The Characteristic Equation

Geometric Algorithms
Lecture 19

CAS CS 132

Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Determine the dimension of the eigenspace of A for the eigenvalue 4.

(try not to do any row reductions)

Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

$$\dim(\text{Nul}(A - 4I))$$

$$A - 4I = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 1 \\ 2 & 4 & 6 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$\text{rank}(A - 4I) = 2$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis

$$\underbrace{\text{rank}(A - 4I)}_{2} + \underbrace{\dim(\text{Nul}(A - 4I))}_{2} = 4$$

Objectives

1. Briefly recap eigenvalues and eigenvectors
2. Get a primer on determinants
3. Determine how to find eigenvalues (not just verify them)

Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

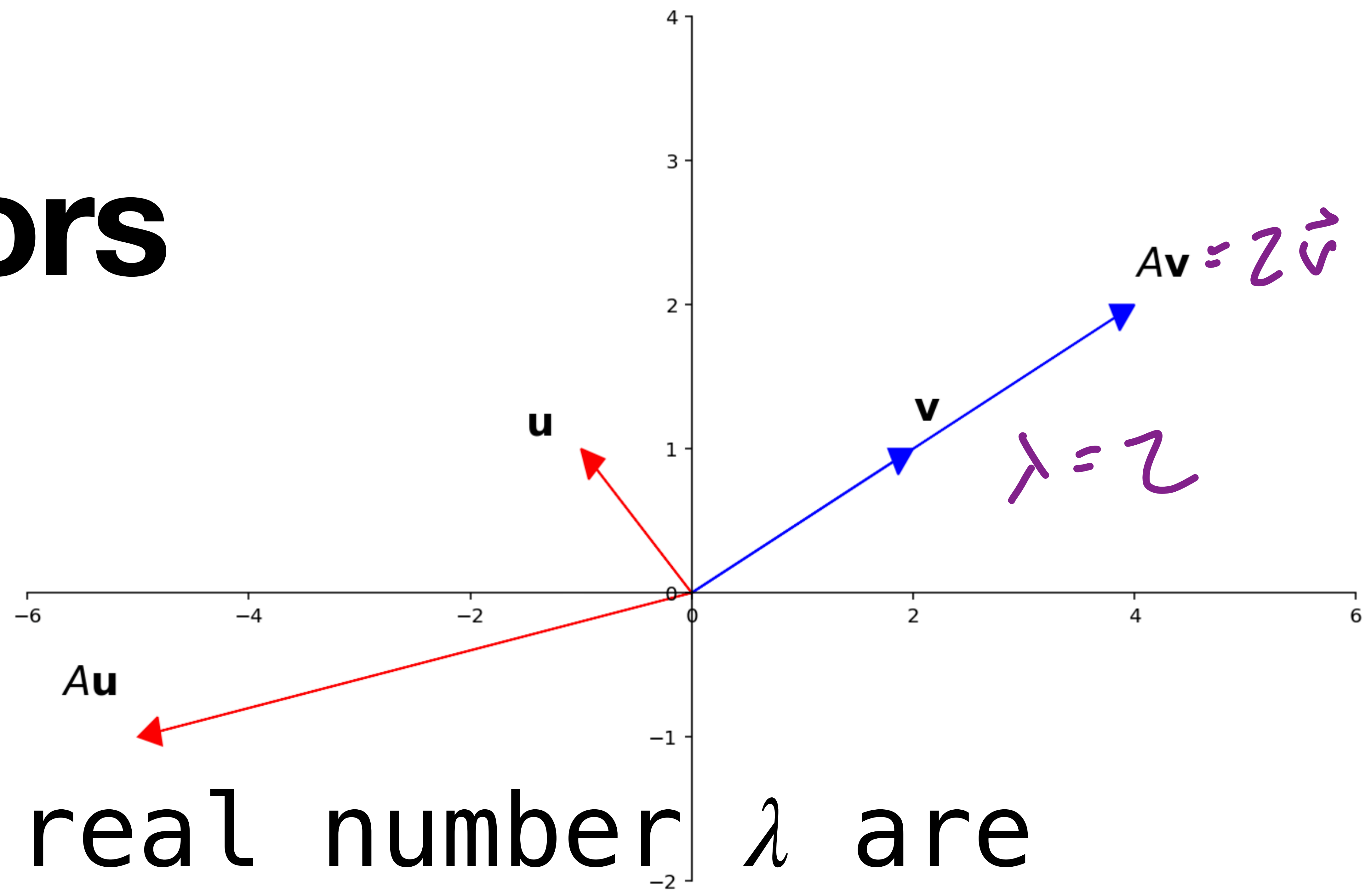
polynomial roots

triangular matrices

multiplicity

Recap

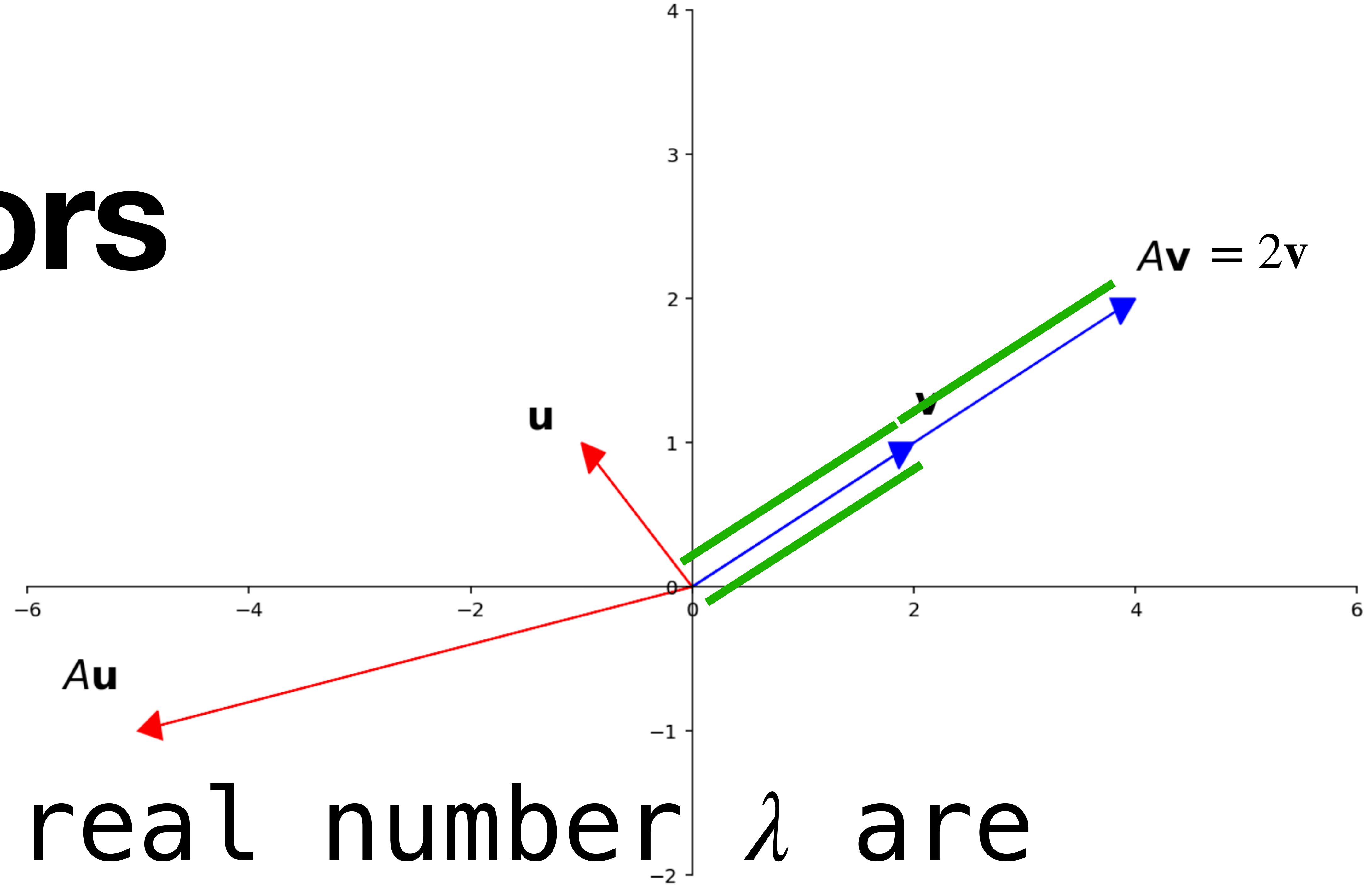
Recall: Eigenvalues/vectors



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

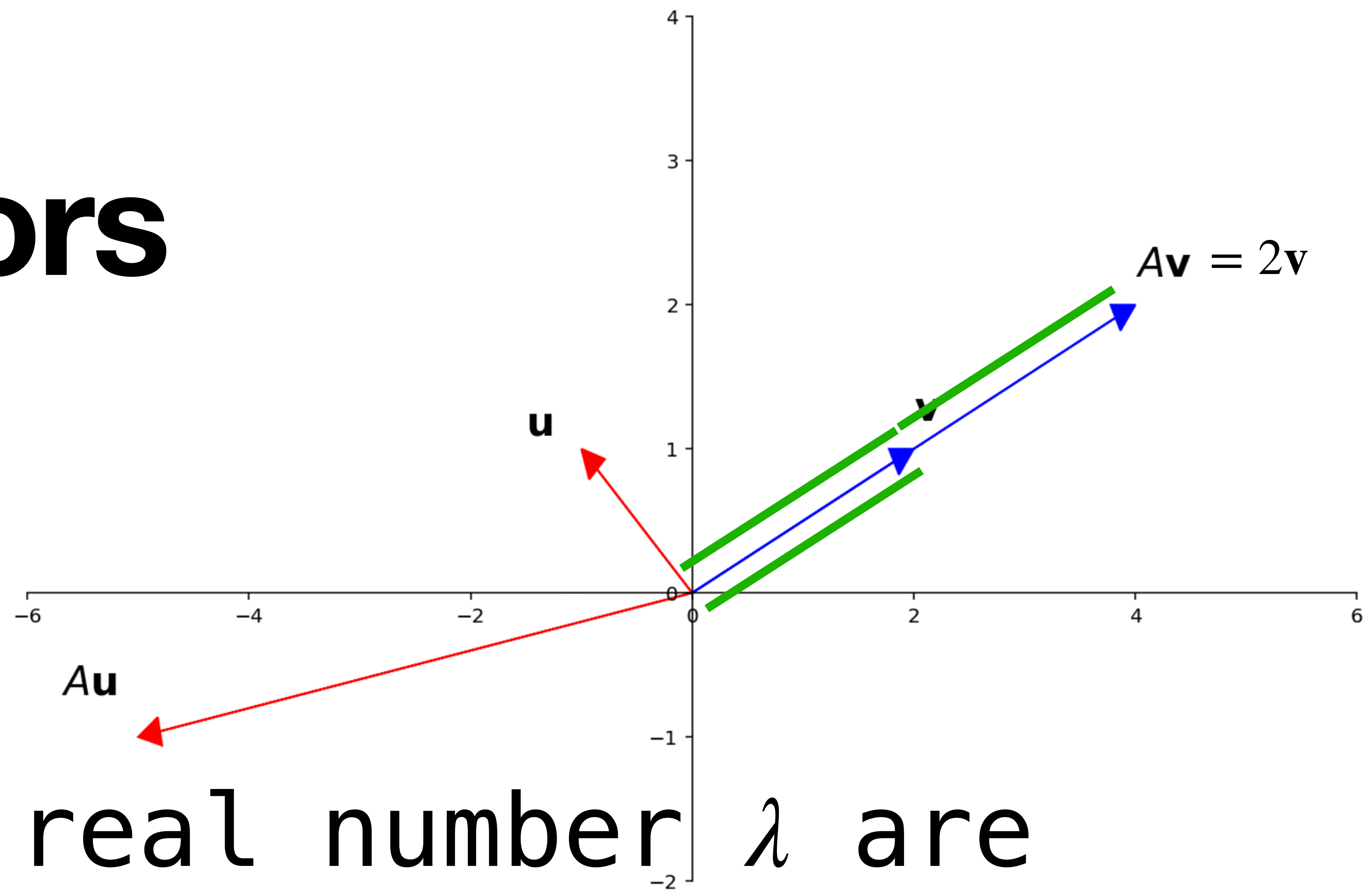
Recall: Eigenvalues/vectors



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

Recall: Eigenvalues/vectors



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

$\mathbf{v}$ is "just scaled" by A, not rotated

Recall: Verifying Eigenvectors

Recall: Verifying Eigenvectors

Question. Determine if $\mathbf{v}$ is an eigenvector of A and determine the corresponding eigenvalues.

Recall: Verifying Eigenvectors

Question. Determine if $\mathbf{v}$ is an eigenvector of A and determine the corresponding eigenvalues.

Solution. Easy. Work out the matrix-vector multiplication.

Recall: Verifying Eigenvectors

Question. Determine if $\mathbf{v}$ is an eigenvector of A and determine the corresponding eigenvalues.

Solution. Easy. Work out the matrix-vector multiplication.

Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix} \times$$

Recall: Verifying Eigenvalues

Recall: Verifying Eigenvalues

Question. Find an eigenvector of A whose corresponding eigenvalue is λ.

Recall: Verifying Eigenvalues

Question. Find an eigenvector of A whose corresponding eigenvalue is λ.

Solution. Find a nontrivial solution to

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

std. state. $\equiv$ eigen vecs $\lambda = 1$

Recall: Verifying Eigenvalues

Question. Find an eigenvector of A whose corresponding eigenvalue is λ.

Solution. Find a nontrivial solution to

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

*If we don't need the vector we can just show that $A - \lambda I$ is **not** invertible (by IMT).*

Recall: Finding Eigenspaces

Recall: Finding Eigenspaces

Question. Find a basis for the eigenspace of A corresponding to λ.

Recall: Finding Eigenspaces

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\mathrm{Nul}(A - \lambda I)$.

sol. to $(A - \lambda I)\vec{x} = 0$

Recall: Finding Eigenspaces

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\mathrm{Nul}(A - \lambda I)$.

(we did this for our recap problem)

How do eigenvectors relate to linear dynamical systems?

Recall: (Closed-Form) Solutions

Recall: (Closed-Form) Solutions

A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

Recall: (Closed-Form) Solutions

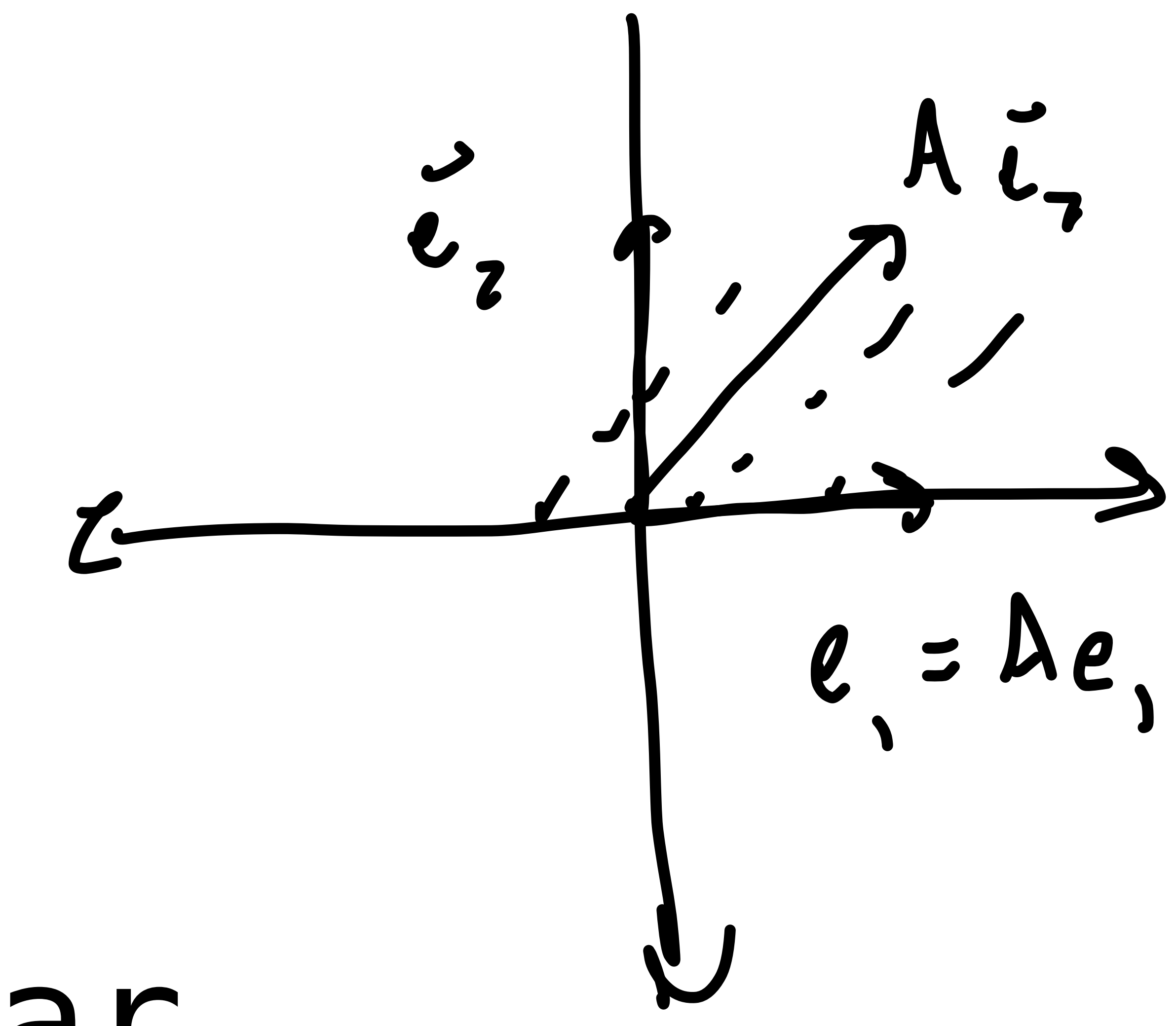
A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

In other word, it does not depend on A^k and is not **recursive**

Example

shearing

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Determine a closed for the above linear dynamical system.

$$v_k = A^k \vec{v}_0 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+k \\ 1 \end{bmatrix}$$

Solutions with Eigenvectors as Initial States

Solutions with Eigenvectors as Initial States

It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

Solutions with Eigenvectors as Initial States

It's easy to give a closed-form solution if the initial state is an eigenvector:

No dependence on A^k or $\mathbf{v}_{k-1}$

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

Solutions with Eigenvectors as Initial States

It's easy to give a closed-form solution if the initial state is an eigenvector:

No dependence on A^k or $\mathbf{v}_{k-1}$

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

The Key Point. This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of $\mathbb{R}^2$:

$$\vec{v} = \vec{b}_1 + \vec{b}_2$$

$$A^k \vec{v} = A^k(\vec{b}_1 + \vec{b}_2) = A^k \vec{b}_1 + A^k \vec{b}_2$$

$$= \lambda_1^k \vec{b}_1 + \lambda_2 \vec{b}_2$$

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows <u>exponentially in λ_1</u>).

Verify:

$v_0 = b_1 + b_2$

$$\frac{v_k}{\lambda_1^k} = \frac{\lambda_1^k b_1}{\lambda_1^k} + \frac{\lambda_2^k b_2}{\lambda_1^k} = \vec{b}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \vec{b}_2, \quad \left(\frac{\lambda_2}{\lambda_1}\right)^k \to 0$$

Eigenbases

Eigenbases

Definition. An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix A is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of A.

Eigenbases

Definition. An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix A is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of A.

We can represent vectors as ***unique*** *linear combinations of eigenvectors.*

Eigenbases

Definition. An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix A is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of A.

We can represent vectors as ***unique*** *linear combinations of eigenvectors.*

Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if A has an eigenbasis $\mathbf{b}_1, \dots, \mathbf{b}_k$, then

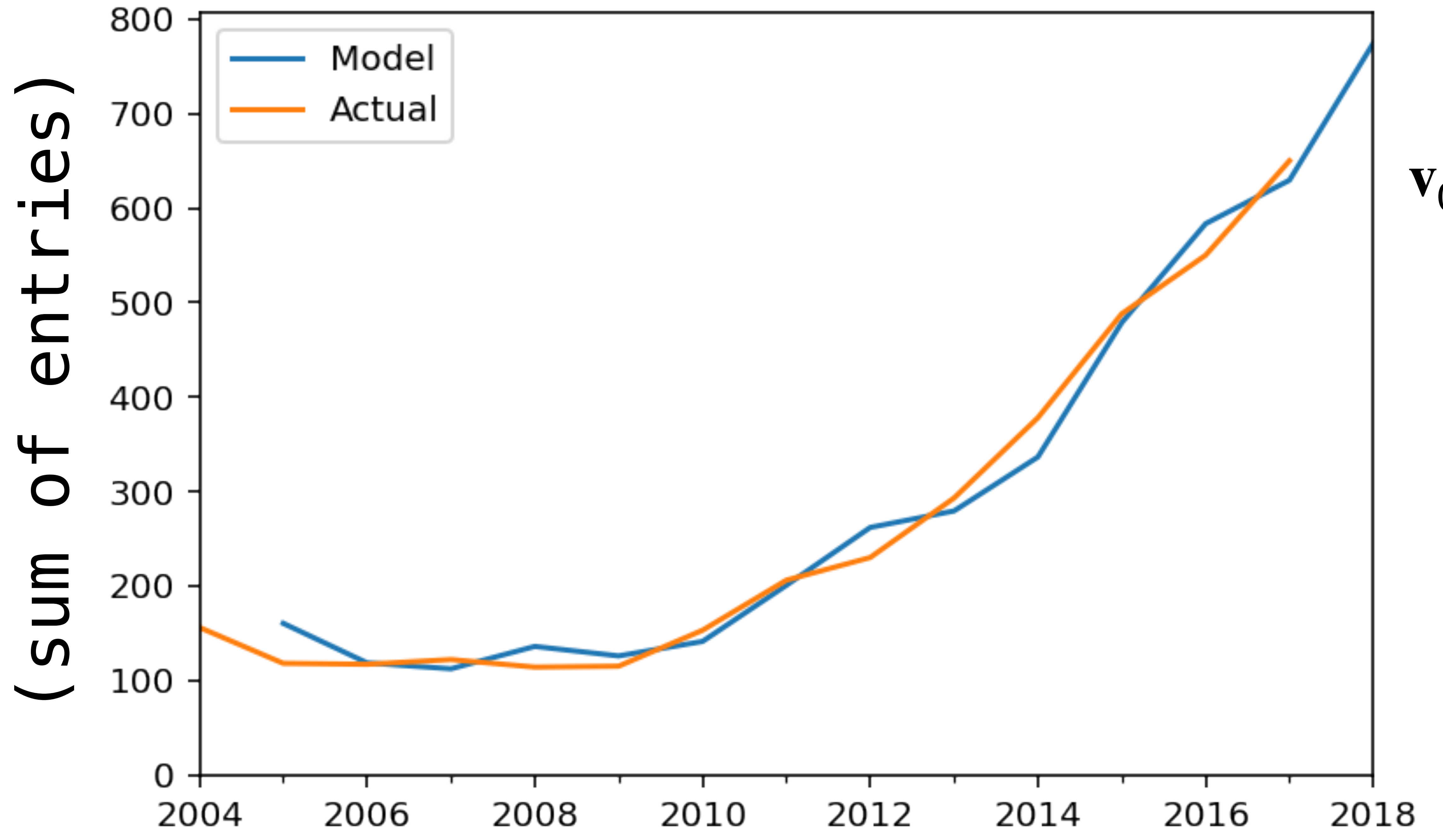
$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \text{\# of year 1 students enrolled in 2024} \\ \text{\# of year 2 students enrolled in 2024} \\ \text{\# of year 3 students enrolled in 2024} \\ \text{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0 \sim \lambda_1^k$$

largest eigenvalue

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the <u>largest eigenvalue</u>.

moving on...

Finding Eigenvalues

Finding Eigenvalues

Question. Determine the eigenvalues of A, along with their associated eigenspaces.

Finding Eigenvalues

Question. Determine the eigenvalues of A, along with their associated eigenspaces.

Solution (Idea). Can we somehow "solve for λ" in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

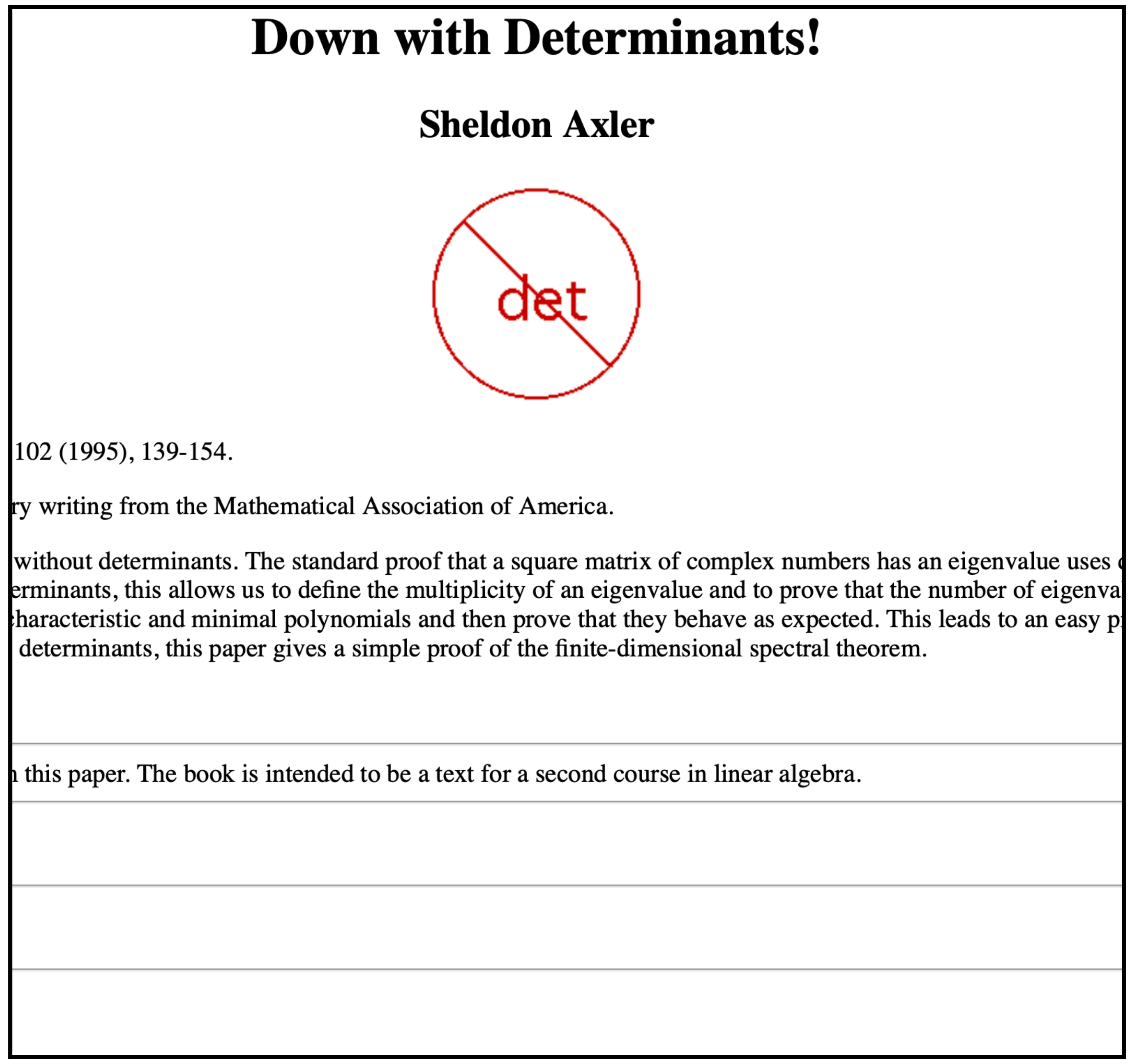
Determinants

An Aside: Determinants are Mysterious

Determinants are strangely polarizing

Some people love them, some people hate them

We'll only scratch the surface...



Down with Determinants!

Sheldon Axler

102 (1995), 139-154.

ry writing from the Mathematical Association of America.

without determinants. The standard proof that a square matrix of complex numbers has an eigenvalue uses
erminants, this allows us to define the multiplicity of an eigenvalue and to prove that the number of eigenva
haracteristic and minimal polynomials and then prove that they behave as expected. This leads to an easy p
determinants, this paper gives a simple proof of the finite-dimensional spectral theorem.

this paper. The book is intended to be a text for a second course in linear algebra.

What kind of thing is the determinant?

What kind of thing is the determinant?

A determinant is a <u>number</u> associated with a matrix.

What kind of thing is the determinant?

A determinant is a <u>number</u> associated with a matrix.

Notation. We will write $\det(A)$ for the determinant of A.

What kind of thing is the determinant?

A determinant is a <u>number</u> associated with a matrix.

Notation. We will write $\det(A)$ for the determinant of A.

In broad strokes, it's a big sum of products of entries of A.

A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
FUNCTION det(A):
  total = 0
  FOR all matrix B we can get by swapping a bunch of rows of A:
    s = 1 IF (# of swaps necessary) is even ELSE -1
    total += s * (product of the diagonal entries of B)
  RETURN total
```

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow^{0} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$(-1)^{0} ad$$

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to^1 \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$(-1)^1 cb$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow^0 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$(-1)^0 aei$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^2 \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

$$(-1)^2 gbf$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^2 \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

$$(-1)^2 dhc$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^1 \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

$$(-1)^1 gec$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^{1} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$(-1)^{1} dbi$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to^1 \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$(-1)^1 ahf$$

Another Perspective

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's row reduce an arbitrary 2×2 matrix:

$$\begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Another Perspective

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Let's row reduce an arbitrary 3×3 matrix:

$$\begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \sim \begin{bmatrix} a & b & c \\ 0 & ae-bd & af-cd \\ 0 & ah-bg & ai-cg \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ 0 & * & * \\ 0 & 0 & (ae-bd)(ai-cg)-(af-cd)(ah-bg) \end{bmatrix}$$

Determinants and Invertibility

Determinants and Invertibility

Theorem. A matrix is invertible if and only if $\det(A) \neq 0$.

Determinants and Invertibility

Theorem. A matrix is invertible if and only if $\det(A) \neq 0$.

So we can yet again extend the IMT:

Determinants and Invertibility

Theorem. A matrix is invertible if and only if $\det(A) \neq 0$.

So we can yet again extend the IMT:

» A is invertible
» $\det(A) \neq 0$
» 0 is not an eigenvalue

These must be all true or all false.

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

Defintion. The **determinant** of a matrix A is given by the above equation, where

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

Defintion. The **determinant** of a matrix A is given by the above equation, where

- U is an echelon form of A

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \ldots U_{nn}$$

Defintion. The **determinant** of a matrix A is given by the above equation, where

- U is an echelon form of A
- s is the number of row swaps used to get U

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

Defintion. The **determinant** of a matrix A is given by the above equation, where

- U is an echelon form of A
- s is the number of row swaps used to get U
- c is the product of all scalings used to get U

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \ldots U_{nn}$$

product of diagonal entries

Defintion. The **determinant** of a matrix A is given by the above equation, where

- U is an echelon form of A
- s is the number of row swaps used to get U
- c is the product of all scalings used to get U

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

product of diagonal entries

0 if A is not invertible

Defintion. The **determinant** of a matrix A is given by the above equation, where

- U is an echelon form of A
- s is the number of row swaps used to get U
- c is the product of all scalings used to get U

Example

$$\frac{(-1)^s}{c} u_{11} u_{22} u_{33}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

$s = 0 \quad c = \not{1}\ 3$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow 3R_3} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{(-1)^0}{3} (1)(-6)(1) = \boxed{-2}$$

Example (Again)

$$\frac{(-1)^s}{c} u_{11} u_{22} u_{33} =$$

$$\frac{(-1)^1}{3}(1)(6)(1) = \boxed{-2}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

$s = \not{0}\ 1 \quad c = \not{1}\ \ \tfrac{1}{\not{2}}(\not{2})(3)$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1/2} \begin{bmatrix} 1 & 2 & -1/2 \\ 1^{-1} & 5^{-2} & 0^{+1/2} \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{bmatrix} 1 & 2 & -1/2 \\ 0 & 3 & 1/2 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow[R_3 \leftarrow 3R_3]{R_2 \leftarrow 2R_2} \begin{bmatrix} 1 & 2 & -1/2 \\ 0 & 6 & 1 \\ 0 & -6 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 2 & -1/2 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The definition holds no matter which sequence of row operations you use.

How To: Determinants

How To: Determinants

Question. Determine the determinant of a matrix A.

How To: Determinants

Question. Determine the determinant of a matrix A.

Solution.

How To: Determinants

Question. Determine the determinant of a matrix A.

Solution.

1. Convert A to an echelon form U.

How To: Determinants

Question. Determine the determinant of a matrix A.

Solution.

1. Convert A to an echelon form U.

2. Keep track of the number of row swaps you used, call this s, and the product of all scalings, call this c

How To: Determinants

Question. Determine the determinant of a matrix A.

Solution.

1. Convert A to an echelon form U.

2. Keep track of the number of row swaps you used, call this s, and the product of all scalings, call this c

3. Determine the product of entries along the diagonal of U, call this P.

How To: Determinants

Question. Determine the determinant of a matrix A.

Solution.

1. Convert A to an echelon form U.

2. Keep track of the number of row swaps you used, call this s, and the product of all scalings, call this c

3. Determine the product of entries along the diagonal of U, call this P.

4. The determinant of A is $\frac{(-1)^s P}{c}$.

The Shorter Version

Beyond small matrices, we'll just use a computer

With NumPy:

numpy.linalg.det(A)

Properties of Determinants

Properties of Determinants (1)

$$\det(AB) = \det(A)\det(B)$$

It follows that AB is invertible if and only if A and B are invertible

(we won't verify this)

Example Question

$A A^{-1} = I$

Use the fact that $\det(AB) = \det(A)\det(B)$ *to give an expression for* $\det(A^{-1})$ *in terms of* $\det(A)$.

Hint. What is $\det(I)$?

$$\det(I) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that A^T is invertible if and only if A is invertible.

(we also won't verify this)

Example Question

If $A^{-1} = A^T$*, then what are the possible values of* $\det(A)$*?*

$\det(A) = 1, -1$

$$\det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{\det(A^T)}$$

$$\det(A^T)^2 = 1$$

Answer

Properties of Determinants (3)

Theorem. If A is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:

upper triangular

$$\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = 6$$

in echelon form

lower triangular

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 5 & -2 & 0 \\ 2 & 1 & 6 \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & 5 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 6 \end{bmatrix} = -12$$

Question

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

Find the determinant of the above matrix.

Answer

Characteristic Equation

What kind of thing is the determinant, really?

What kind of thing is the determinant, really?

The determinant of a matrix A is an <u>arithmetic expression</u> written in terms of the entries of A.

What kind of thing is the determinant, really?

The determinant of a matrix A is an arithmetic expression written in terms of the entries of A.

But a matrix may not have numbers as entries.

What kind of thing is the determinant, really?

The determinant of a matrix A is an arithmetic expression written in terms of the entries of A.

But a matrix may not have numbers as entries.

We might think of the matrix $A - \lambda I$ has having *polynomials* as entries.

What kind of thing is the determinant, really?

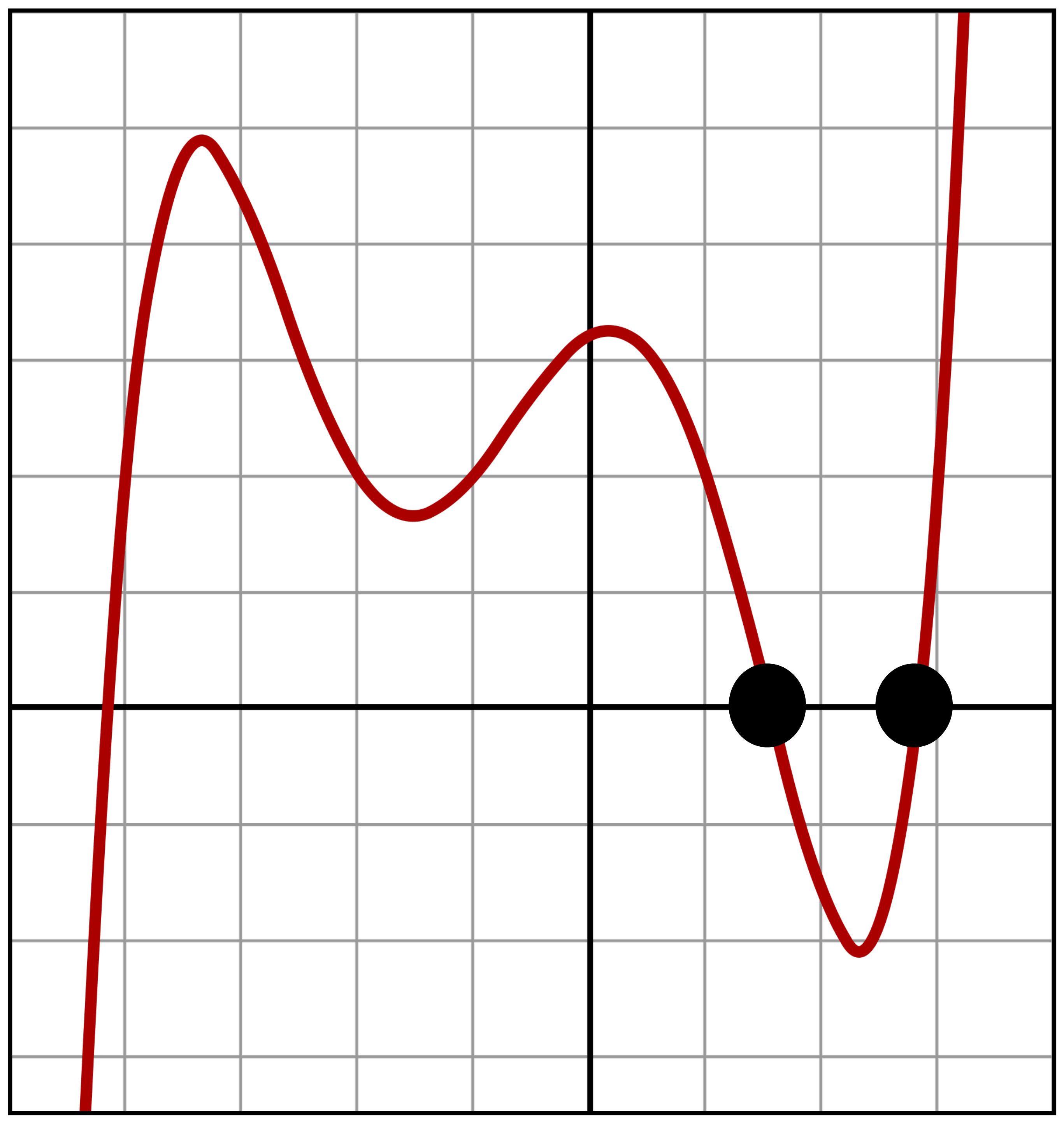
The determinant of a matrix A is an <u>arithmetic expression</u> written in terms of the entries of A.

But a matrix may not have numbers as entries.

We might think of the matrix $A - \lambda I$ has having *polynomials* as entries.

Then $\det(A - \lambda I)$ is a **polynomial.**

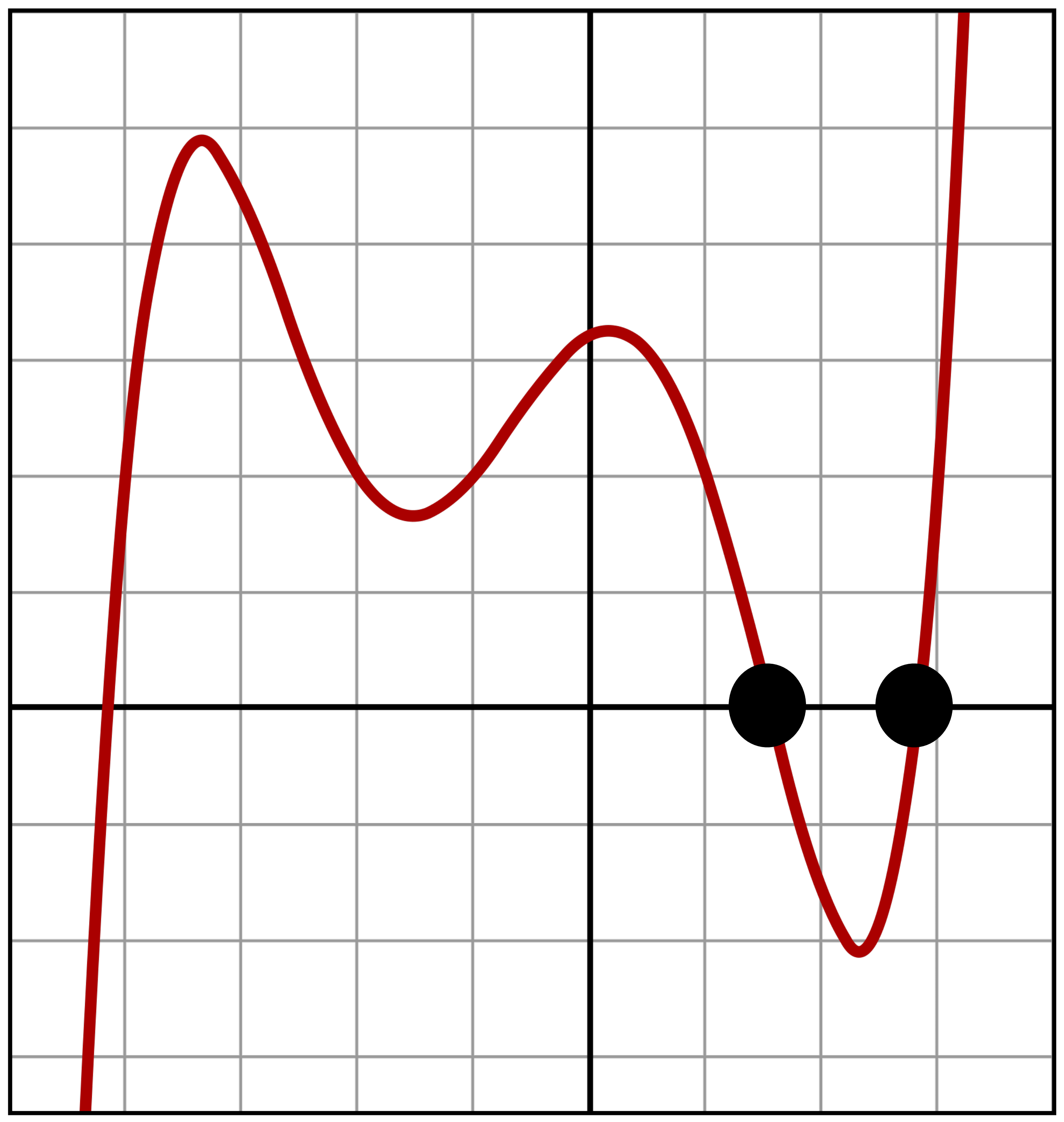
Reminder: Polynomial Roots



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

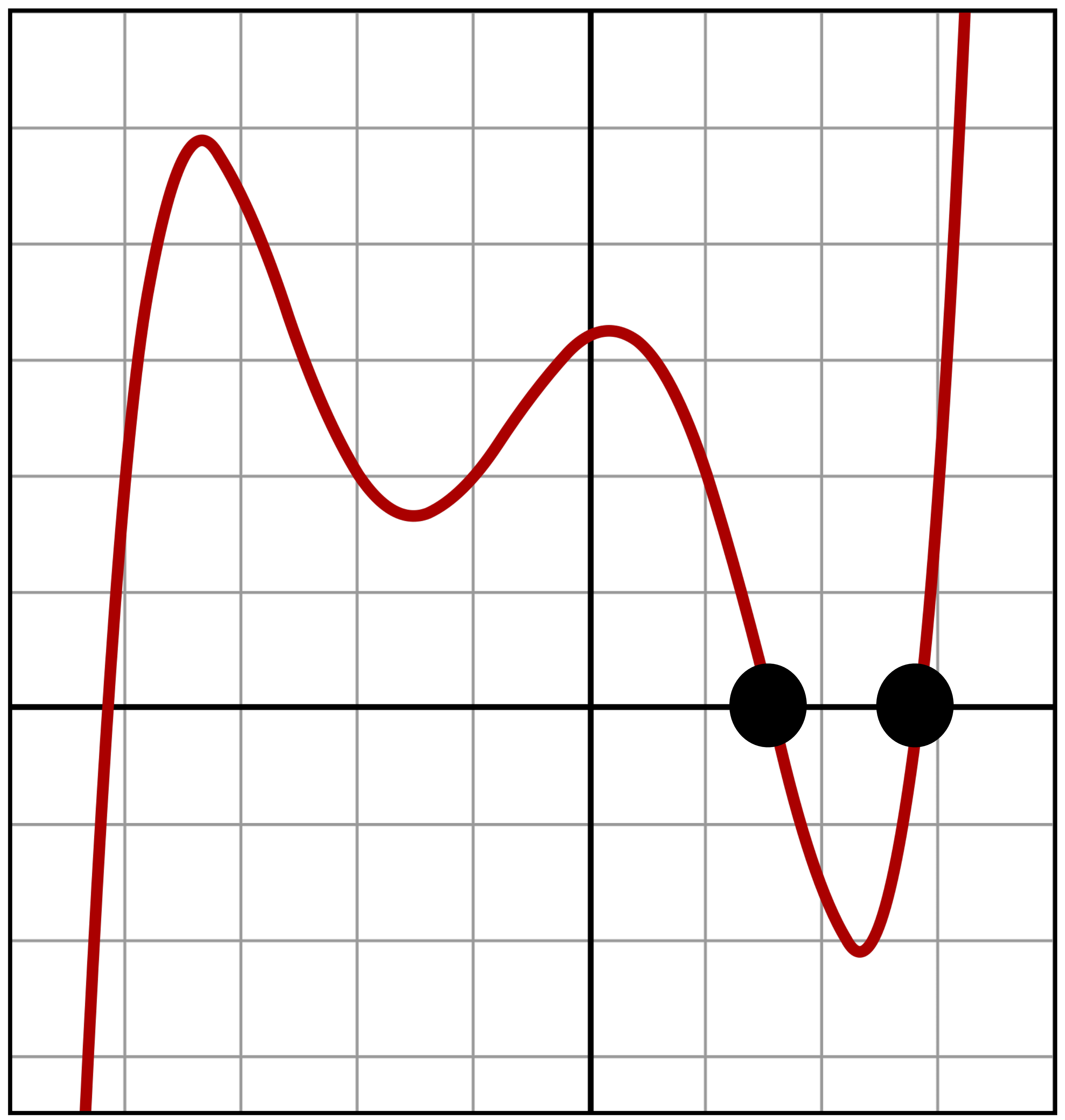


https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

(A polynomial may have many roots)



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

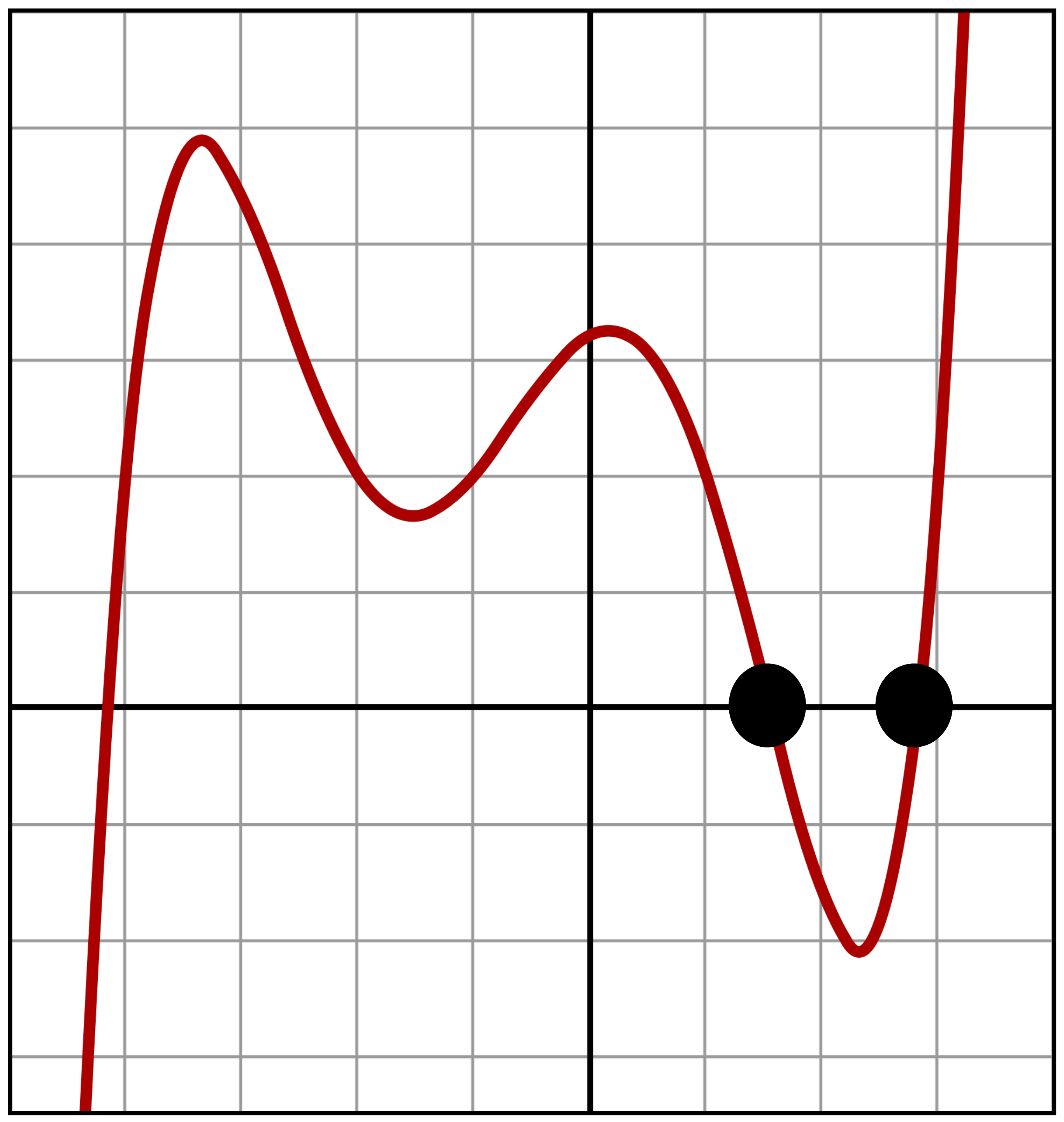
Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

(A polynomial may have many roots)

If r is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

Characteristic Polynomial

Characteristic Polynomial

Definition. The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* λ.

Characteristic Polynomial

Definition. The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* λ.

This is a polynomial with the eigenvalues of A as roots.

Characteristic Polynomial

Definition. The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* λ.

This is a polynomial with the eigenvalues of A as roots.

So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

Example: 2×2 **Matrix**

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1$$

$$= \lambda^2 - \lambda - 1$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{1 \pm \sqrt{1+4}}{2} = \lambda_1 = \frac{1+\sqrt{5}}{2} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

A Special Linear Dynamical System

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^k \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}F_{k+1}\\ F_k\end{bmatrix}$$

$$\mathbf{v}_{k+1} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix}1\\ 0\end{bmatrix}$$

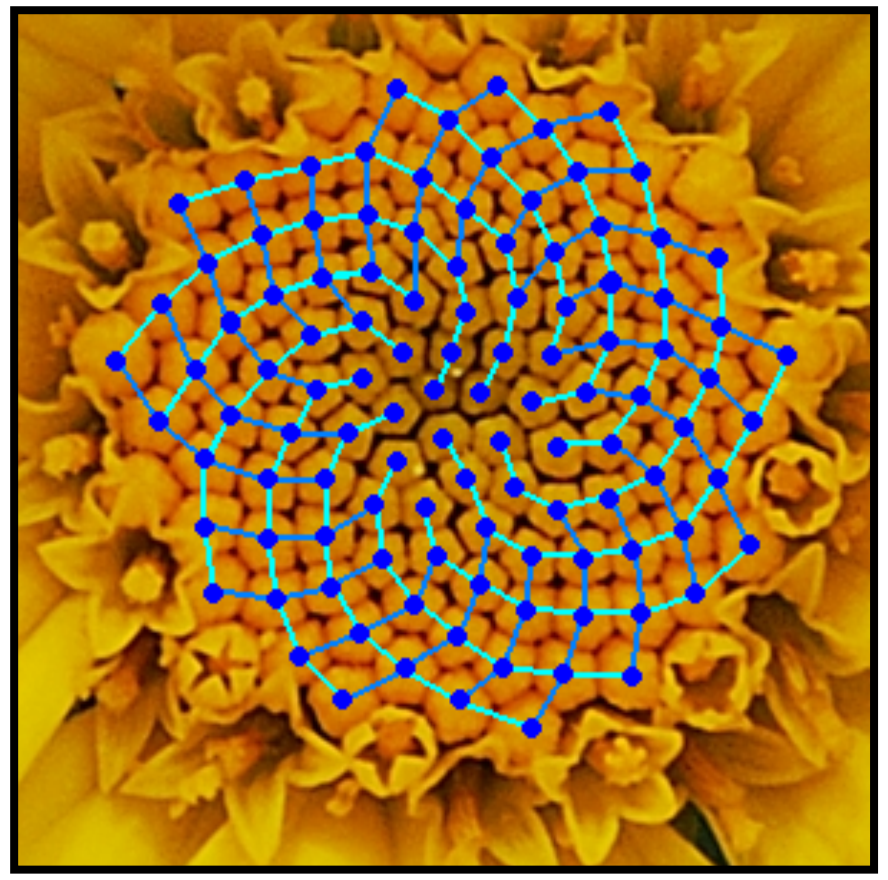
Consider the system given by the above matrix.

What does this system represent?:

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} \quad \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix} \quad \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}3\\ 2\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}5\\ 3\end{bmatrix} \cdots$$

Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

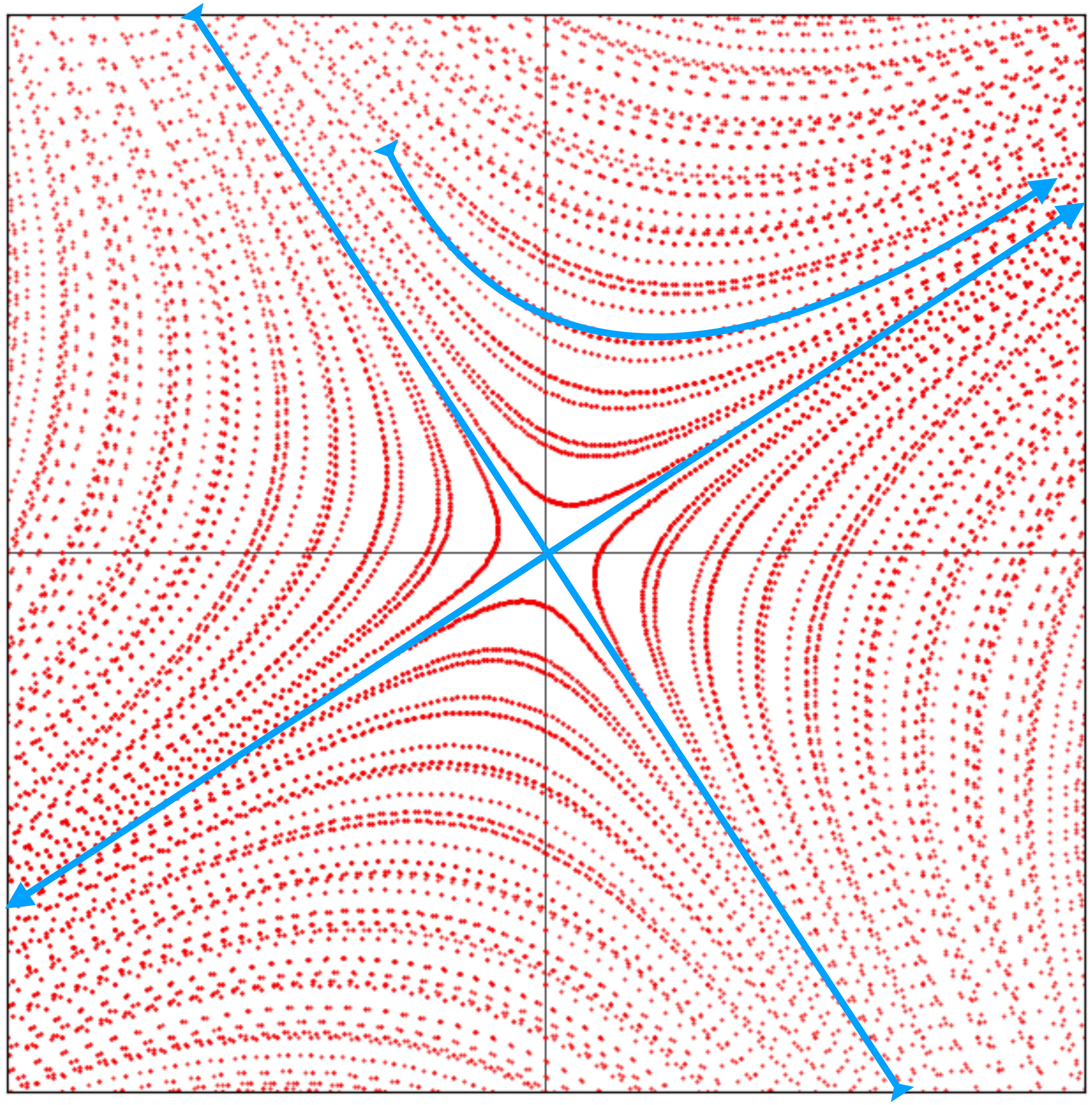
```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature, engineering, etc.

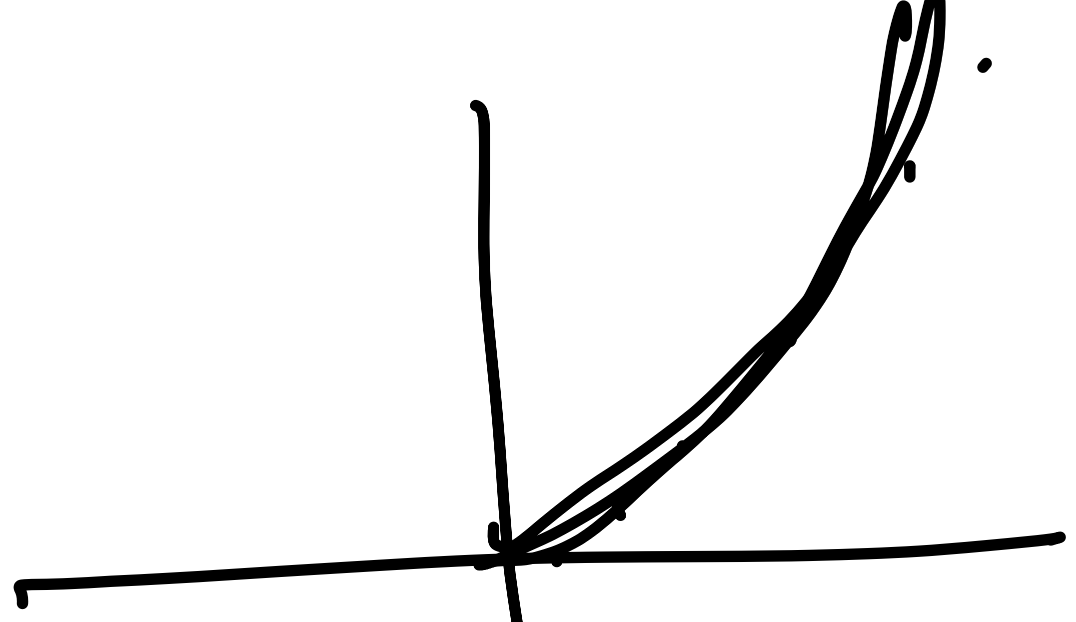
https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG

Recall: The Picture



The eigenvalue of this matrix is the golden ratio

Golden Ratio



$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \frac{F_{k+1}}{F_k} \to \varphi \text{ as } k \to \infty$$

This is the largest eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

The "long term behavior" is the Fibonacci sequence is defined by the golden ratio:

$$F_k = \Theta(\varphi^k)$$

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

$$\begin{bmatrix} 1-\lambda & 0 & * & * \\ 0 & -\lambda & * & * \\ 0 & 0 & 1-\lambda & * \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix}$$

$$(1-\lambda)^2(-\lambda)(4-\lambda)$$

$$\lambda = 1, 0, 4$$

How To: Finding Eigenvalues

How To: Finding Eigenvalues

Question. Find all eigenvalues of the matrix A.

How To: Finding Eigenvalues

Question. Find all eigenvalues of the matrix A.

Solution. Find the roots of the characteristic polynomial of A.

An Observation: Multiplicity

$$\lambda^1(\lambda - 1)^2(\lambda - 4)^1$$ multiplicities

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$.

The multiplicity is an upper bound on "how large" the eigenspace is.

Example

Let A be a 5×5 matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$.

» What is $\text{rank}(A)$?

» What is the minimum possible rank of $A - I$?