

The Characteristic Equation

Geometric Algorithms

Lecture 19

Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Determine the dimension of the eigenspace of A for the eigenvalue 4.

(try not to do any row reductions)

Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Objectives

1. Briefly recap eigenvalues and eigenvectors
2. Get a primer on determinants
3. Determine how to find eigenvalues (not just verify them)

Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

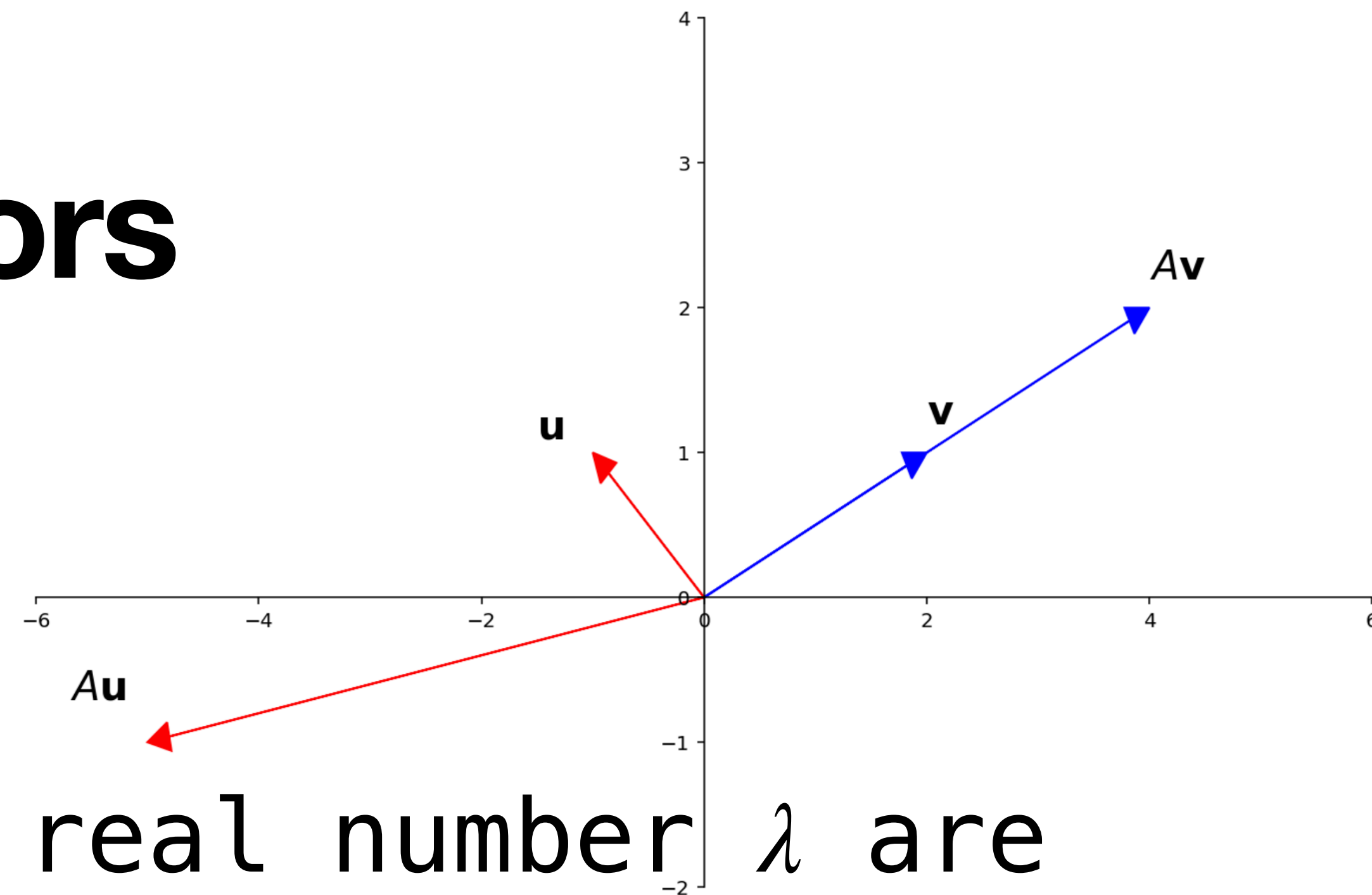
polynomial roots

triangular matrices

multiplicity

Recap

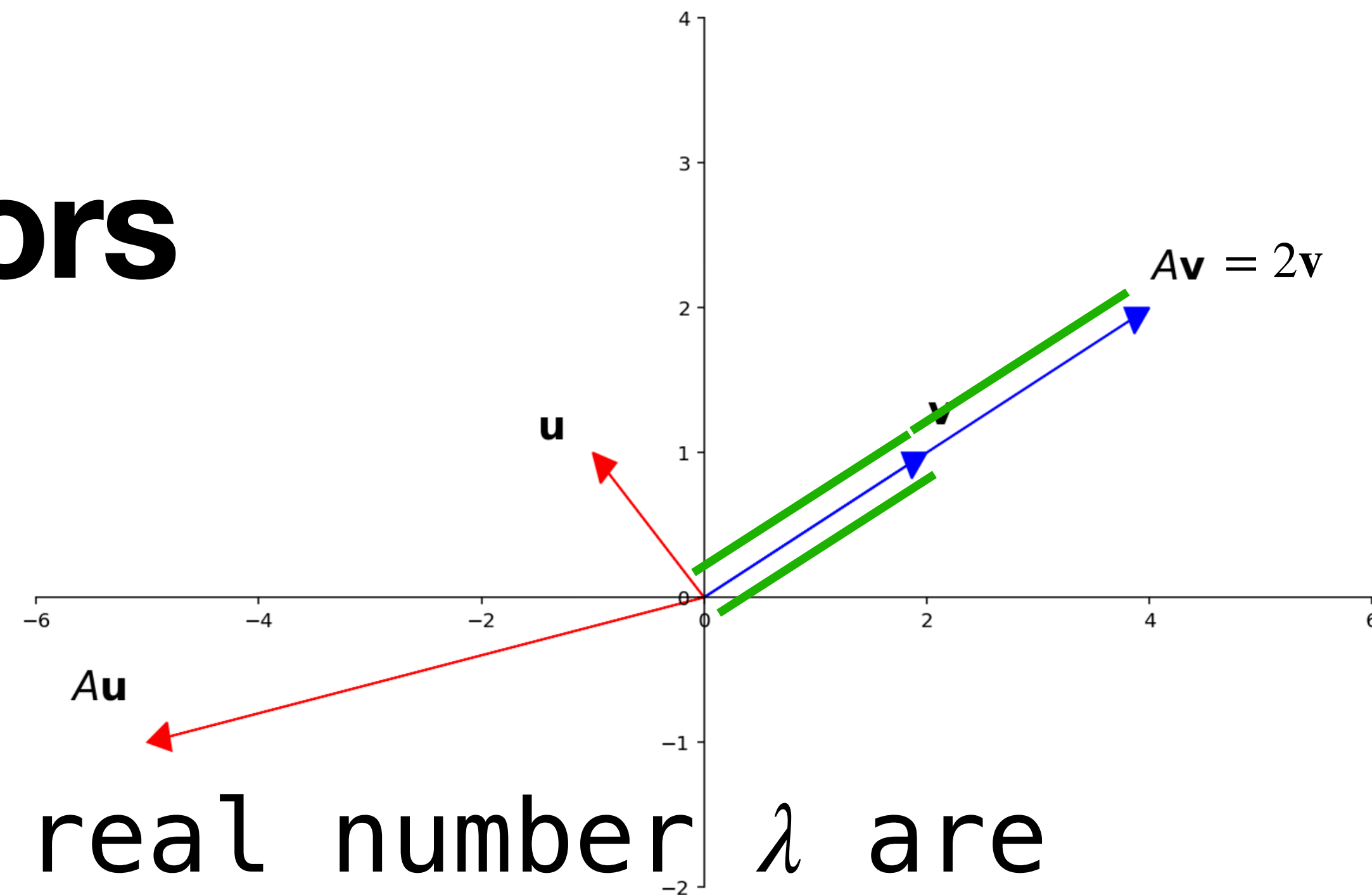
Recall: Eigenvalues/vectors



A *nonzero* vector \mathbf{v} in \mathbb{R}^n and real number λ are an **eigenvector** and **eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

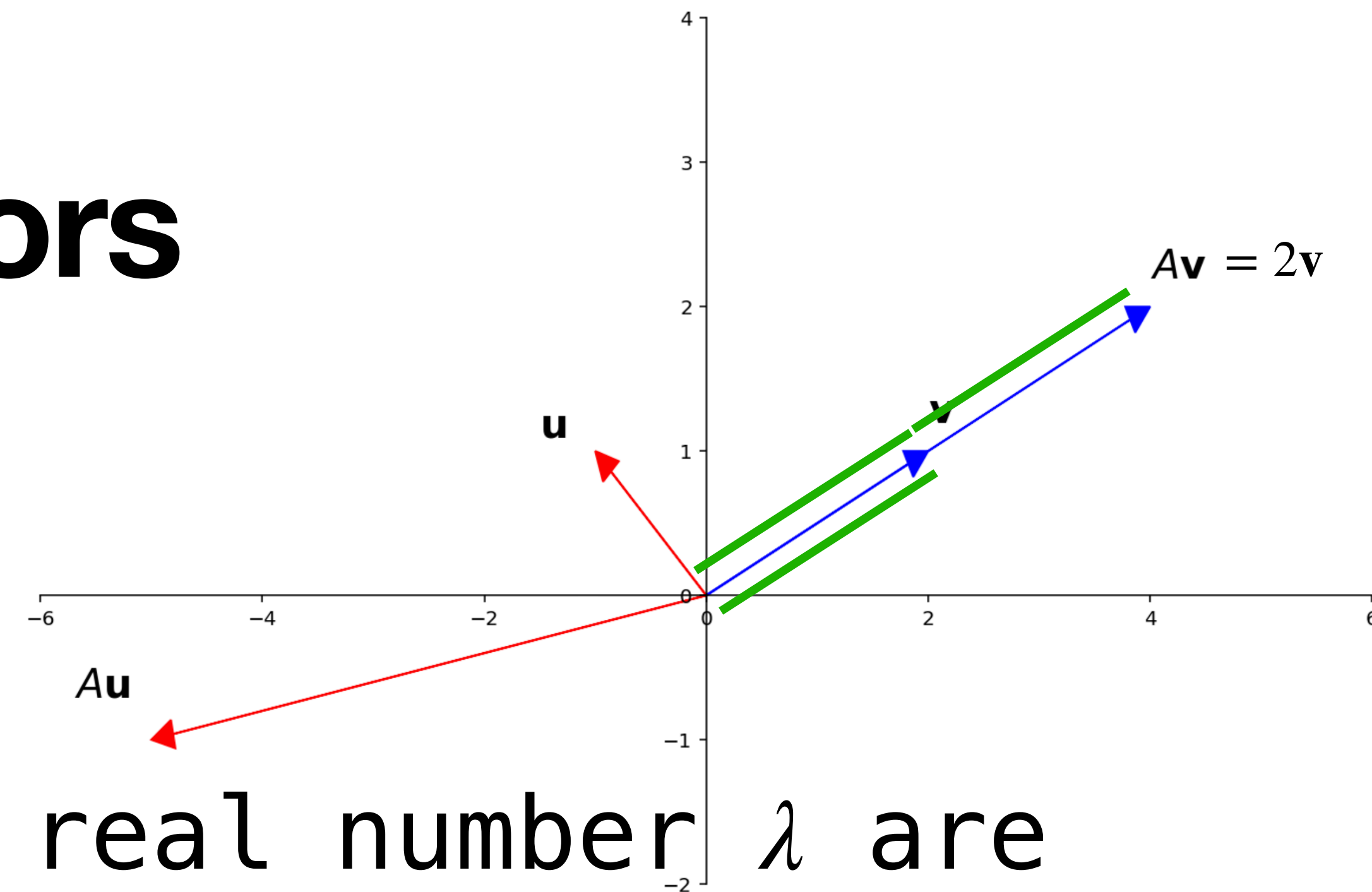
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$$Av = \lambda v$$

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A *nonzero* vector \mathbf{v} in \mathbb{R}^n and real number λ are an **eigenvector** and **eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

\mathbf{v} is "just scaled" by A , not rotated

Recall: Verifying Eigenvectors

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Solution. Easy. Work out the matrix–vector multiplication.

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix} \quad \times$$

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$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

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*If we don't need the vector we can just show that $A - \lambda I$ is **not** invertible (by IMT).*

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Solution. Find a basis for $\text{Nul}(A - \lambda I)$.

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(we did this for our recap problem)

How do eigenvectors relate
to linear dynamical systems?

Recall: (Closed-Form) Solutions

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A (closed-form) solution of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for \mathbf{v}_k which is does **not** contain A^k or previously defined terms

Recall: (Closed-Form) Solutions

A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for \mathbf{v}_k which is **not** contain A^k or previously defined terms

In other word, it does not depend on A^k and is **not recursive**

Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed form for the above linear dynamical system.

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

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The Key Point. This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k \mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of \mathbb{R}^2 :

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state \mathbf{v}_0 , if \mathbf{v}_0 can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 > \lambda_2 \dots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows exponentially in λ_1).

Verify:

Eigenbases

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Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

Theorem. For a linear dynamical system A with initial state \mathbf{v}_0 , if A has an eigenbasis $\mathbf{b}_1, \dots, \mathbf{b}_k$, then

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for some constant c_1 , where where λ_1 is the **largest eigenvalue of A** and \mathbf{b}_1 **is its eigenvector**.

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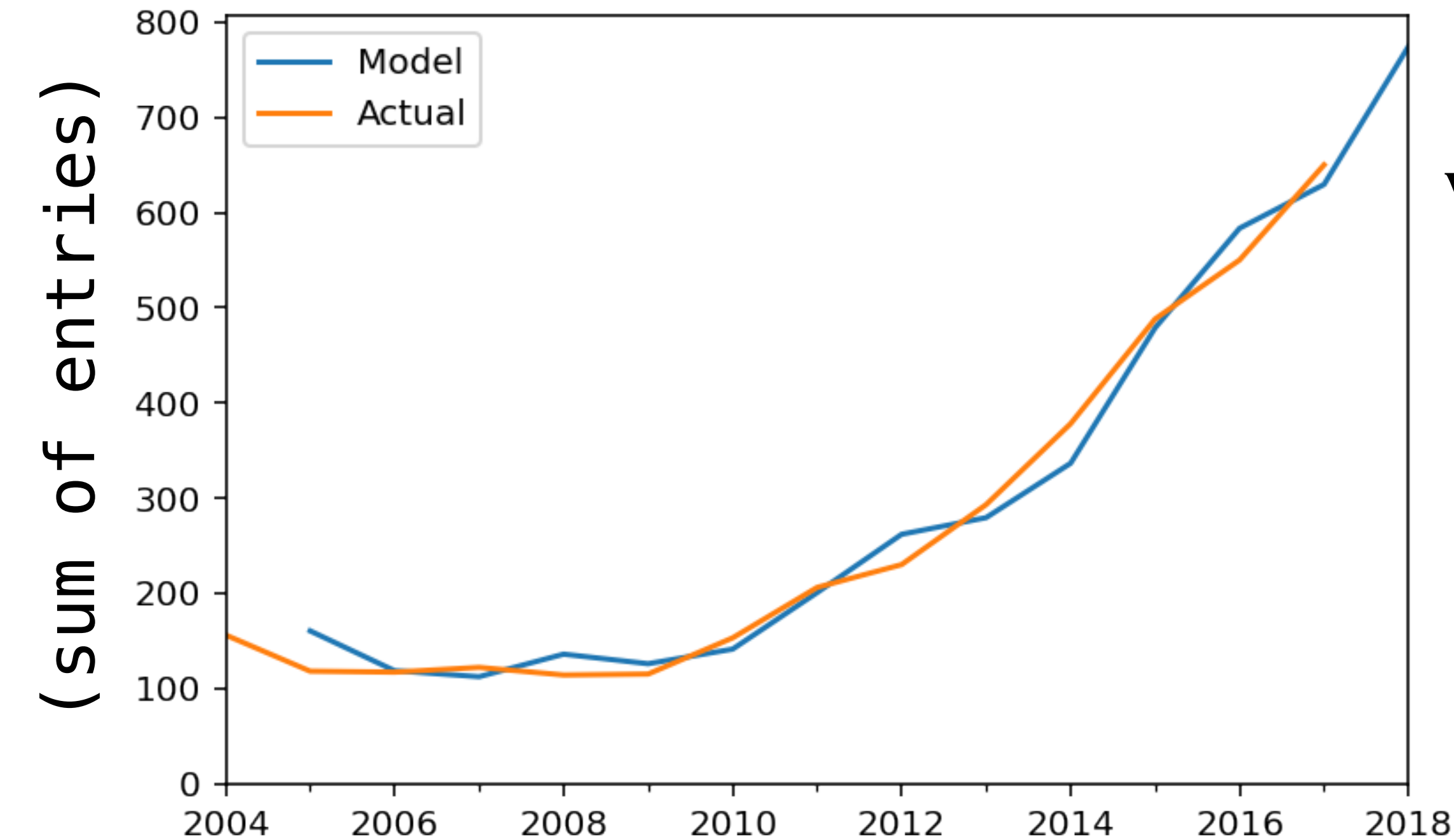
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The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \# \text{ of year 1 students enrolled in 2024} \\ \# \text{ of year 2 students enrolled in 2024} \\ \# \text{ of year 3 students enrolled in 2024} \\ \# \text{ of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the largest eigenvalue.

moving on . . .

Finding Eigenvalues

Finding Eigenvalues

Question. Determine the eigenvalues of A , along with their associated eigenspaces.

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Solution (Idea). Can we somehow "solve for λ " in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Determinants

An Aside: Determinants are Mysterious


Determinants are
strangely polarizing

Some people love them,
some people hate them

We'll only scratch the
surface...

Down with Determinants!

Sheldon Axler



102 (1995), 139-154.

ry writing from the Mathematical Association of America.

without determinants. The standard proof that a square matrix of complex numbers has an eigenvalue uses \det . Without determinants, this allows us to define the multiplicity of an eigenvalue and to prove that the number of eigenvalues equals the dimension of the space. Characteristic and minimal polynomials and then prove that they behave as expected. This leads to an easy proof of the finite-dimensional spectral theorem.

in this paper. The book is intended to be a text for a second course in linear algebra.

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In broad strokes, it's a big sum of products of entries of A .

A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
1 FUNCTION det(A):  
2   total = 0  
3   FOR all matrix B we can get by swapping a bunch of rows of A:  
4     s = 1 IF (# of swaps necessary) is even ELSE -1  
5     total += s * (product of the diagonal entries of B)  
6   RETURN total
```

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$(-1)^0 ad$$

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The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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The Determinant of 3×3 matrices

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{2} \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

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The Determinant of 3×3 matrices

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Another Perspective

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's row reduce an arbitrary 2×2 matrix:

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Theorem. A matrix is invertible if and only if $\det(A) \neq 0$.

So we can yet again extend the IMT:

- » A is invertible
- » $\det(A) \neq 0$
- » 0 is not an eigenvalue

These must be all true or all false.

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \cdots U_{nn}$$

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$c = 0$ if A is not invertible

Definition. The determinant of a matrix A is given by the above equation, where

- U is an echelon form of A
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Example

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

Example (Again)

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

The definition holds no matter
which sequence of row
operations you use.

How To: Determinants

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4. The determinant of A is $\frac{(-1)^s P}{c}$.

The Shorter Version

Beyond small matrices, we'll just use a computer

With NumPy:

`numpy.linalg.det(A)`

Properties of Determinants

Properties of Determinants (1)

$$\det(AB) = \det(A) \det(B)$$

It follows that AB is invertible if and only if A and B are invertible

(we won't verify this)

Example Question

Use the fact that $\det(AB) = \det(A)\det(B)$ to give an expression for $\det(A^{-1})$ in terms of $\det(A)$.

Hint. What is $\det(I)$?

Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that A^T is invertible if and only if A is invertible.

(we also won't verify this)

Example Question

If $A^{-1} = A^T$, then what are the possible values of $\det(A)$?

Answer

Properties of Determinants (3)

Theorem. If A is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:

Question

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

Find the determinant of the above matrix.

Answer

Characteristic Equation

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We might think of the matrix $A - \lambda I$ as having *polynomials* as entries.

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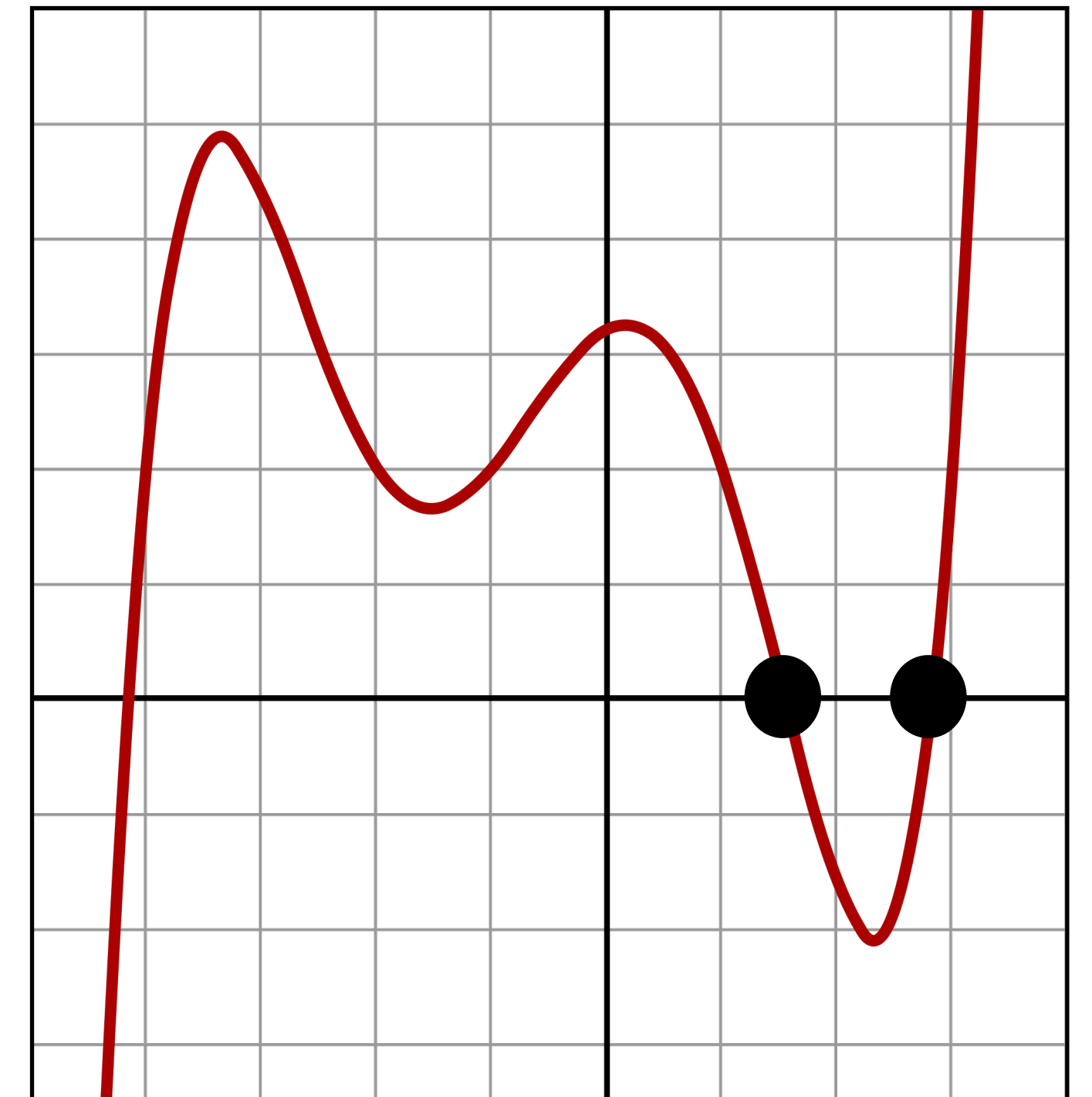
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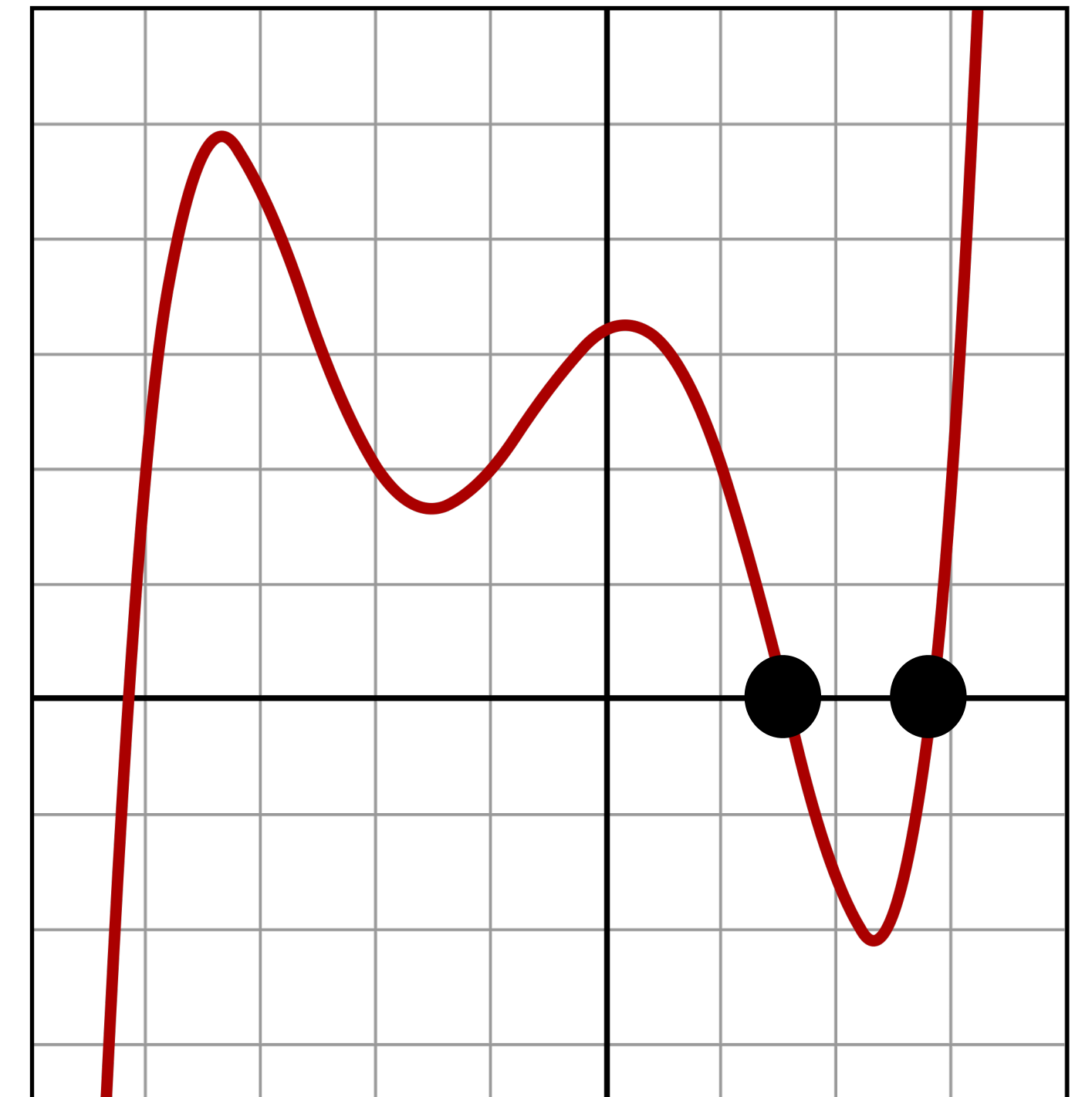
Then $\det(A - \lambda I)$ is a **polynomial**.

Reminder: Polynomial Roots



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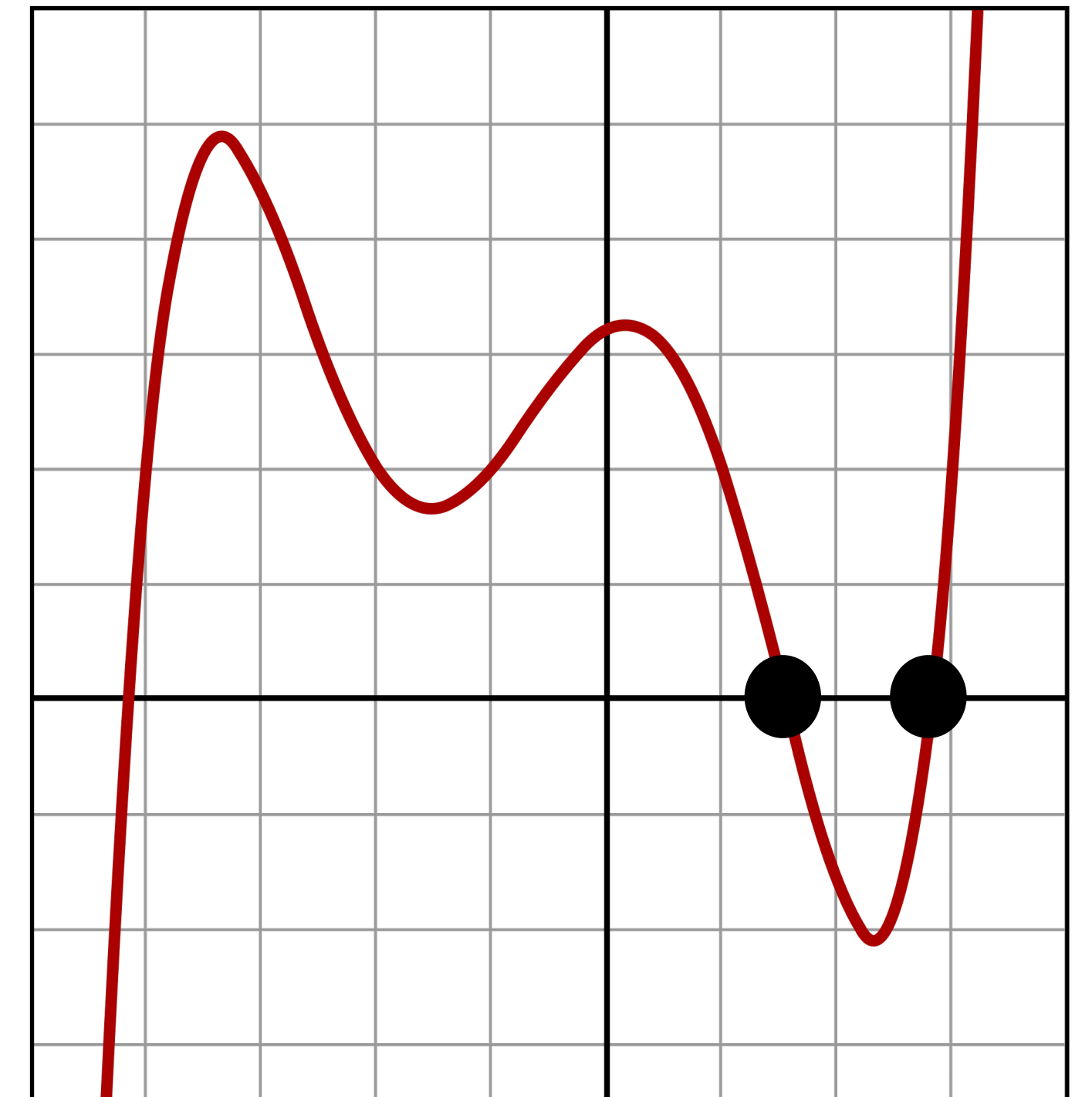
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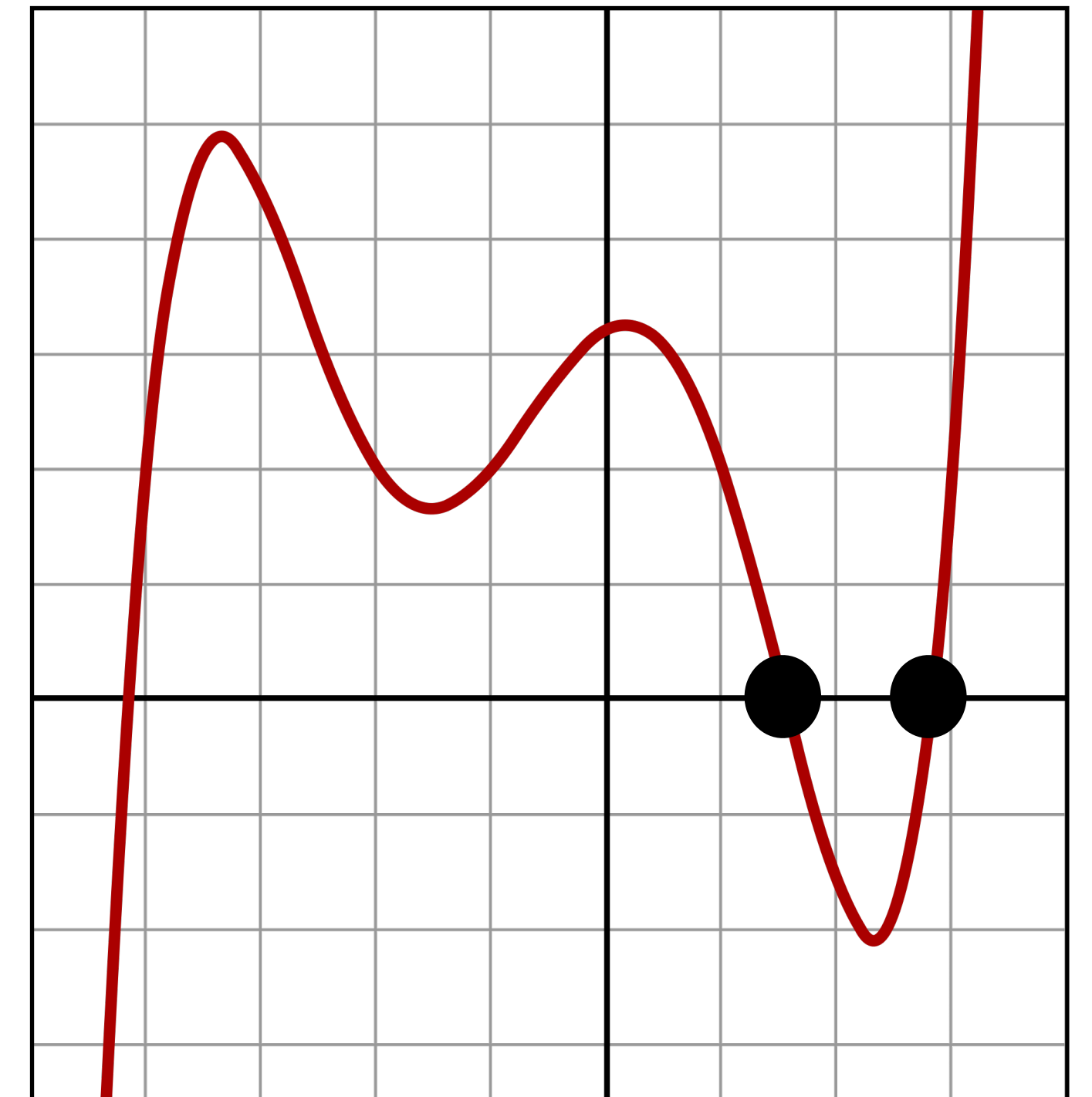
Reminder: Polynomial Roots

A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

(A polynomial may have many roots)

If r is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



Characteristic Polynomial

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Definition. The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* λ .

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Definition. The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* λ .

This is a polynomial with the eigenvalues of A as roots.

So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

Example: 2×2 Matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the system given by the above matrix.

What does this system represent?:

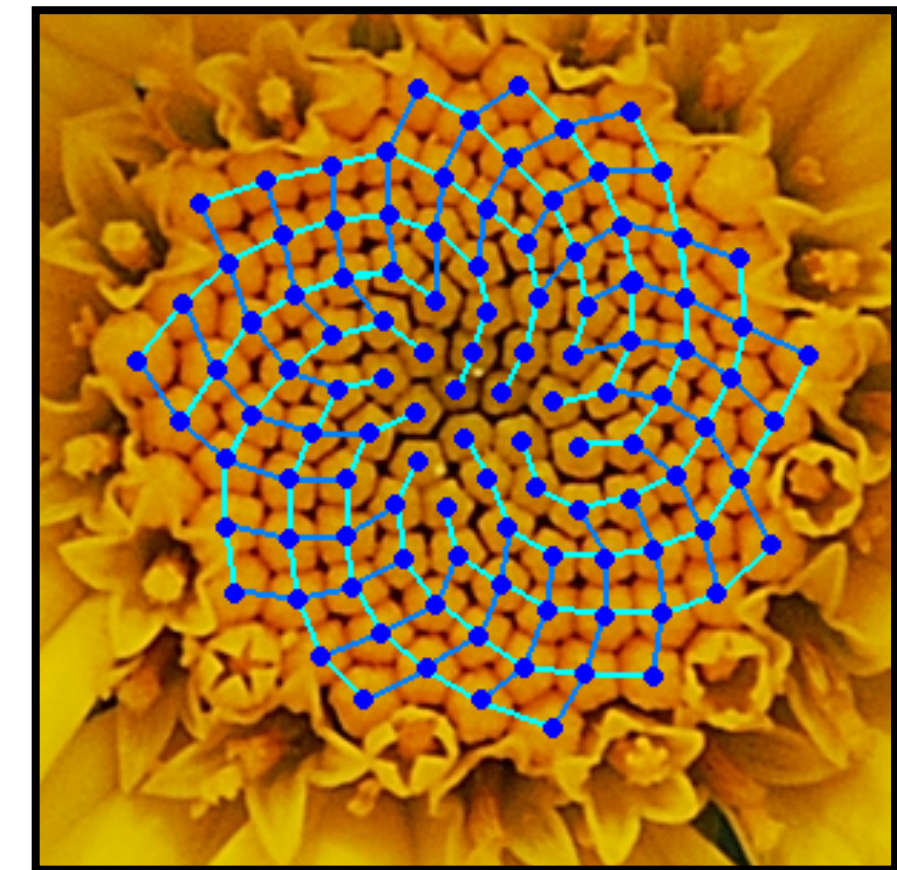
Fibonacci Numbers

$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

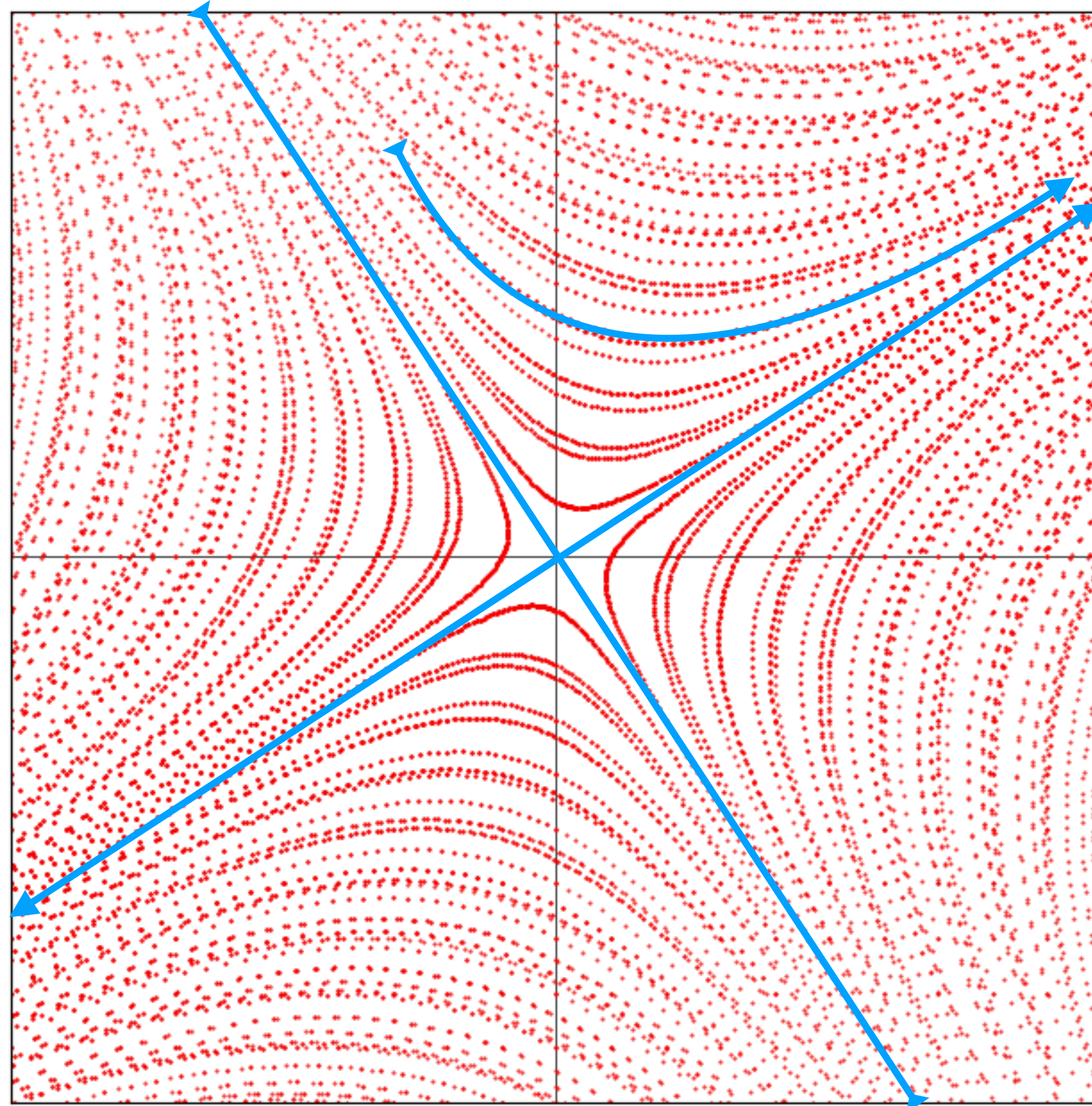
```
define fib(n):  
  curr, next ← 0, 1  
  repeat n times:  
    curr, next ← next, curr + next  
  return curr
```



The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature, engineering, etc.

Recall: The Picture



The eigenvalue of
this matrix is the
golden ratio

Golden Ratio

$$\varphi = \frac{1 + \sqrt{5}}{2} \quad \frac{F_{k+1}}{F_k} \rightarrow \varphi \text{ as } k \rightarrow \infty$$

This is the largest eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

The "long term behavior" is the Fibonacci sequence is defined by the golden ratio:

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

How To: Finding Eigenvalues

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Question. Find all eigenvalues of the matrix A .

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Solution. Find the roots of the characteristic polynomial of A .

An Observation: Multiplicity

$$\lambda^1 (\lambda - 1)^2 (\lambda - 4)^1 \text{ multiplicities}$$

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$.

The multiplicity is an upper bound on "how large" the eigenspace is.

Example

Let A be a 5×5 matrix with characteristic polynomial $(x - 1)^3(x - 3)(x + 5)$.

» What is $\text{rank}(A)$?

» What is the minimum possible rank of $A - I$?