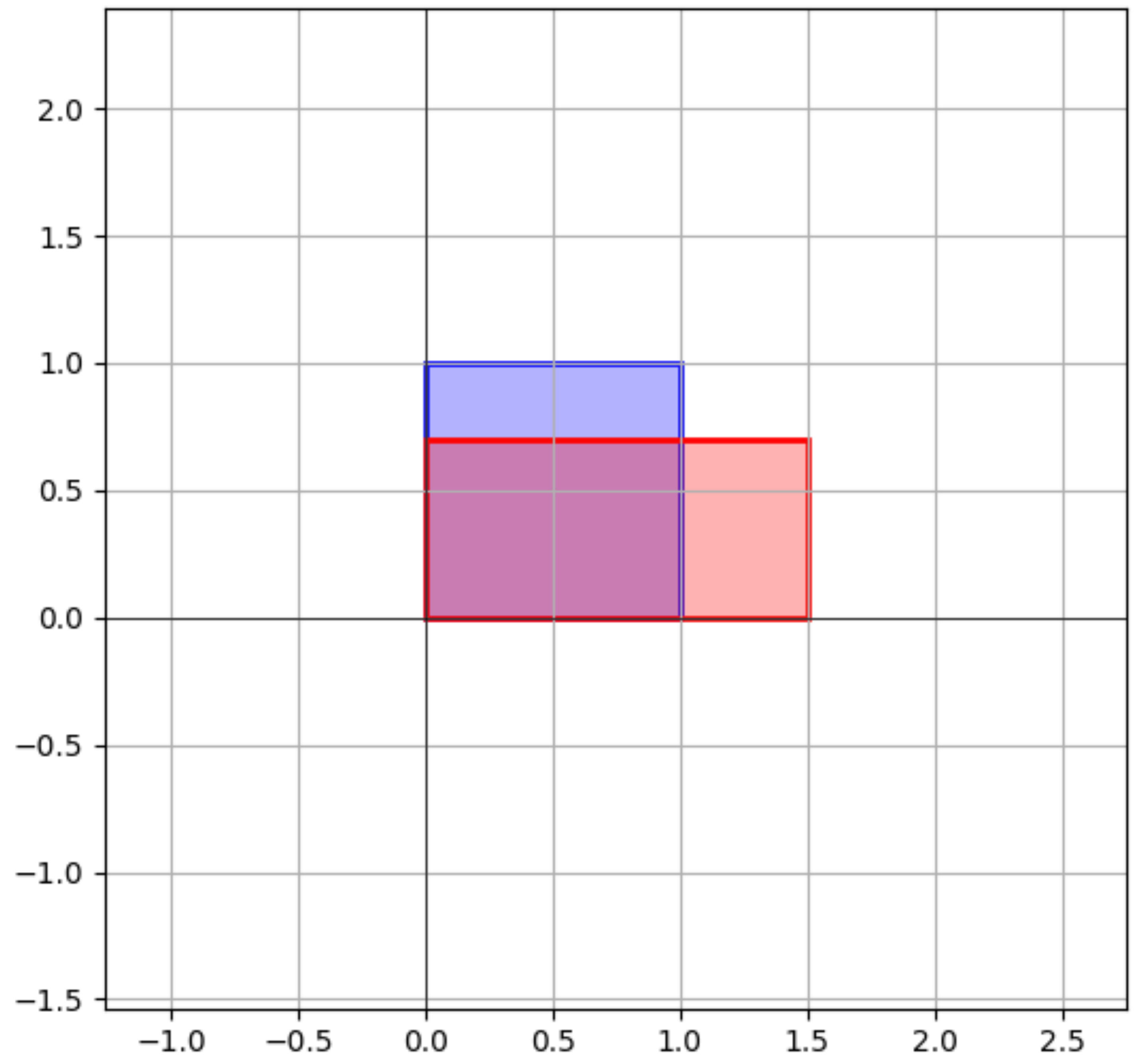
Eigenvectors of A are stretched by A without changing their direction.

The amount they are stretched is called the **eigenvalue.**

Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

It transforms each entry individually and then combines them.



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Eigenbases (Informal)

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of* A. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

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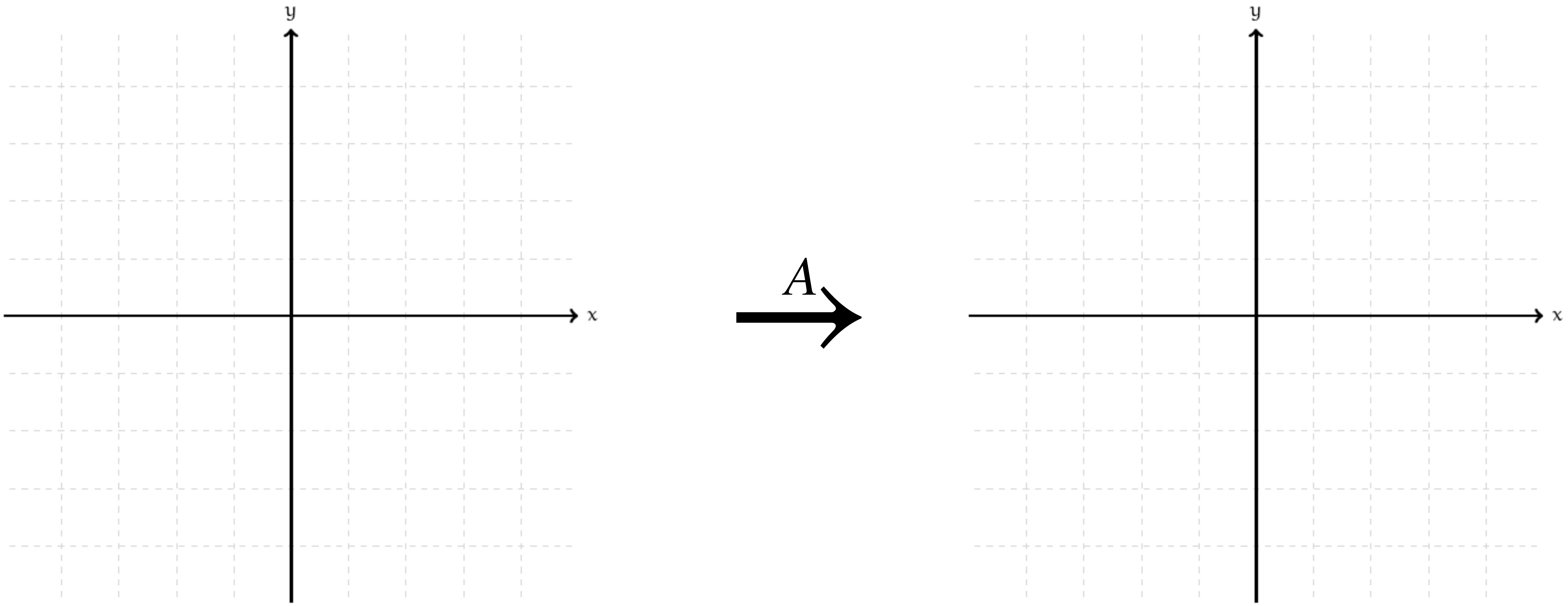
$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

It's "easy to describe" how A transforms $\mathbf{v}$.

It *transforms each "component" individually and then combines them.*

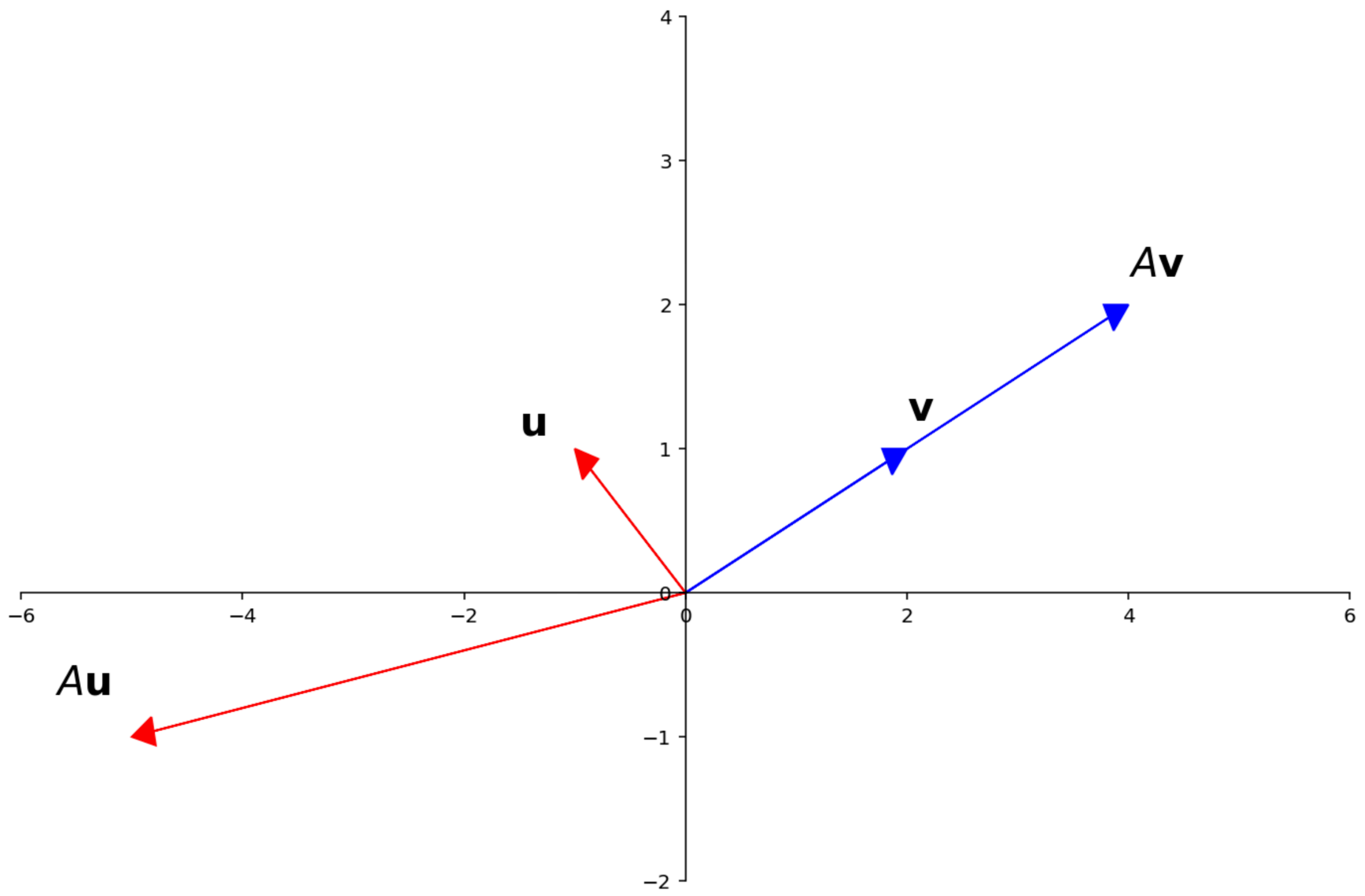
Verify:

Eigenbases (Pictorially)



Eigenvalues and Eigenvectors

Formal Definition



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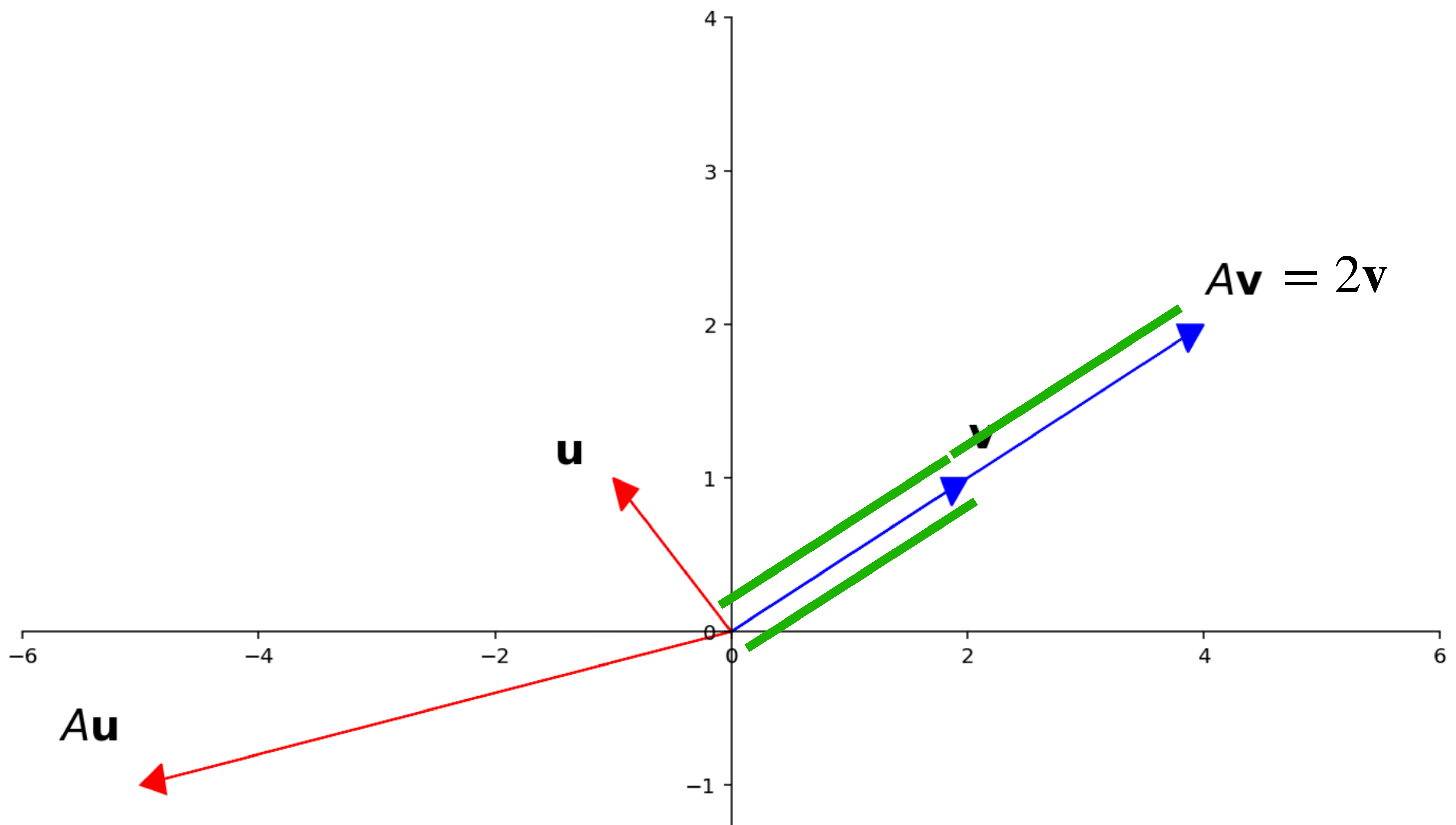

A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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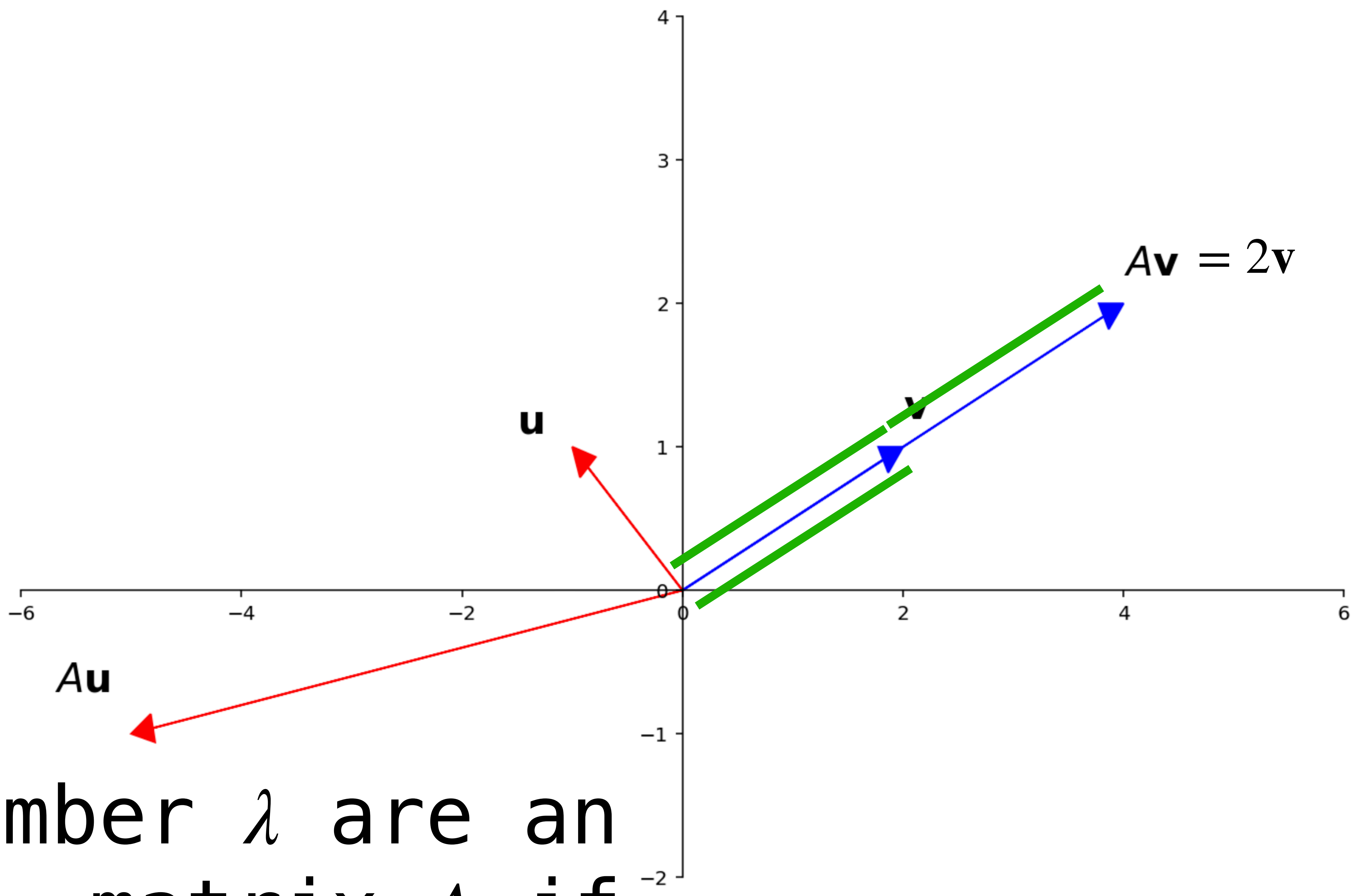

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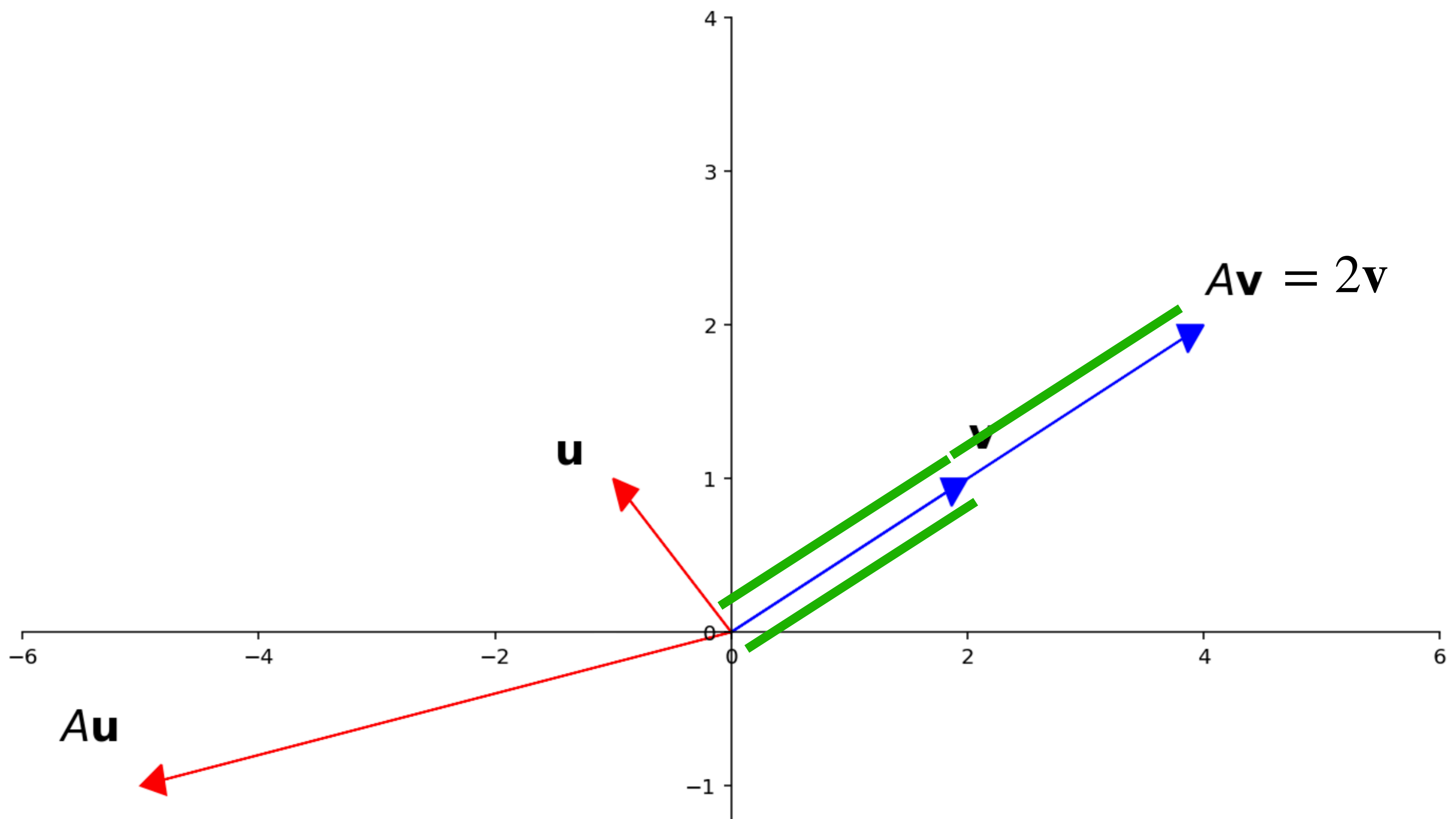

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Note. Eigenvectors <u>must</u> be nonzero, but it is possible for 0 to be an eigenvalue.

What if 0 is an eigenvalue?

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If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$$

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If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

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In other words,

» $\mathbf{v} \in \mathsf{Nul}(A)$
» $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

Extending the IMT (Again)

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Theorem. A $n \times n$ matrix is invertible if and only if it <u>does not</u> have 0 as an eigenvalue.

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To reiterate. An eigenvalue 0 is equivalent to

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Theorem. A $n \times n$ matrix is invertible if and only if it does not have 0 as an eigenvalue.

To reiterate. An eigenvalue 0 is equivalent to

- $A\mathbf{x} = \mathbf{0}$ has no nontrivial solutions
- the columns of A are linearly dependent
- $\mathsf{Col}(A) \neq \mathbb{R}^n$
- ...

Example: Unequal Scaling

```
Let's determine it's eigenvalues
and eigenvectors:
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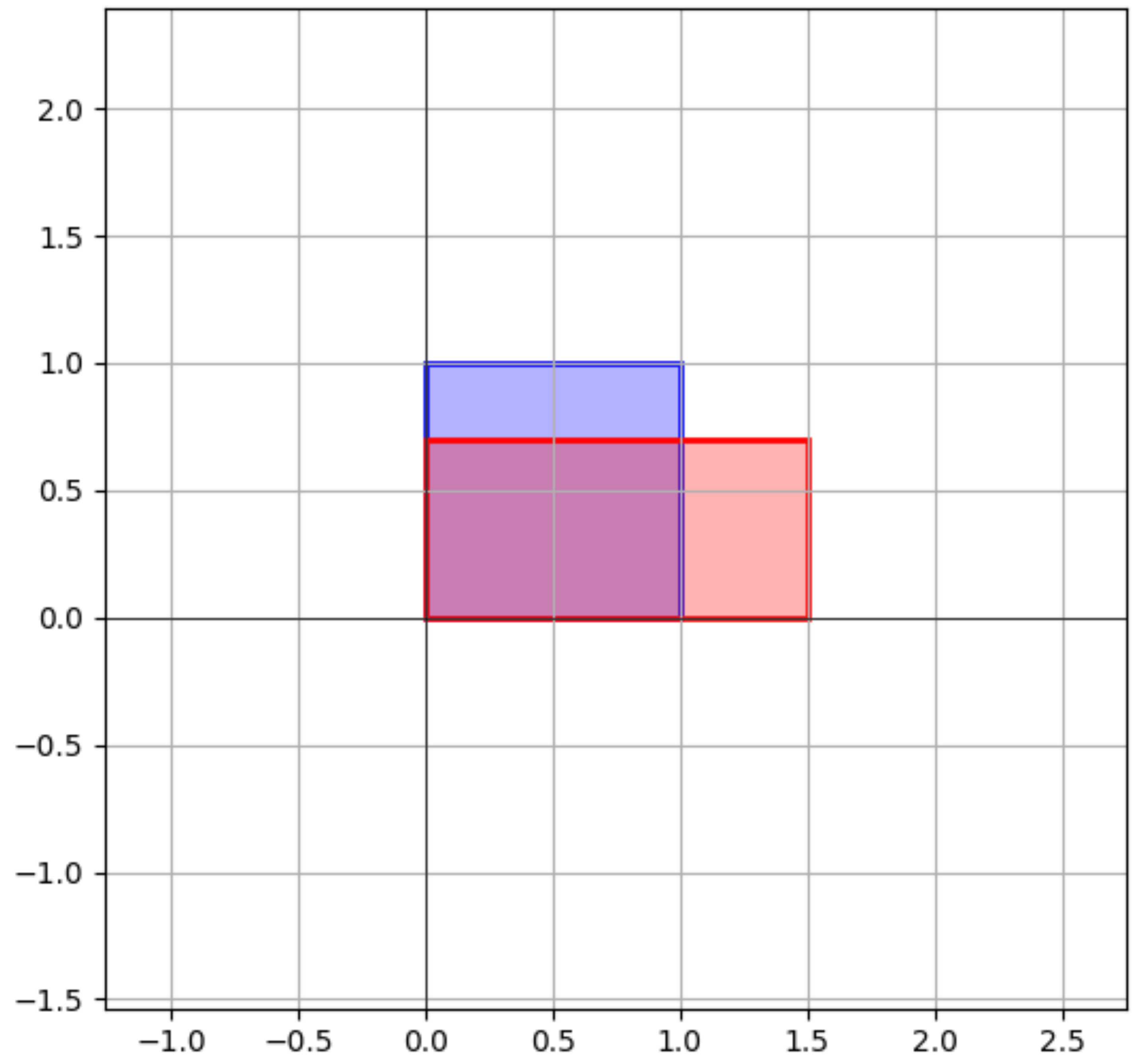
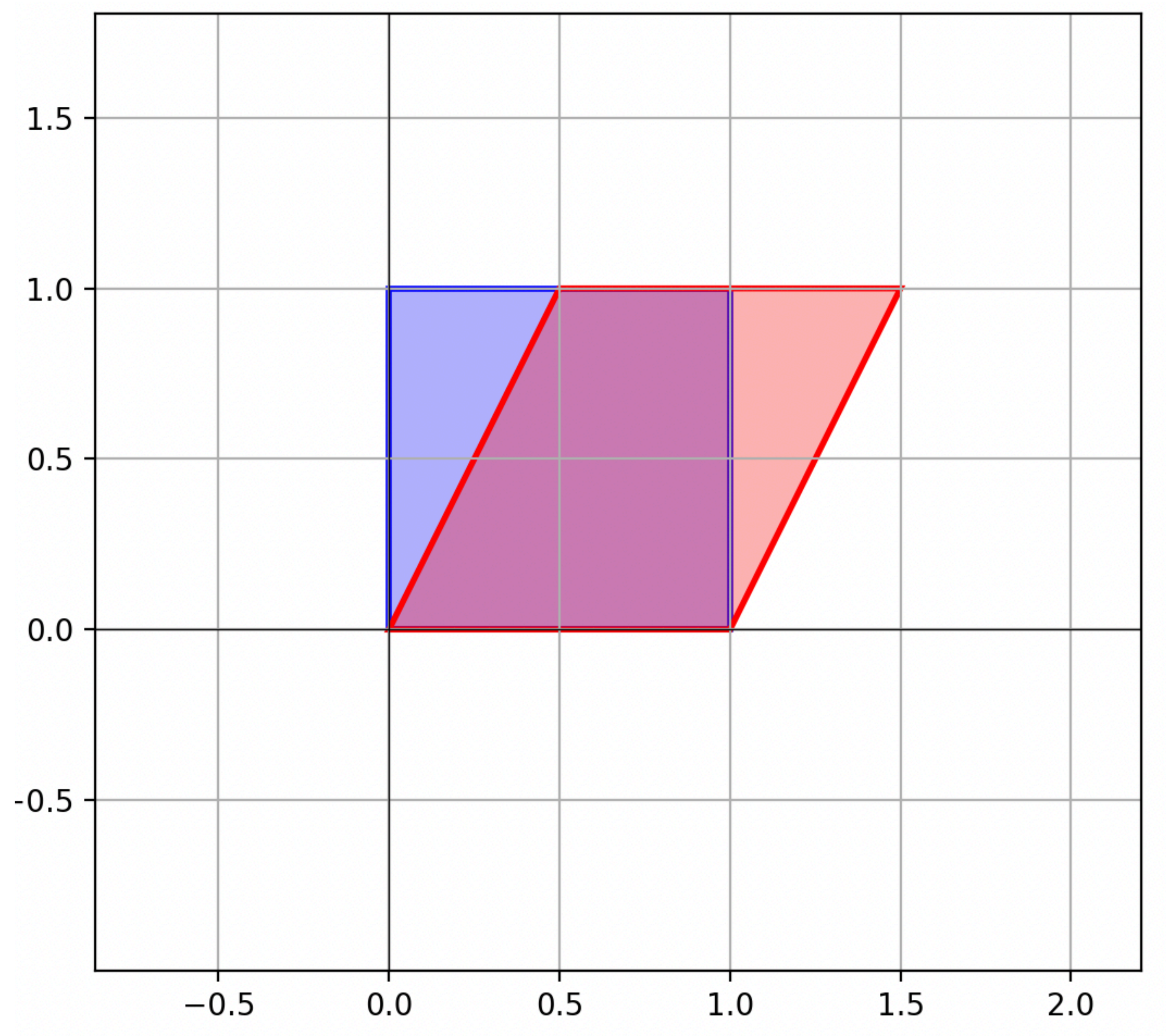


$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Example: Shearing

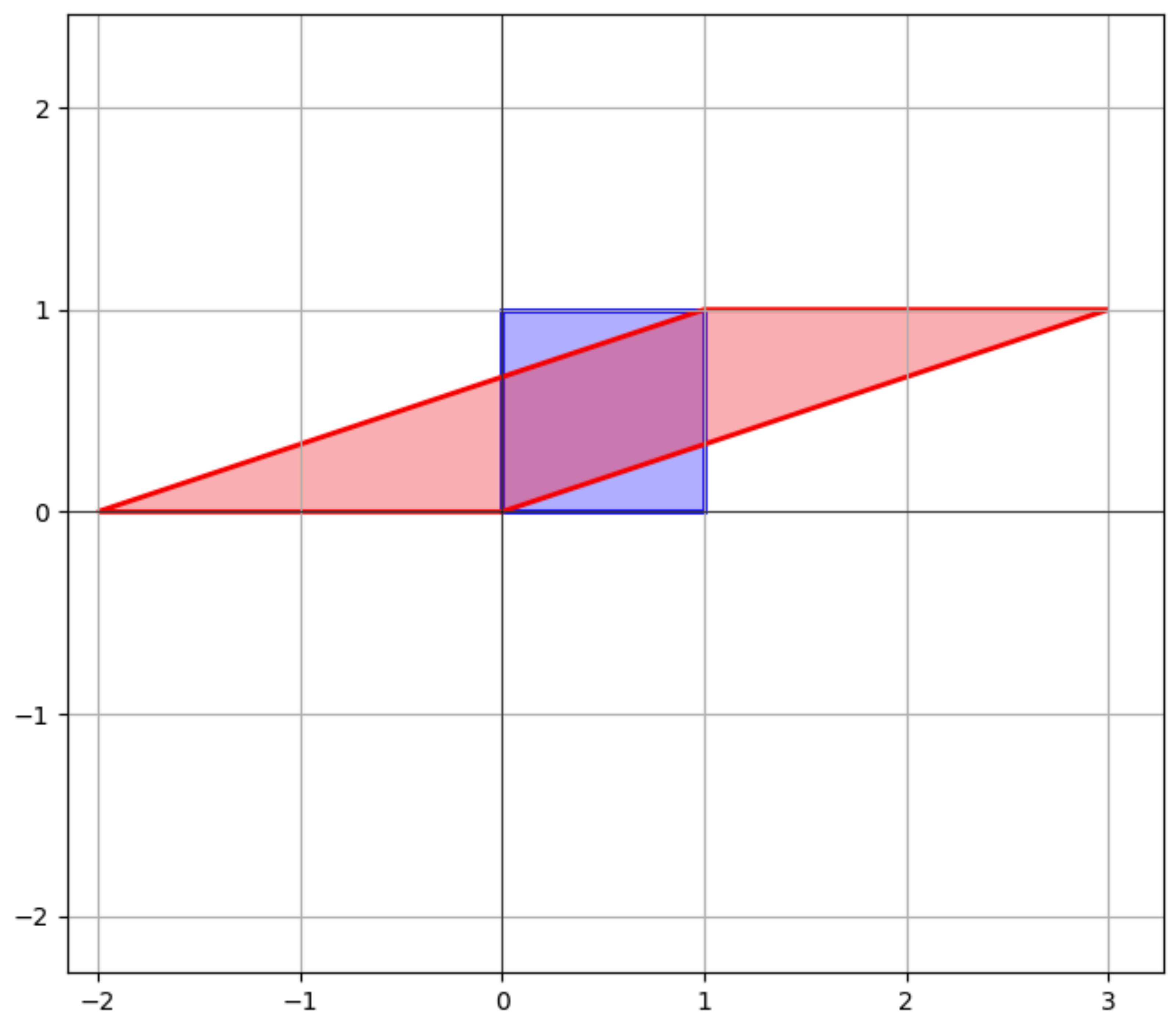
Let's determine it's eigenvalues and eigenvectors:



$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$



How do we verify eigenvalues and eigenvectors?

Verifying Eigenvectors

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

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Solution. Easy. Work out the matrix-vector multiplication.

Verifying Eigenvectors

$$\begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

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Before we go over how to do this...

Verifying Eigenvalues (Warm Up)

Question. Verify that 1 is an eigenvalue of

$$\begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

Hint. Recall our discussion of Markov Chains.

Solution:

Steady-States and Eigenvectors

Steady–state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue 1.

How did we find steady–state vectors?:

Steady-States and Eigenvectors

$$\mathbf{v} \text{ is a steady-state vector}^{*} \equiv \mathbf{v} \in \mathrm{Nul}(A - I)$$

*It must also be a probability vector

Verifying Eigenvalues

This is harder...

Question. Show that λ is an eigenvalue of A.

Solution:

Verifying Eigenvalues

$$\mathbf{v} \texttt{ is an eigenvector for } \lambda \quad \equiv \quad \mathbf{v} \in \textsf{Nul}(A - \lambda I)$$

Verifying Eigenvalues

This is harder...

Question. Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Solution:

Problem

Verify that 2 *is an eigenvalue of* $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Answer

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

How many eigenvectors can a matrix have?

Linear Independence of Eigenvectors

Theorem.* If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

So an $n \times n$ matrix can have at most n eigenvalues.

Why?:

*We won't prove this.

Eigenspace

Fact. The set of eigenvectors for a eigenvalue λ of $A \in \mathbb{R}^{n \times n}$ form a subspace of $\mathbb{R}^n$.

Verify:

Eigenspace

Definition. The set of eigenvectors for a eigenvalue λ of A is called the **eigenspace** of A corresponding to λ.

It is the same as $\mathrm{Nul}(A - \lambda I)$.

How To: Basis of an Eigenspace

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\mathrm{Nul}(A - \lambda I)$.

We know how to do this.

Example

$$\begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -1 \\ -4 & 0 & 5 \end{bmatrix}$$

Determine a basis for the eigenspace corresponding to the eigenvalue 1*:*

How do we find eigenvalues?

How do we find eigenvalues?

We'll cover this next time...

Eigenvalues of Triangular Matrices

Theorem. The eigenvalues of a triangular matrix are its entries along the diagonal.

Verify:

Example

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Determine the eigenvectors and values of the above matrix:

Linear Dynamical Systems

Recall: Linear Dynamical Systems

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Definition. A **(discrete time) linear dynamical system** is described by a $n \times n$ matrix A. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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A tells us how our system evolves over time.

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Recall: State Vectors

$$
\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}
$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number k time steps

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$$\boxed{\mathbf{v}_k = A^k\mathbf{v}_0}$$

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It's also difficult computationally because matrix multiplication is expensive

(Closed-Form) Solutions

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In other word, it does not depend on A^k and is not **recursive**

Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed for the above linear dynamical system.

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

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The Key Point. **This is still true of sums of eigenvectors.**

Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of $\mathbb{R}^2$:

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows <u>exponentially in λ_1</u>).

Verify:

Eigenbases

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Not all matrices have eigenbases.

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Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \dots, \mathbf{b}_k$, then

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for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

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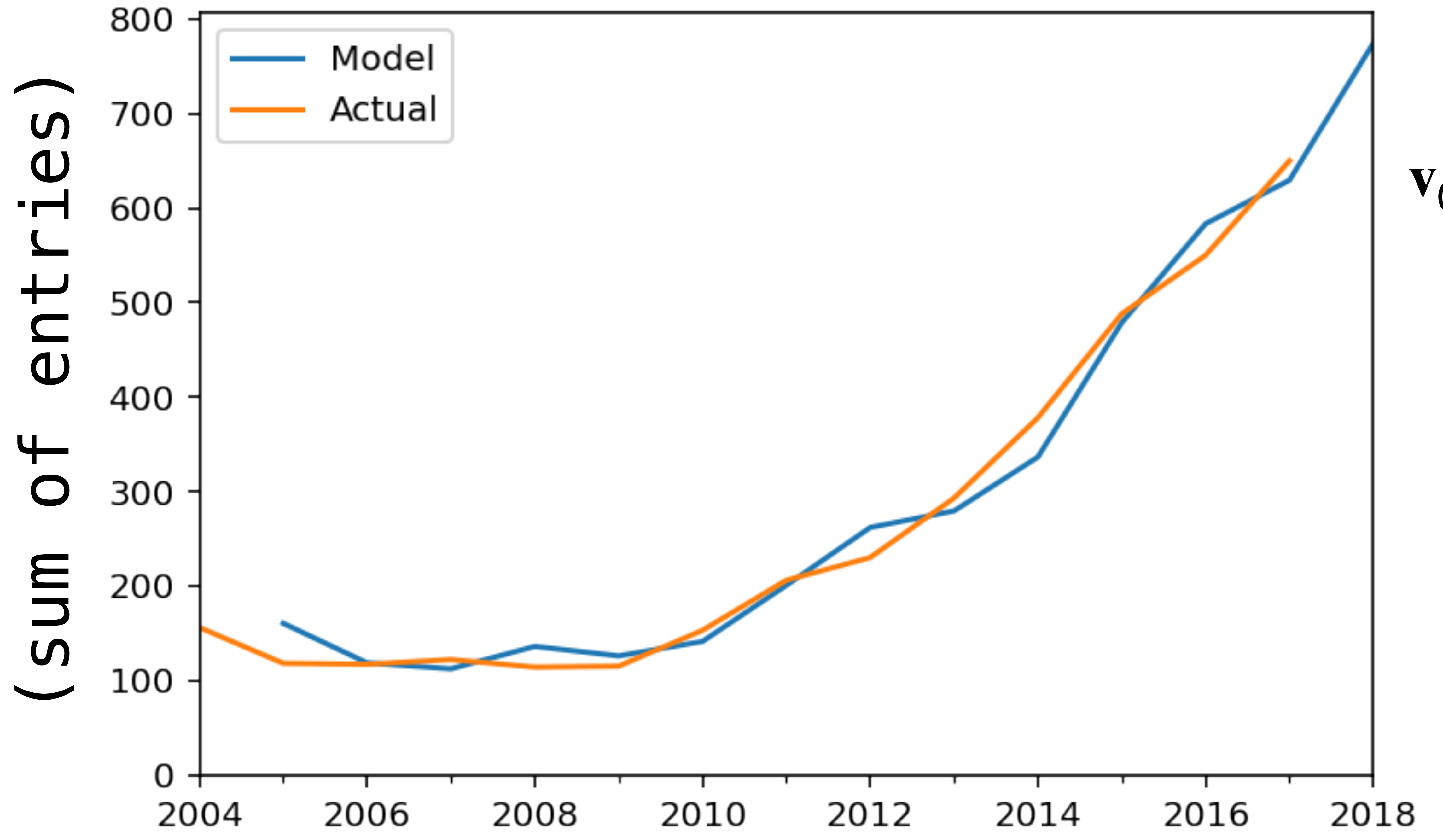
for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \text{\# of year 1 students enrolled in 2024} \\ \text{\# of year 2 students enrolled in 2024} \\ \text{\# of year 3 students enrolled in 2024} \\ \text{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the <u>largest eigenvalue</u>.

Another Example: Golden Ratio

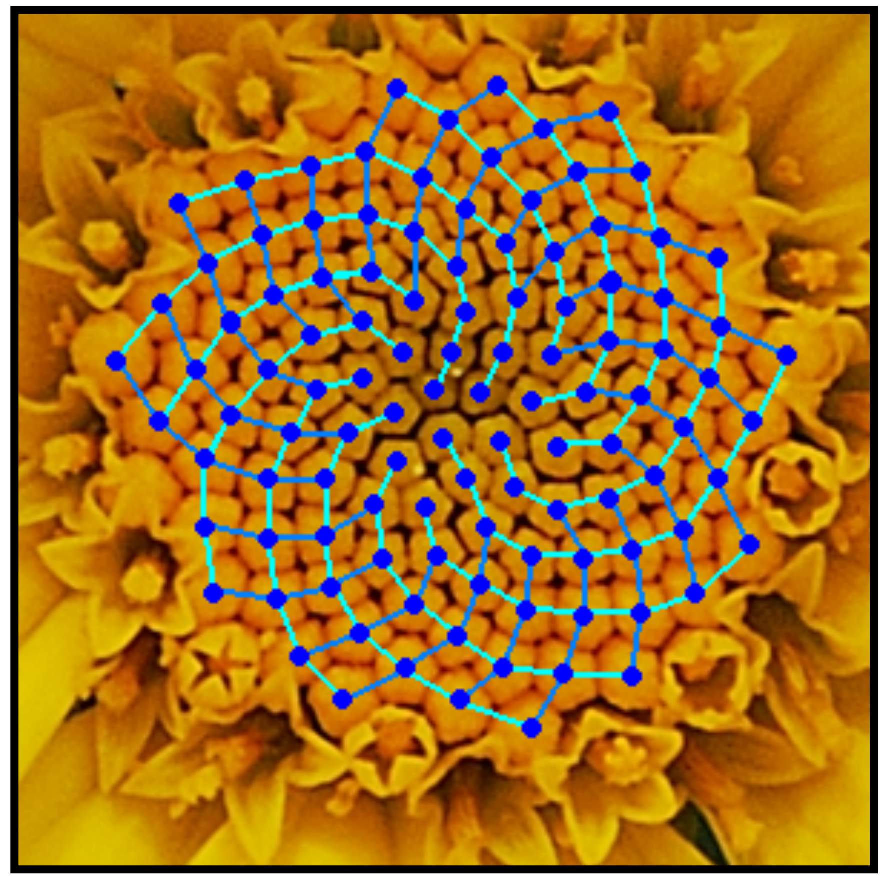
A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

`Consider the system given by the above matrix.`

`What does this matrix represent?:`

Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature.

https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG

Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2}$$

The "long term behavior" is the Fibonacci sequence is defined by the golden ratio.

This is the largest eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**