Diagonalization

Geometric Algorithms
Lecture 20

CAS CS 132

Objectives

1. Finish our discussion on the characteristic polynomial
2. Motivate diagonalization via linear dynamical systems and changes of coordinate systems
3. Describe how to diagonalize a matrix

Keywords

multiplicity

similar matrices

diagonalizable matrices

change of basis

eigenbasis

Recap: Characteristic Polynomial

Recall: Determinants and Invertibility

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$\equiv$ λ is an eigenvalue of A

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polynomial in λ

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How To: Finding Eigenvalues

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Question. Determine the eigenvalues of A.

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Solution. Find the *roots* of the characteristic polynomial of A, which is

$$\det(A - \lambda I)$$

viewed as a *polynomial* in λ.

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We'll also use

numpy.linalg.eig(A)

Example

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda) - (-4)$$

$$= -3 - \lambda + 3\lambda + \lambda^2 + 4 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \quad \checkmark$$

$$\lambda = -1$$

$$\begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 4-6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

solve $(A - (-1)I)\vec{x} = \vec{0}$

$$A + I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2} x_2$

x_2 is free

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eig. vect. with eig. val. -1

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes <u>pre-factored</u>:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -3 & 0 & 6 \\ 0 & -\lambda & 1 & 1 \\ 0 & 0 & (1-\lambda) & 2 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= \boxed{(1-\lambda)^2(-\lambda)(4-\lambda)}$$

An Observation: Multiplicity

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This is called the **(algebraic) multiplicity** of the root

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Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$ (and at least 1)

The multiplicity is an upper bound on "how large" the eigenspace is

all possible sol. to $(A-I)\vec{x}=0$

$A\vec{x}=\vec{x}$

$\shortparallel$

$\mathrm{Nul}(A-I)$ = eigspace of $\lambda=1$

Example

Let A be a 5×5 matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$

» *What is* $\mathrm{rank}(A)$? 5, since 0 is not an eigenvalue

» *What is the minimum possible rank of $A-I$?*

$\mathrm{rank}(A-I) + \dim(\mathrm{Nul}(A-I)) = 5$

$\shortparallel$

dim of eig. space of 1 ≤ 3

so

$\mathrm{rank}(A-I) \geq 2$

Practice Problem

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

Determine the eigenvalues and an eigenbasis for the above matrix

Challenge: Show that any 2×2 matrix with positive entries must have 2 distinct eigenvalues (Hint: discriminant)

Answer

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5+4 \\ -4+8 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \checkmark$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix} = (5-\lambda)(2-\lambda) - 4$$

$$= 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$$

$\lambda = 6$:

Solve $(A - 6I)\vec{x} = \vec{0}$ $\quad A - 6I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2$, x_2 is free

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5+1 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$\lambda = 1$:

$$A - I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/4 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -1/4\, x_2$, x_2 is free $\quad \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Motivating Diagonalization via Linear Dynamical Systems

Recall: Eigenbasis

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Definition. An eigenbasis of H for the matrix A is a basis of H made up of eigenvectors of A

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We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$

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We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$

The Question. **When can we describe any vector in $\mathbb{R}^n$ as a unique linear combination of eigenvectors of A?**

Recall: Linear Dynamical Systems

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$$

$$\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$$

$$\mathbf{v}_4 = A\mathbf{v}_3 = A^4\mathbf{v}_0$$

$$\vdots$$

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$

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multiplying by A changes the state.

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$

demo

Eigenbases and Closed-Form solutions

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Given $\mathbf{v}_k = A\mathbf{v}_{k-1} = A^k\mathbf{v}_0$, if

$$\mathbf{v}_0 = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$$

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then

$$A^k\mathbf{v}_0 = \alpha_1\lambda_1^k\mathbf{b}_1 + \alpha_2\lambda_2^k\mathbf{b}_2 + \alpha_3\lambda_3^k\mathbf{b}_3$$

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closed-form solution

Verify: $A^k(\alpha_1 \vec{b}_2 + \alpha_2 \vec{b}_2 + \alpha_3 \vec{b}_3) = \alpha_1 A^k \vec{b}_1 + \alpha_2 A^k \vec{b}_2 + \alpha_3 A^k \vec{b}_2$

$= \alpha_1 \lambda_1^k \vec{b}_1 + \alpha_2 \lambda_2^k \vec{b}_2 + \alpha_3 \lambda_3^k \vec{b}_3$

Application: Eigenbases and Limiting Behavior

Theorem. If A has an eigenbasis with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k \mathbf{u}$ for some vector $\mathbf{u}$.

In the long term, the system grows <u>exponentially in λ_1</u>.

if $\lambda_1 < 0$ then $\vec{v}_k \to \vec{0}$ as $k \to \infty$

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Sometimes, A behaves simply on $\mathscr{B}$, as in the case of <u>eigenbases</u>.

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Given a basis $\mathscr{B}$ for $\mathbb{R}^n$, we only need to know how $A \in \mathbb{R}^n$ behaves on $\mathscr{B}$.

Sometimes, A behaves simply on $\mathscr{B}$, as in the case of <u>eigenbases</u>.

What we're really doing is <u>changing our coordinate system</u> to expose a behavior of A.

Recap: Change of Basis

Recall: Bases define Coordinate Systems

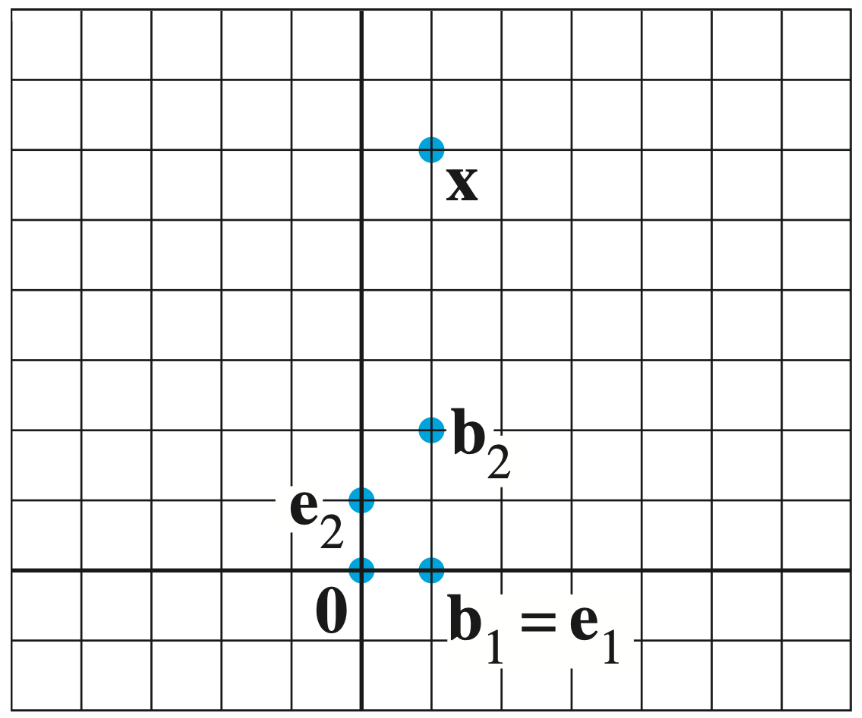

FIGURE 1 Standard graph paper.

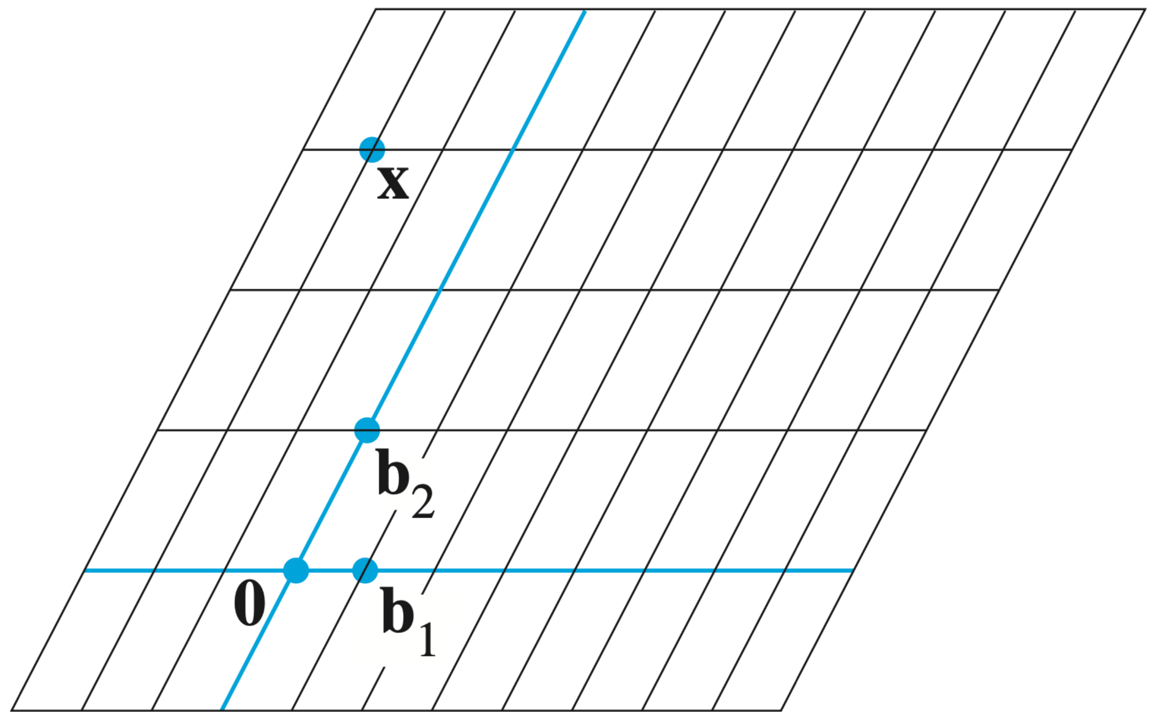

FIGURE 2 $\mathcal{B}$-graph paper.

Linear Algebra and its Applications (Lay, Lay, McDonald)

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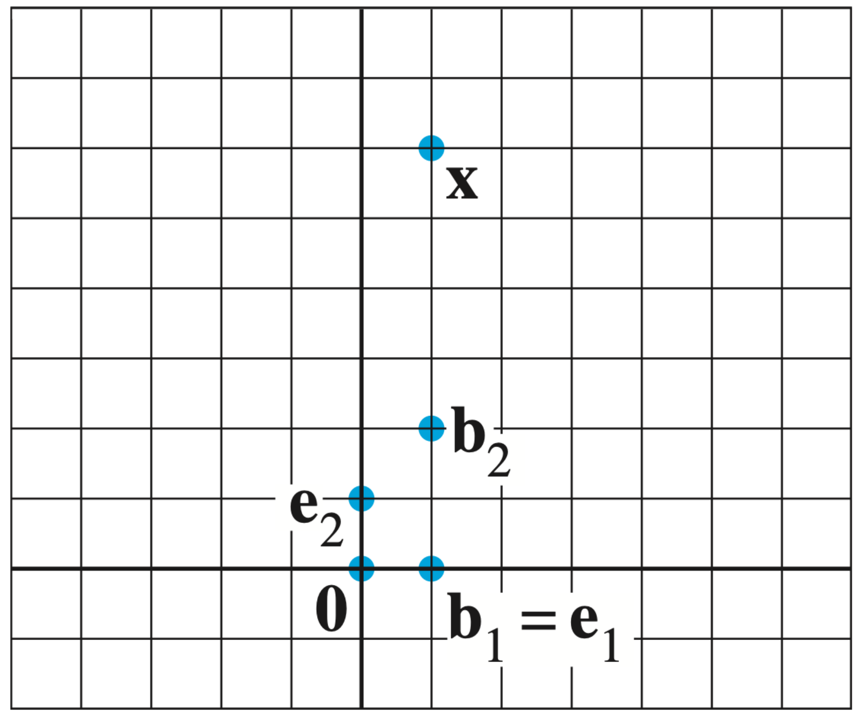

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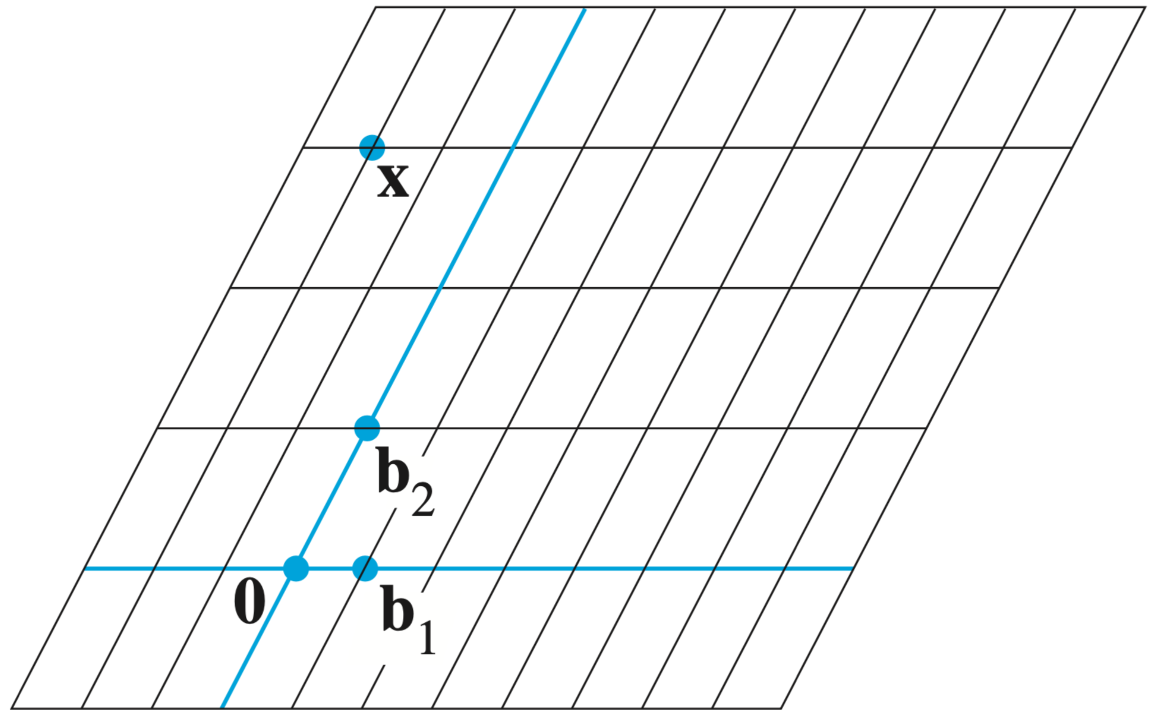

FIGURE 2 $\mathcal{B}$-graph paper.

Given a basis $\mathscr{B}$ of $\mathbb{R}^n$, there is **exactly one way** to write every vector as a linear combination of vectors in $\mathscr{B}$

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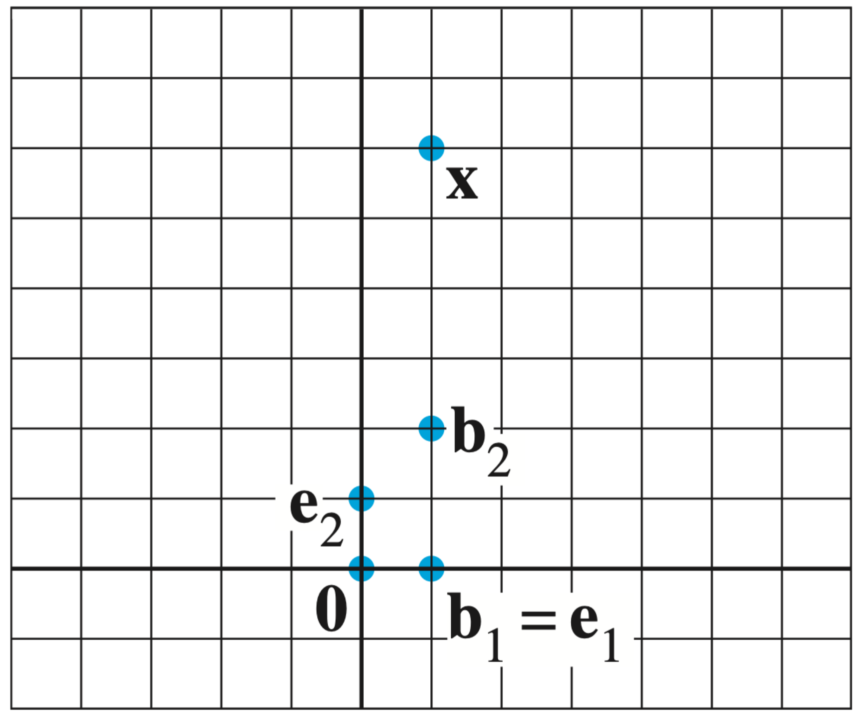

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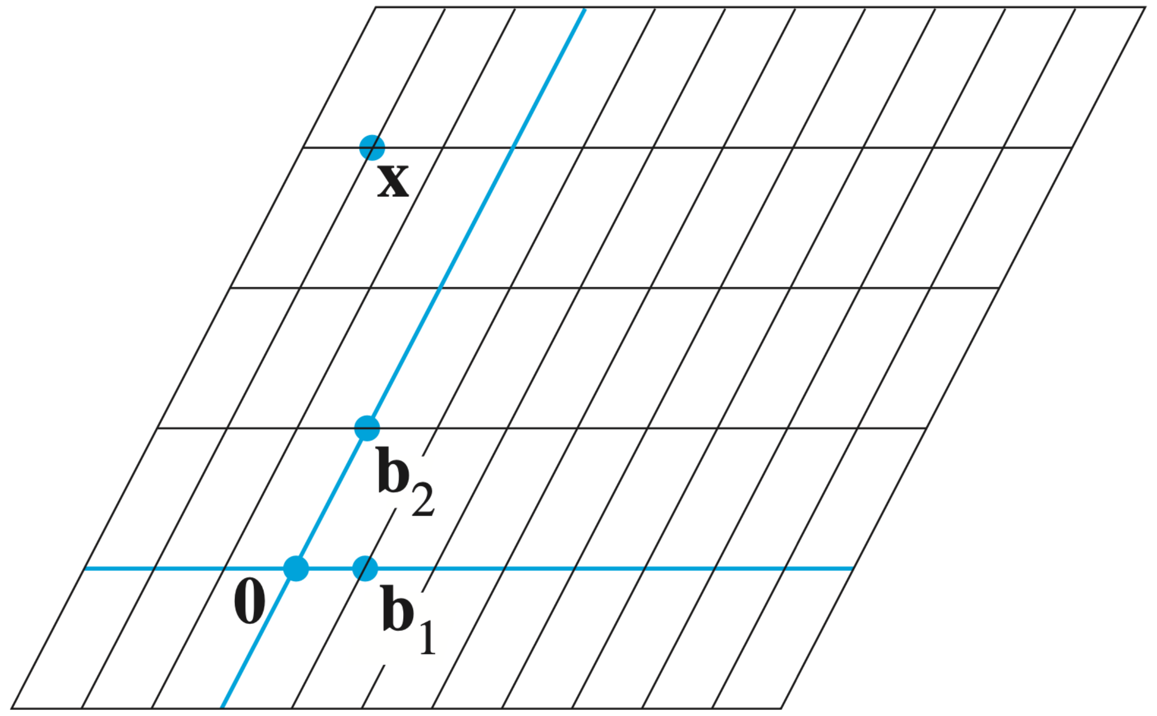

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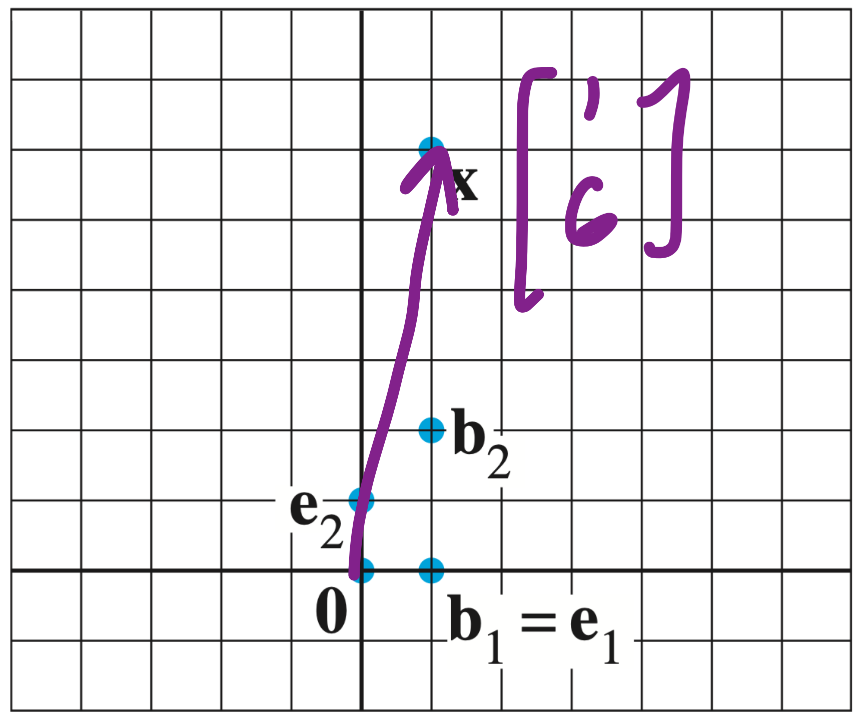

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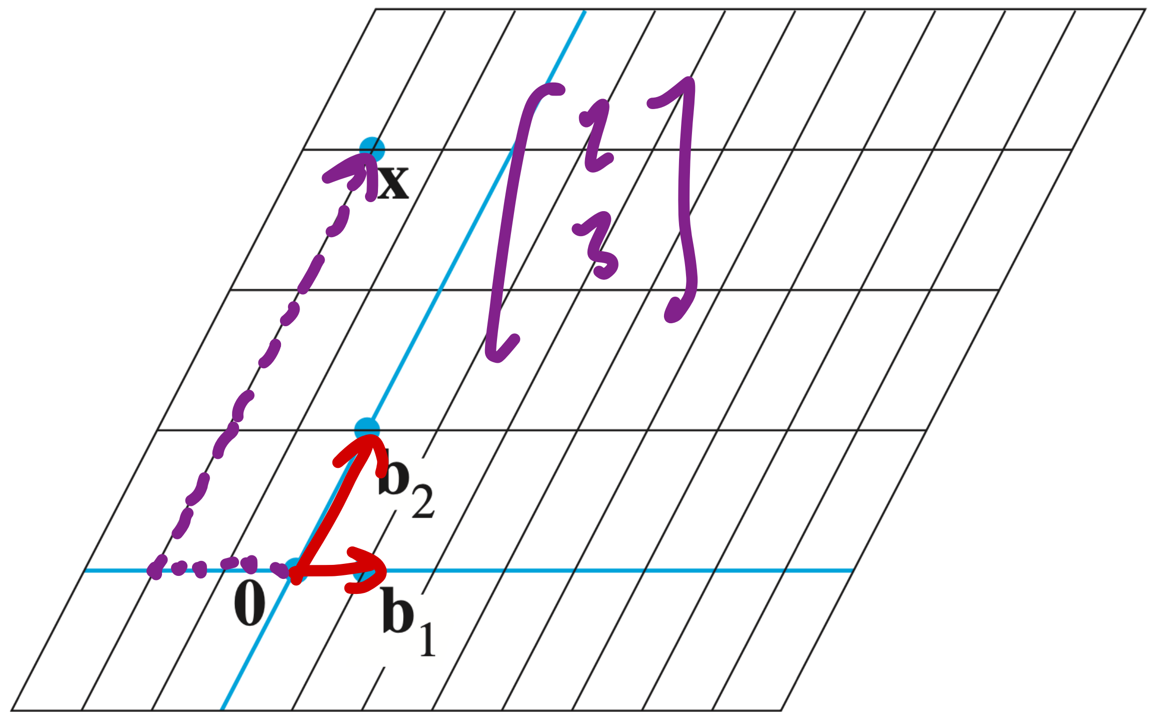

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Every basis provides a way to write down *coordinates* of a vector

$\mathscr{B}$ defines a "different grid for our graph paper"

Recall: Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a $\mathbb{R}^n$ and let $\mathscr{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be a basis of $\mathbb{R}^n$ where

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_n\mathbf{b}_n$$

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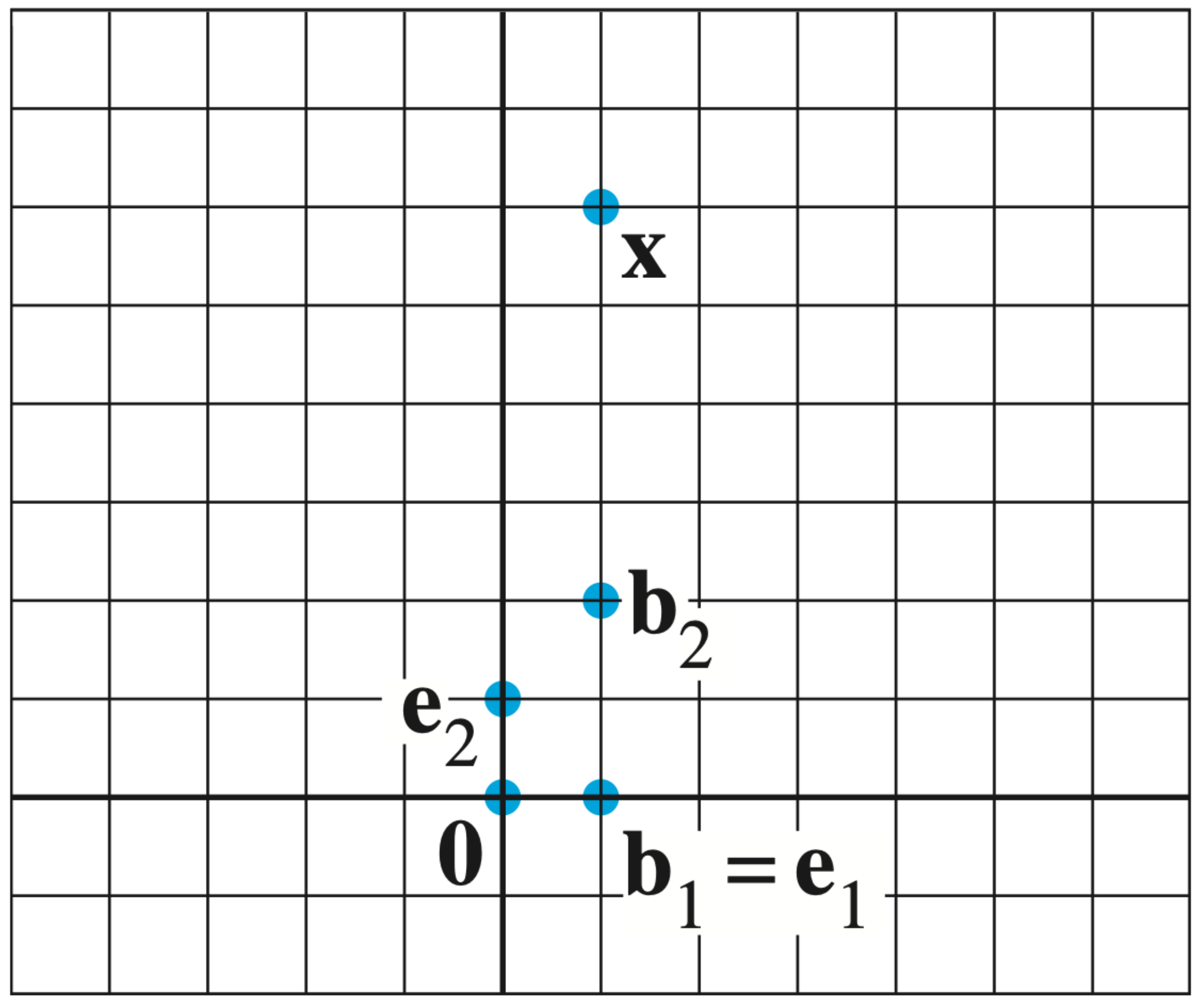
Definition. The **coordinate vector of $\mathbf{v}$ relative to $\mathscr{B}$** is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Recall: Coordinate Vectors

$$\begin{bmatrix}1\\6\end{bmatrix} = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix}$$



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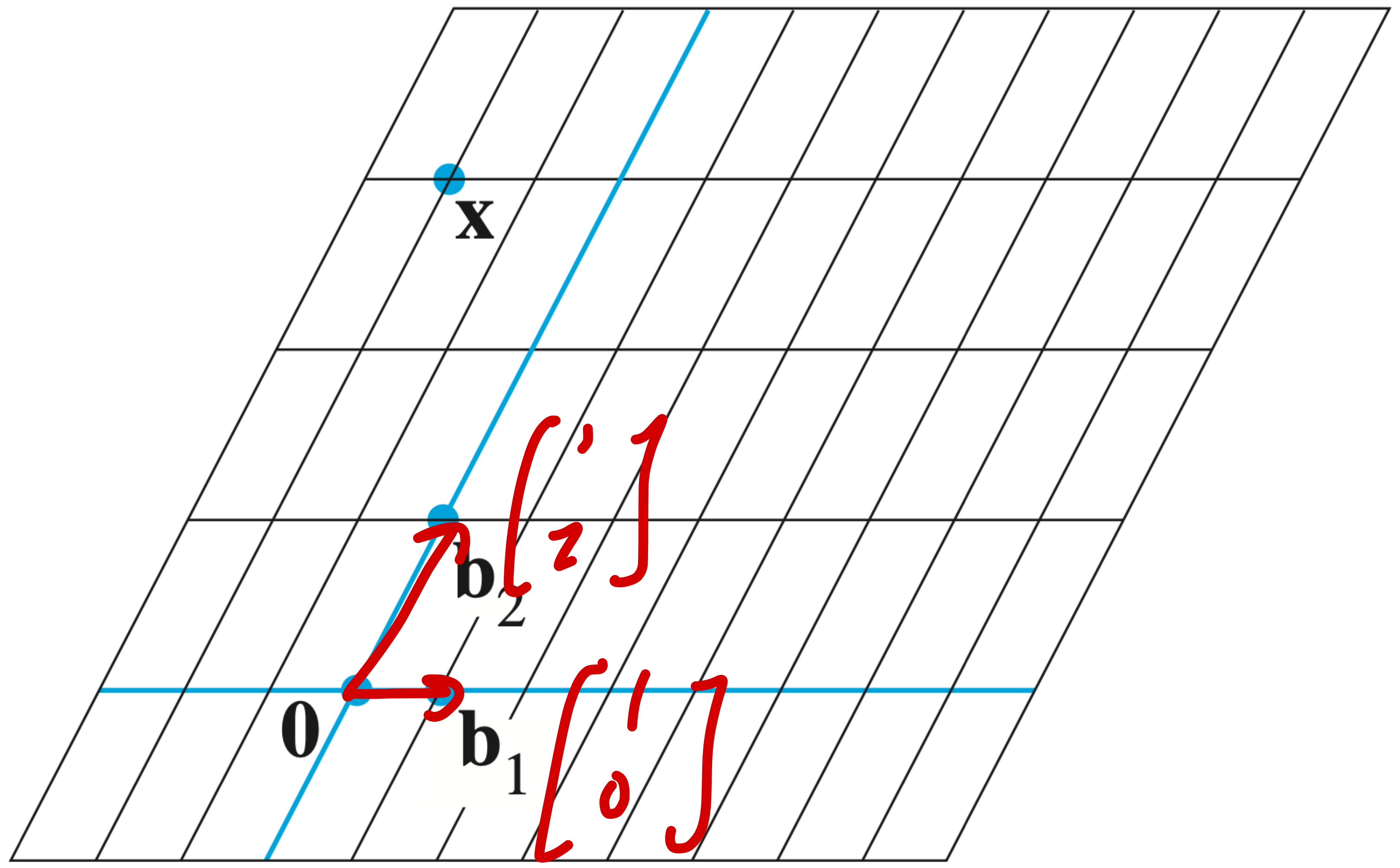


FIGURE 2 $\mathcal{B}$-graph paper.

$$\begin{bmatrix}1\\6\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$$

Question (Conceptual)

$b_1 = b_1 + 0b_2 \ldots + 0b_n$

We know that if a $n \times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]$ is invertible, then the columns of B form a basis $\mathscr{B}$ of $\mathbb{R}^n$

What is the matrix that implements the transformation

$C[\vec{b}_1 \ldots \vec{b}_n] = I$

$C B B^{-1} = I B^{-1}$

$C = B^{-1}$

$$\mathbf{x} \overset{C}{\mapsto} [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$C\vec{b}_1 = [\vec{b}_1]_B$

$= \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} = \vec{e}_1$

$C\vec{b}_2 = \vec{e}_2$

$C b_n = \vec{e}_n$

where $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$?

Change of Basis Matrix

Theorem. If $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ form a basis of $\mathbb{R}^n$, then

$$[\mathbf{x}]_{\mathscr{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]^{-1}\mathbf{x}$$

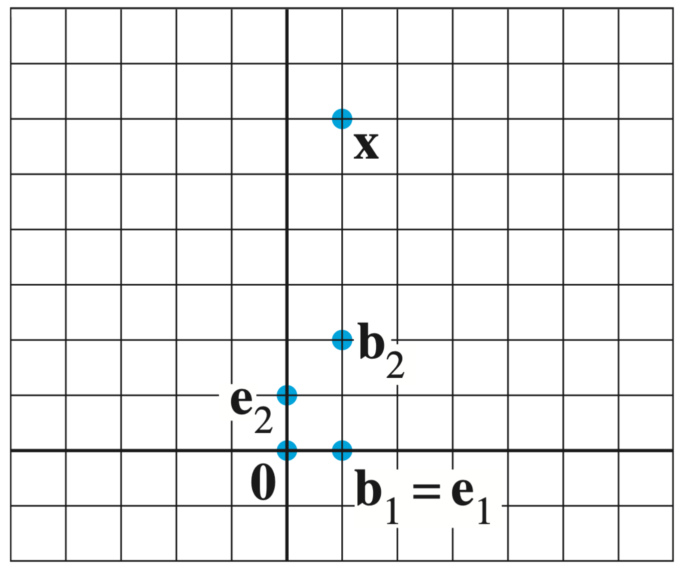
Matrix inverses perform changes of bases.

How To: Change of Basis

Question. Given a basis $\mathscr{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of $\mathbb{R}^n$, find the matrix which implements $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$.

Solution. Construct the matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]^{-1}$.

Example



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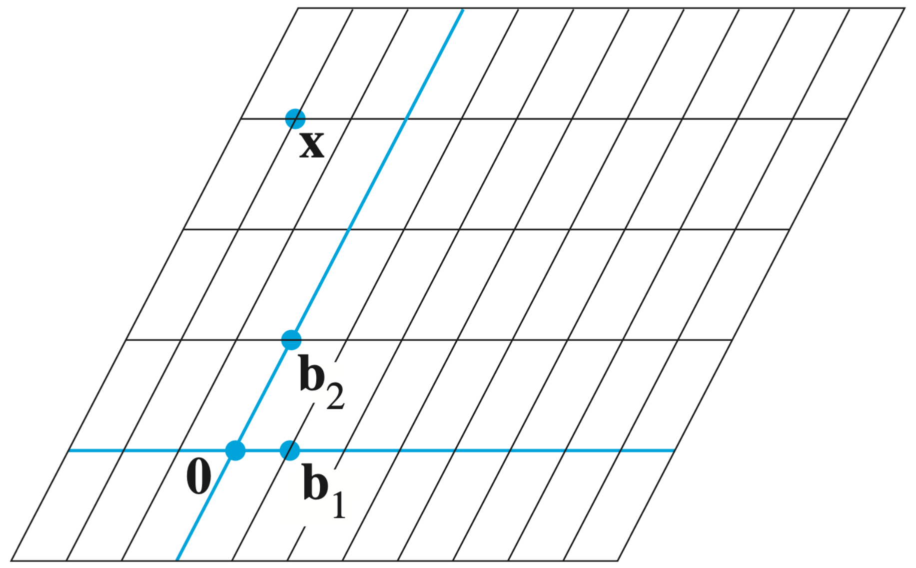

FIGURE 2 $\mathcal{B}$-graph paper.

Write the change-of-bases matrix for the basis $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right)$

Diagonalization

ex. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonal Matrices

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ex. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Definition. A $n \times n$ matrix A is **diagonal** if

$i \neq j$ if and only if $A_{ij} = 0$

Diagonal Matrices

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Only the diagonal entries can be nonzero ex. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Diagonal Matrices

np.diag([1, -0.4, 22, 0])

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Only the diagonal entries can be nonzero

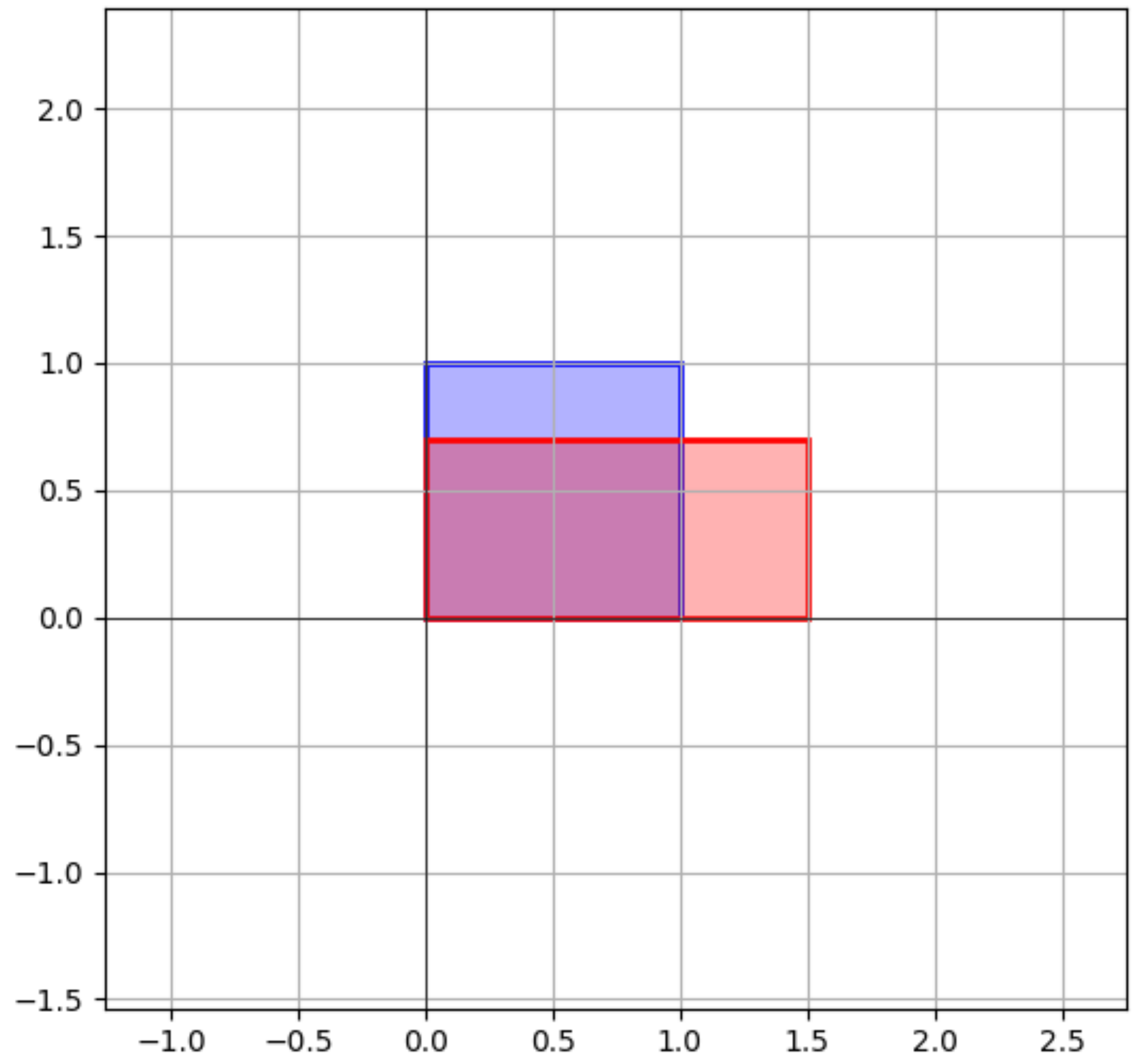
Diagonal matrices are scaling matrices

Recall: Unequal Scaling

The scaling matrix *affects each component of a vector in a simple way*

The diagonal entries <u>scale</u> each corresponding entry

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

High level question:
When do matrices "behave" like scaling matrices "up to" change of basis?

Scaling and Eigenvectors

Scaling and Eigenvectors

The idea. Matrices behave like scaling matrices on eigenvectors.

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathscr{B}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

Scaling and Eigenvectors

The idea. Matrices behave like scaling matrices on eigenvectors.

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$\cancel{A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{B}}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

The fundamental question:

Can we expose this behavior in terms of a *matrix factorization*?

Recall: Matrix Factorization

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A **factorization** of a matrix A is an equation which expresses A as a product of one or more matrices, e.g.,

$$A = PBP^{-1}$$

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A **factorization** of a matrix A is an equation which expresses A as a product of one or more matrices, e.g.,

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Factorizations can:

- make working with A easier
- expose important information about A

"undo" the chge of basis
$P = (P^{-1})^{-1}$

Similar Matrices

change of bases into the columns of P

$$A = \overbrace{P}\underline{B}\overbrace{P^{-1}}$$

Definition. A matrix A is **similar** to a matrix B if there is some invertible matrix P such that $A = PBP^{-1}$

A and B are the same up to a change of basis

Similar Matrices and Eigenvalues

Theorem. Similar matrices have the same eigenvalues.

Verify: $A = PBP^{-1}$

$\det(A - \lambda I) =$

$\det(PBP^{-1} - \lambda I) =$

$\det(PBP^{-1} - \lambda P P^{-1} I) =$

$\det(P(BP^{-1} - \lambda P^{-1} I)) =$

$\det(P(B - \lambda \cancel{P^{-1}} I \cancel{P})P^{-1}) =$

$\det(P(B - \lambda I)P^{-1}) =$

$= \cancel{\det}(P)\det(B - \lambda I)\cancel{\det}(P^{-1})$

$1/\det(P)$

$= \det(B - \lambda I)$

Diagonalizable Matrices

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Diagonalizable matrices are the same as scaling matrices up to a change of basis

Important: Not all Matrices are Diagonalizable

This is very different from the LU factorization

We will need to figure out which matrices are diagonalizable

Question. *Is the zero matrix diagonalizable?*

$$P0P^{-1} = P0 = 0$$

Application: Matrix Powers

only take the power of B

Theorem. If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$

It may be easier to take the power of B (as in the case of diagonal matrices)

Verify: $A^k = P B \cancel{P^{-1}} \cancel{P} B \cancel{P^{-1}} \cancel{P} B \cancel{P^{-1}} \cdots \cancel{P} B P^{-1}$

$P B^k P^{-1}$

How To: Matrix Powers

Question. Given A is diagonalizable, determine A^k

Solution. Find it's diagonalization PDP^{-1} and then compute PD^kP^{-1}

Remember that

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

But how do we find the diagonalization...

Diagonalization and Eigenvectors

Suppose we have a diagonalization

$$A = PDP^{-1}$$

What do we know about it?

Columns of P are eigenvectors

$P\vec{e}_1 = \vec{p}_1$

$$A = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}$$

Verify:

$A\vec{p}_1 = PD(P^{-1}\vec{p}_1) = PD\vec{e}_1 = P\begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} = P\lambda_1\vec{e}_1$

$\lambda_1 P\vec{e}_1 = \lambda_1 p_1$

Columns of P are eigenvectors

$$A = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]^{-1}$$

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In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

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In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

And the entries of D are the **eigenvalues** associated to each eigenvector

Columns of P are eigenvectors

$$A = \underbrace{[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]}_{\text{eigenbasis}} \overbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}^{\text{eigenvalues}} [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}$$

In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

And the entries of D are the **eigenvalues** associated to each eigenvector

A diagonalization exposes a lot of information about A

The Diagonalization Theorem

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Theorem. A matrix is diagonalizable if and only if it has an eigenbasis

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(we just did the hard part, if a matrix is diagonalizable then it has an **eigenbasis**)

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(we just did the hard part, if a matrix is diagonalizable then it has an **eigenbasis**)

We can use the same recipe to go in the other direction, given an eigenbasis, we can **build a diagonalization**

Diagonalizing a Matrix

High Level

$$A = PDP^{-1}$$

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The columns of P form an <u>eigenbasis</u> for A

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$$A = PDP^{-1}$$

The columns of P form an eigenbasis for A

The diagonal of D are the eigenvalues for each column of P

The matrix P^{-1} is a change of basis to this eigenbasis of A

Step 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Find all the eigenvalues of A

Find the roots of $\det(A - \lambda I)$

e.g.

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

Step 2: Eigenvectors

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

Find **bases** of the corresponding eigenspaces

e.g.

$$\mathrm{Nul}(A - I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Nul}(A + 2I) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 3: Construct P

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

If there are n eigenvectors from the previous step they form an **eigenbasis**

Build the matrix with these vectors as the columns

e.g.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 5: Construct D

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Build the matrix with eigenvalues as diagonal entries

Note the order. It should be the same as the order of columns of P

e.g.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Step 6: Invert P

Find the inverse of P (we know how to do this)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Putting it Together

$$\underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}$$

How to: Diagonalizing a Matrix

Question. Find a diagonalization of $A \in \mathbb{R}^n$, or determine that A is not diagonalizable

Solution.

1. Find the eigenvalues of A, and bases for their eigenspaces. If these eigenvectors don't form a basis of $\mathbb{R}^n$, then A is **not diagonalizable**
2. Otherwise, build a matrix P whose columns are the eigenvectors of A
3. Then build a diagonal matrix D whose entries are the eigenvalues of A *in the same order*
4. Invert P
5. The diagonalization of A is PDP^{-1}

We know how to do every step, its a matter of putting it all together

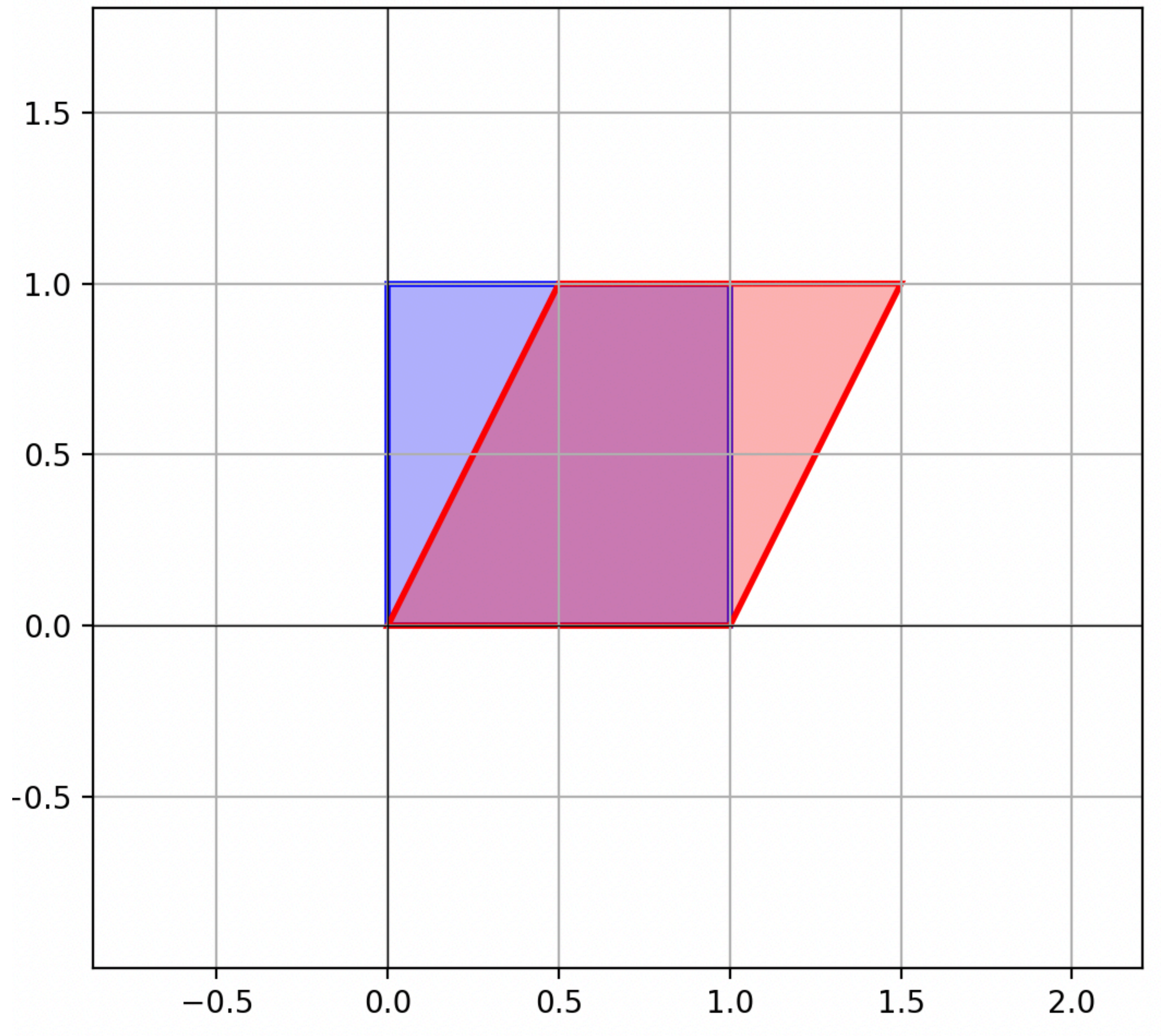
Example of Failure: Shearing

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The shearing matrix has a single eigenvalue with an eigenspace of dimension 1

We can't build an eigenbasis of $\mathbb{R}^2$ for A

In other words, A is not diagonalizable



Important case: Distinct Eigenvalues

ex. $\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Theorem. If an $n \times n$ matrix has has n distinct eigenvalues, then it is diagonalizable

This is because eigenvectors with distinct eigenvalues are *linearly independent*

Example

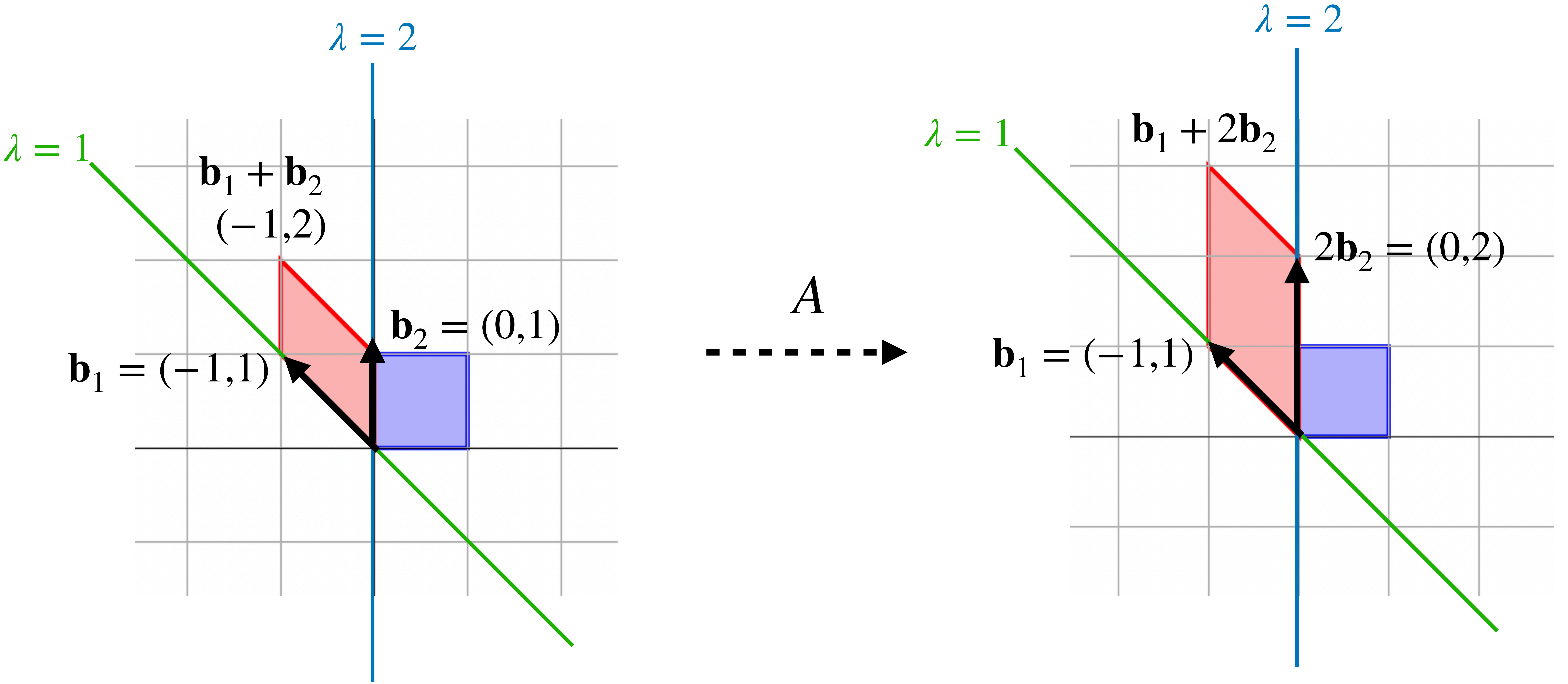
$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

Find a diagonalization of the above matrix

The Picture

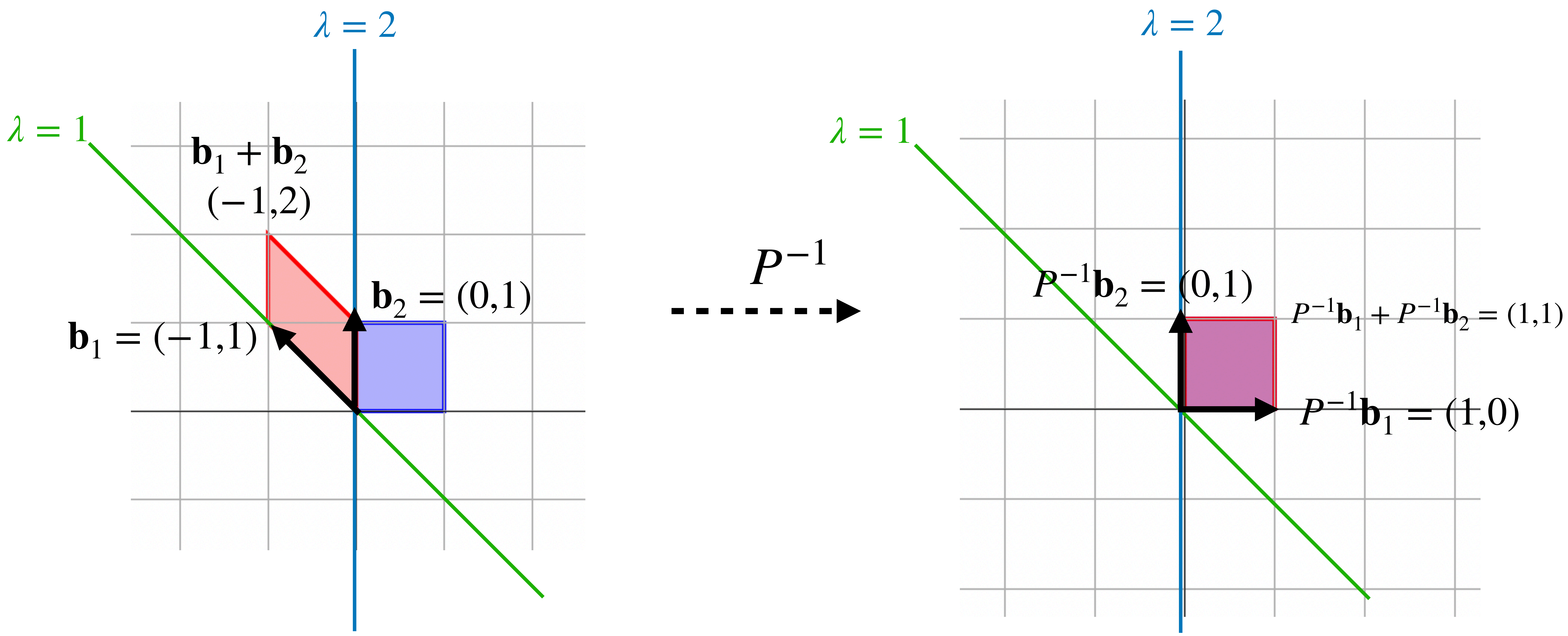
Example (Geometric)

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$



Example (Geometric)

$$P^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

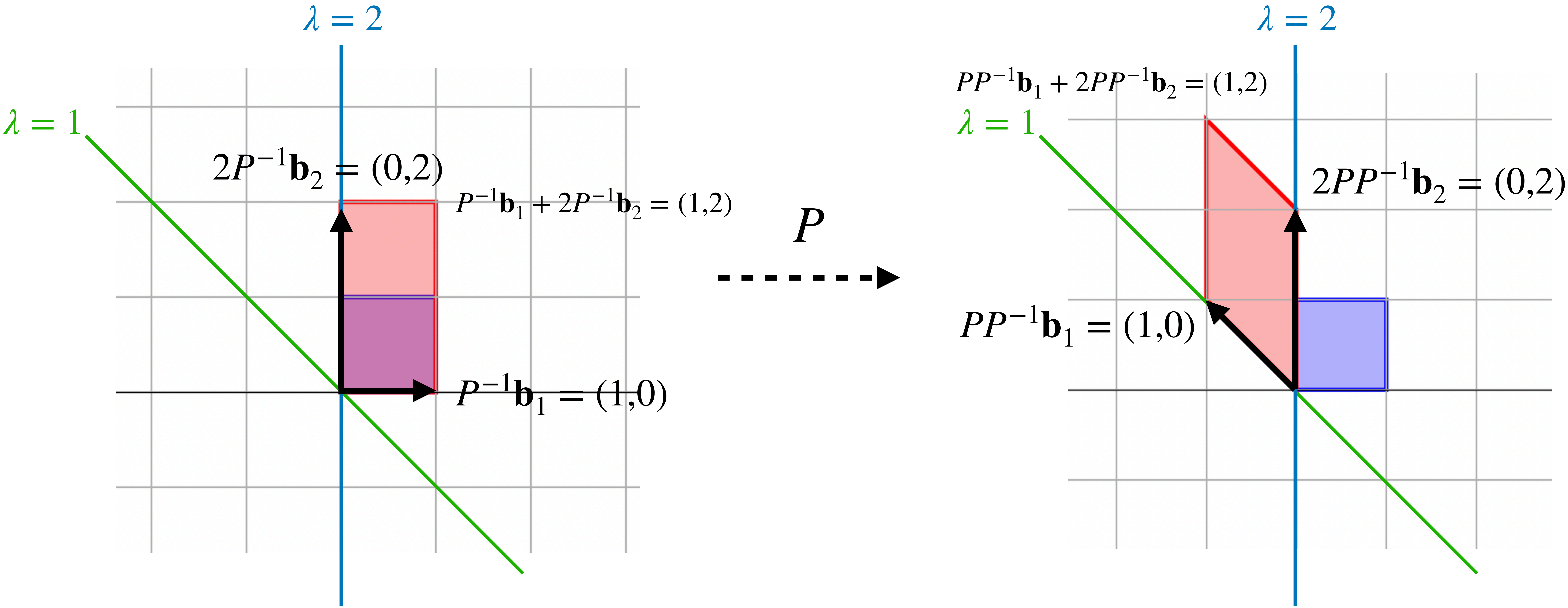


Example (Geometric)

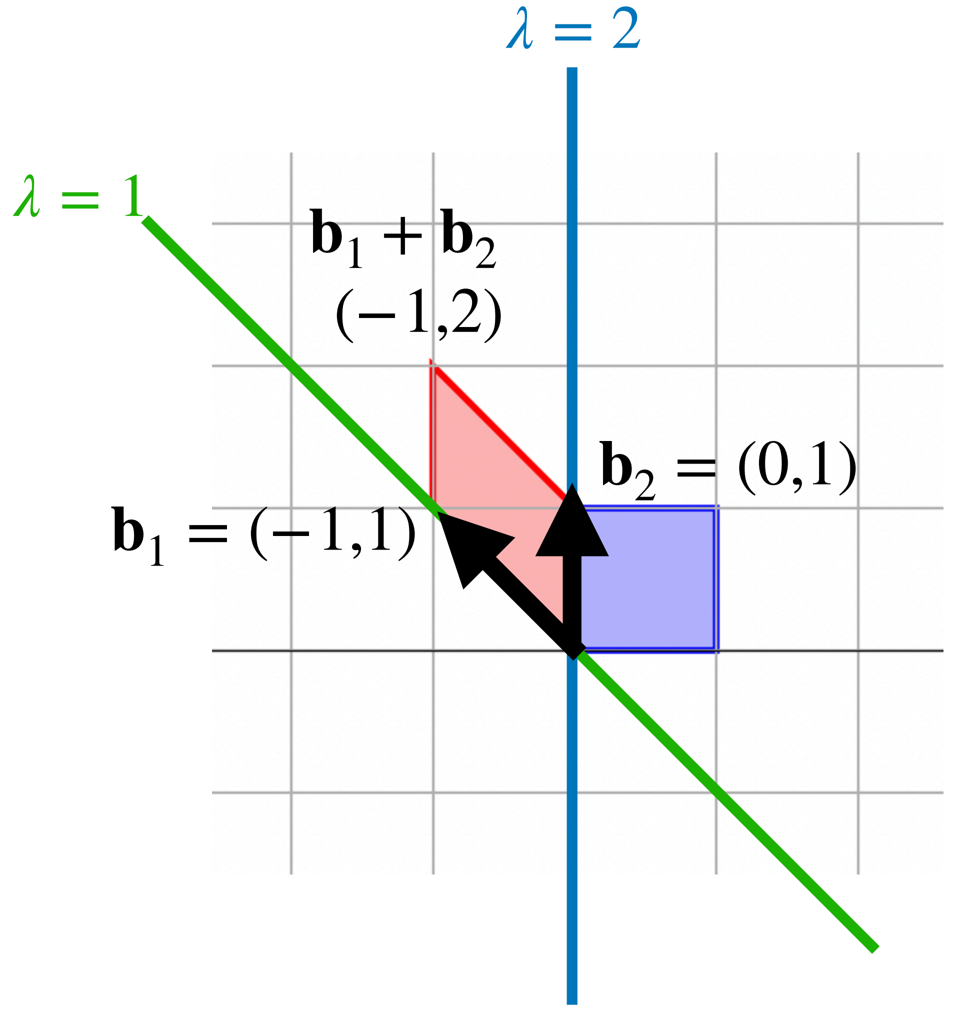
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example (Geometric)

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$



Example (Geometric)



$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$