Analytic Geometry in $\mathbb{R}^n$

Geometric Algorithms
Lecture 21

CAS CS 132

Practice Problem

Let A be a 4×4 matrix with eigenvalues 3 and -2 where $\dim(\mathrm{Nul}(A + 2I)) = 3$.

***True** or **False**: A must be diagonalizable.*

$\Updownarrow$

A has an eigenbasis

$v_1, \dots, v_k$ are eigenvectors for distinct eigenvalues

then $\{v_1 \dots v_k\}$ is LI

Answer: True

The set of eigenvectors we get from the diagonalization procedure is of size 4, which means there is an eigenbasis of $\mathbb{R}^4$ for A.

Objectives

1. Recall what we learned in algebra class
2. Connect the familiar notions of lengths, distances, and angles to inner products
3. Begin discussing the fundamental concept of orthogonality

Keywords

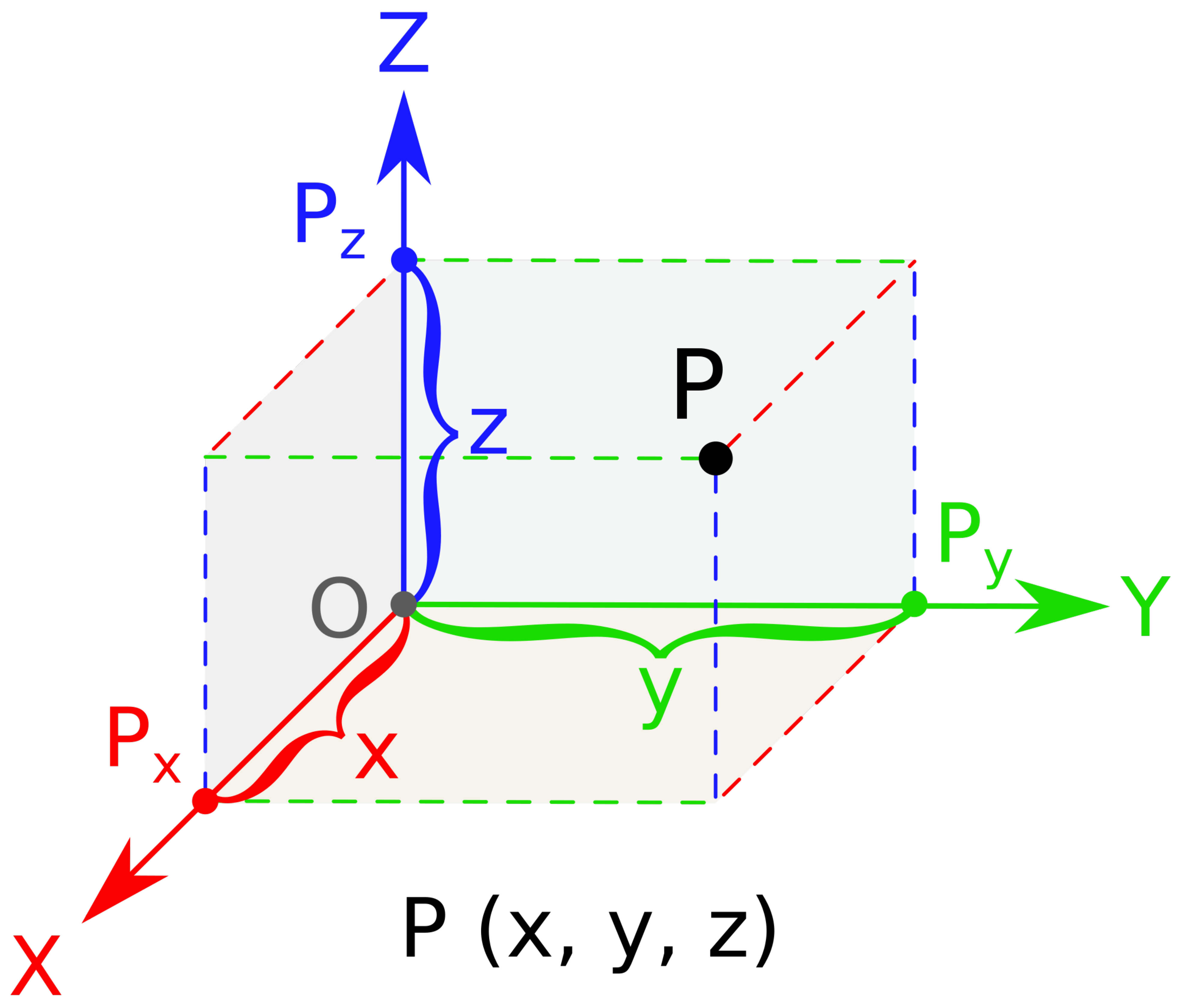
inner product

norm

orthogonal

Motivation

What is Analytic Geometry?



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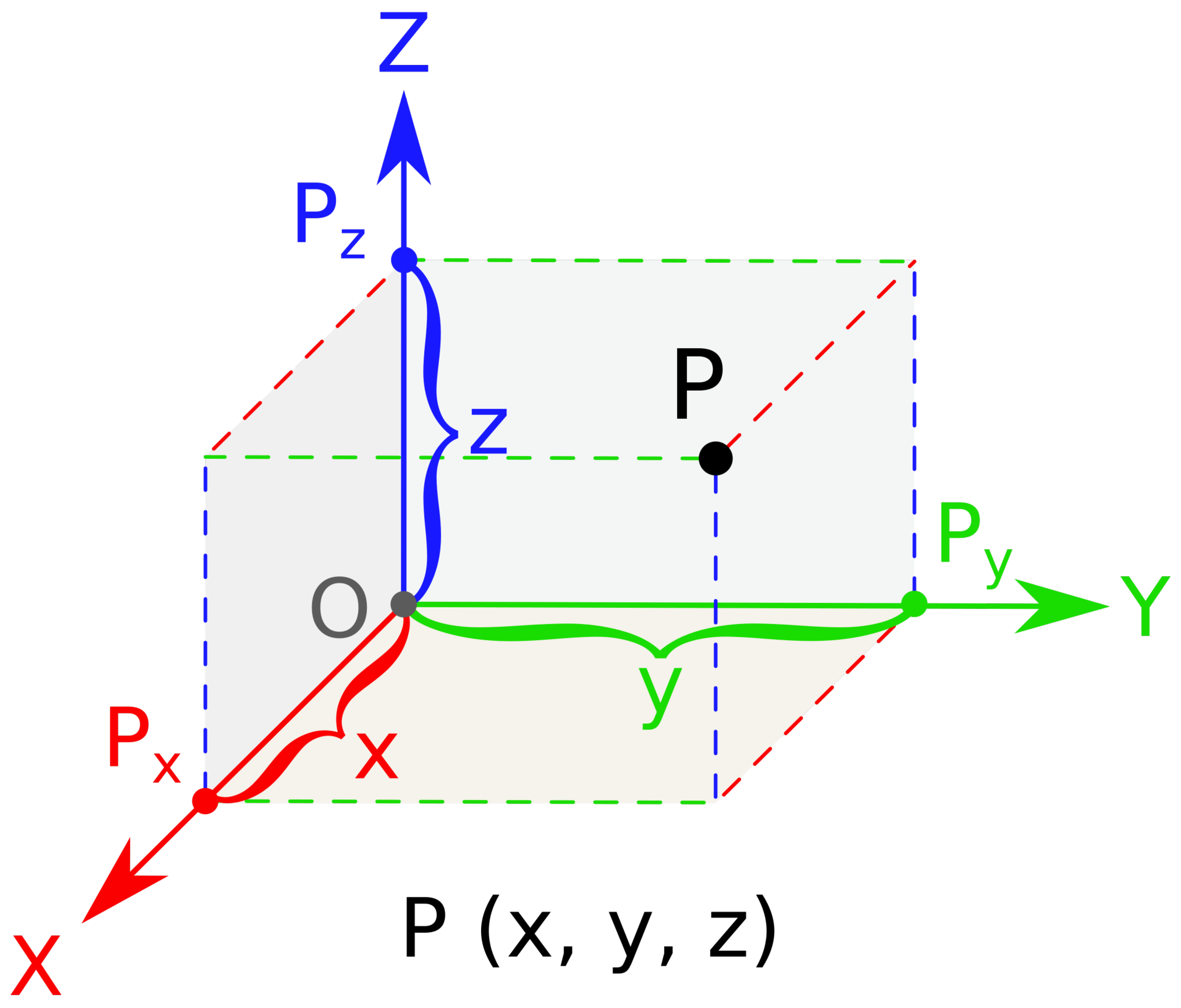
What is Analytic Geometry?

Analytic geometry is the study of space using a coordinate system.



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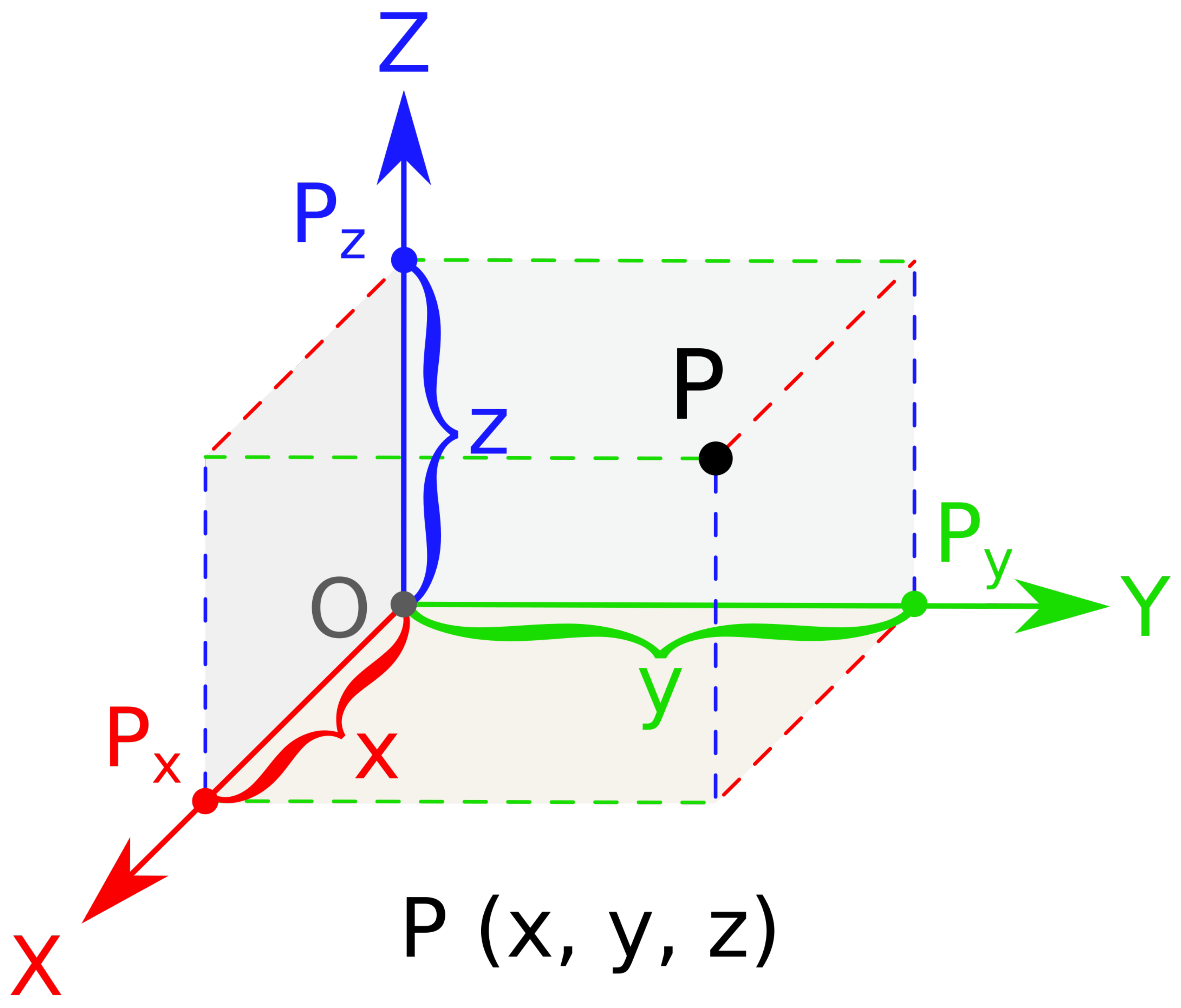
What is Analytic Geometry?

Analytic geometry is the study of space using a coordinate system.

We're interested in equations about lines, curves, shapes, angles, etc.



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Analytic geometry is the study of space using a coordinate system.

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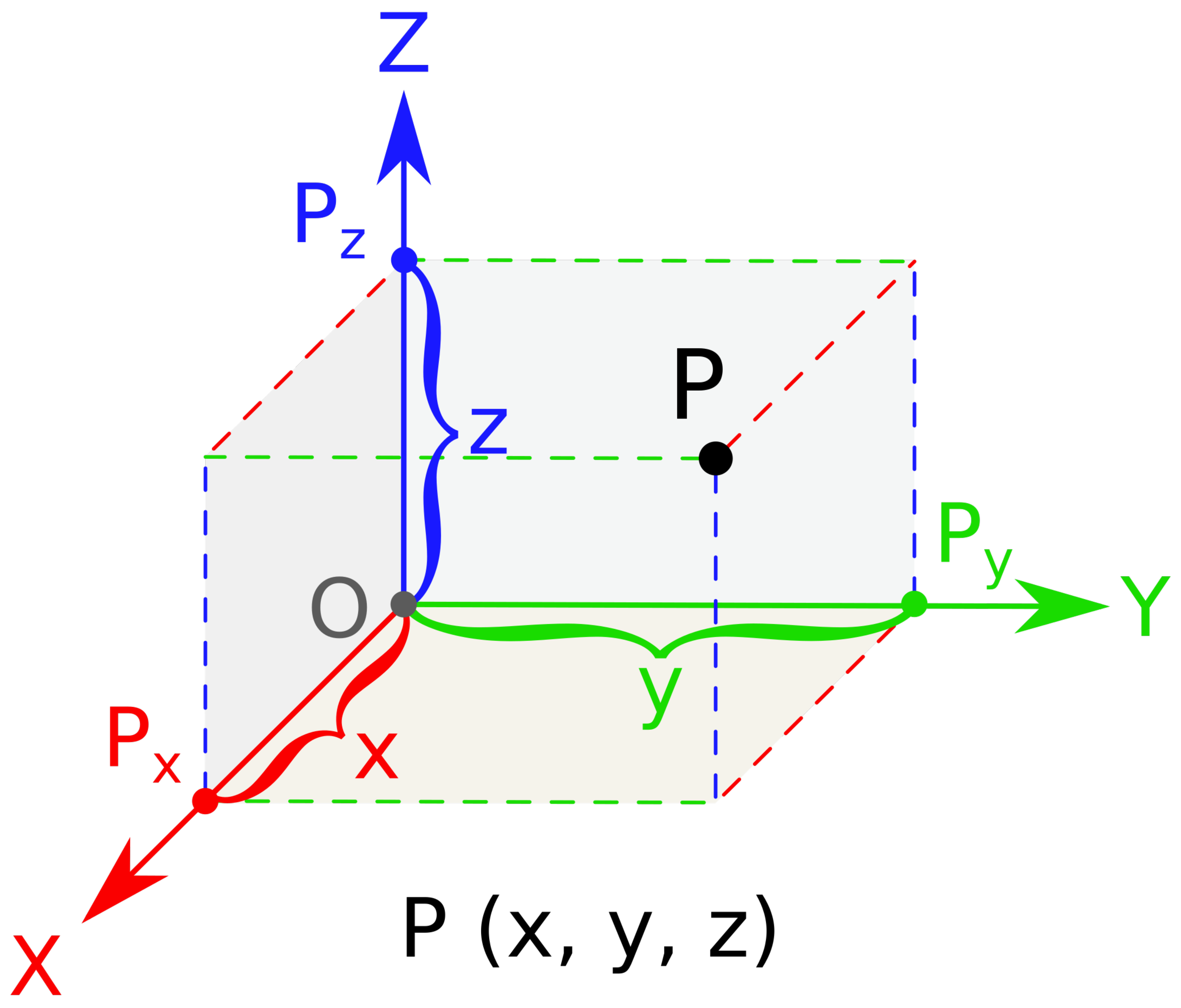
The fundamental concepts are:



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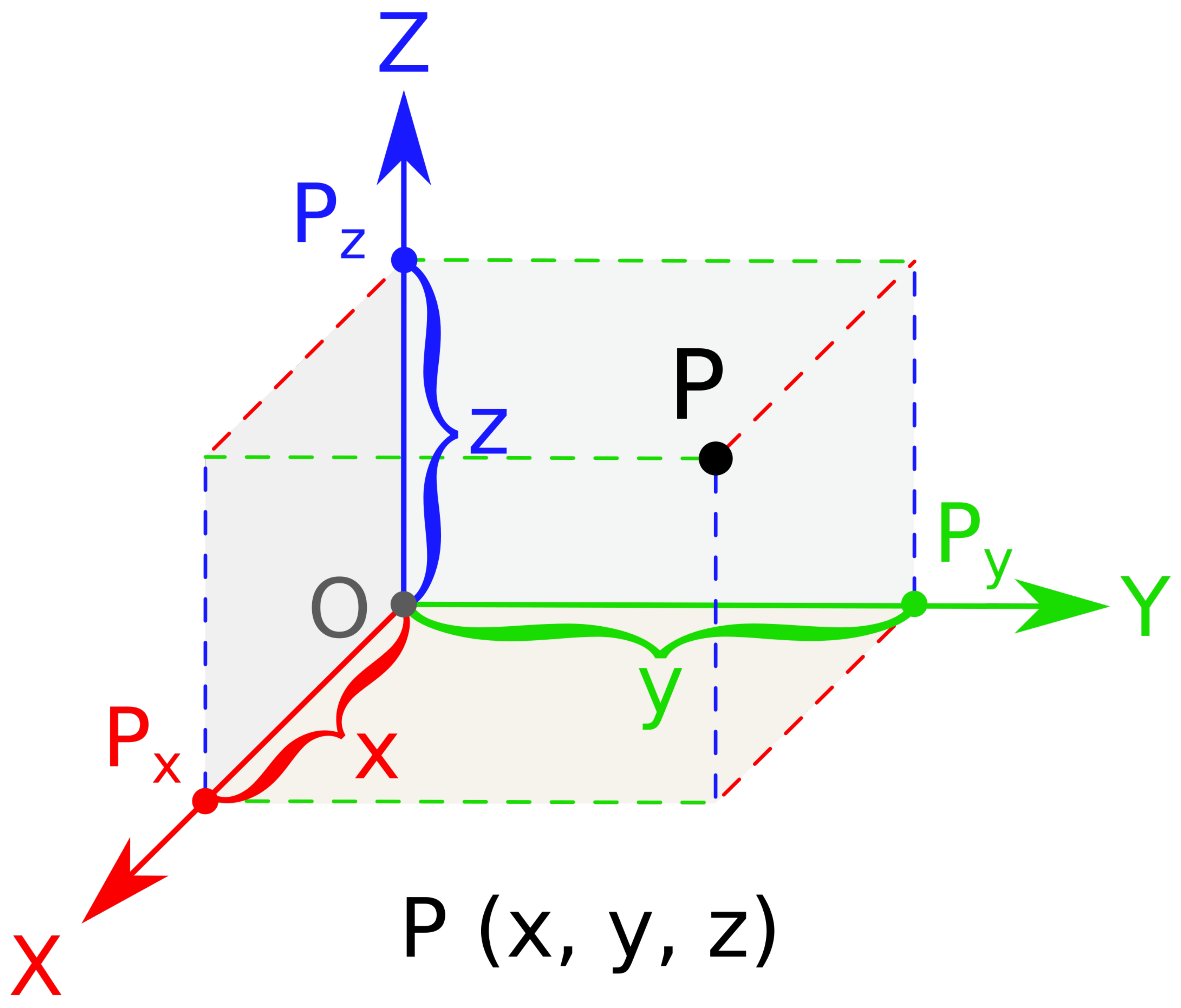
What is Analytic Geometry?

Analytic geometry is the study of space using a coordinate system.

We're interested in equations about lines, curves, shapes, angles, etc.

The fundamental concepts are:

» distance
» position
» area
» angle



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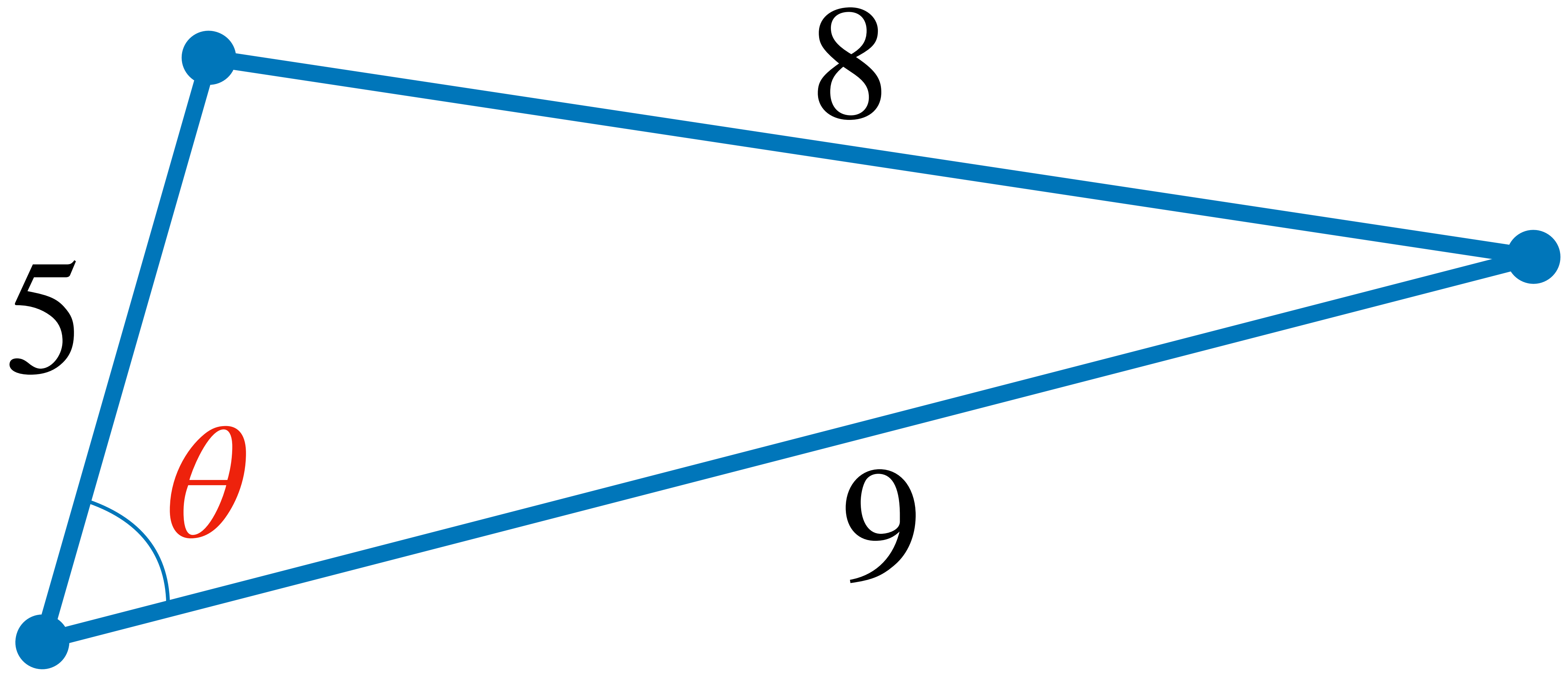
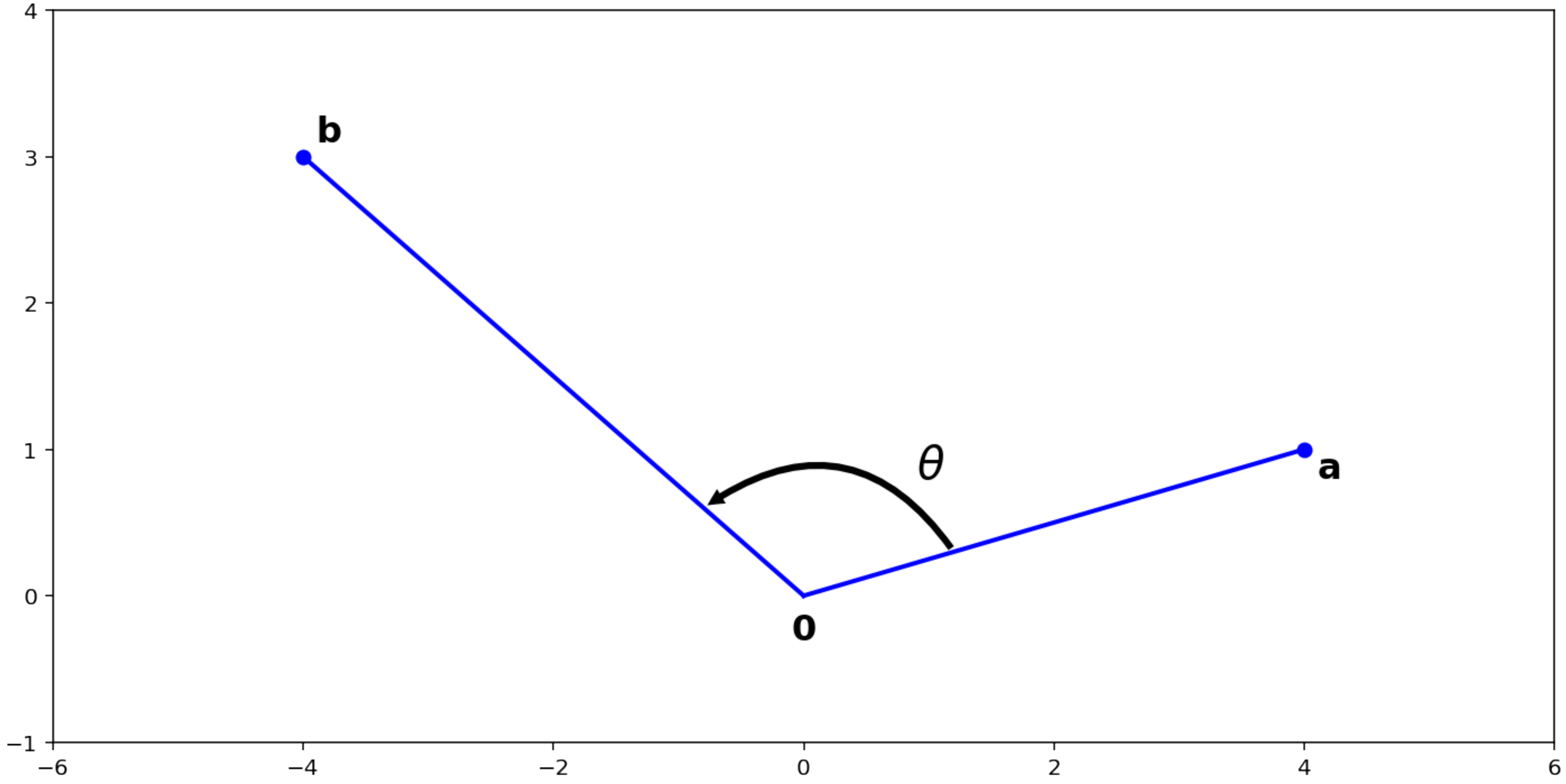
A Potentially Familiar Example



What is the value of θ*?*

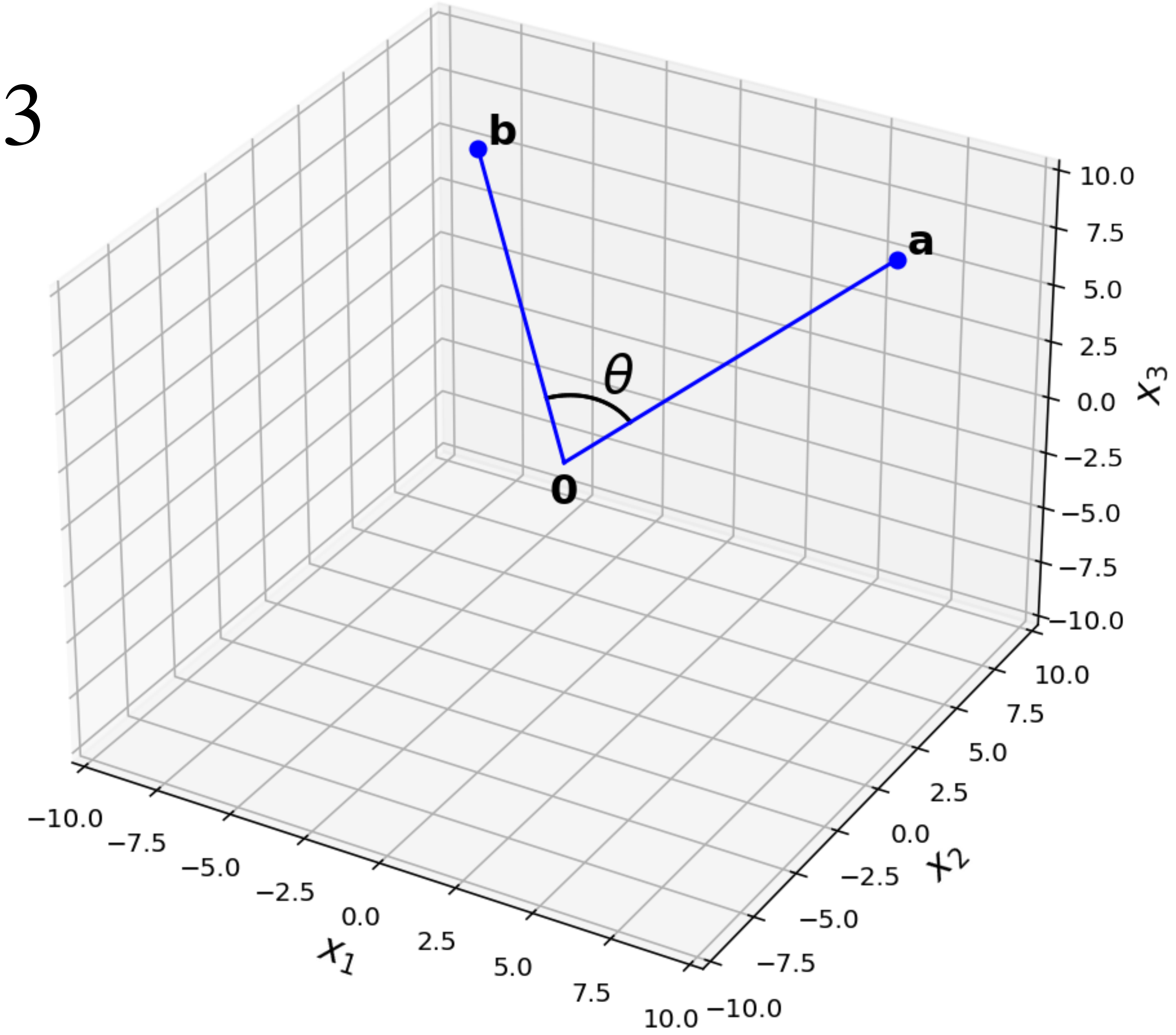
https://www.mathsisfun.com/algebra/trig-cosine-law.html

Angles in $\mathbb{R}^2$



What is the value of θ?

Angles in $\mathbb{R}^3$



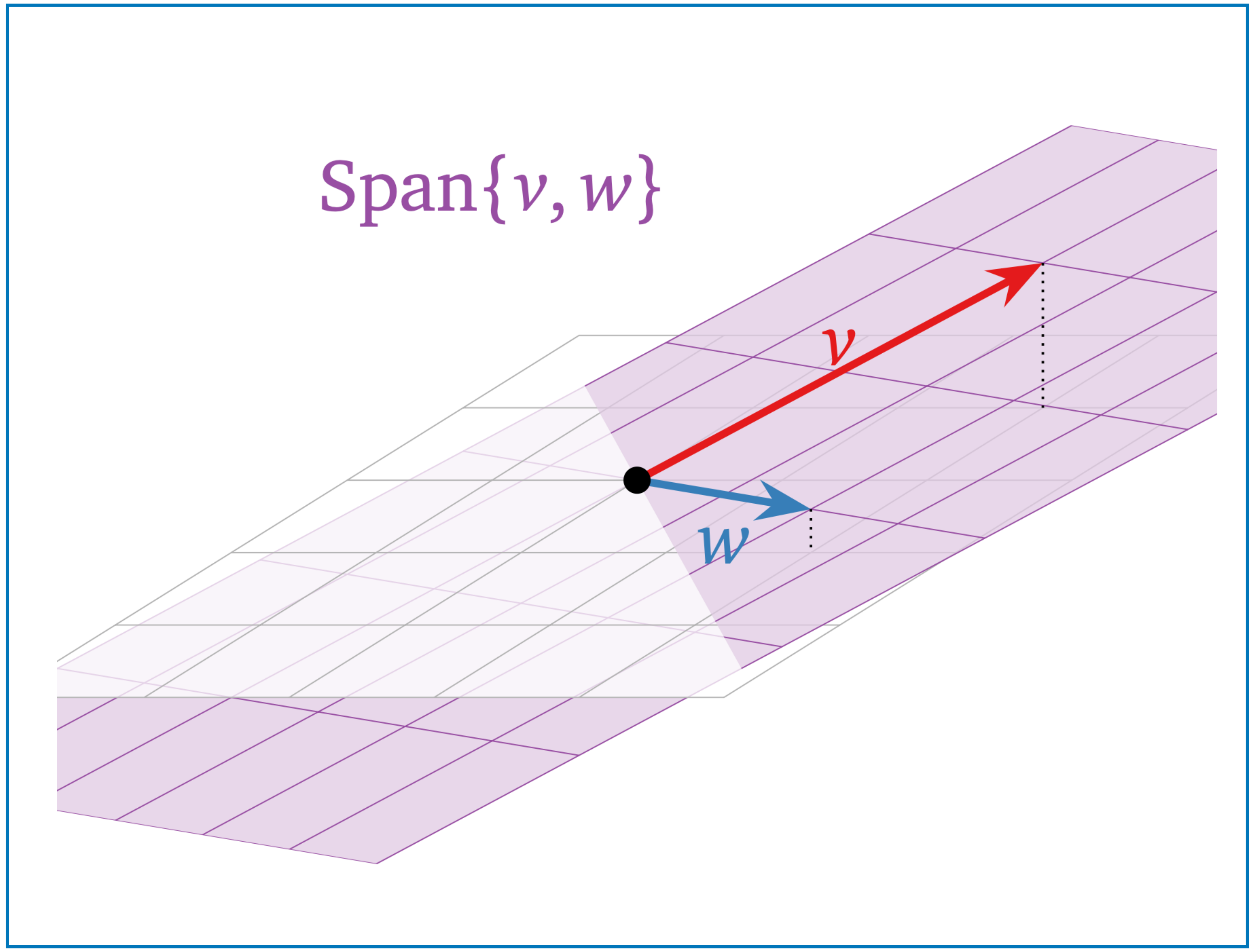
What is the value of θ?

The First Key Idea

Angles make sense in *any* dimension.

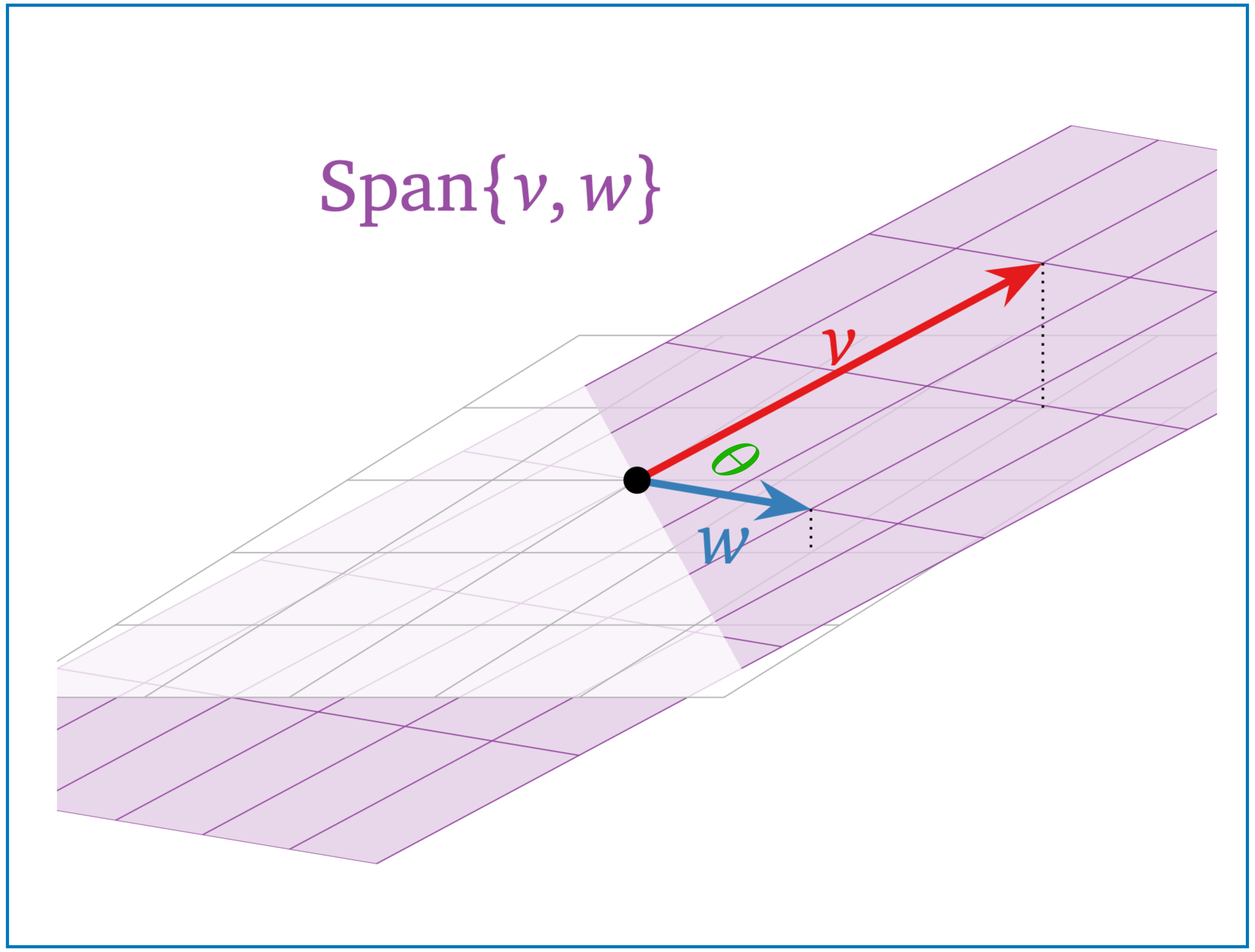
Any pair of vectors in $\mathbb{R}^n$ span a (2D) plane.

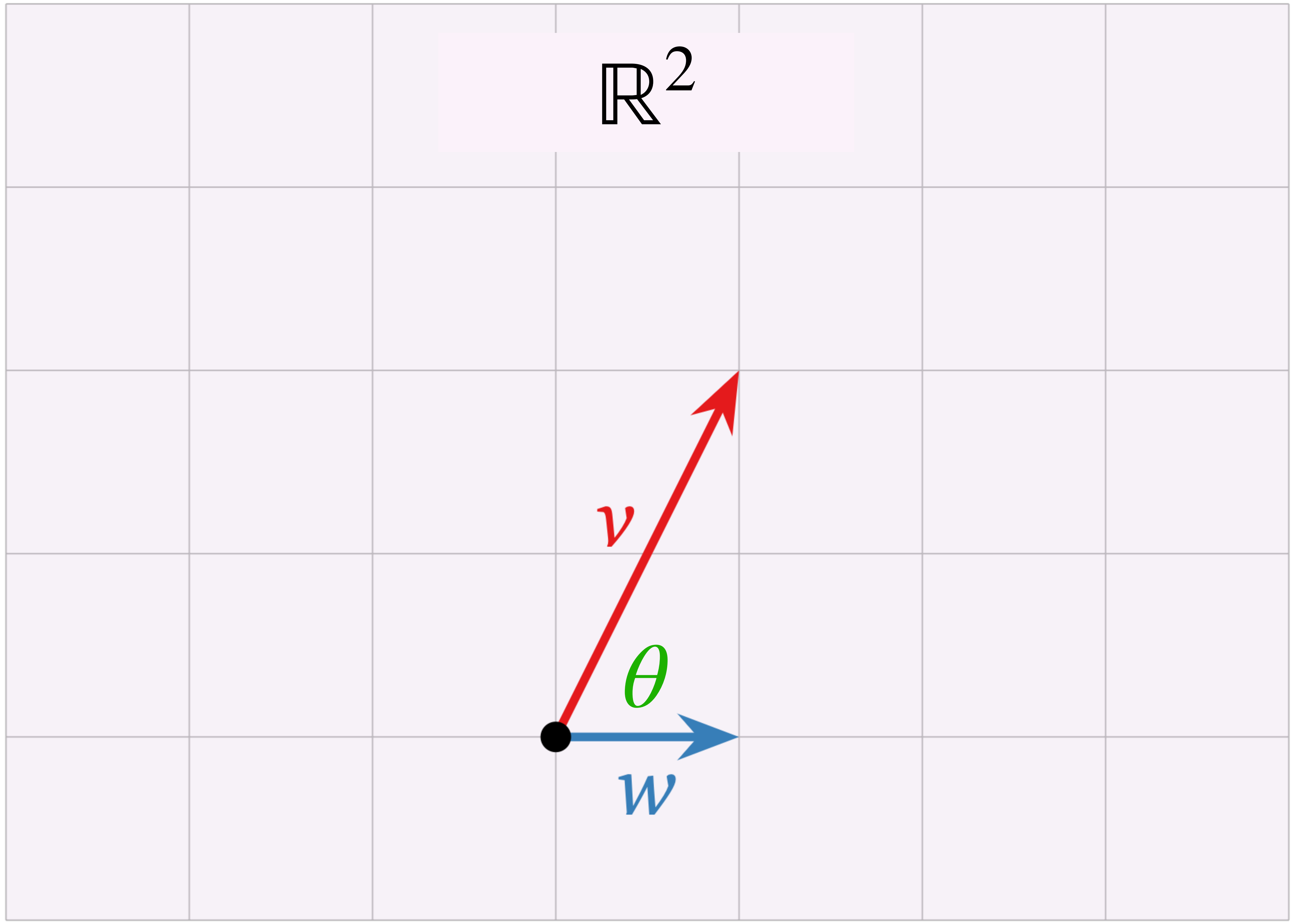
(We could formalize this via change of bases)



The Picture

We can do "normal" analytic geometry here





change of basis from span$\{v, w\}$ to $\mathbb{R}^2$

A Fundamental Question

Doing this change of basis every time we want to do geometry is a lot of work...

Can we do it directly using ideas we've been learning?

Recall: Inner Products

Recall: Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Recall: Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Inner Products

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Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

a.k.a. dot product

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All of the basic concepts of analytic geometry can be defined *in terms of inner products*.

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Definition (Advanced). An **inner product space** is a vector space with an inner product function.

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All of the basic concepts of analytic geometry can be defined *in terms of inner products*.

Definition (Advanced). An **inner product space** is a vector space with an inner product function.

Inner product spaces (like $\mathbb{R}^n$) are places where you can do analytic geometry.

The Fundamental Question

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How do we do analytic geometry,
given we have an inner product?
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Inner Products

Recall: Inner Products (Again)

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

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Example

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0.5 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0.5 \\ -1 \\ 3 \end{bmatrix} = (-3)2 + 2(0.5) + 1(-1) + 4(3)$$

$$= \cancel{-6} \quad \cancel{1} + \cancel{-1} + \cancel{12}$$

$$= \boxed{6}$$

Algebraic Properties of Inner Products

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry)

$$\left.\begin{array}{l} \bullet\ (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w}) \\ \bullet\ (\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) \end{array}\right\} \text{linearity in the first argument}$$

exercise: linearity in second arg.

- $\mathbf{u} \cdot \mathbf{u} \geq 0$ (nonnegativity)
- $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

Verifying Additivity

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_2 \end{bmatrix}$$

$$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3$$

$$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2 + u_3 w_3 + v_3 w_3$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$$

Homogeneity in the Right Argument

$$\langle \mathbf{v}, c\mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle$$

Verify:

$$\langle \vec{v}, c\vec{u} \rangle = \langle c\vec{u}, \vec{v} \rangle \quad \text{(symmetry)}$$

$$= c\langle \vec{u}, \vec{v} \rangle \quad \text{(homogen. in left)}$$

An Aside: What is this linear transformation?

$$\underset{\mathbb{R}^3}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} \mapsto \overset{\vec{r}}{\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^1$$

Let's find the matrix for this transformation:

$$\vec{x} \mapsto \begin{bmatrix} 3 & 5 & 7 \end{bmatrix} \vec{x} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto 3 \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto 7$$

$$\vec{x} \mapsto \vec{r}^T \vec{x} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto 5$$

Algebraic Properties of Inner Products

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry)
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$
- $(\alpha \mathbf{u}) \cdot v = \alpha(\mathbf{u} \cdot \mathbf{v})$

linearity in the first argument

- $\mathbf{u} \cdot \mathbf{u} \geq 0$ (nonnegativity)
- $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

Nonnegativity

$$\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} v_i^2$$

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`Squared values are` <u>`always nonnegative`</u>`.`

Nonnegativity

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Squared values are <u>always nonnegative</u>.

Therefore $\langle \mathbf{v}, \mathbf{v} \rangle$ is always nonnegative.

Nonnegativity

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Squared values are always nonnegative.

Therefore $\langle \mathbf{v}, \mathbf{v} \rangle$ is always nonnegative.

Question. What happens when we scale a vector to make it longer?

Nonnegativity and Scaling

$$\langle c\mathbf{v}, c\mathbf{v}\rangle = c^2\langle \mathbf{v}, \mathbf{v}\rangle = c^2 \sum_{i=1}^{n} v_i^2$$

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If $c > 0$ then $\langle c\mathbf{v}, c\mathbf{v}\rangle > \langle \mathbf{v}, \mathbf{v}\rangle$.

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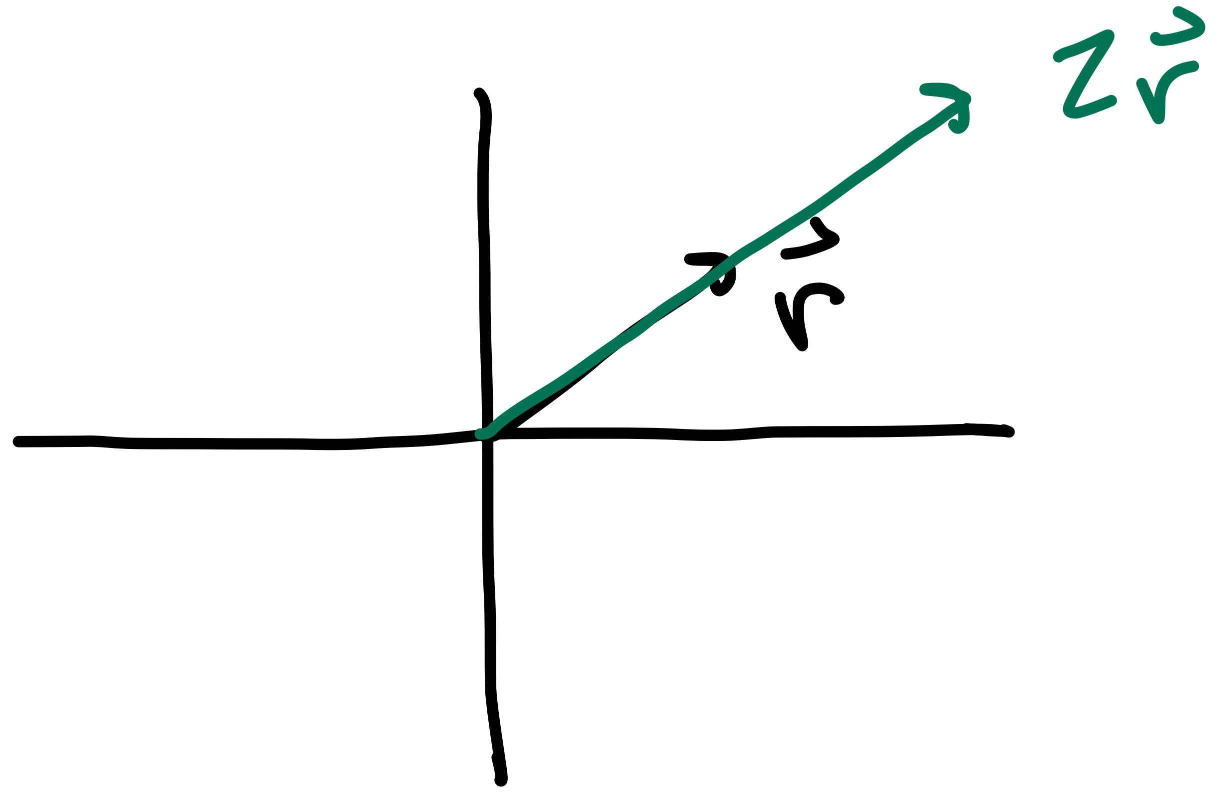
If $c > 0$ then $\langle c\mathbf{v}, c\mathbf{v}\rangle > \langle \mathbf{v}, \mathbf{v}\rangle$.

Increasing the length of a vector increases its inner product with itself.

$\langle 2\vec{r}, 2\vec{r}\rangle = 4\langle \vec{r}, \vec{r}\rangle$



Nonnegativity and Scaling

$$\langle c\mathbf{v}, c\mathbf{v}\rangle = c^2\langle \mathbf{v}, \mathbf{v}\rangle = c^2 \sum_{i=1}^{n} v_i^2$$

If $c > \overset{1}{\cancel{0}}$ then $\langle c\mathbf{v}, c\mathbf{v}\rangle > \langle \mathbf{v}, \mathbf{v}\rangle$.

Increasing the length of a vector increases its inner product with itself.

This means $\langle \mathbf{v}, \mathbf{v}\rangle$ is capturing some notion of magnitude.

The Fundamental Question

`How does this all connect back to distances and angles?`

Question

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$$

Simplify the expression $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$ *using the properties of inner products.*

Answer: $\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle$

(linear in 1^{st} arg)

$$\langle \boxed{u+v}, u-v \rangle = \langle \boxed{u}, u-v \rangle + \langle \boxed{v}, u-v \rangle$$

(lin. in 2^{nd} arg)

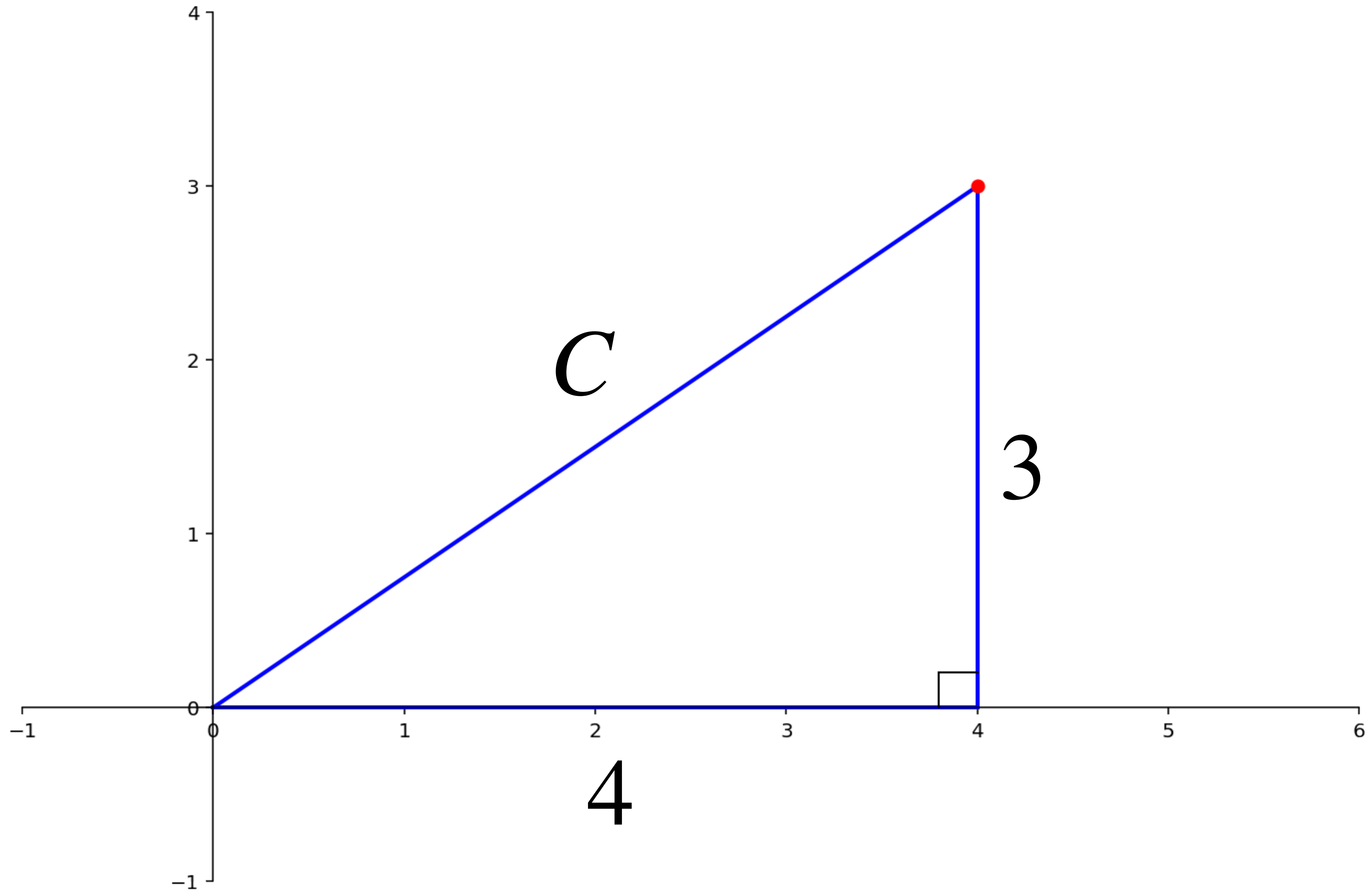
$$= \langle u,u \rangle - \langle u,v \rangle + \langle v,u \rangle - \langle v,v \rangle$$

(symmetry)

$$= \langle u,u \rangle - \langle v,v \rangle \cancel{- \langle u,v \rangle} \cancel{+ \langle u,v \rangle}$$

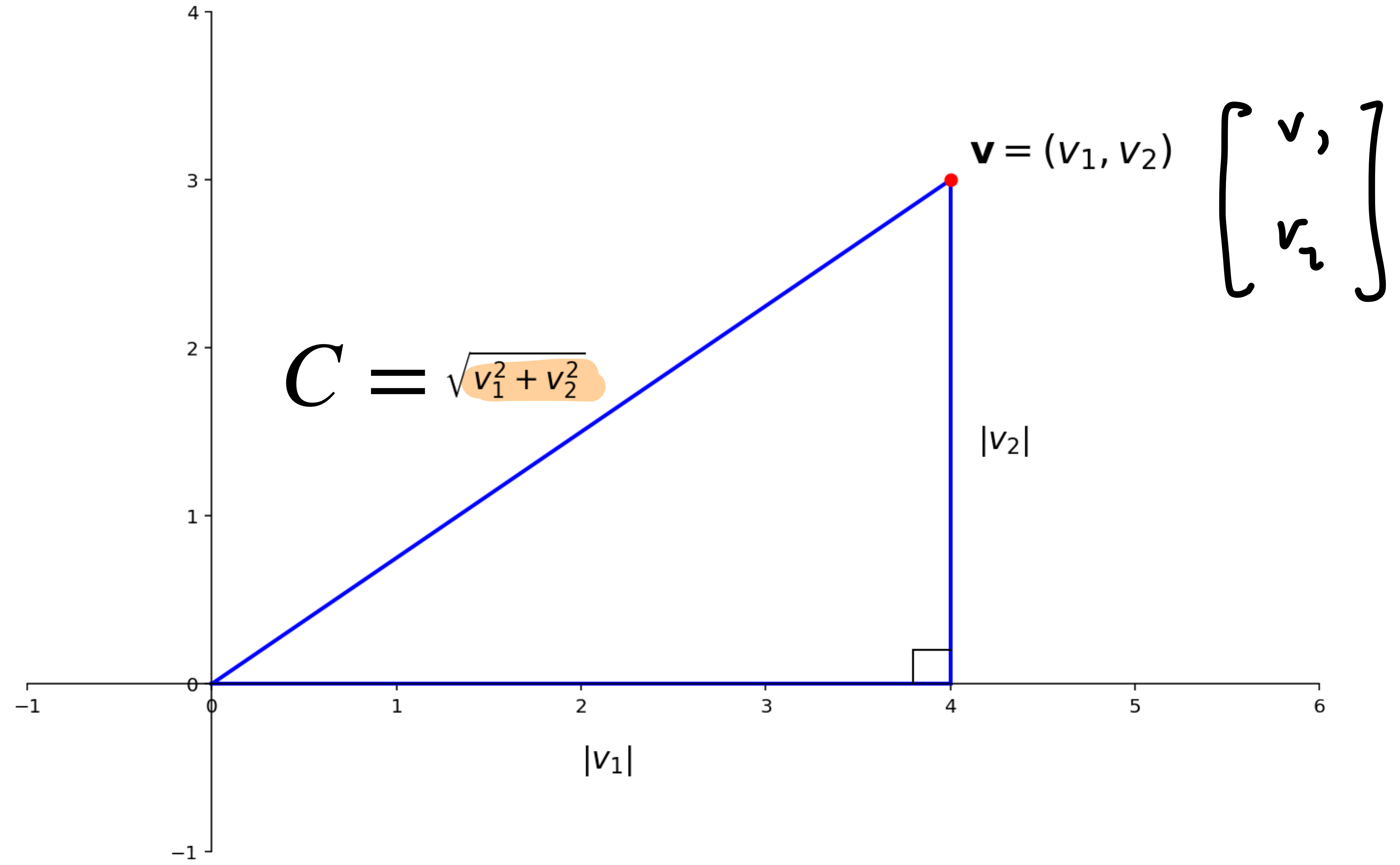
Norms (Lengths/Distances)

Another Potentially Familiar Question



How long is the side C?

Pythagorean Theorem



Theorem (Pythagoras). *For a right triangle, the square of the length of the hypotenuse is the sum of the squares of the lengths of the remaining to sides.*

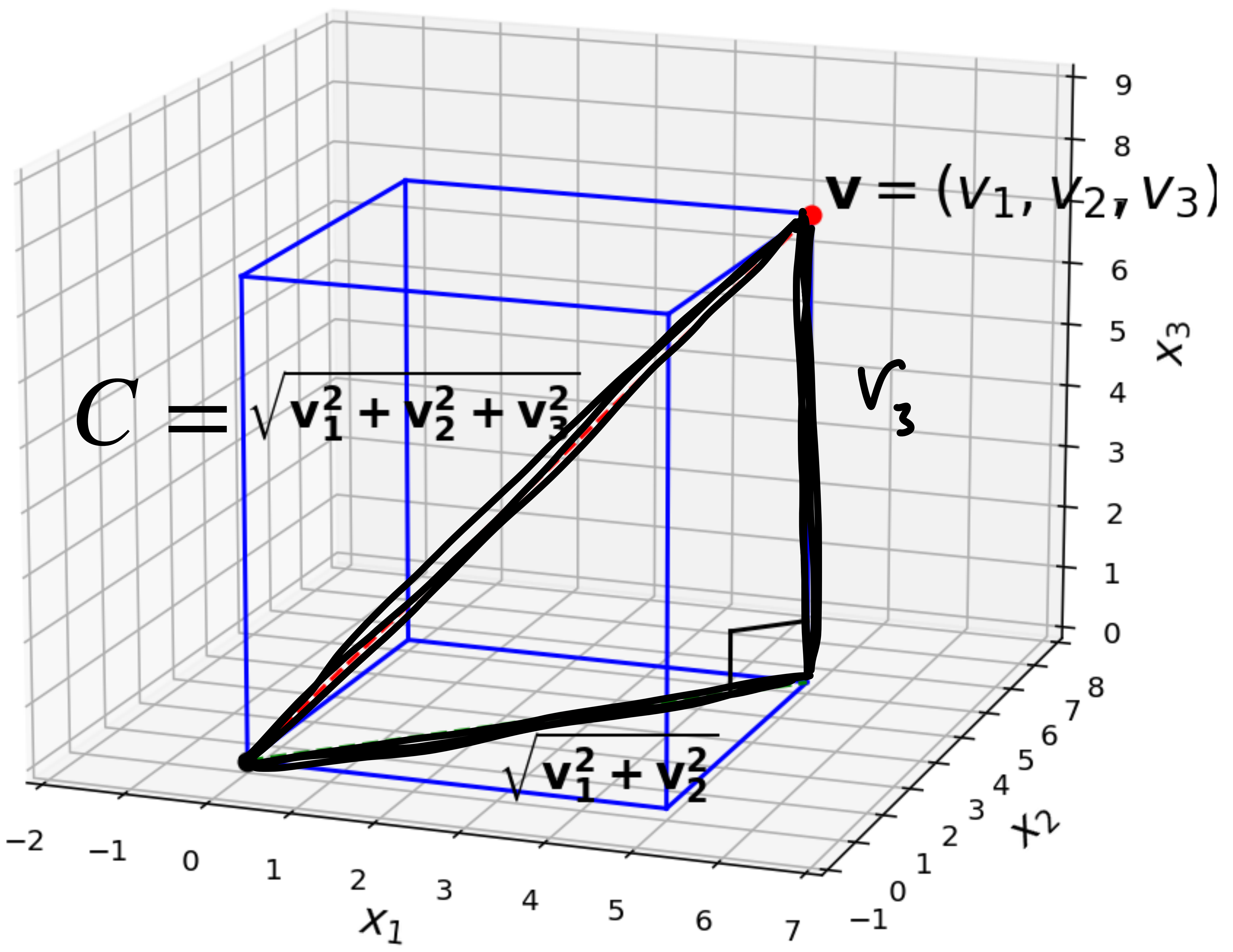
This still works in $\mathbb{R}^3$

Theorem (Pythagoras). $C = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Verify:

$$\sqrt{\left(\sqrt{v_1^2+v_2^2}\right)^2 + v_3^2} =$$

$$\sqrt{v_1^2 + v_2^2 + v_3^2} = C$$



Norm

Definition. The (ℓ^2) **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \left\| \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

The norm of a vector is the square root of the sum of the squares of its entries.

Norms and Inner Products

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

The norm of a vector is the square root of the inner product with itself.

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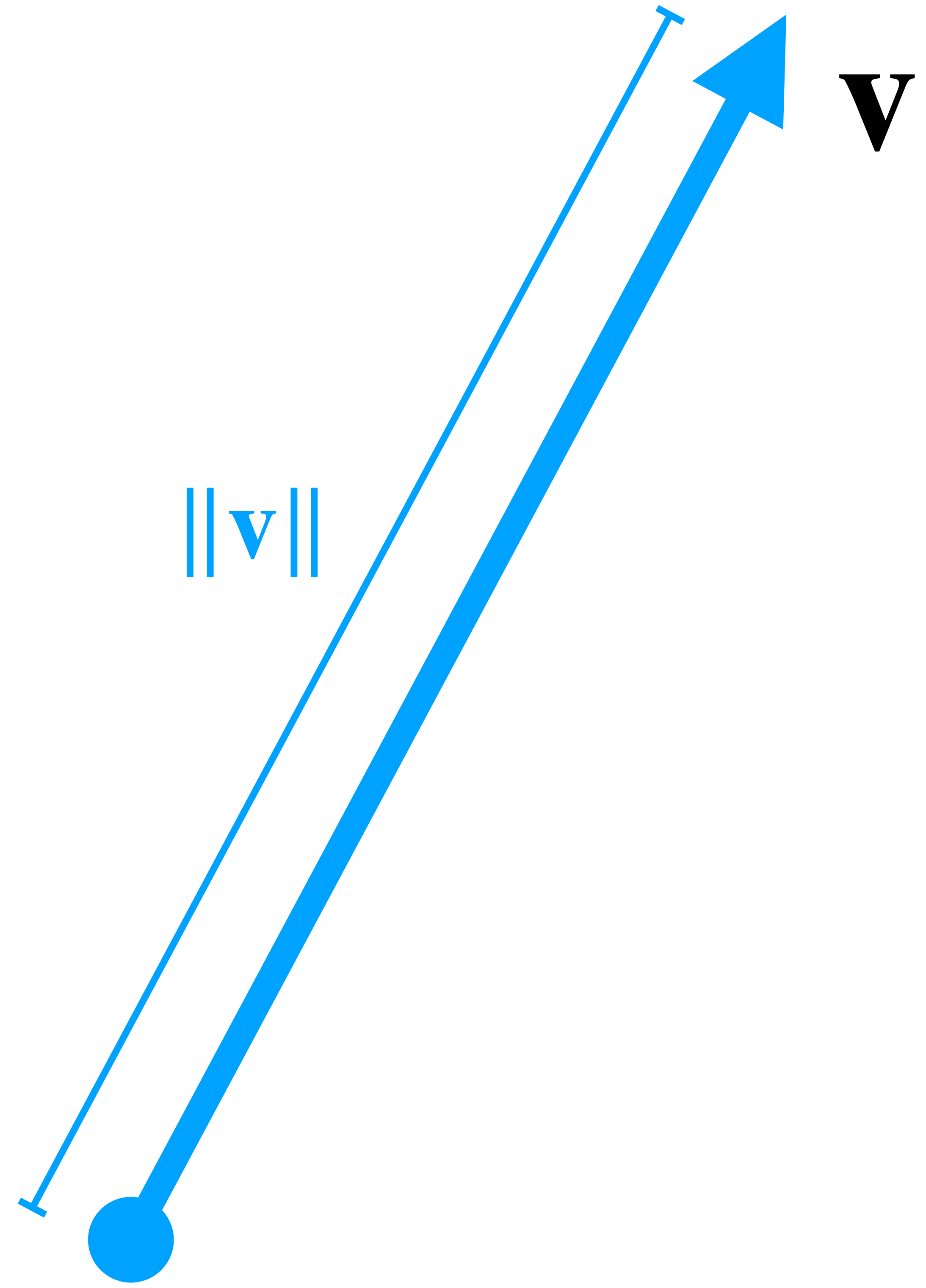
The norm of a vector is the square root of the inner product with itself.

It's important that $\mathbf{v}^T\mathbf{v}$ is nonnegative.

Norms and Distance

Norms give us a notion of <u>length</u>.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ this is our existing notion of length.

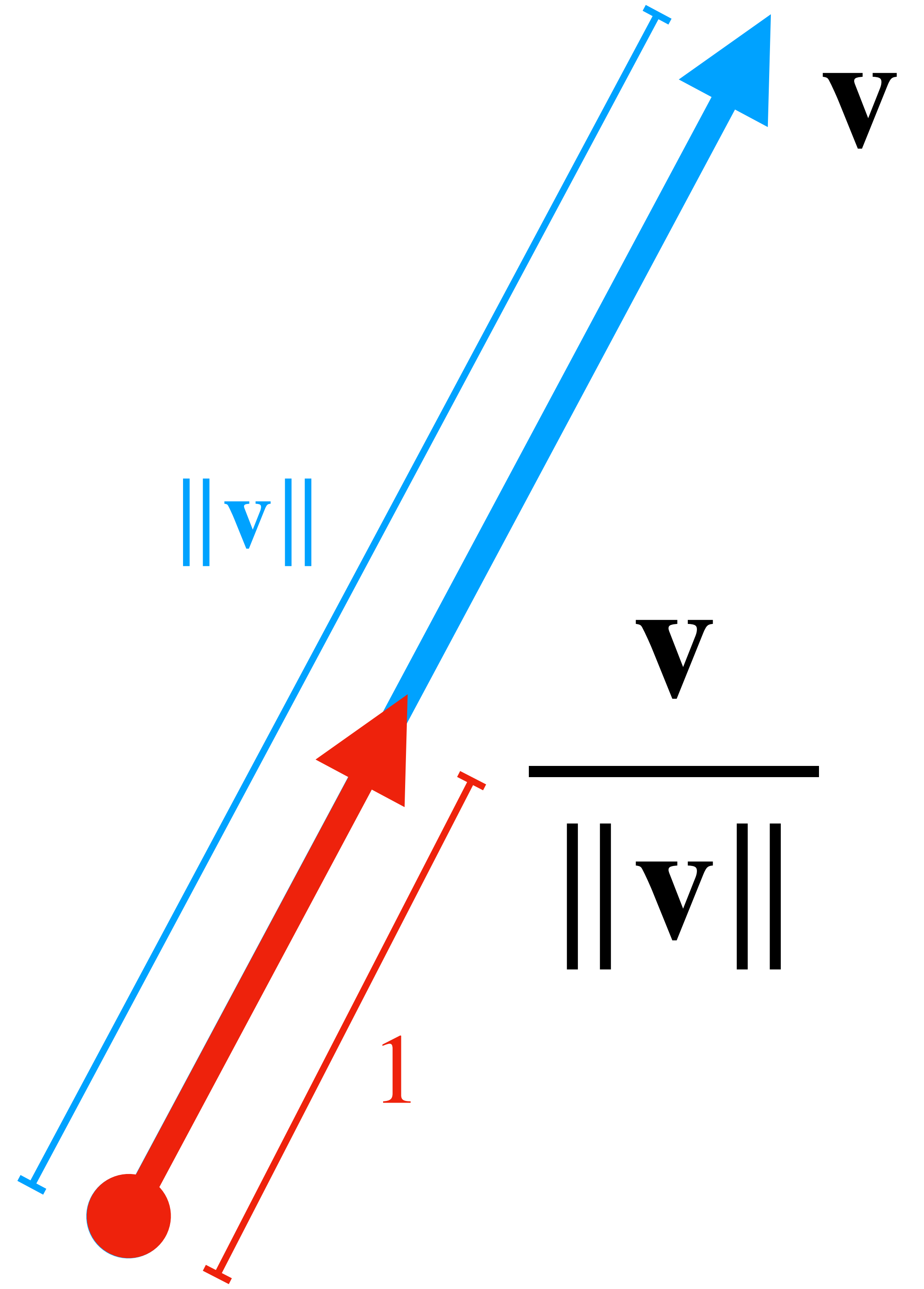


ℓ^2 Normalization

Definition. A **unit vector** is a vector $\mathbf{v}$ such that $\|v\| = 1$.

We often *normalize* vectors if we only care about their direction:

$$\mathbf{v} \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$



How To: Normalizing Vectors

Question. Find the unit vector which points in the same direction as $\mathbf{u}$.

Solution. Compute $\|\mathbf{u}\|$. The unit vector is then

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Example

Find the unit vector in the same direction as $\vec{r} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

$$\|\vec{r}\| = \sqrt{1^2 + (-2)^2 + (2)^2 + 0} = \sqrt{9} = 3$$

$$\frac{\vec{r}}{\|\vec{r}\|} = \frac{\vec{r}}{3} = \vec{u} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

$$\|\vec{u}\| = \sqrt{(1/3)^2 + (-2/3)^2 + (2/3)^2 + 0}$$

$$= \sqrt{1/9 + 4/9 + 4/9}$$

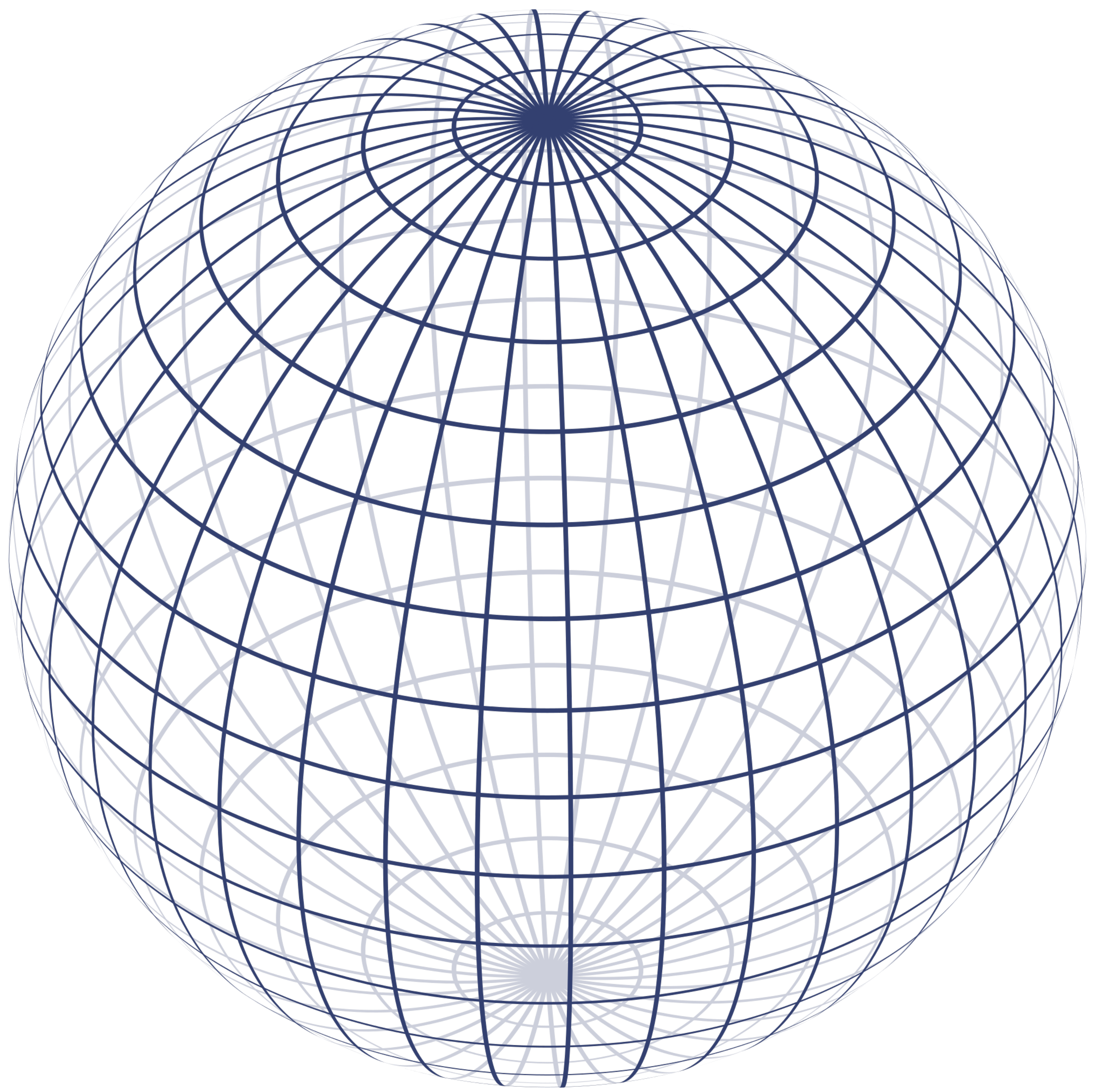
$$= \sqrt{1} = 1$$

The Unit Sphere

Definition. The unit n–sphere is the collection of all unit vectors in $\mathbb{R}^n$.

Vector norms allow us to talk about spheres in higher dimensions.

A sphere is a collection of points equidistant from a center point.



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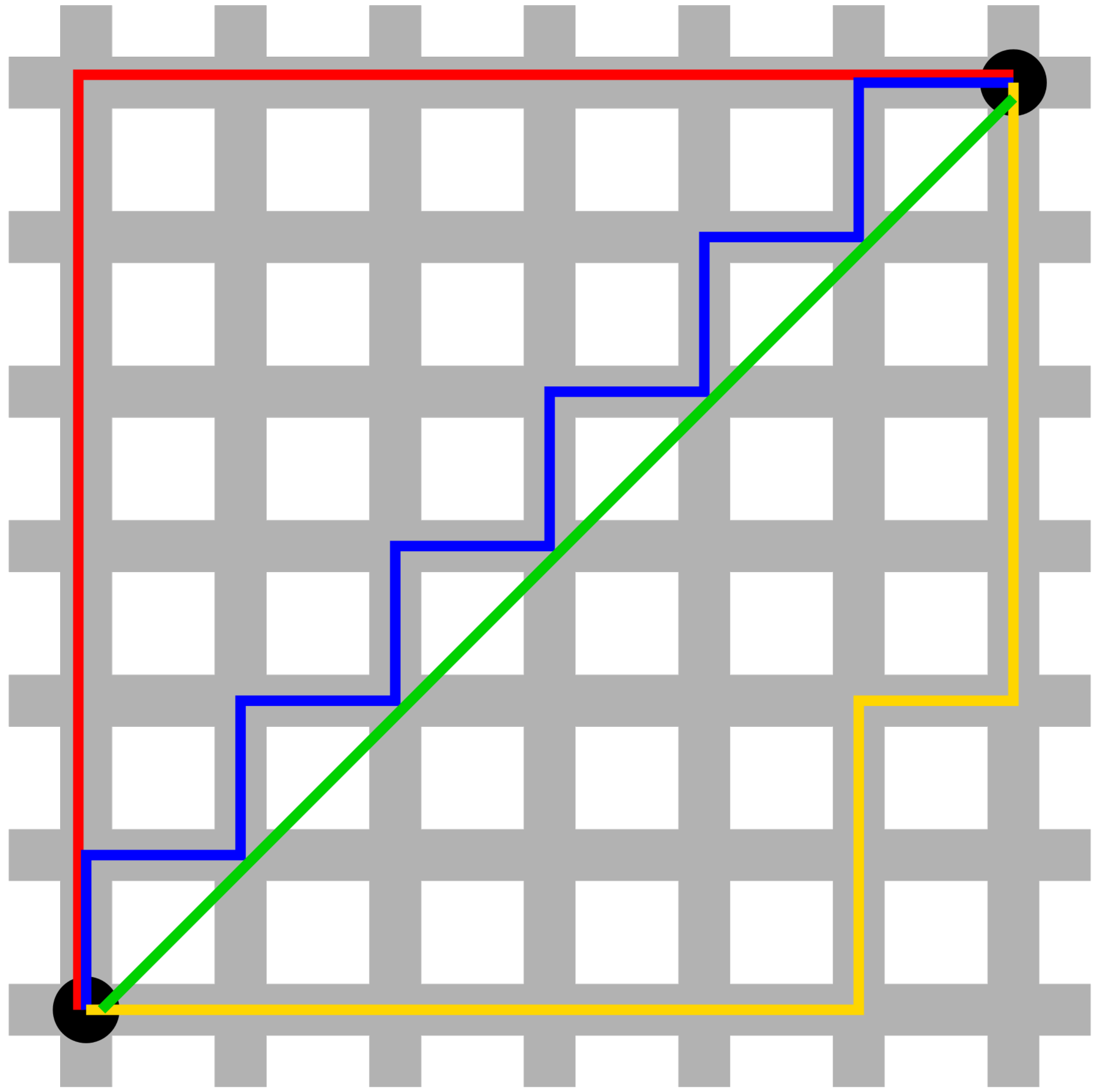
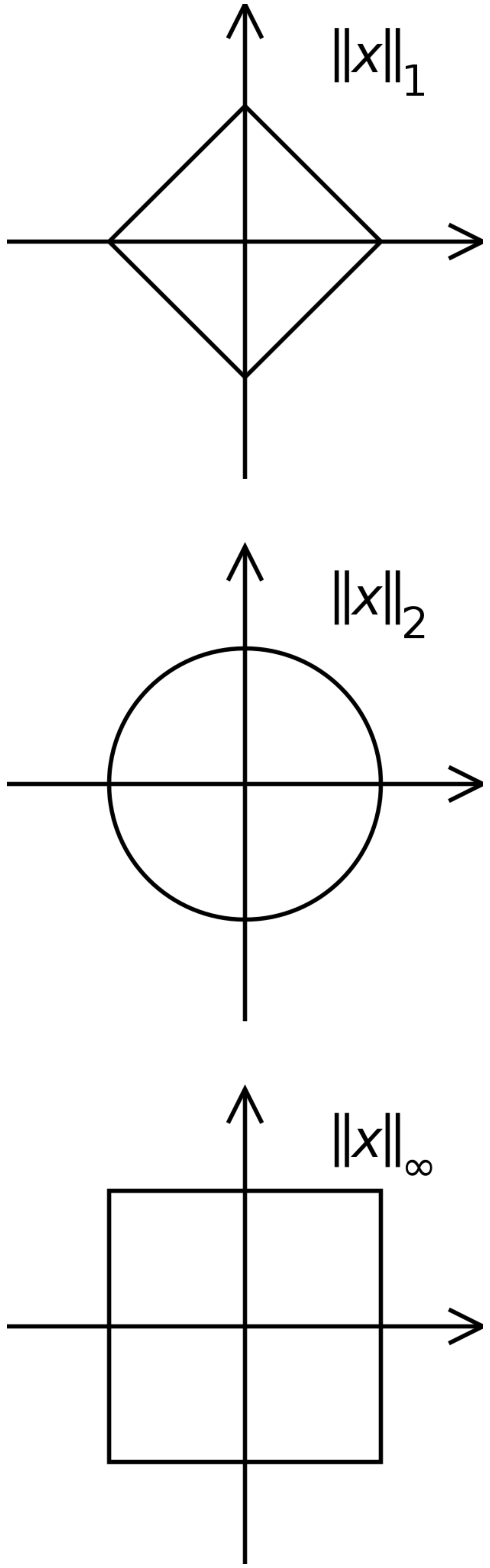
An Aside: Other Notions of Distance





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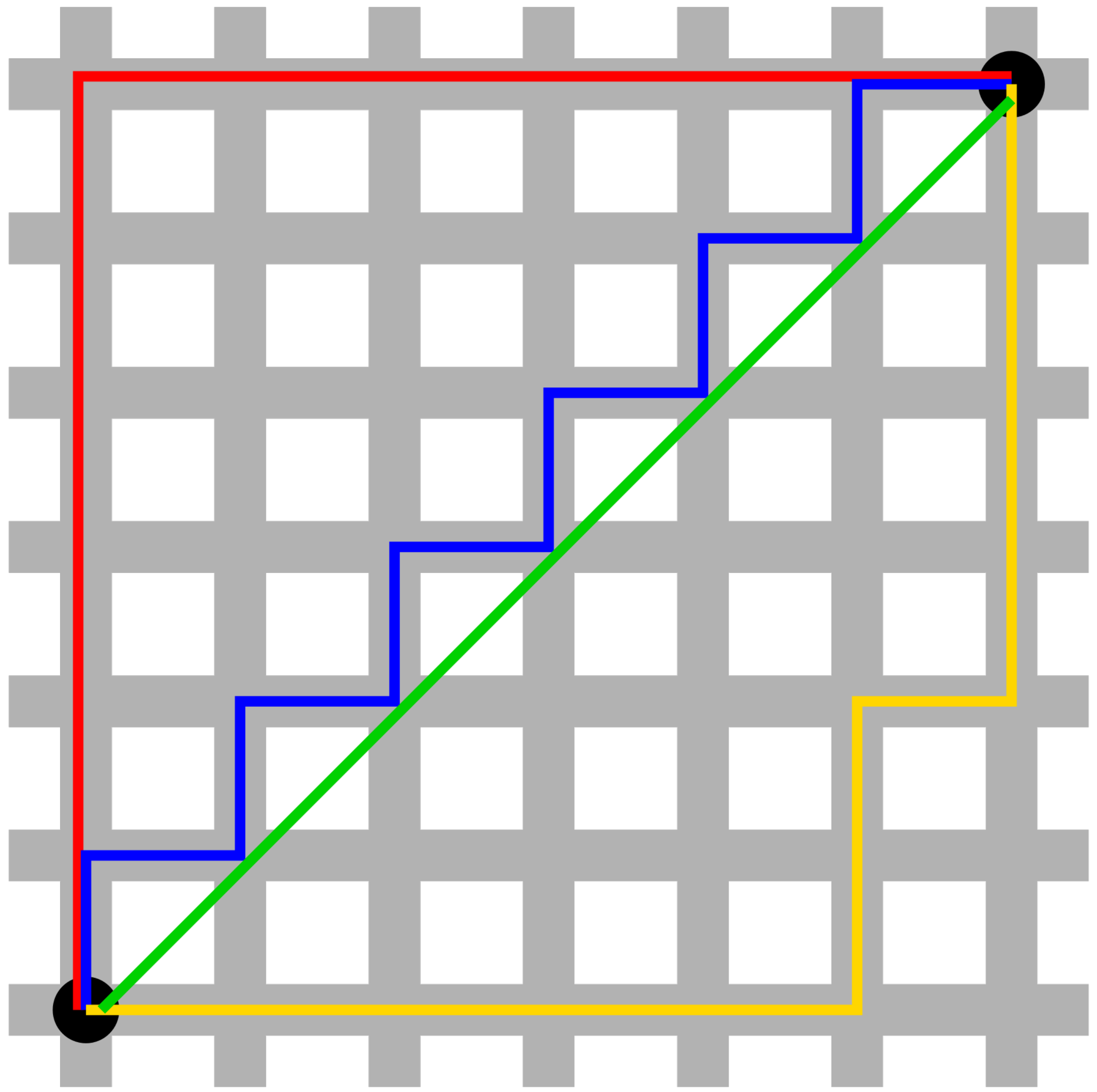
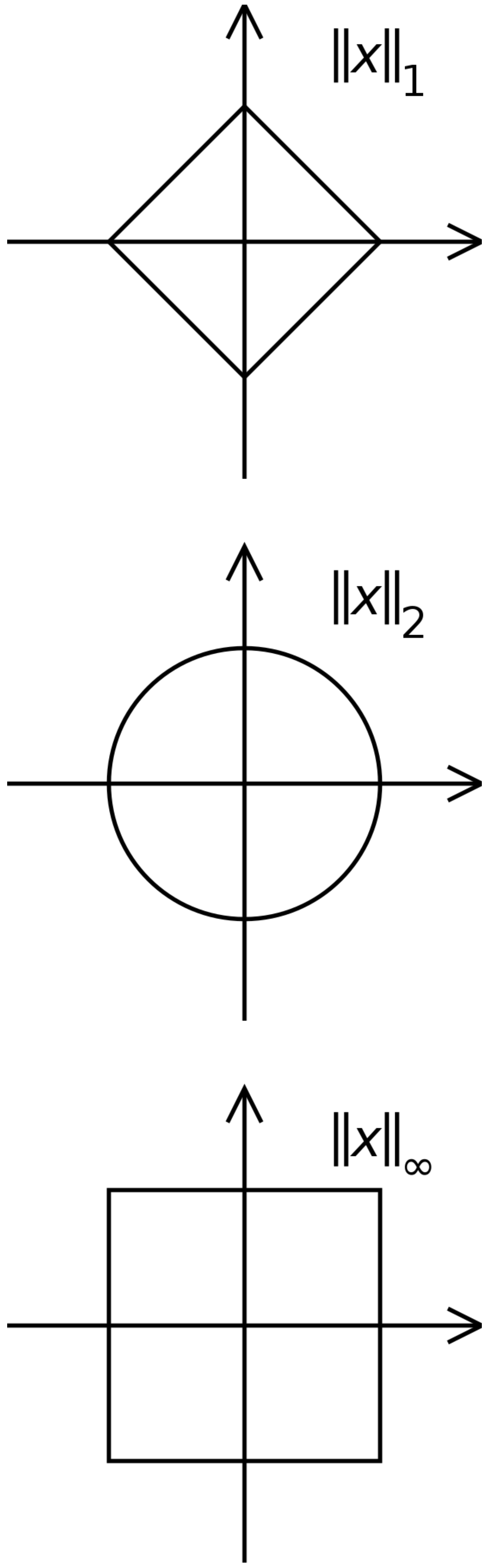
https://commons.wikimedia.org/wiki/File:Manhattan_distance.svg

An Aside: Other Notions of Distance

Why are we talking about norms and inner products so generally?





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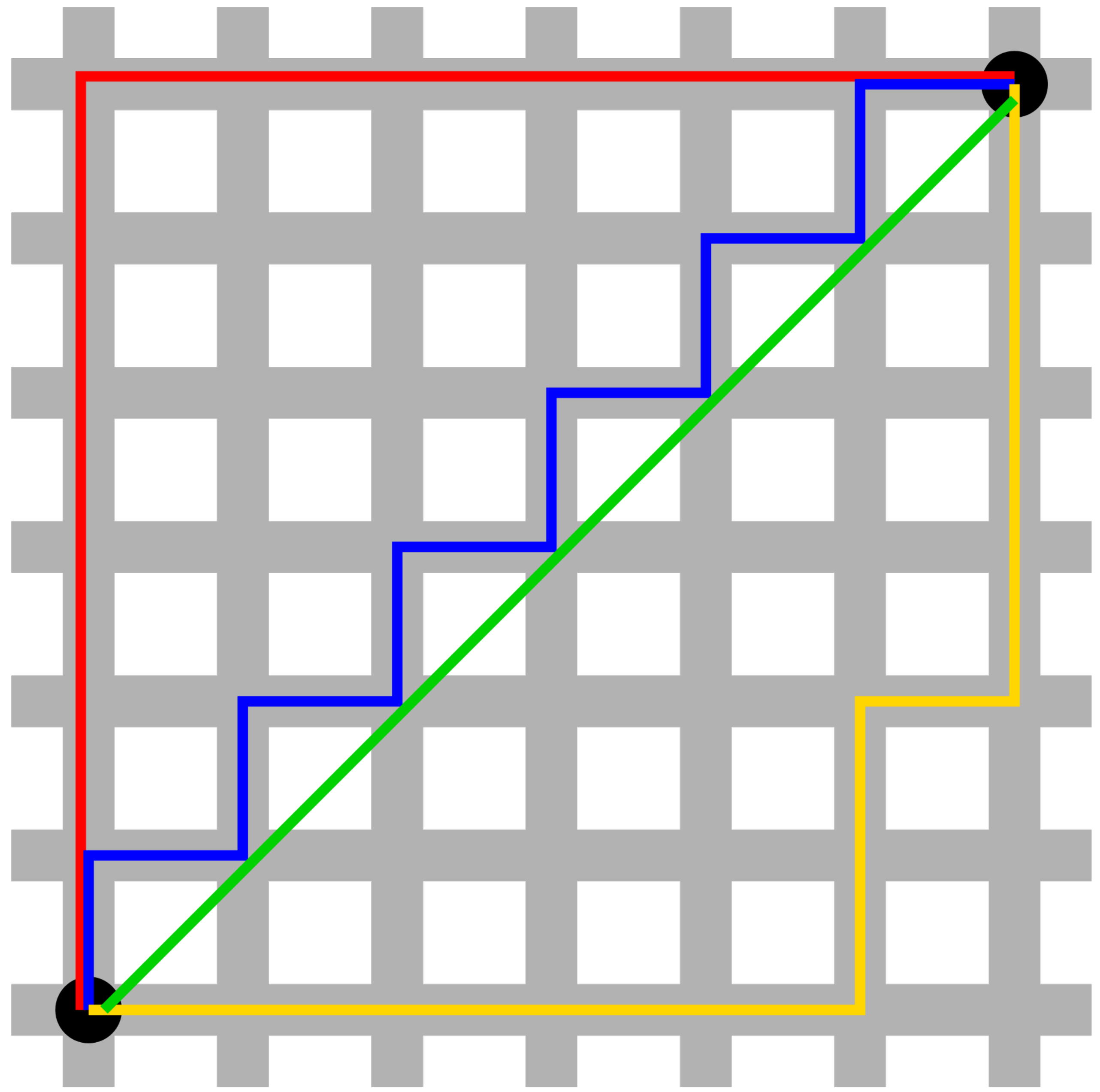
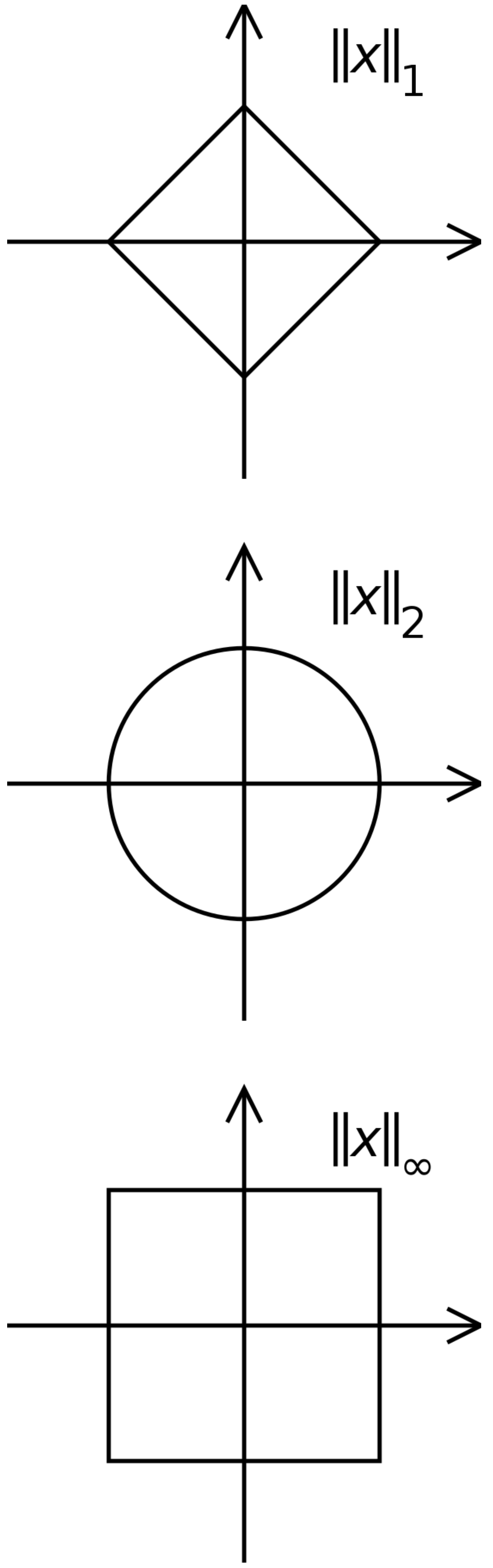
An Aside: Other Notions of Distance

Why are we talking about norms and inner products so generally?

Because there are other inner products and norms.





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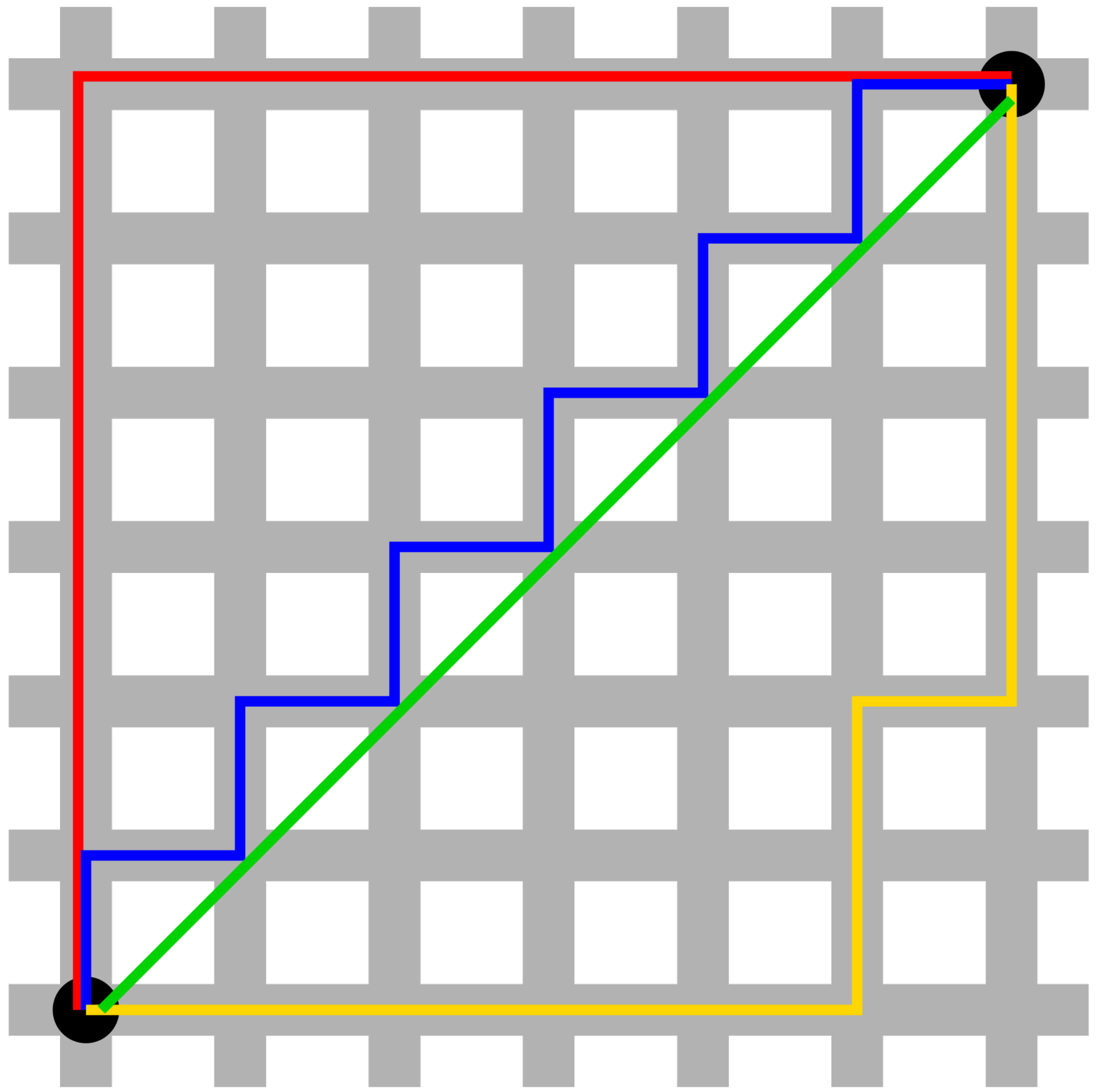
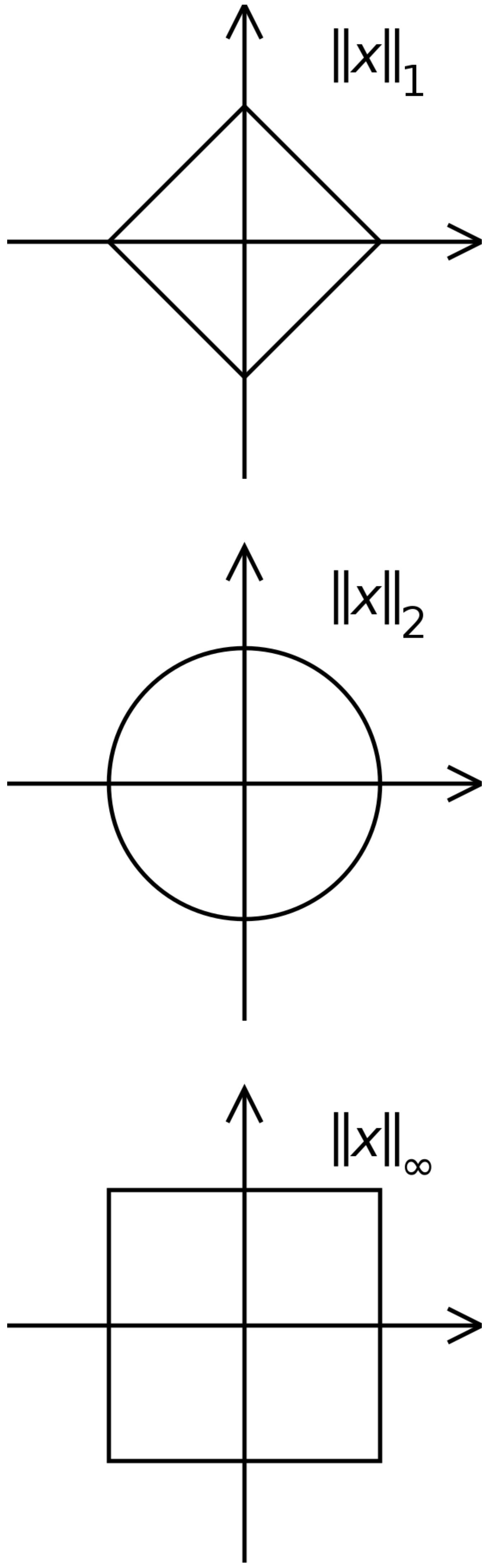
e.g., Manhattan distance





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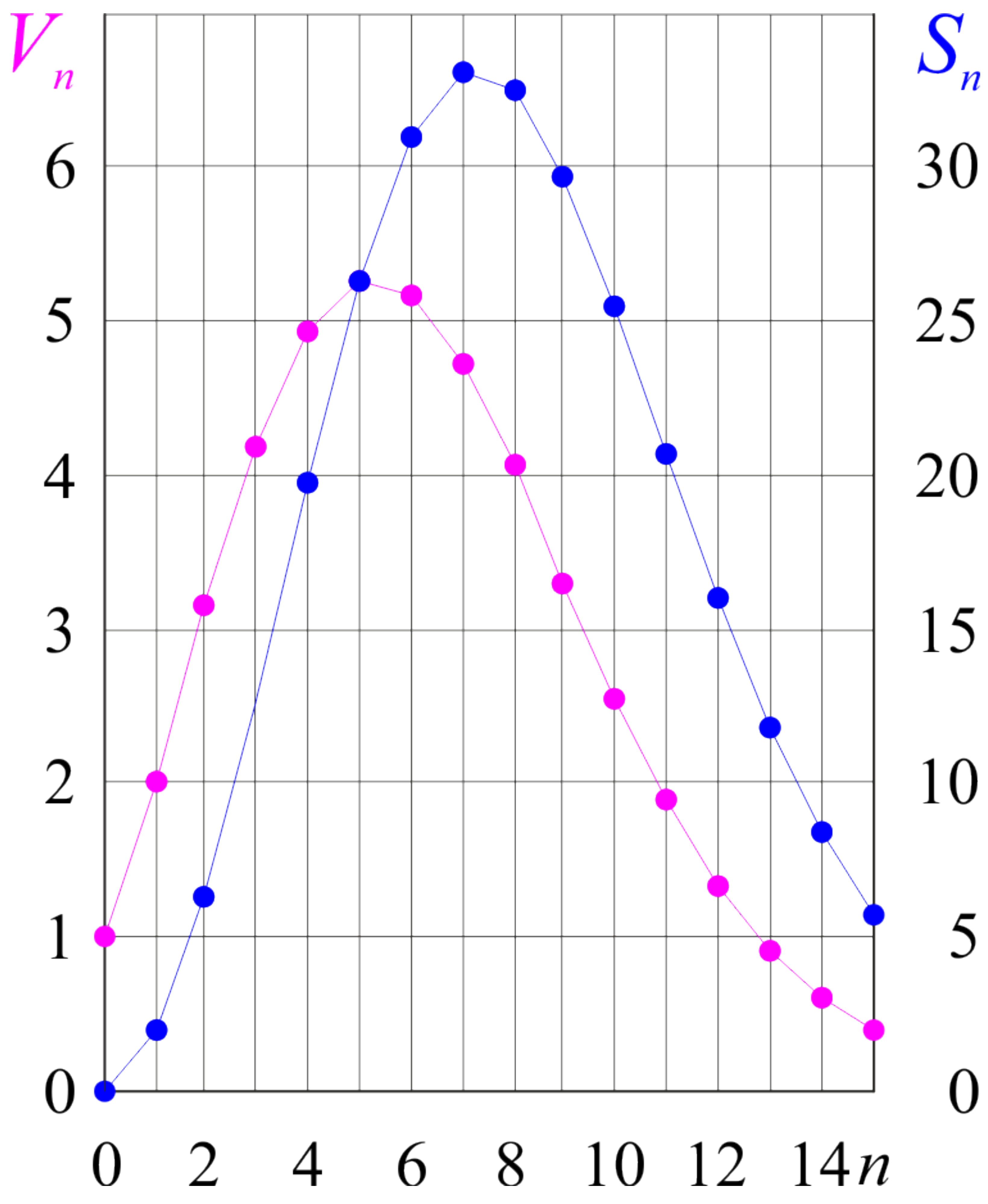
Another Aside: Surface Area and Volume

With a bit of calculus, we can calculate the surface area and volume of the unit n–sphere.

And the result is bizarre...

the infinite dimensional unit sphere has no volume or surface area...



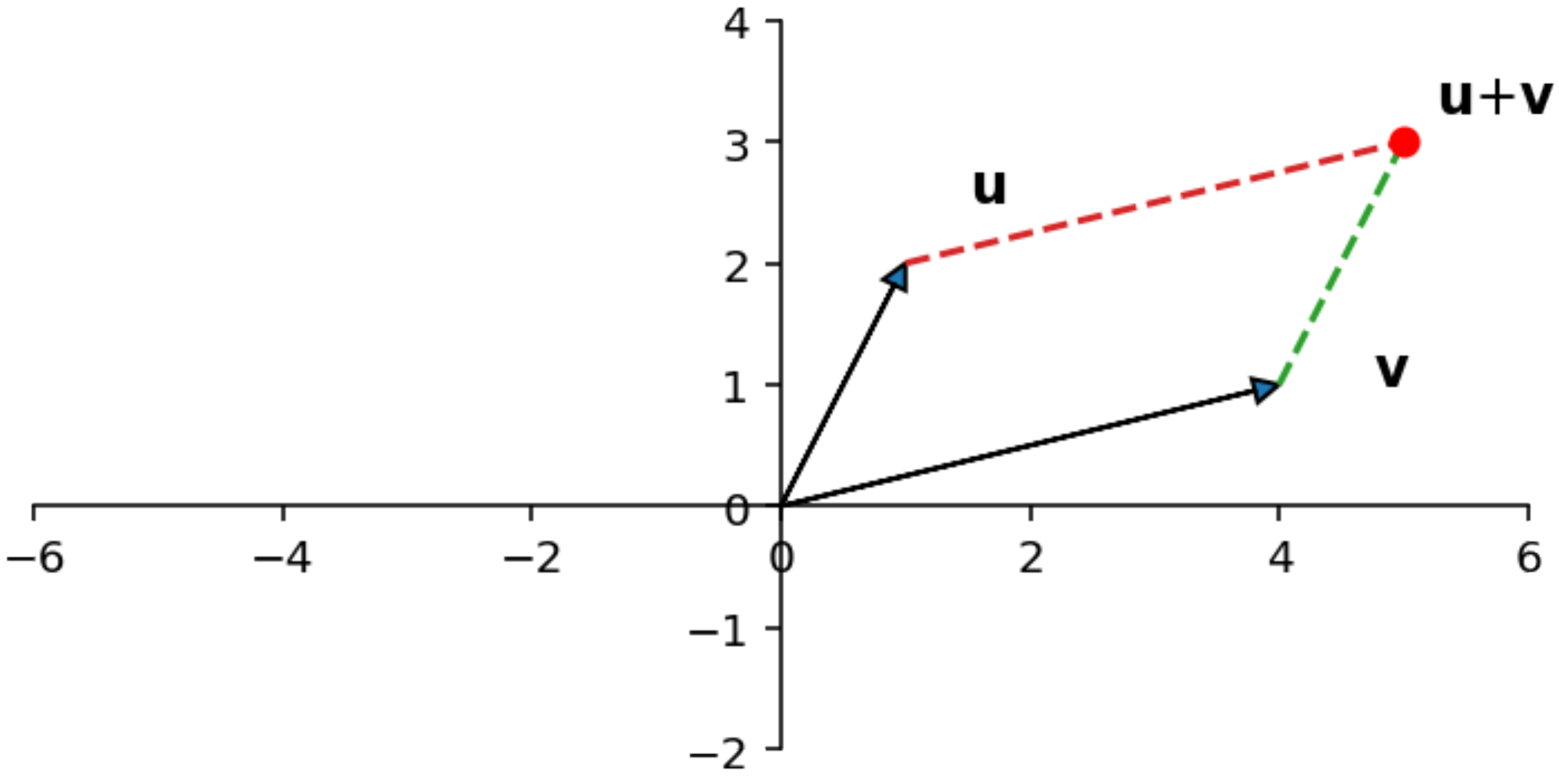
https://commons.wikimedia.org/wiki/File:Graphs_of_volumes_(V)_and_surface_areas_(S)_of_n-balls_of_radius_1.png moving on...

Distance

If we know how to calculate lengths of vectors, we know how to calculate distances.

Recall: Vector Addition

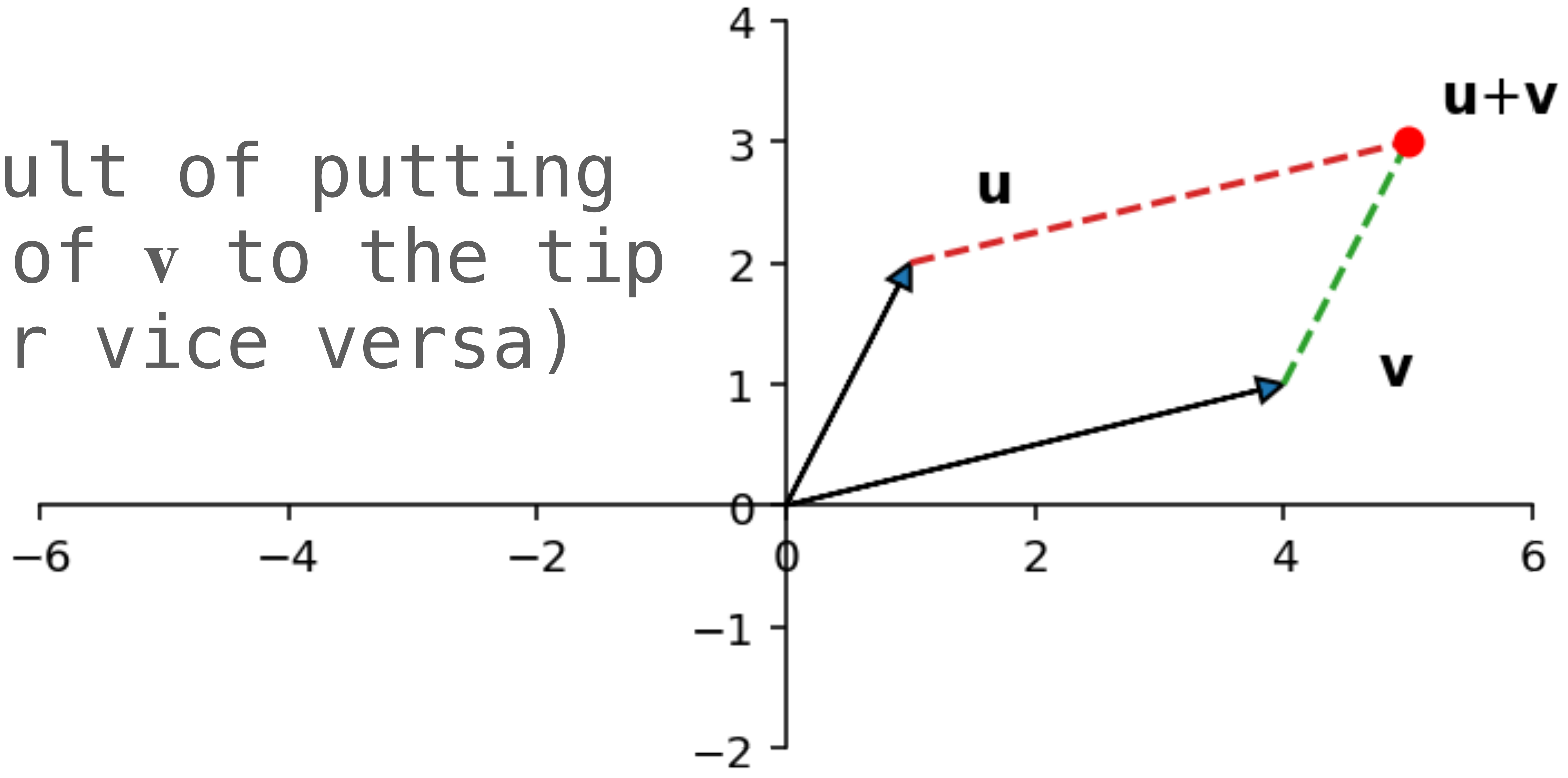
tip-to-tail rule:



Recall: Vector Addition

tip–to–tail rule:

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



Recall: Vector Addition

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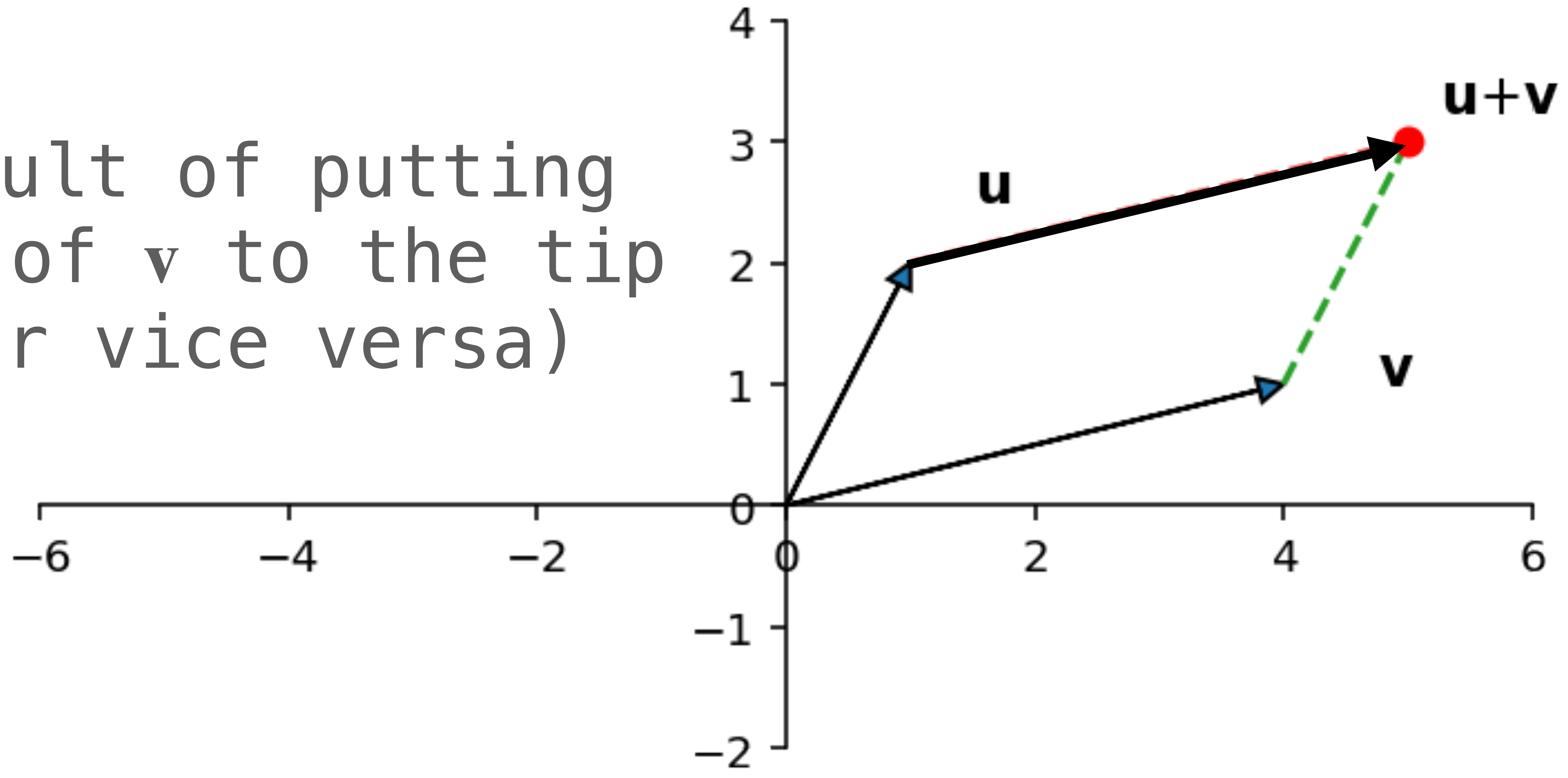

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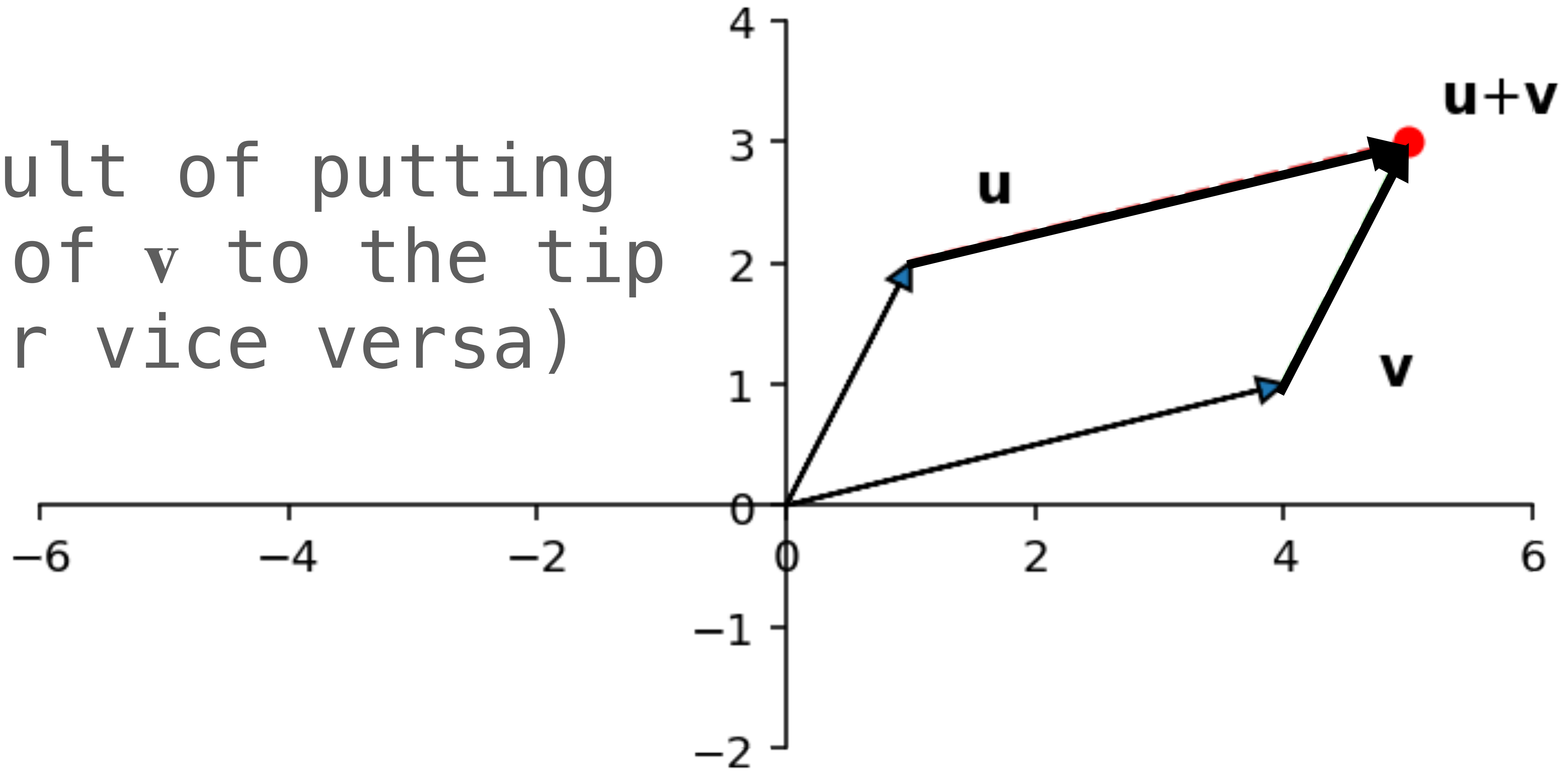
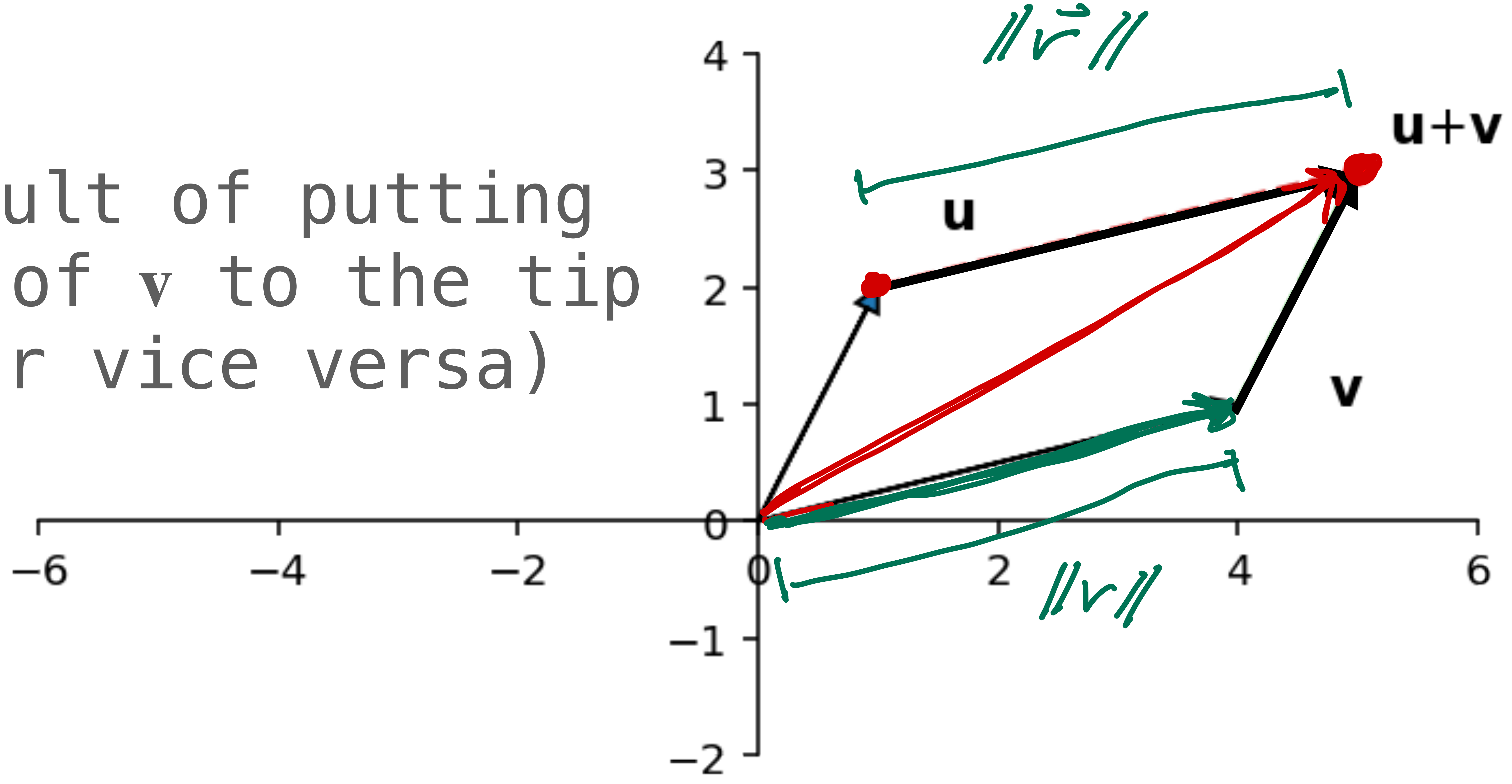


Recall: Vector Addition

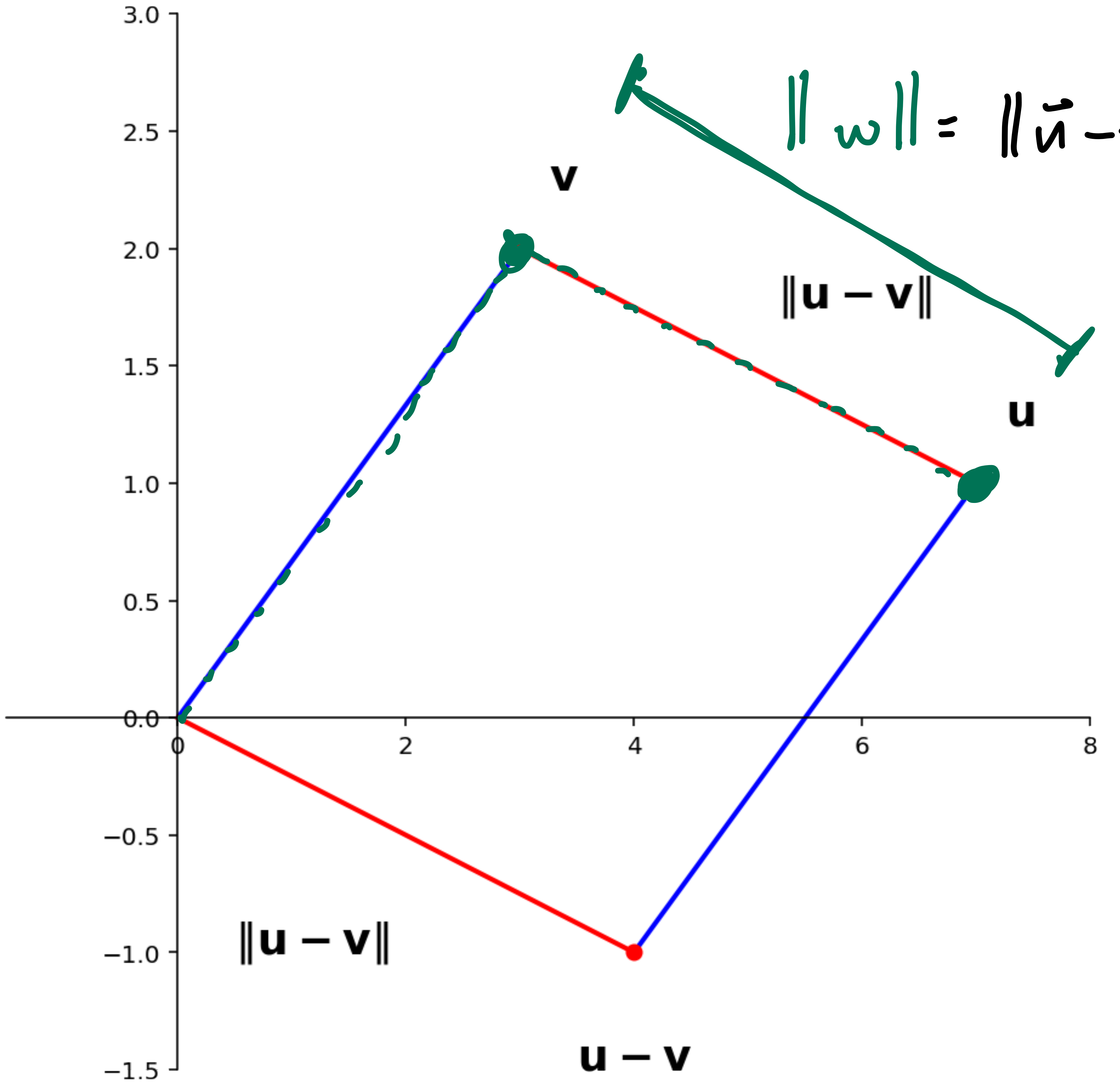
tip-to-tail rule:

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)

The distance between $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ is the length of $\mathbf{v}$



Distance (Pictorially)



$\vec{v} + \vec{w} = \vec{u}$

$\vec{w} = \vec{u} - \vec{v}$

$\|w\| = \|\vec{u} - \vec{v}\|$

$\cancel{\vec{v}} + (\vec{u} - \cancel{\vec{v}}) = \vec{u}$

Distance (Algebraically)

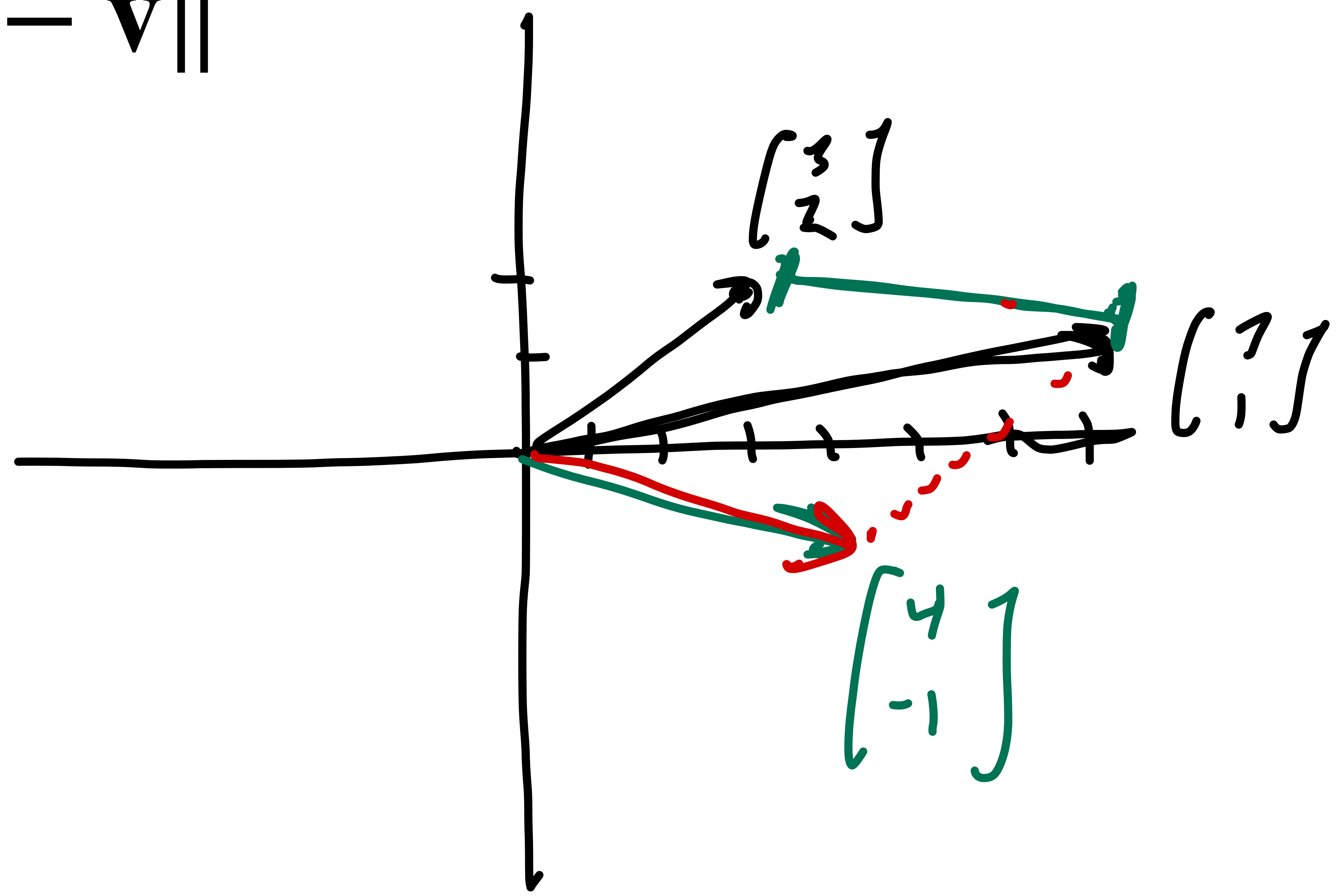
Definition. The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is given by

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g., $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

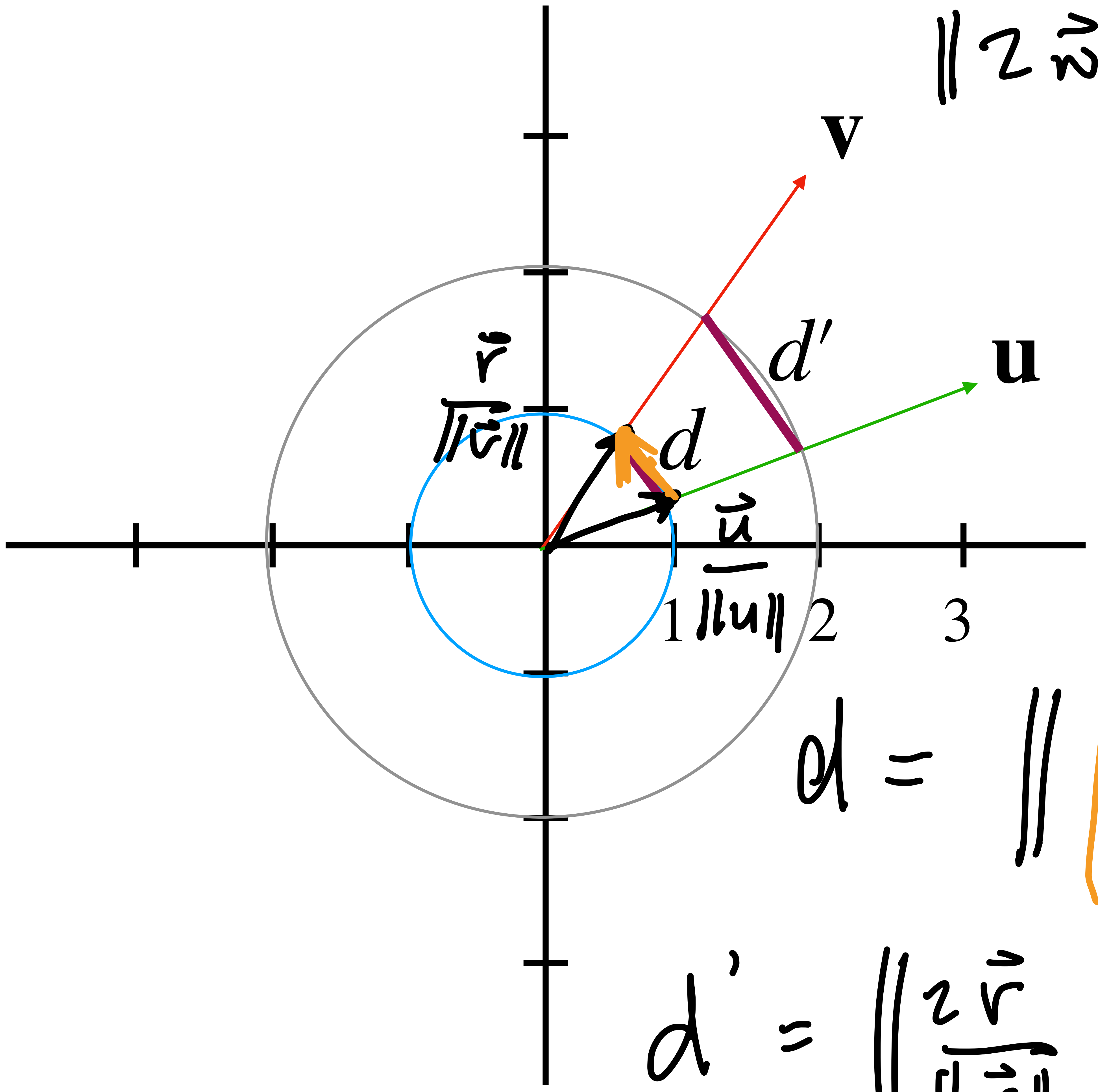
$$\vec{u} - \vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\vec{u} - \vec{v}\| = \sqrt{16+1} = \sqrt{17}$$



Question

$$\frac{\vec r}{\|\vec r\|} \in \mathbb{R}^2$$



$$\|2\vec w\| = \sqrt{\langle 2w, 2w\rangle} = \sqrt{4\langle w, w\rangle} = 2\sqrt{\langle w, w\rangle} = 2\|w\|$$

$$d = \left\| \boxed{\frac{\vec r}{\|\vec r\|} - \frac{\vec u}{\|\vec u\|}} \right\|$$

$$d' = \left\| \frac{2\vec r}{\|\vec r\|} - \frac{2\vec u}{\|\vec u\|} \right\| = \left\| 2\left(\frac{\vec r}{\|\vec r\|} - \frac{\vec u}{\|\vec u\|}\right) \right\|$$

Find an expression for the distance d.

$$d' = 2d$$

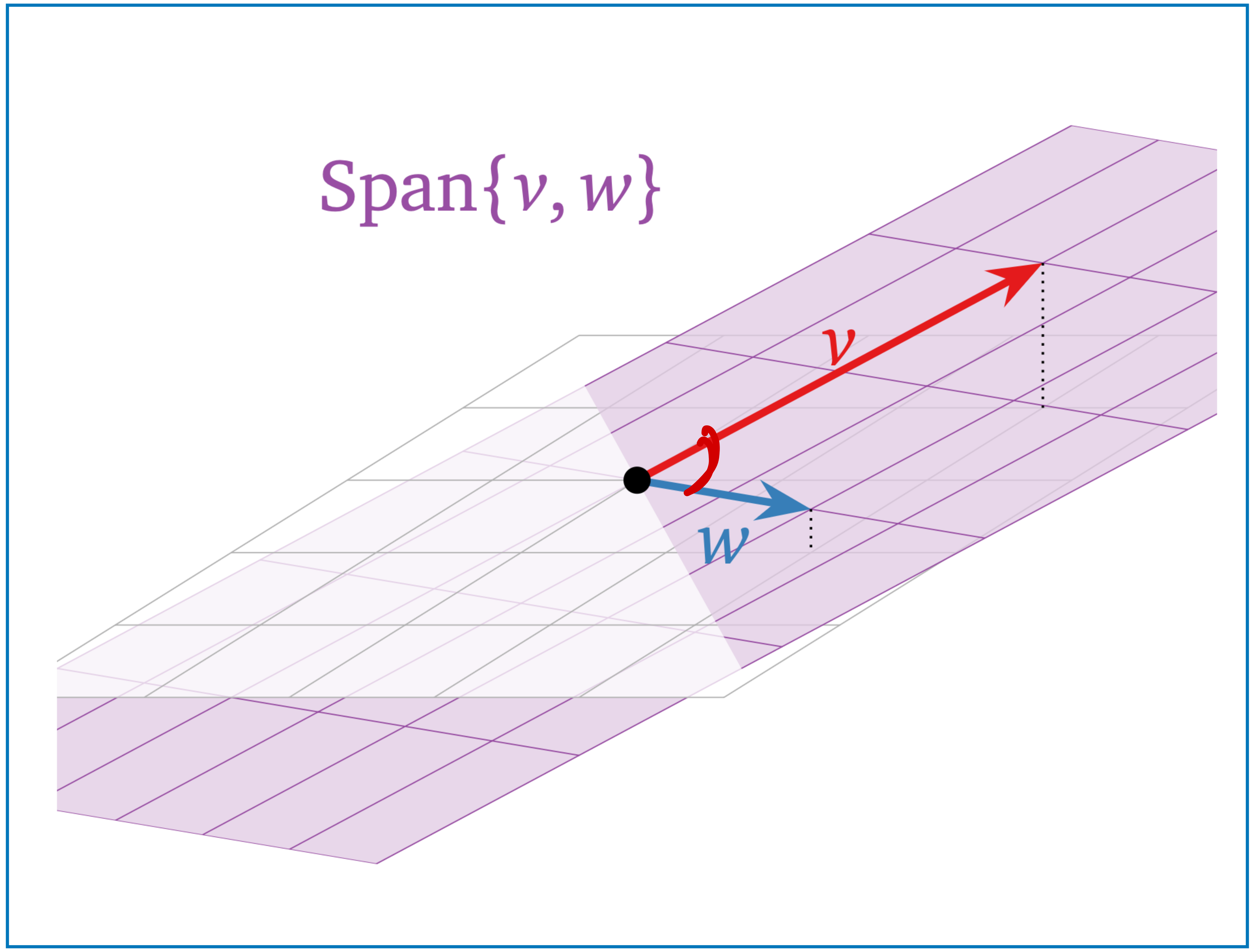
Challenge. Find an expression for d'.

Answer

Angles

Again, Angles still make sense

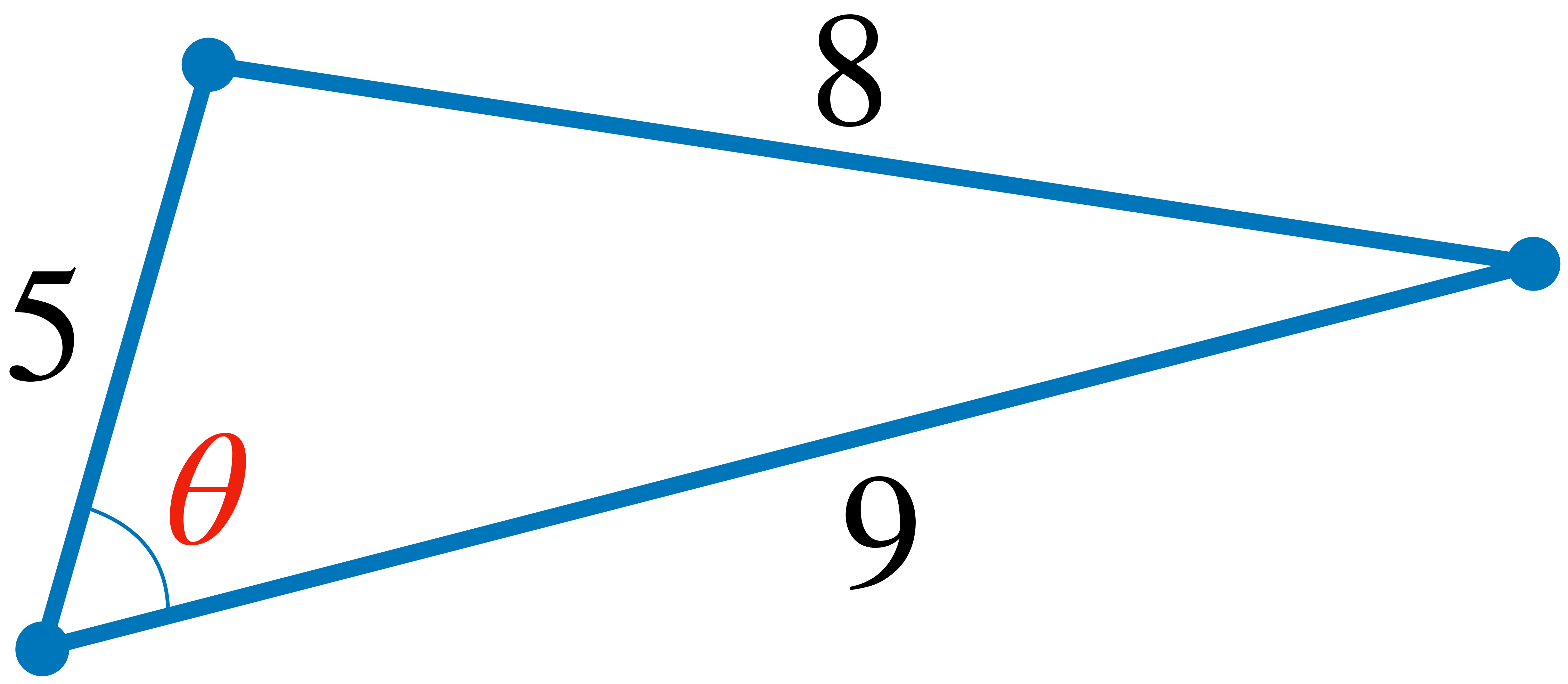
Any pair of vectors in $\mathbb{R}^n$ span a (2D) plane.



Fundamental Question

How do we determine the angle between any two vectors?

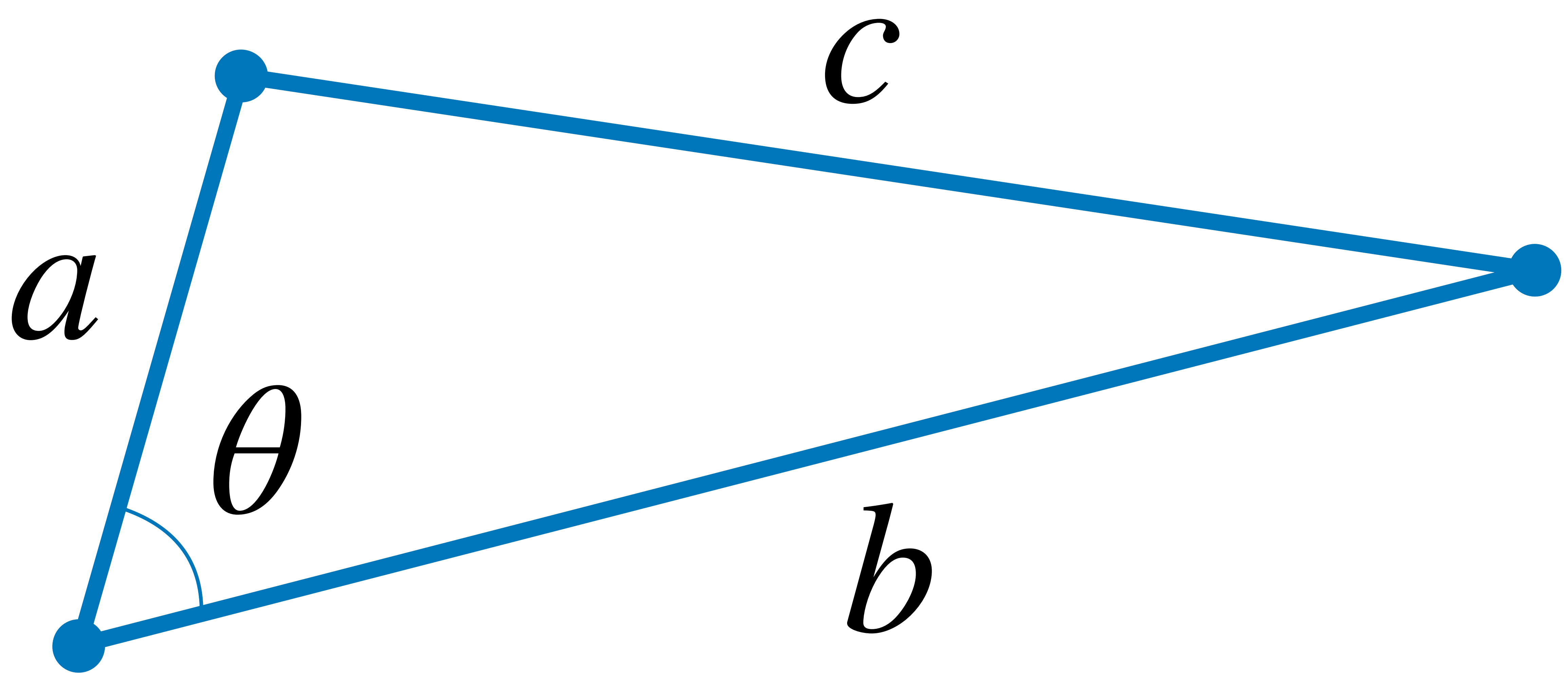
Recall: A Potentially Familiar Example



What is the value of θ*?*

https://www.mathsisfun.com/algebra/trig-cosine-law.html

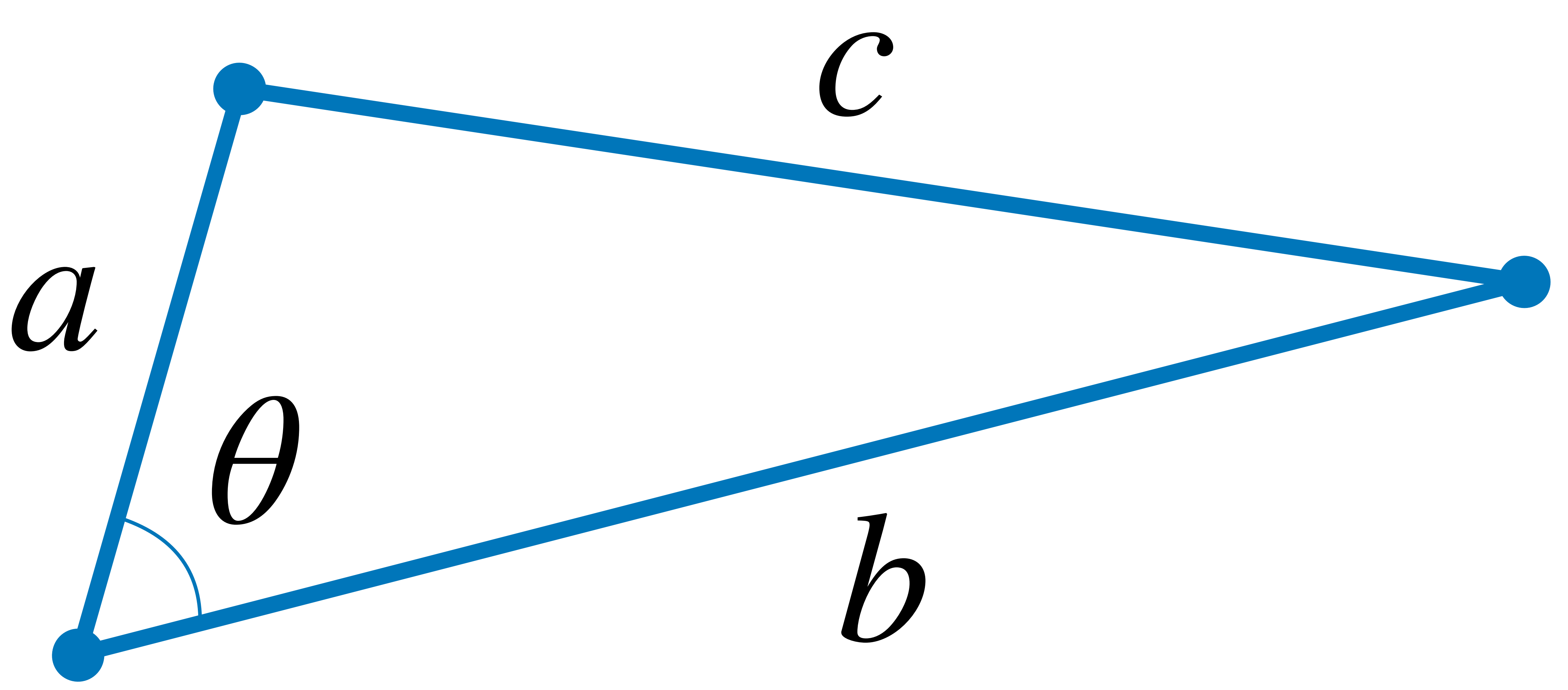
Law of Cosines



Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Law of Cosines

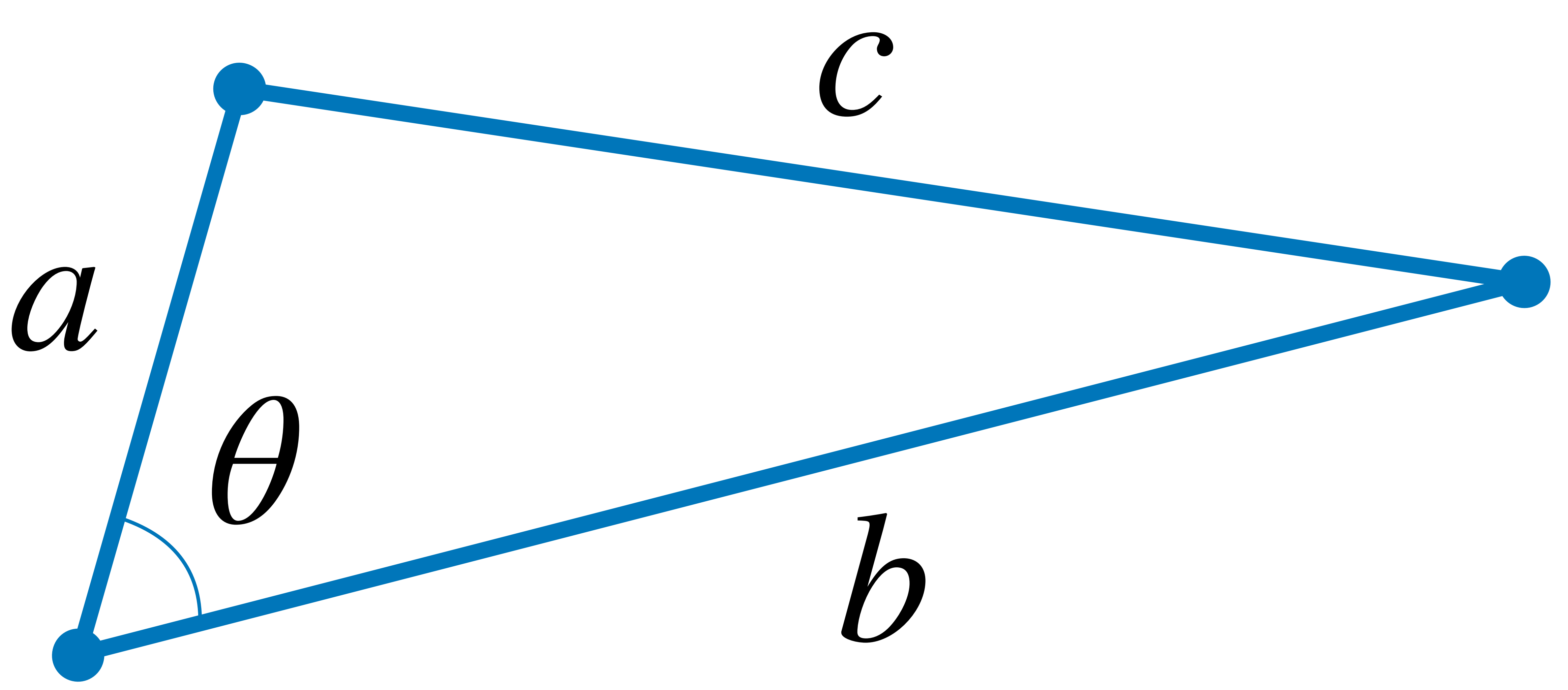


Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Generalized the Pythagorean Theorem

Law of Cosines



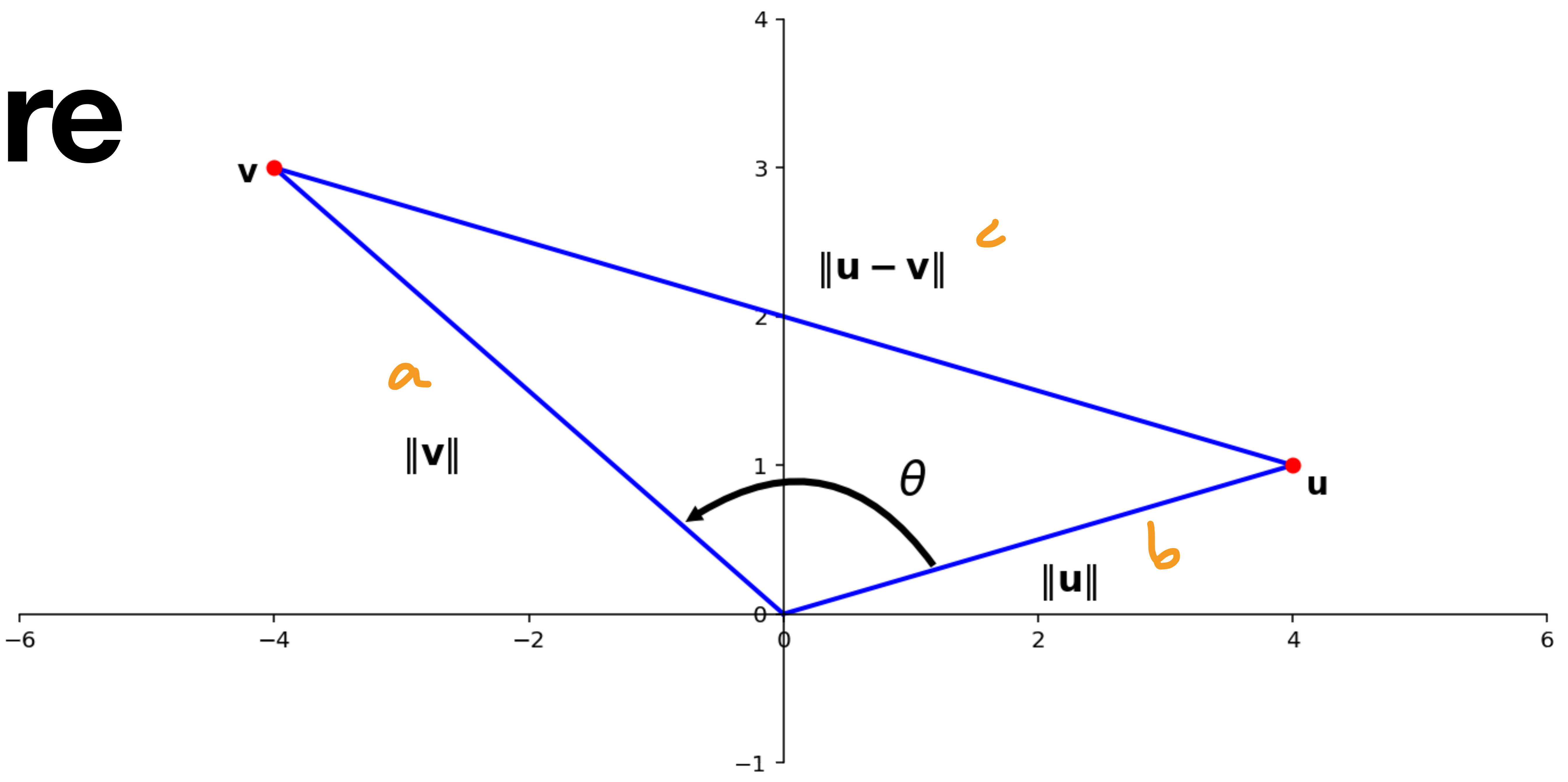
Theorem.

0 exactly when $\theta = 90^\circ$

$$\boxed{c^2 = a^2 + b^2} \boxed{- 2ab\cos\theta}$$

Generalized the Pythagorean Theorem

The Picture



In more "vector"-y terms:

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

$c^2 \quad a^2 \quad b^2 \quad 2ab\cos\theta$

Isolating θ

$$\cos\theta = \frac{c^2 - a^2 - b^2}{2ab} \qquad\qquad \theta = \cos^{-1}\left(\frac{c^2 - a^2 - b^2}{2ab}\right)$$

`We might remember these equations...`

Isolating θ

$\|w\| = \sqrt{\langle w, w\rangle}$ $\|w\|^2 = \langle w, w\rangle$

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

Let's isolate θ in this equation:

$\|u - v\|^2 - \|u\|^2 - \|v\|^2 = -2\|u\|\|v\| \cos\theta$

$\langle u - v, u - v\rangle - \langle u, u\rangle - \langle v, v\rangle = -2\|u\|\|v\| \cos\theta$

$\cancel{\langle u, u\rangle} - \langle u, v\rangle - \langle v, u\rangle + \cancel{\langle v, v\rangle} \cancel{- \langle u, u\rangle} \cancel{- \langle v, v\rangle}$

$\cancel{-}2\langle u, v\rangle = \cancel{-}2\|u\|\|v\| \cos\theta$

$\cos\theta = \left\langle \frac{\vec{u}}{\|\vec{u}\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle$

Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

$$\cos\theta = \left\langle \frac{\mathbf{u}}{||\mathbf{u}||}, \frac{\mathbf{v}}{||\mathbf{v}||} \right\rangle$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

How To: Angles

Question. Find the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution. Compute $\cos^{-1}\left(\frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}\right)$ (with a calculator).

Example

Find the angle between the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -7 \\ -2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 8 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

Example: Step 1

Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

$$\|\mathbf{u}\| = \sqrt{1^2 + 3^2 + (-7)^2 + (-2)^2} = 7.93$$

$$\|\mathbf{v}\| = \sqrt{8^2 + (-2)^2 + 4^2 + 6^2} = 10.95$$

Example: Step 2

Normalize the vectors.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{bmatrix} 0.13 \\ 0.38 \\ -0.88 \\ -0.25 \end{bmatrix} \qquad \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 0.73 \\ -0.18 \\ 0.36 \\ 0.54 \end{bmatrix}$$

Example: Step 3

Find their inner product.

$$\left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle = (0.13 \cdot 0.73) + (0.38 \cdot -0.18) + (-0.88 \cdot 0.36) + (-0.25 \cdot 0.54)$$

$$= -0.44$$

Example: Step 4

```
Compute the angle.
```

$$\theta = \cos^{-1}(-0.44) \approx 116^\circ$$

Orthogonality (Perpendicularity)

A Simpler Fundamental Question

How do we determine if angle between any two vectors is 90°?

Orthogonality

Orthogonality

Definition (Informal). Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if then angle between them is 90°.

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Orthogonal and perpendicular are the same thing.

But it doesn't connect back to inner products.

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Orthogonal and perpendicular are the same thing.

But it doesn't connect back to inner products.

(and it's difficult to compute with)

Recall: Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

$$\cos\theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

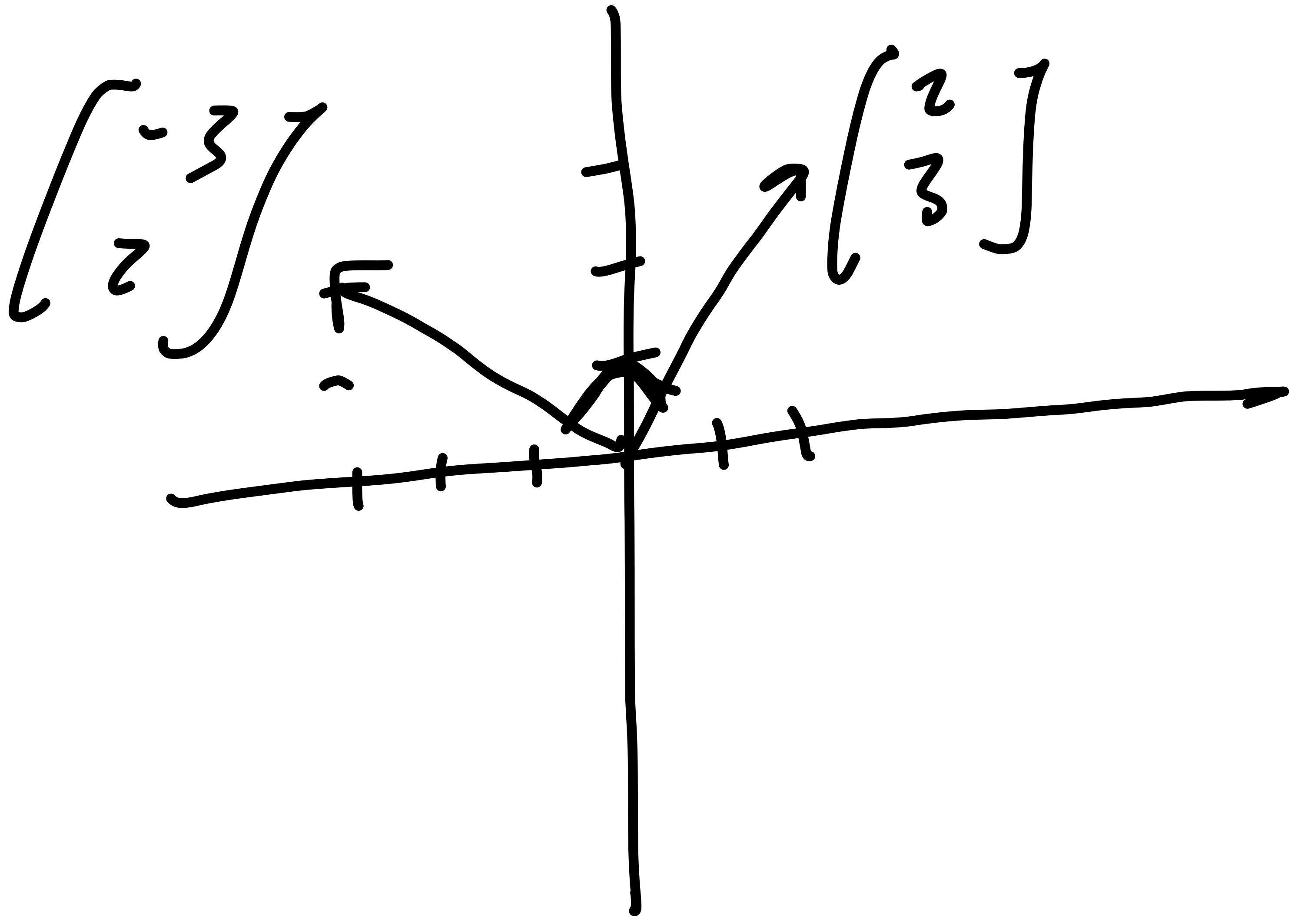
Orthogonality

Definition (Actual). Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

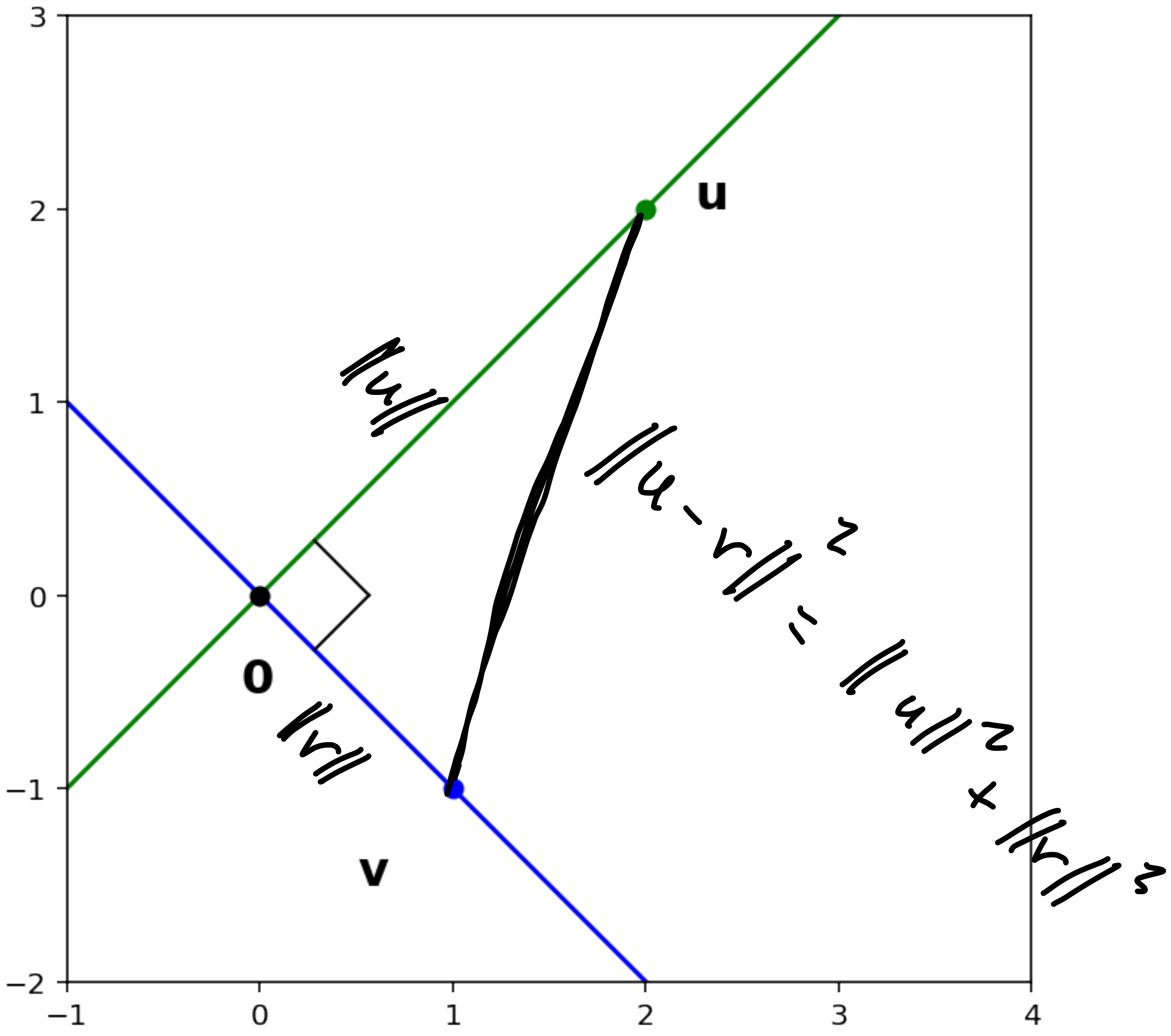
This definition gives an easy computational way to determine orthogonality.

<u>Example.</u>

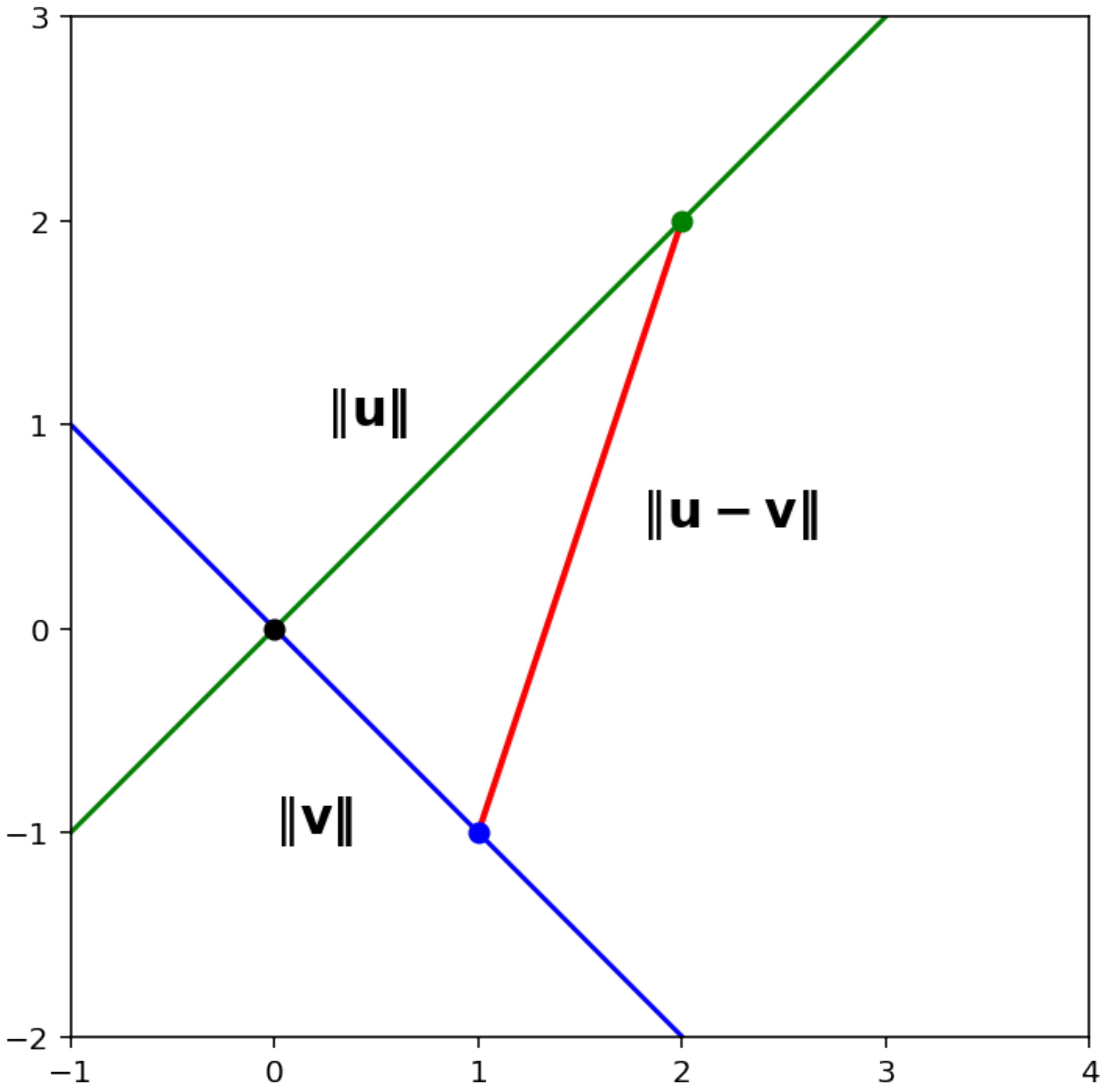
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



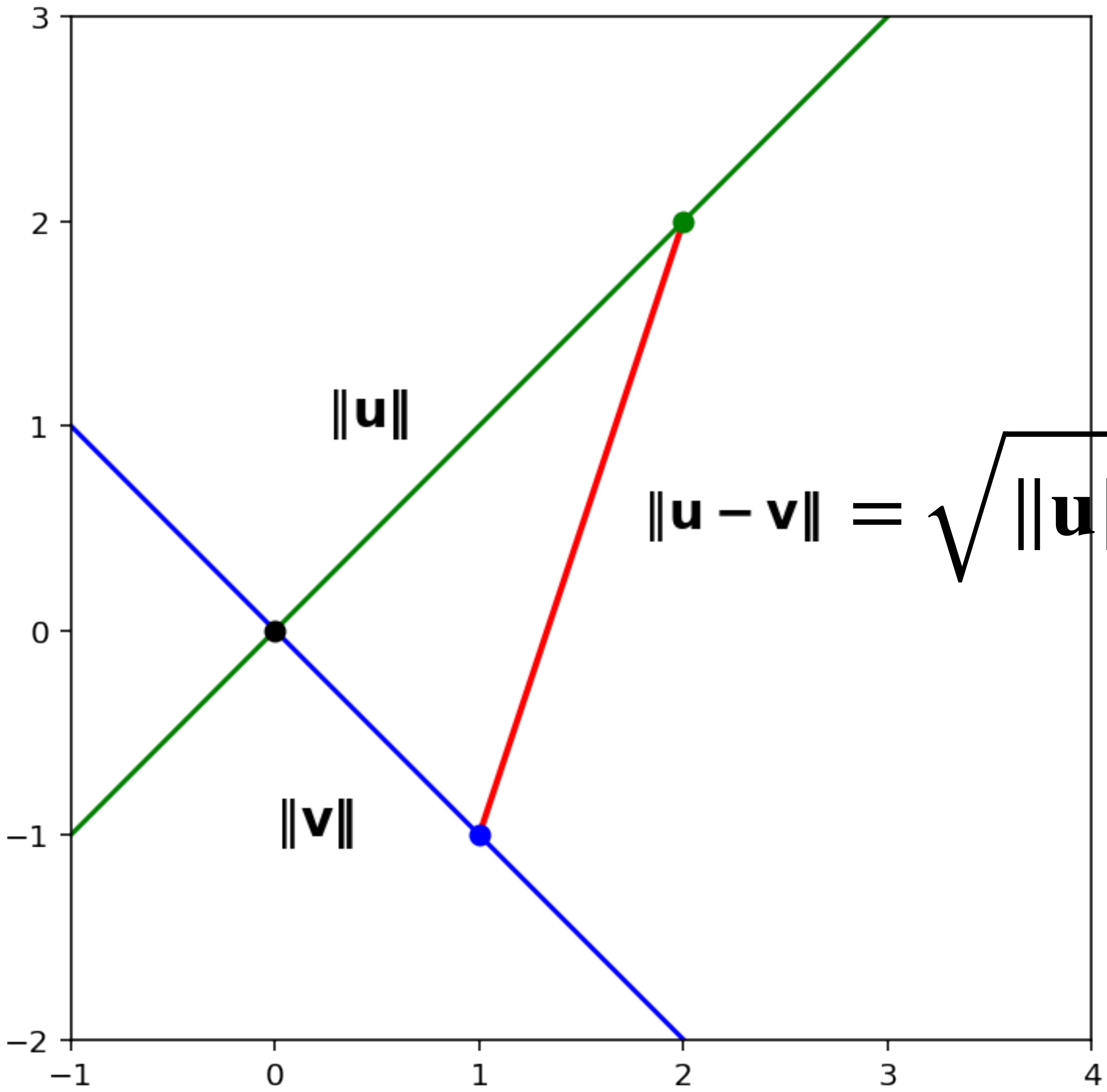
Derivation by Picture



Derivation by Picture



Derivation by Picture



$$\|\mathbf{u}-\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}$$

(Pythagoras)

Derivation by Algebra

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal exactly when

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

Let's simplify this a bit:

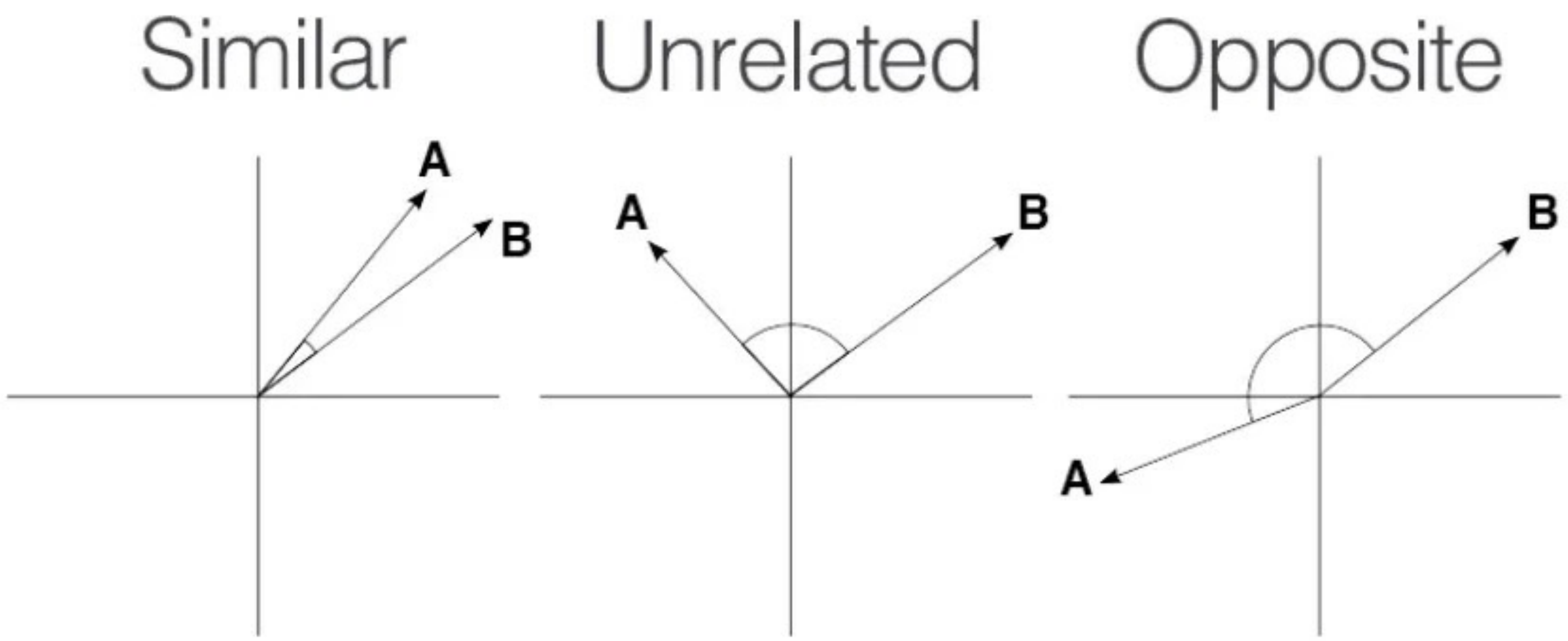
How To: Orthogonality

Question. Determine if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

Solution. Determine if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. If yes, then they are perpendicular. If no, then they are not.

Application: Cosine Similarity

High Level



Data points are <u>very big vectors</u>.

Similar vectors "point in nearly the same direction."

https://medium.com/@milana.shxanukova15/cosine-distance-and-cosine-similarity-a5da0e4d9ded

Example: Netflix Users

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix} \quad \text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix} \quad \text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{c} \texttt{comedy} \\ \texttt{drama} \\ \texttt{horror} \\ \texttt{romance} \end{array}$$

A Netflix user might be represented as a vectors whose ith entry is the number of movies they've watched in a particular genre.

Who are more likely to share similar interests in movies?

Cosine Similarity

Definition. The **cosine similarity** of two vectors is the cosine of the angle between them.

If its close to 0, then two Netflix users watch very different movies.

If its close to 1, then two Netflix users watch very similar movies.

Example: Netflix Users

`comedy`
`drama`
`horror`
`romance`

$$\text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix}$$

$$\text{sim}(\text{user}_2, \text{user}_3) \approx 0.52$$

$$\text{sim}(\text{user}_1, \text{user}_2) \approx 0.92$$

$$\text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix}$$

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{sim}(\text{user}_1, \text{user}_3) \approx 0.32$$

Other Examples

- ***Document similarity***
  - Documents $\mapsto$ word count vectors
  - Similar documents should use similar words
- ***Word2Vec***
  - Words $\mapsto$ vector *somehow*
  - This underlies modern natural language processing (NLP)

Summary

We can talk about distances and angles in $\mathbb{R}^n$.

Every basic geometric concept connects to inner products.

Once we can talk about distances and angles we can talk about similarity.