

# Analytic Geometry in $\mathbb{R}^n$

**Geometric Algorithms**

**Lecture 21**

# Practice Problem

Let  $A$  be a  $4 \times 4$  matrix with eigenvalues 3 and  $-2$  where  $\dim(\text{Nul}(A + 2I)) = 3$ .

**True or False:**  $A$  must be diagonalizable.



$A$  has eigenbasis

$$A \vec{v} = \lambda_1 \vec{v}, \quad A \vec{w} = \lambda_2 \vec{w} \quad \lambda_1 \neq \lambda_2 \implies \{ \vec{v}, \vec{w} \} \text{ is LI}$$

# Answer: True

The set of eigenvectors we get from the diagonalization procedure is of size 4, which means there is an eigenbasis of  $\mathbb{R}^4$  for  $A$ .

# Objectives

1. Recall what we learned in algebra class
2. Connect the familiar notions of lengths, distances, and angles to **inner products**
3. Begin discussing the fundamental concept of orthogonality

# Keywords

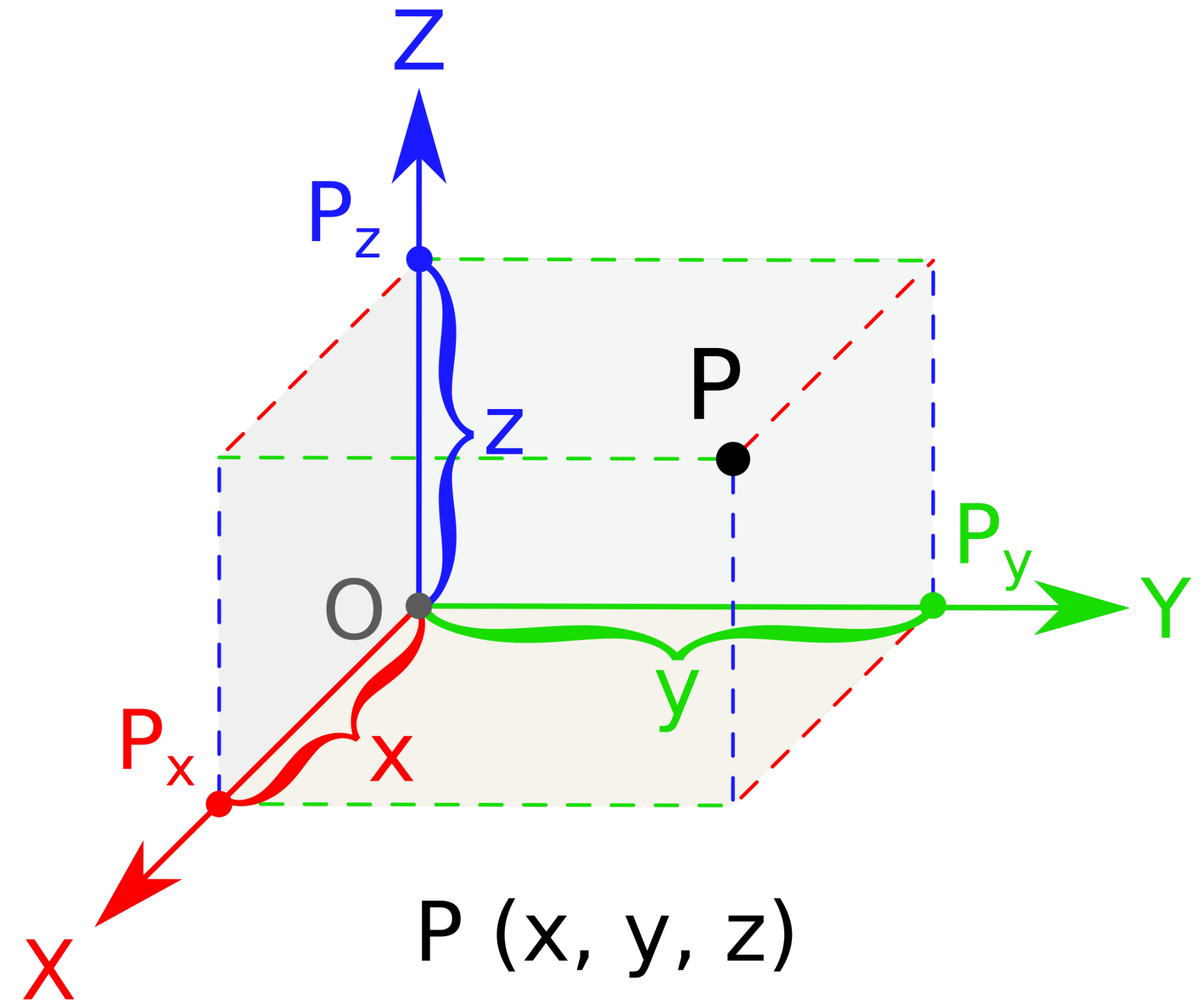
inner product

norm

orthogonal

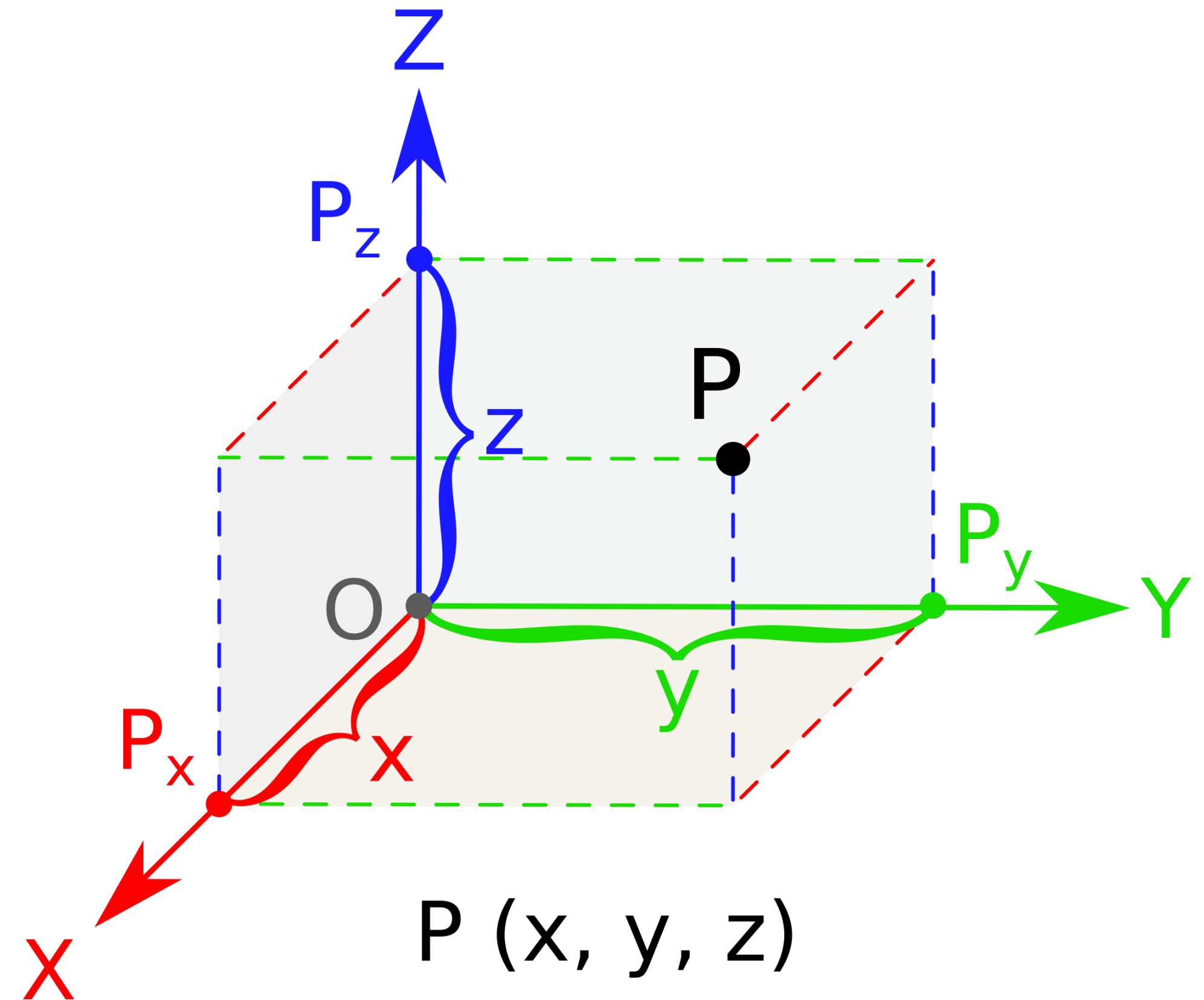
# Motivation

# What is Analytic Geometry?



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Analytic geometry is the study of space using a coordinate system.

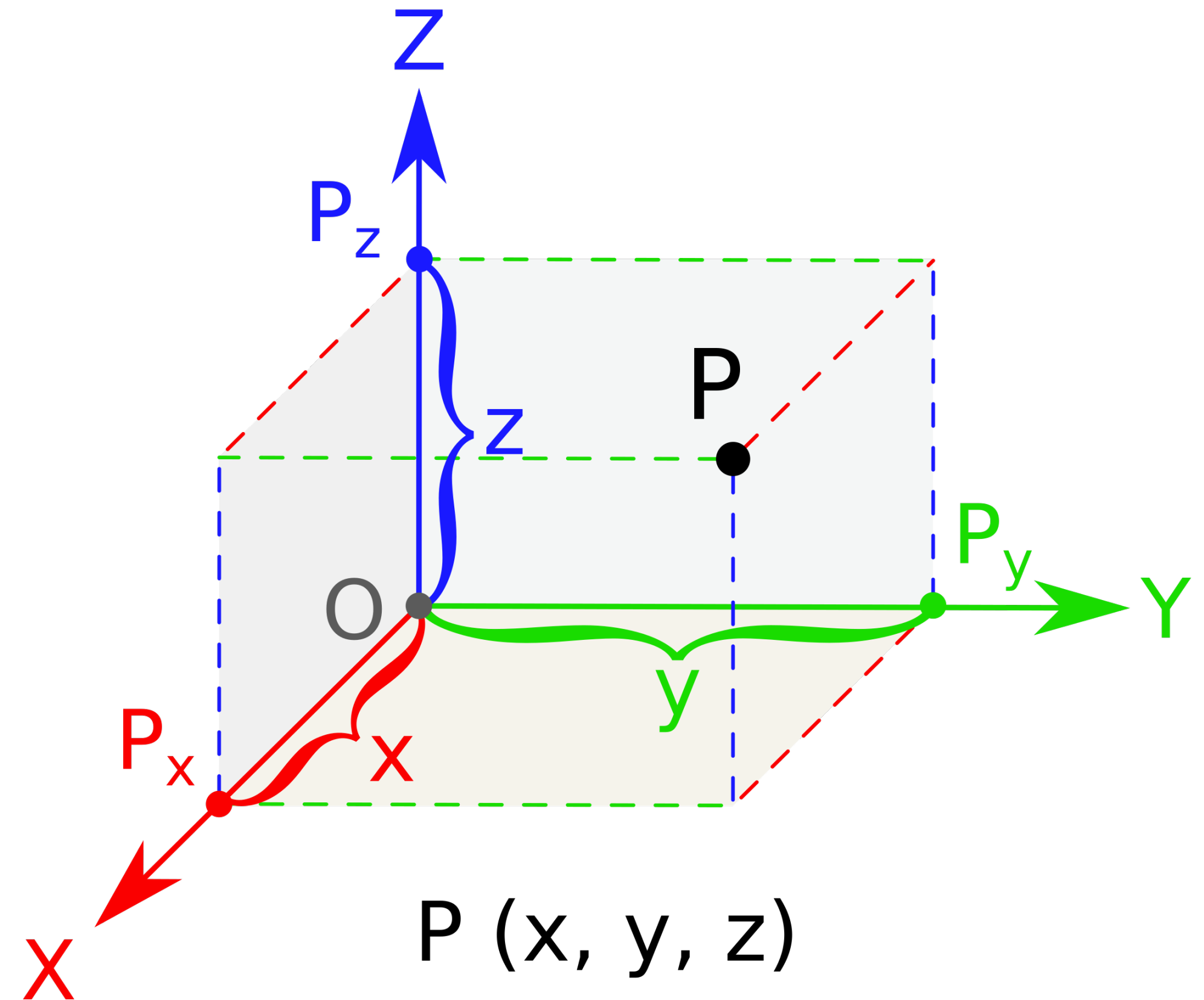




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We're interested in equations about lines, curves, shapes, angles, etc.

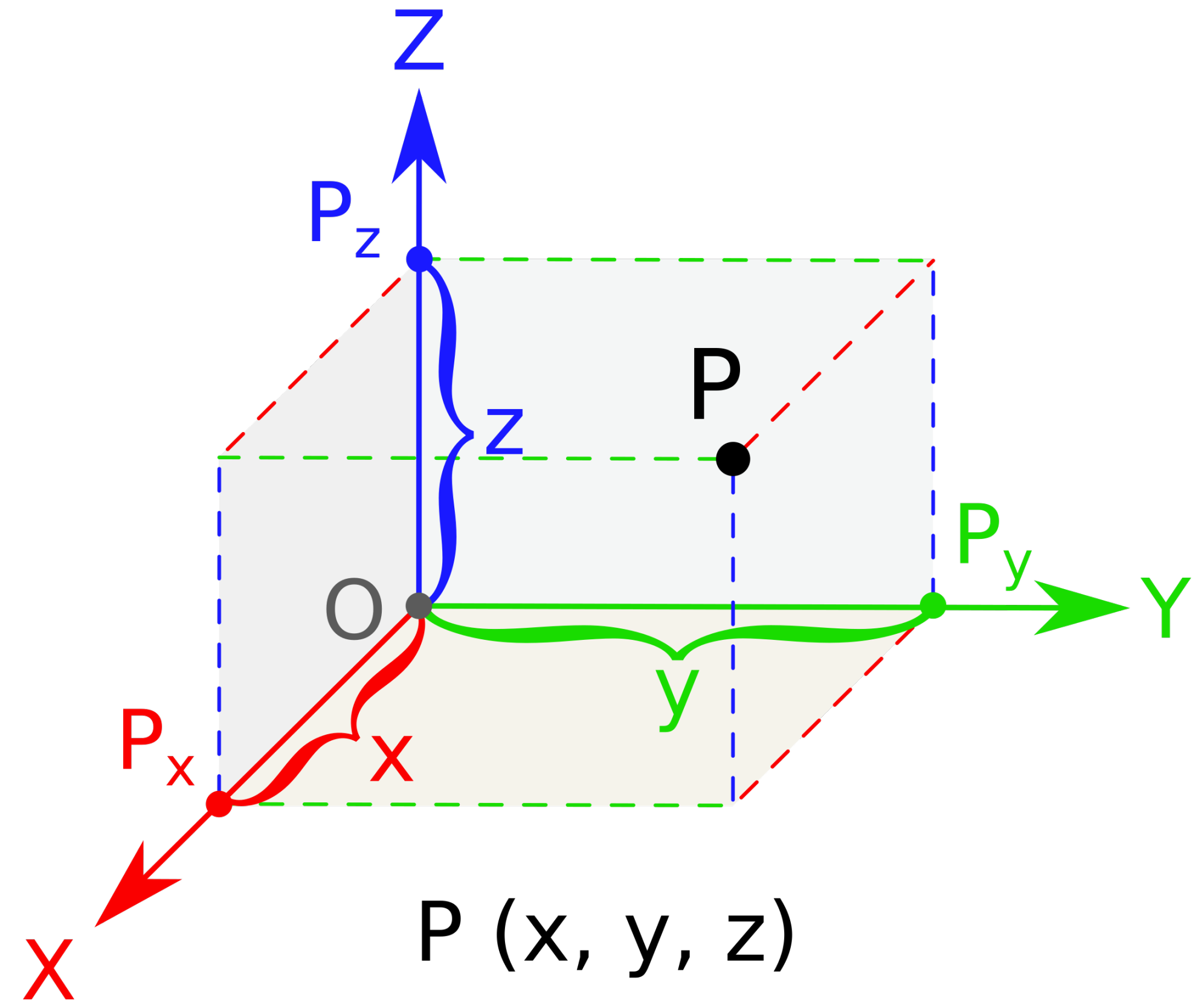


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We're interested in equations about lines, curves, shapes, angles, etc.

The fundamental concepts are:



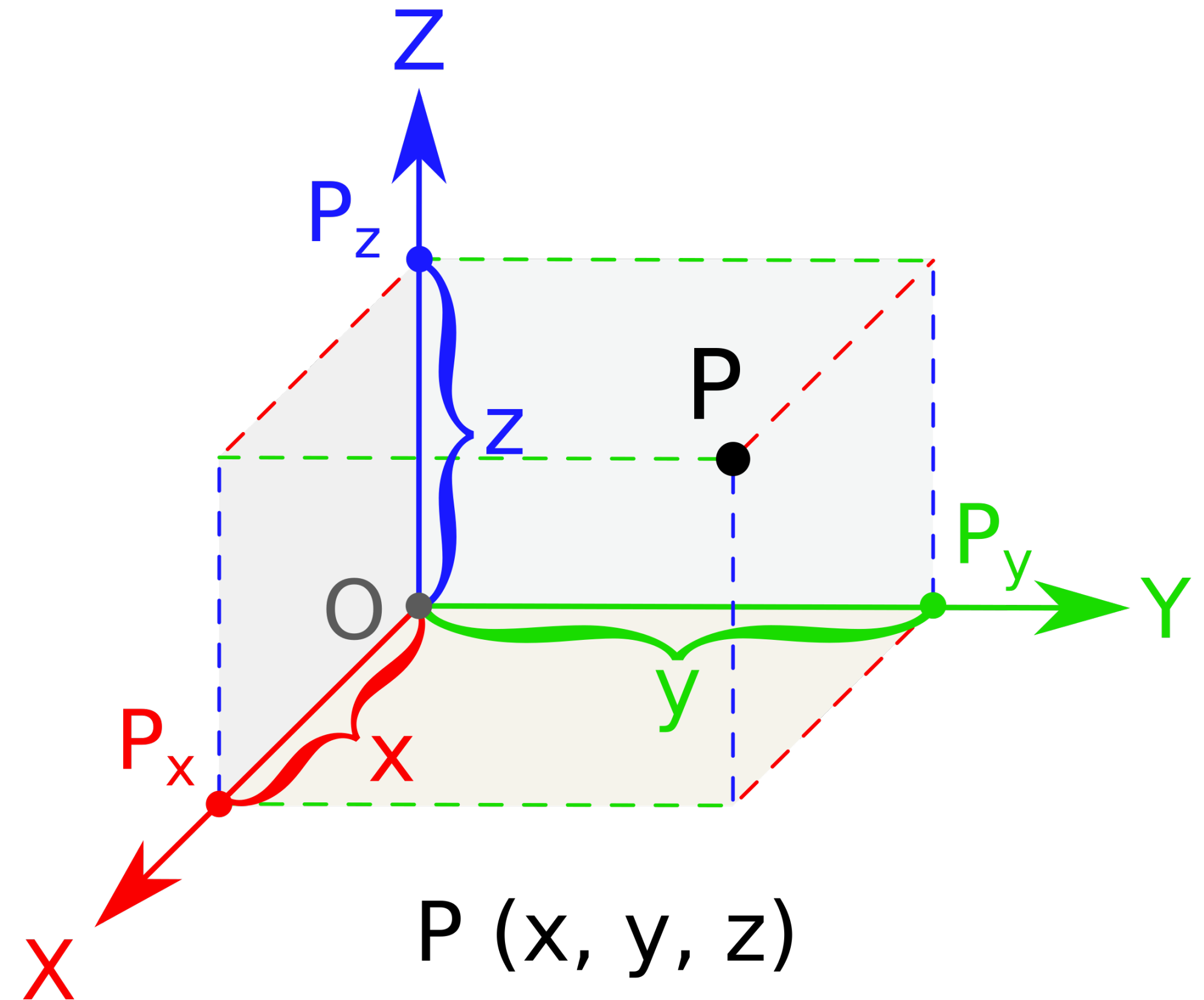
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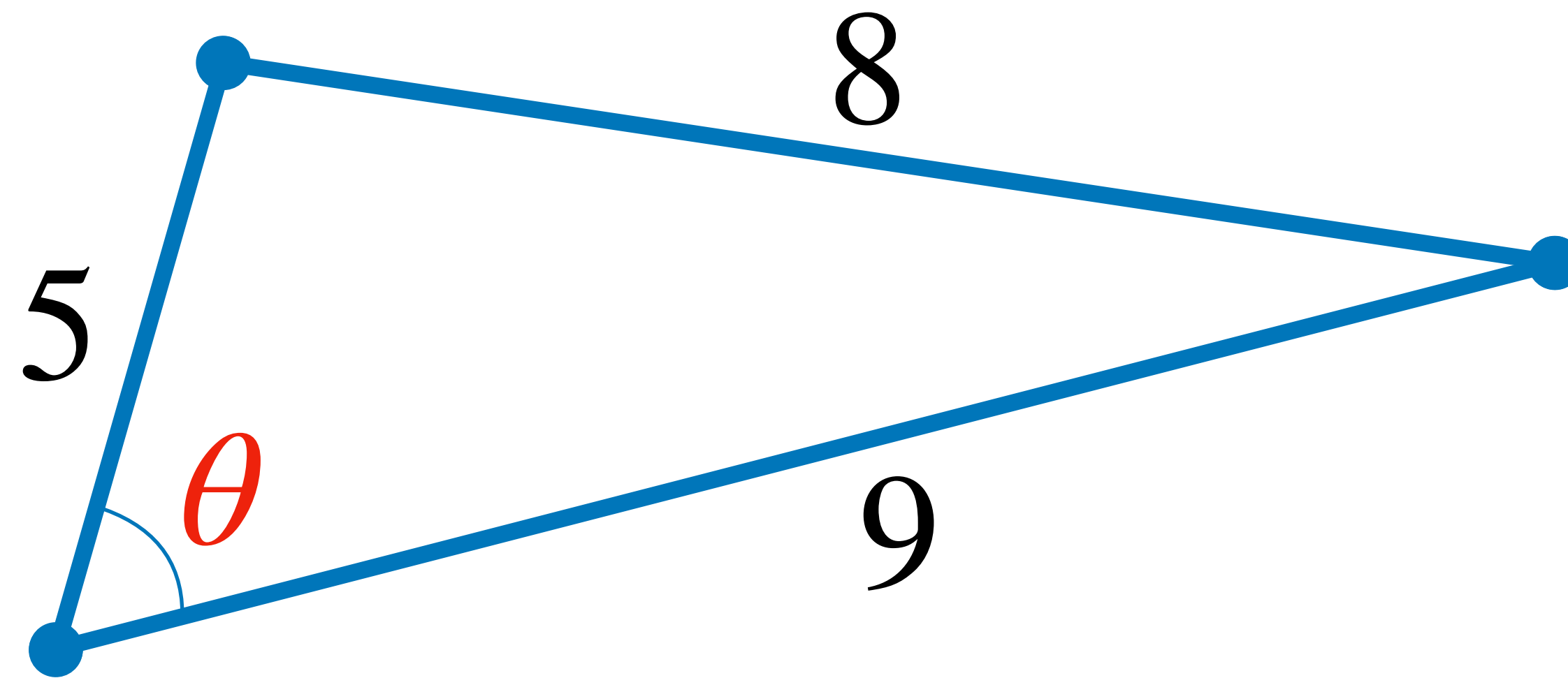
We're interested in equations about lines, curves, shapes, angles, etc.

The fundamental concepts are:

- » distance
- » position
- » area
- » angle

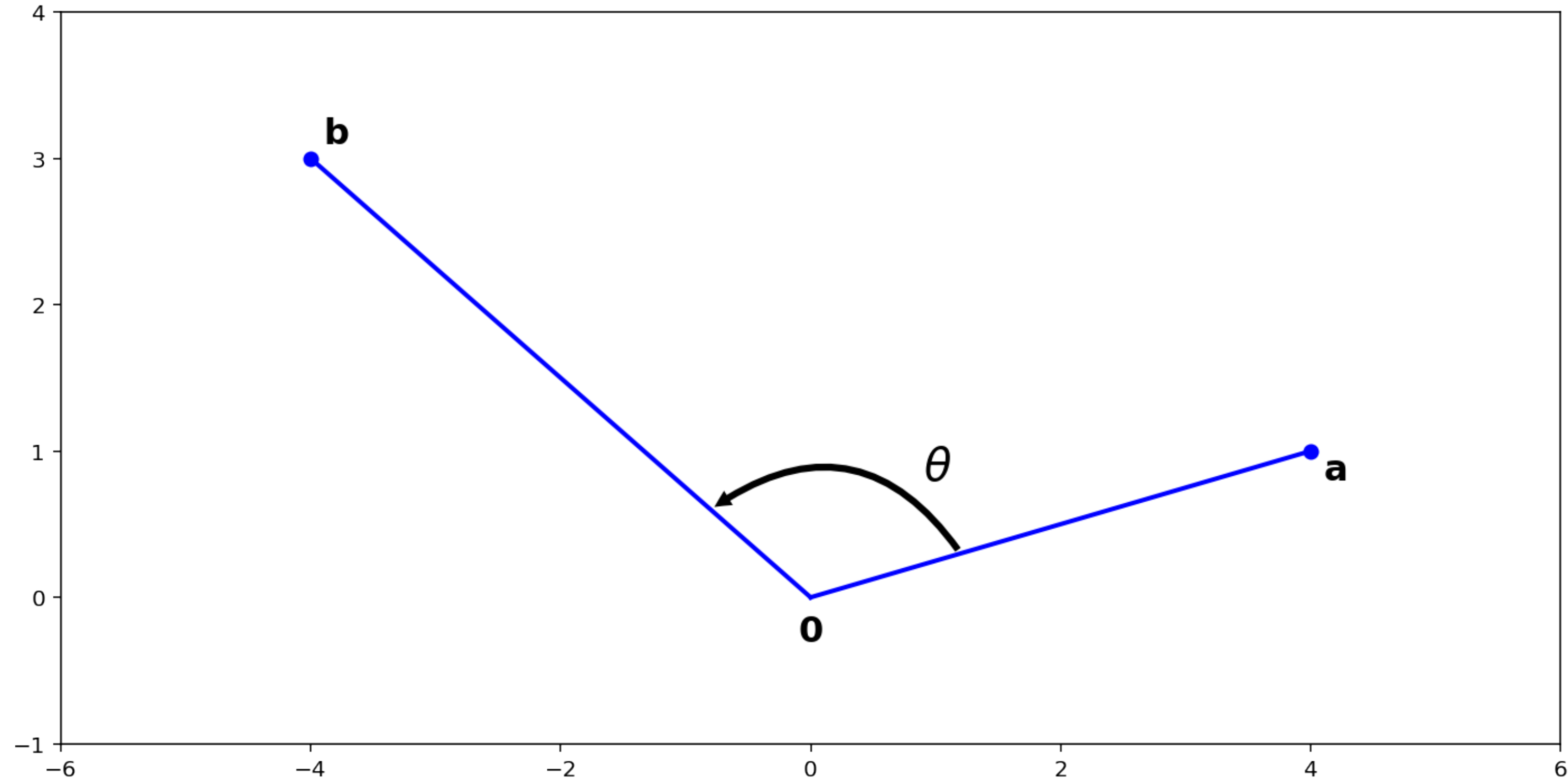


# A Potentially Familiar Example



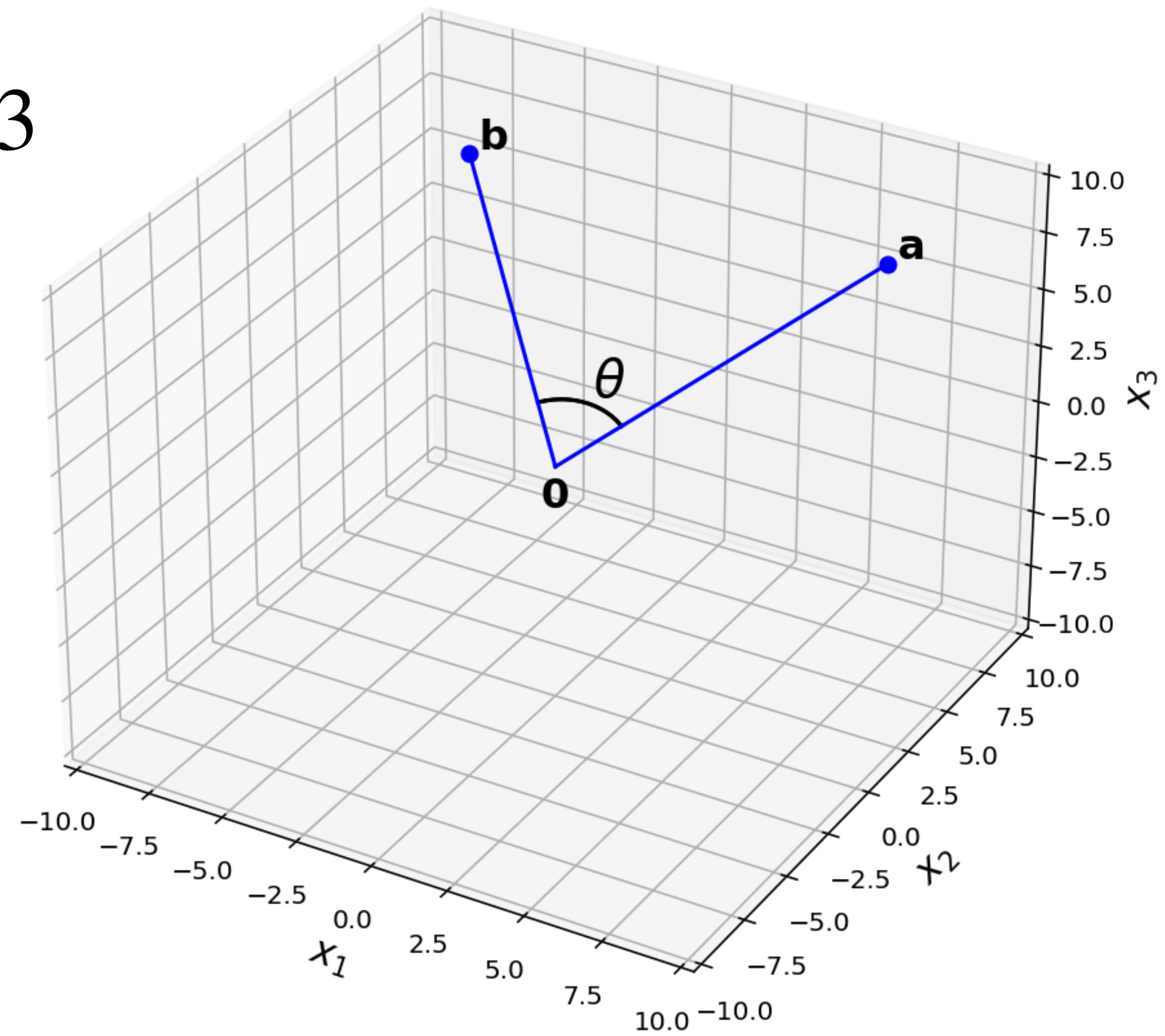
*What is the value of  $\theta$ ?*

# Angles in $\mathbb{R}^2$



*What is the value of  $\theta$ ?*

# Angles in $\mathbb{R}^3$



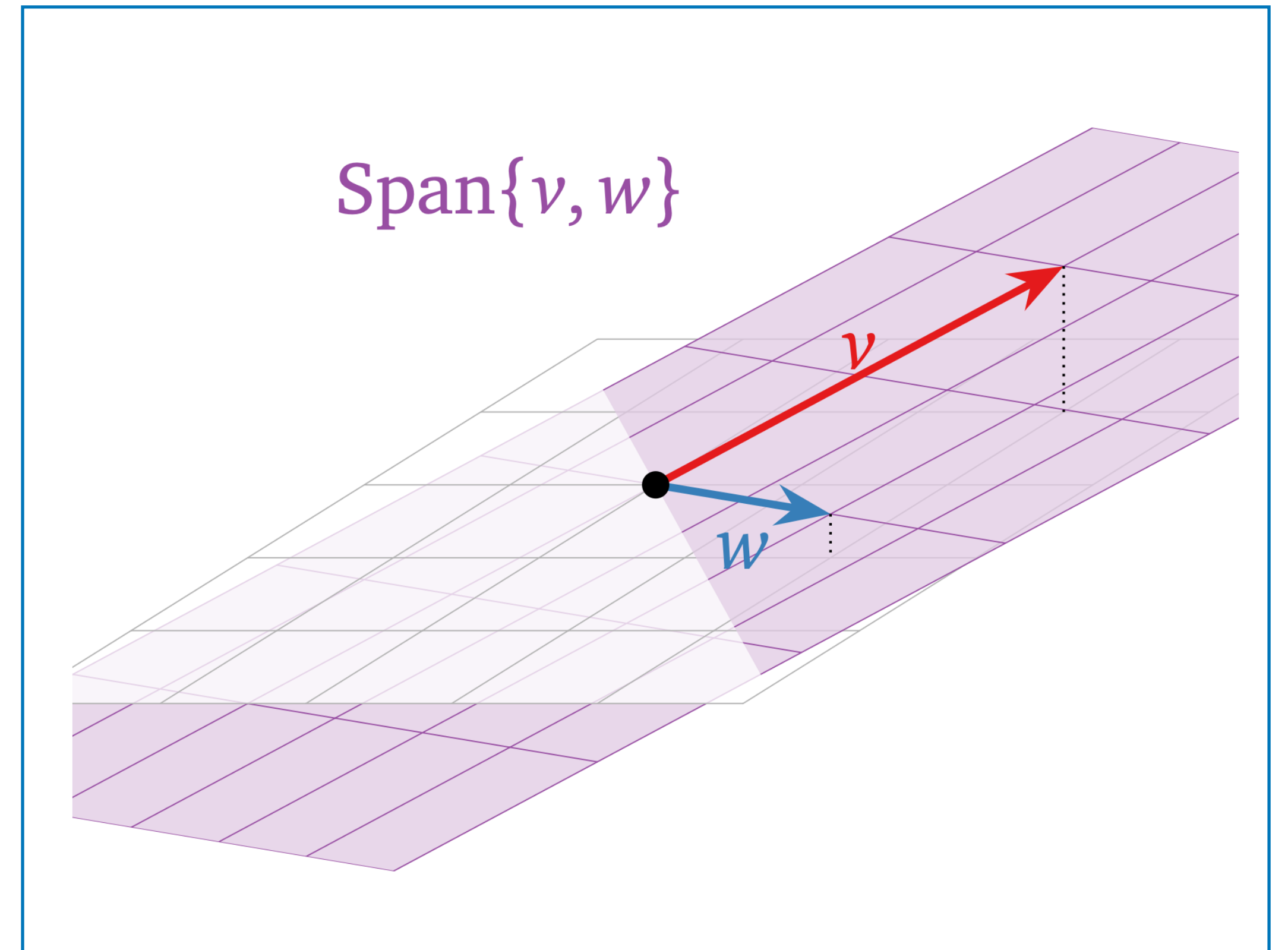
*What is the value of  $\theta$ ?*

# The First Key Idea

Angles make sense in *any* dimension.

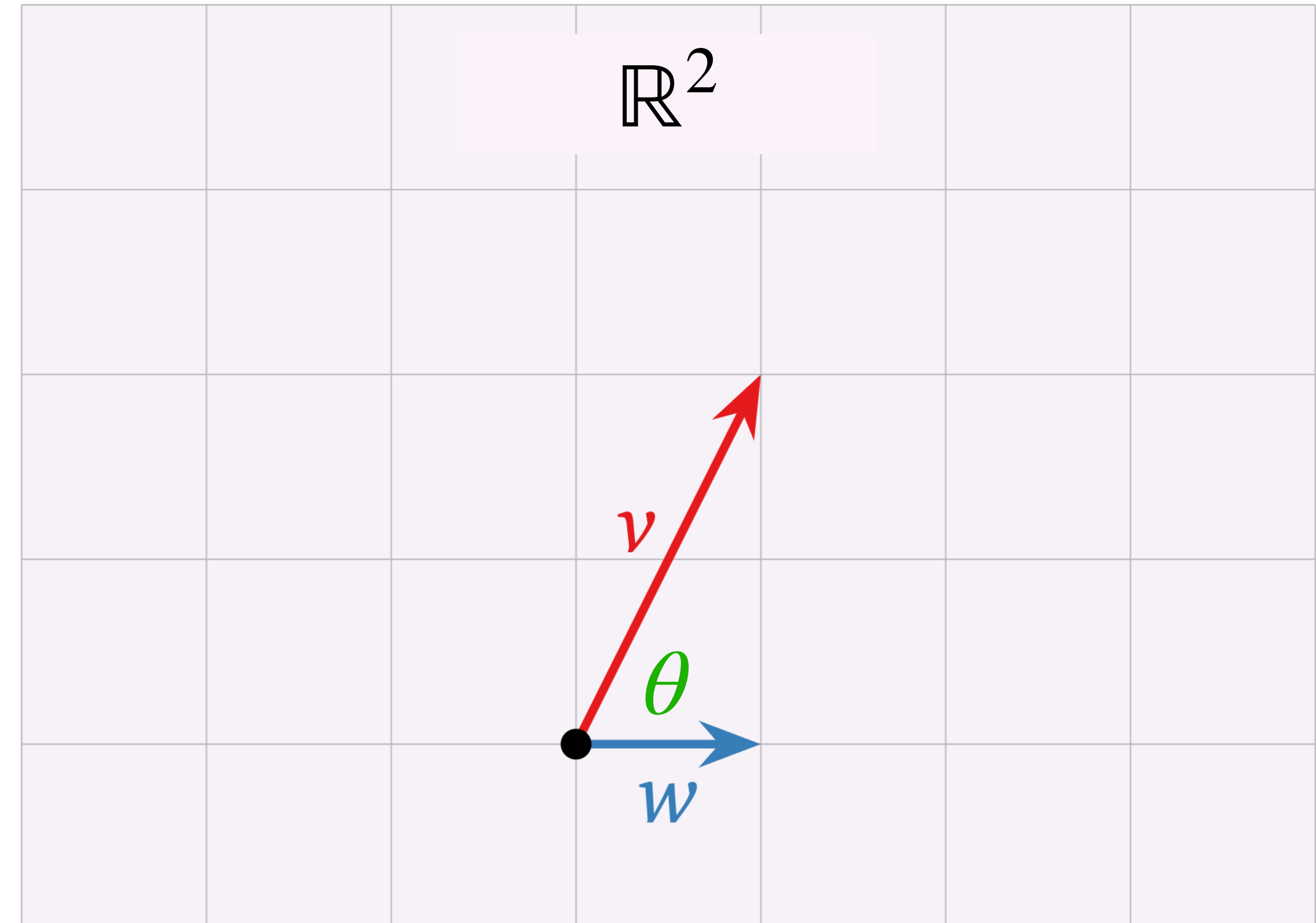
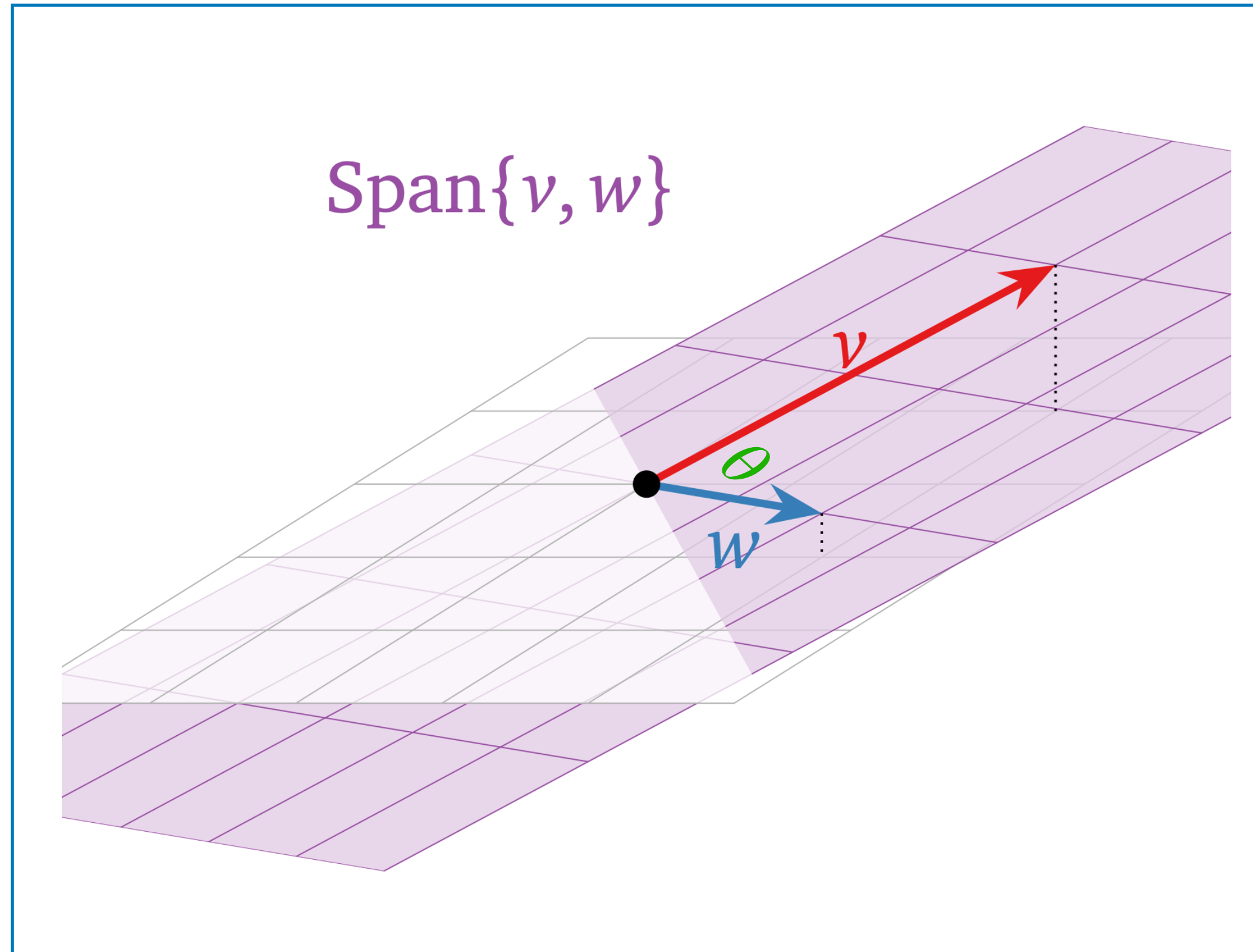
**Any pair of vectors in  $\mathbb{R}^n$  span a (2D) plane.**

*(We could formalize this via change of bases)*



# The Picture

*We can do "normal" analytic geometry here*



change of basis from  $\text{span}\{v, w\}$  to  $\mathbb{R}^2$



# A Fundamental Question

Doing this change of basis every time we want to do geometry is a lot of work...

**Can we do it directly using ideas we've been learning?**

# Recall: Inner Products

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$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

# Recall: Inner Products

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

**Definition.** The **inner product** of two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

# Recall: Inner Products

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**Definition.** The **inner product** of two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  is **a.k.a. dot product**

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

# **The Second Key Idea**

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All of the basic concepts of analytic geometry can be defined *in terms of inner products*.

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**Definition (Advanced).** An **inner product space** is a vector space with an inner product function.

**Inner product spaces (like  $\mathbb{R}^n$ ) are places where you can do analytic geometry.**

# The Fundamental Question

How do we do analytic geometry,  
given we have an inner product?

# Inner Products

# Recall: Inner Products (Again)

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

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# Example

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0.5 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} & [-3 \quad 2 \quad 1 \quad 4] \begin{bmatrix} 2 \\ 0.5 \\ -1 \\ 3 \end{bmatrix} = (-3)(2) + 2(0.5) \\ & -6 + 1 - 1 + 12 \quad + 1(-1) \\ & \quad \quad \quad + 4(3) \\ & \boxed{6} \end{aligned}$$

# Algebraic Properties of Inner Products

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  (symmetry)
  - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$
  - $(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$
- } linearity in the first argument
- $\mathbf{u} \cdot \mathbf{u} \geq 0$  (nonnegativity)
  - $\mathbf{u} \cdot \mathbf{u} = 0$  if and only if  $\mathbf{u} = \mathbf{0}$

exercise: also  
linear in second  
argument

# Verifying Additivity

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left\langle \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle$$

$$(u_1 + v_1)w_1 + (u_2 + v_2)w_2 + (u_3 + v_3)w_3 =$$

$$u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + u_3w_3 + v_3w_3$$

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

# Homogeneity in the Right Argument

$$\langle \mathbf{v}, c\mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle$$

Verify:  $\langle \vec{v}, c\vec{u} \rangle = \sum_{i=1}^n v_i(cu_i) = c \sum_{i=1}^n v_i u_i = c \langle \vec{v}, \vec{u} \rangle$



# An Aside: What is this linear transformation?

$$\begin{matrix} \mathbb{R}^3 \\ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{matrix} \mapsto \begin{matrix} \mathbb{R}^1 \\ \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \end{matrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Let's find the matrix for this transformation:

$$A = [3 \quad 5 \quad 7]$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto 3 \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto 5$$

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto 7$$

# Algebraic Properties of Inner Products

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  (symmetry)
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# Nonnegativity

$$\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^n v_i^2$$

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Therefore  $\langle \mathbf{v}, \mathbf{v} \rangle$  is always nonnegative.

**Question.** What happens when we scale a vector to make it longer?

# Nonnegativity and Scaling

$$\langle c\mathbf{v}, c\mathbf{v} \rangle = c^2 \langle \mathbf{v}, \mathbf{v} \rangle = c^2 \sum_{i=1}^n v_i^2$$

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If  $c > 0$  then  $\langle c\mathbf{v}, c\mathbf{v} \rangle > \langle \mathbf{v}, \mathbf{v} \rangle$ .



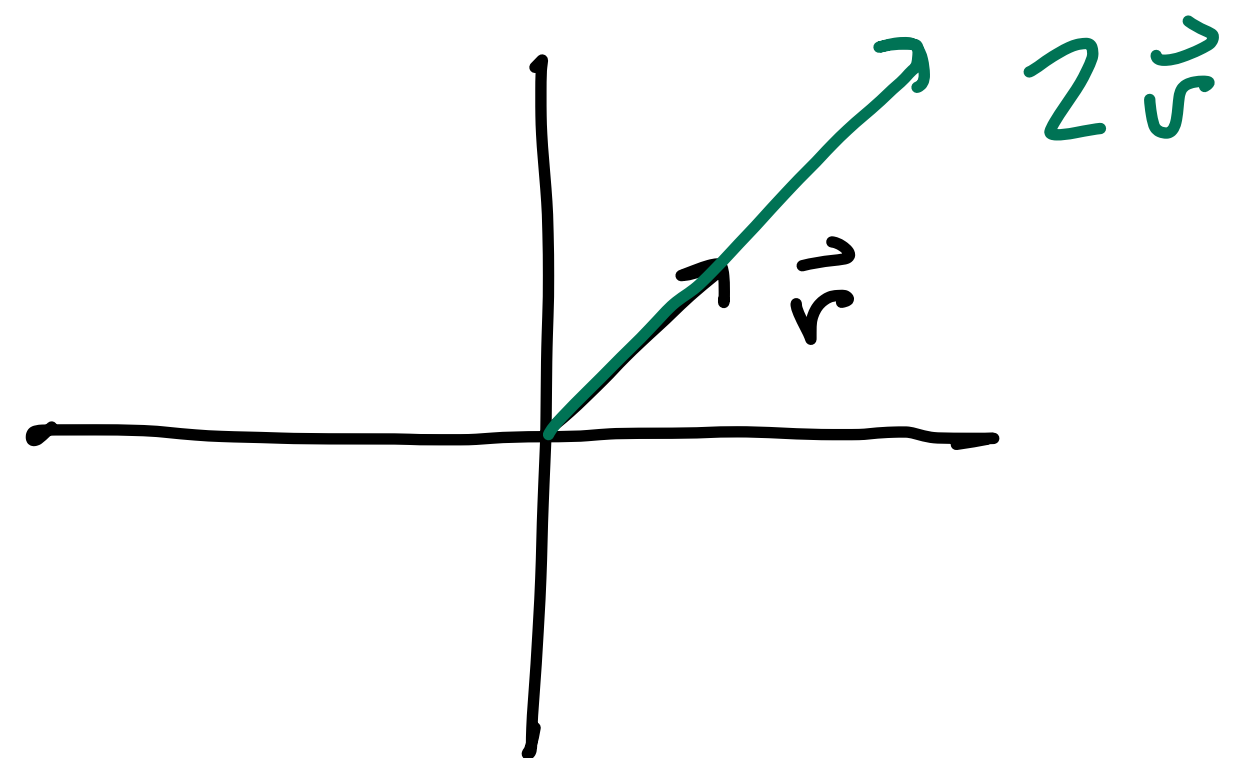
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*Increasing the length of a vector increases its inner product with itself.*

$$\langle 2\vec{v}, 2\vec{v} \rangle = 2^2 \langle \vec{v}, \vec{v} \rangle = 4\langle \vec{v}, \vec{v} \rangle$$



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*Increasing the length of a vector increases its inner product with itself.*

**This means  $\langle \mathbf{v}, \mathbf{v} \rangle$  is capturing some notion of magnitude.**

# Question

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$$

*Simplify the expression  $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$  using the properties of inner products.*

**Answer:**  $\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle$

$$\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \langle \vec{u}, \vec{u} - \vec{v} \rangle + \langle \vec{v}, \vec{u} - \vec{v} \rangle$$

$$= \langle \vec{u} - \vec{v}, \vec{u} \rangle + \langle \vec{u} - \vec{v}, \vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle - \cancel{\langle \vec{u}, \vec{v} \rangle} + \cancel{\langle \vec{u}, \vec{v} \rangle} - \langle \vec{v}, \vec{v} \rangle$$

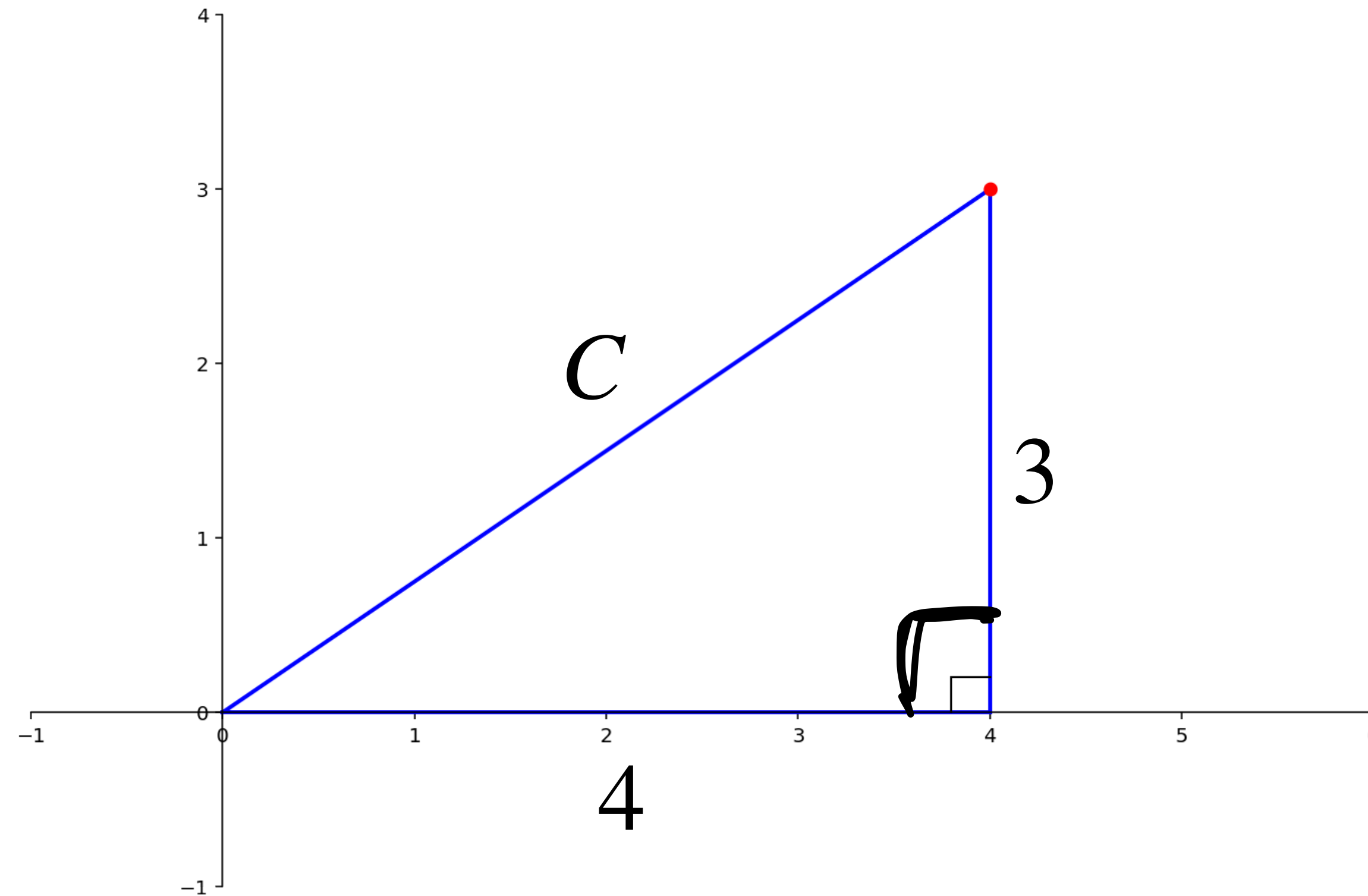
$$= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle$$

# Norms (Lengths/Distances)

# The Fundamental Question

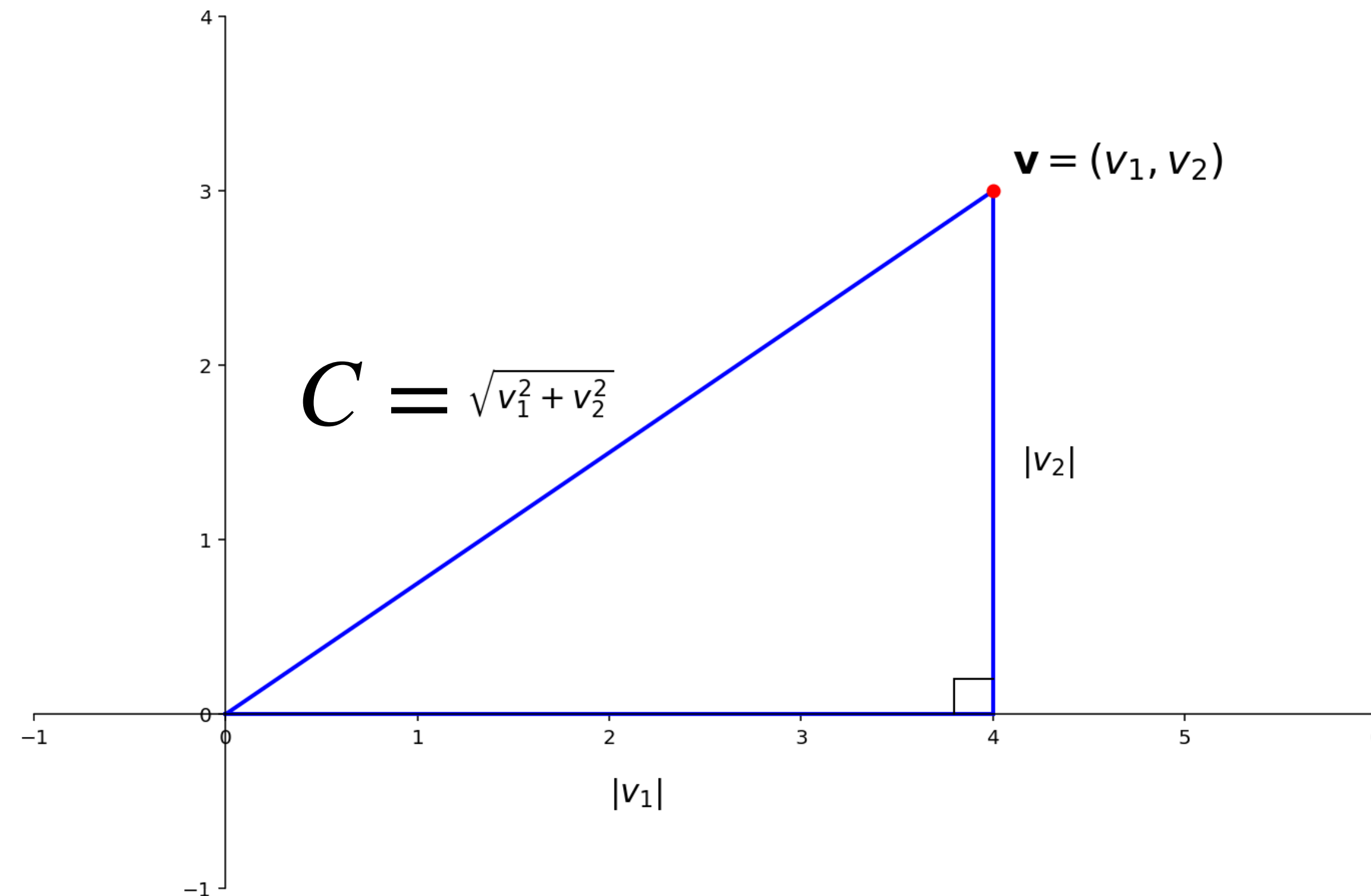
How does this all connect back to distances and angles?

# Another Potentially Familiar Question



*How long is the side C?*

# Pythagorean Theorem



**Theorem (Pythagoras).** *For a right triangle, the square of the length of the hypotenuse is the sum of the squares of the lengths of the remaining two sides.*

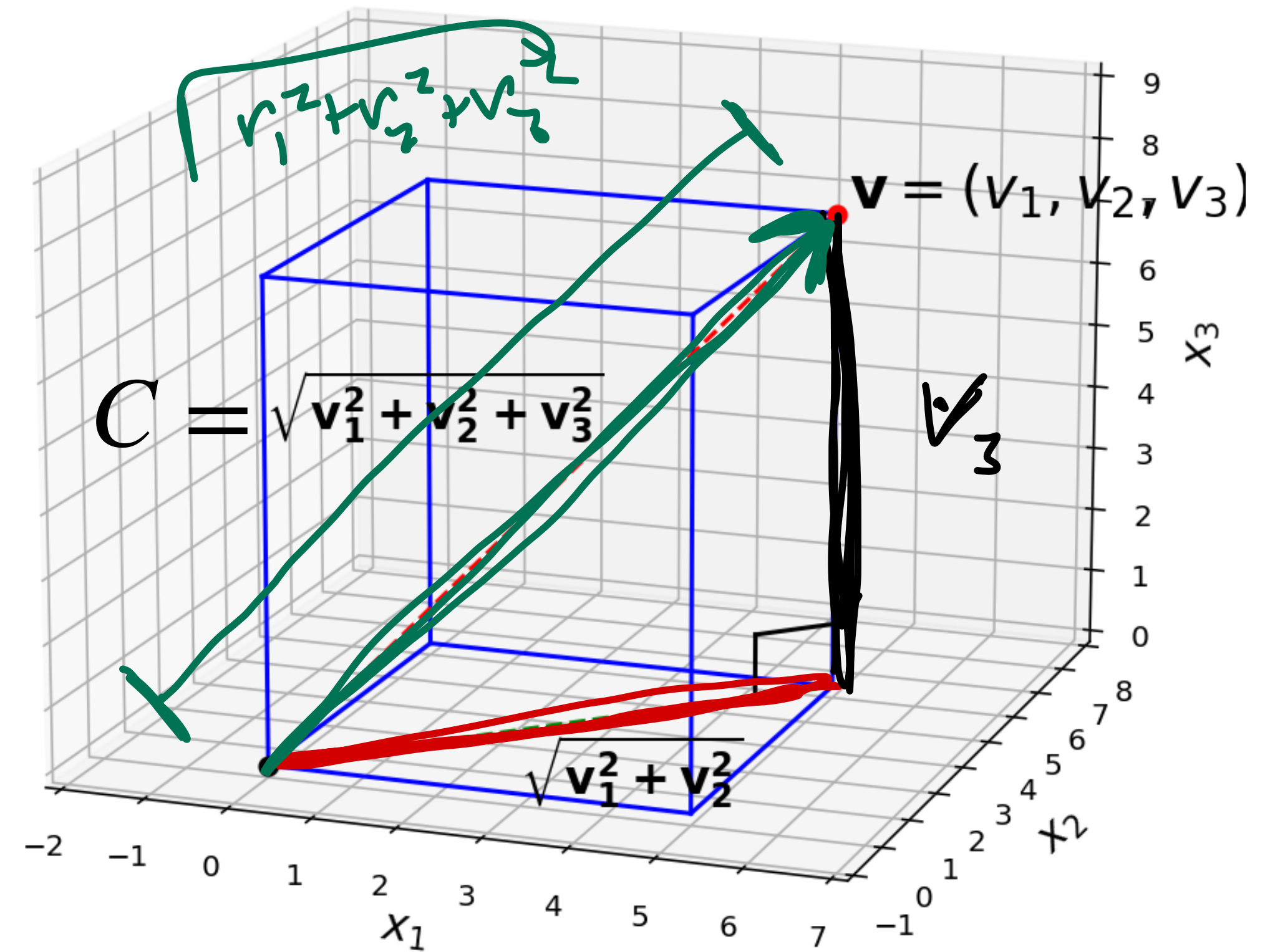


# This still works in $\mathbb{R}^3$

**Theorem (Pythagoras).**  $C = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Verify:

$$\sqrt{\left(\sqrt{v_1^2 + v_2^2}\right)^2 + v_3^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$



# Norm

**Definition.** The ( $\ell^2$ ) norm of a vector  $\mathbf{v}$  in  $\mathbb{R}^n$  is

$$\|\mathbf{v}\| = \left\| \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = \sqrt{\sum_{i=1}^n v_i^2} = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

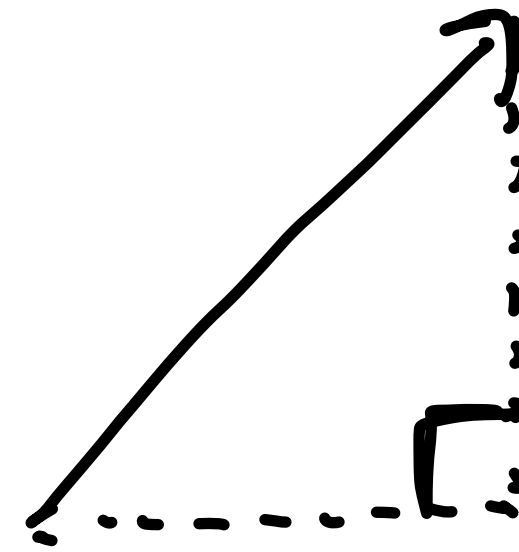
*The norm of a vector is the square root of the sum of the squares of its entries.*

# Norms and Inner Products

**Definition.** The  $\ell^2$  norm of a vector  $\mathbf{v}$  in  $\mathbb{R}^n$  is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

*The norm of a vector is the square root of the inner product with itself.*



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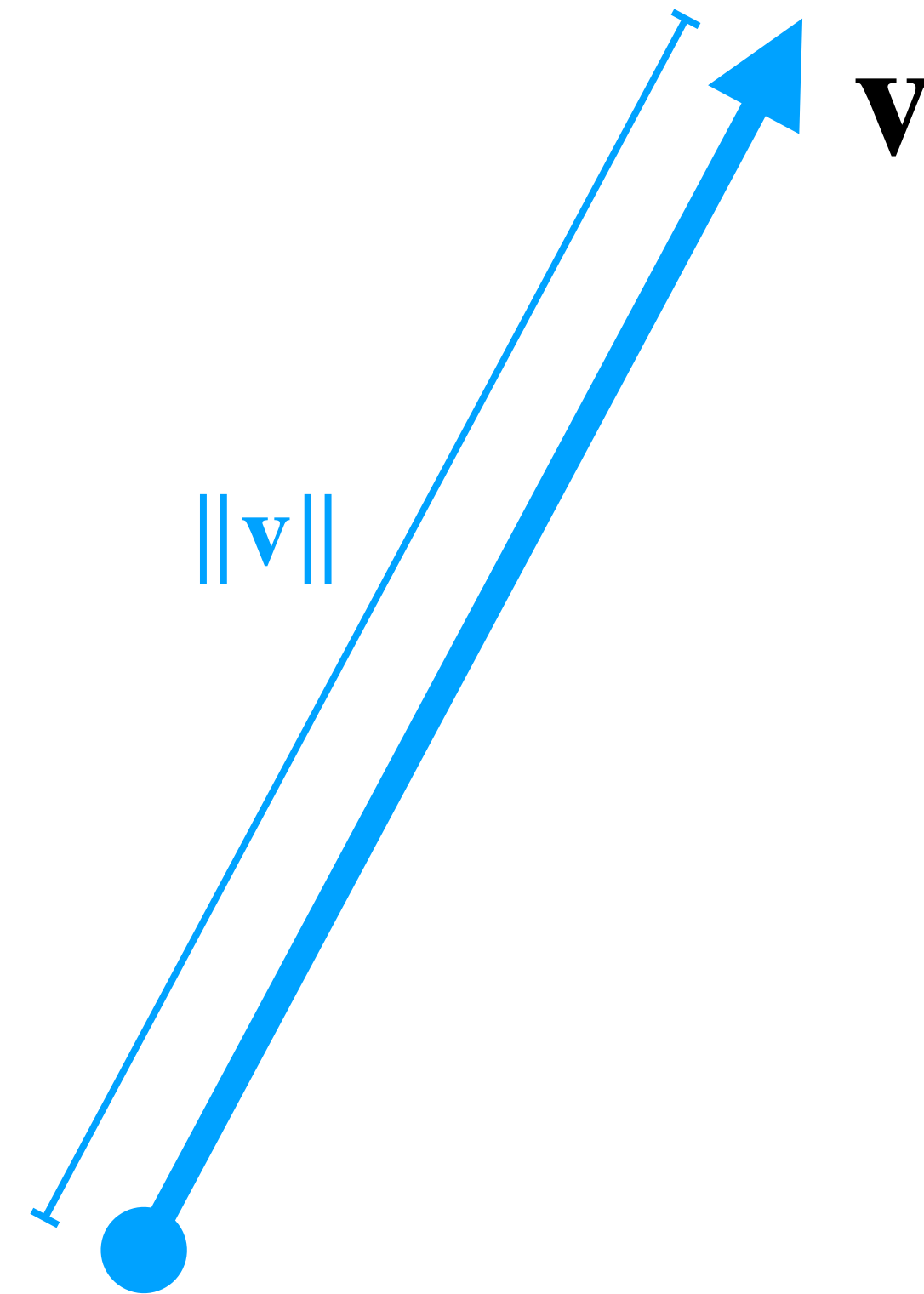
*The norm of a vector is the square root of the inner product with itself.*

**It's important that  $\mathbf{v}^T \mathbf{v}$  is nonnegative.**

# Norms and Distance

Norms give us a notion of length.

In  $\mathbb{R}^2$  and  $\mathbb{R}^3$  this is our existing notion of length.

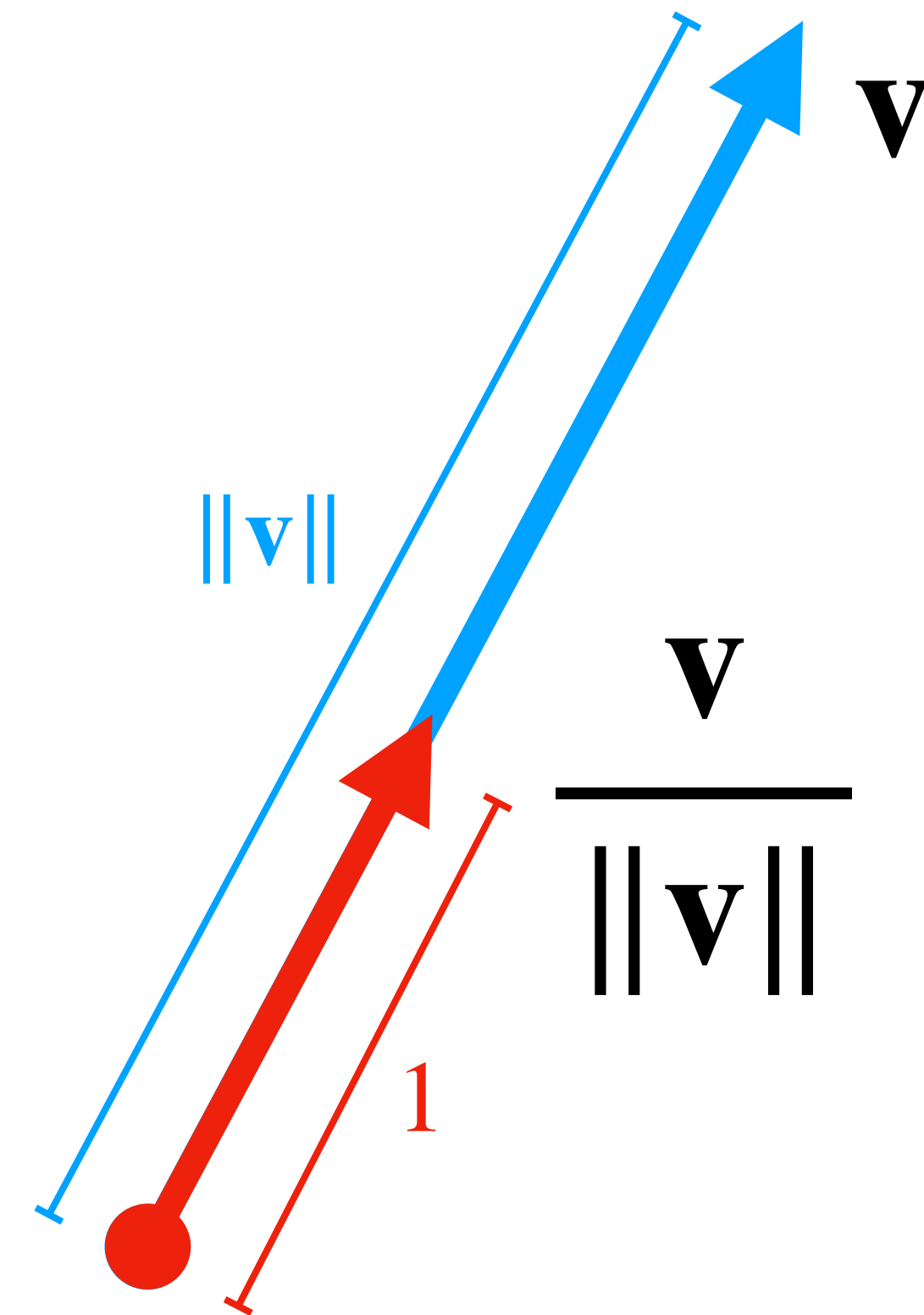


# $\ell^2$ Normalization

**Definition.** A **unit vector** is a vector  $\mathbf{v}$  such that  $\|\mathbf{v}\| = 1$ .

We often *normalize* vectors if we only care about their direction:

$$\mathbf{v} \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$



# How To: Normalizing Vectors

**Question.** Find the unit vector which points in the same direction as  $\mathbf{u}$ .

**Solution.** Compute  $\|\mathbf{u}\|$ . The unit vector is then

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}$$

# Example

Find the unit vector in the same direction as  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = 3$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \|\vec{u}\| &= \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 0} \\ &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0} = \sqrt{\frac{9}{9}} = 1 \end{aligned}$$

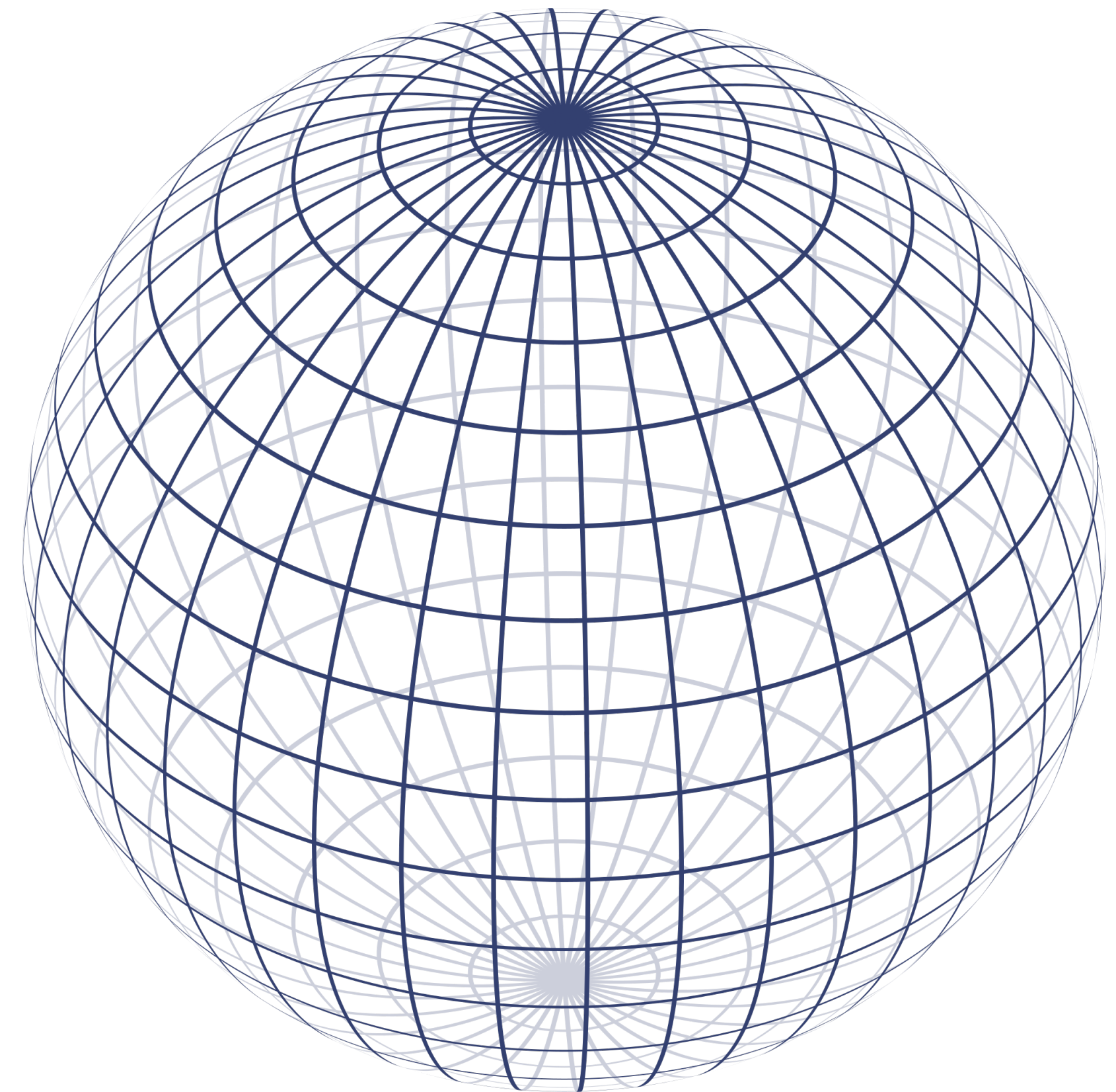


# The Unit Sphere

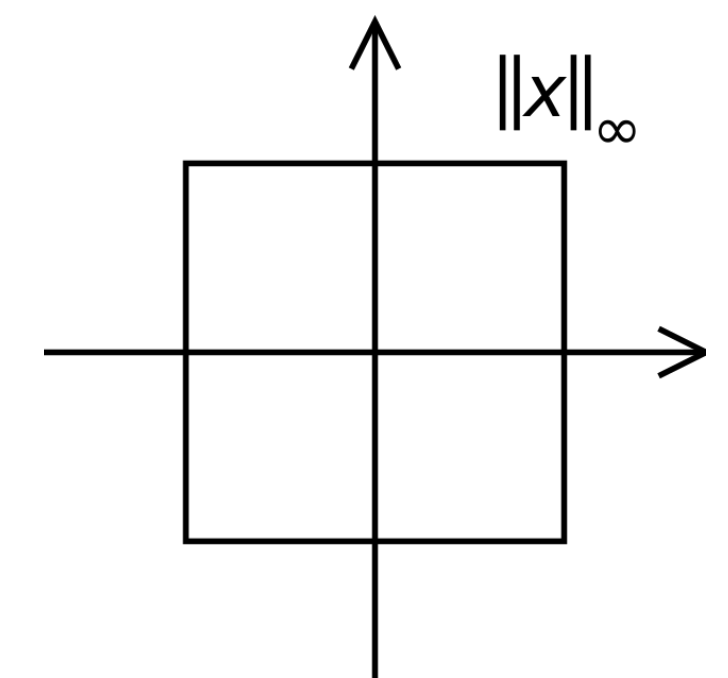
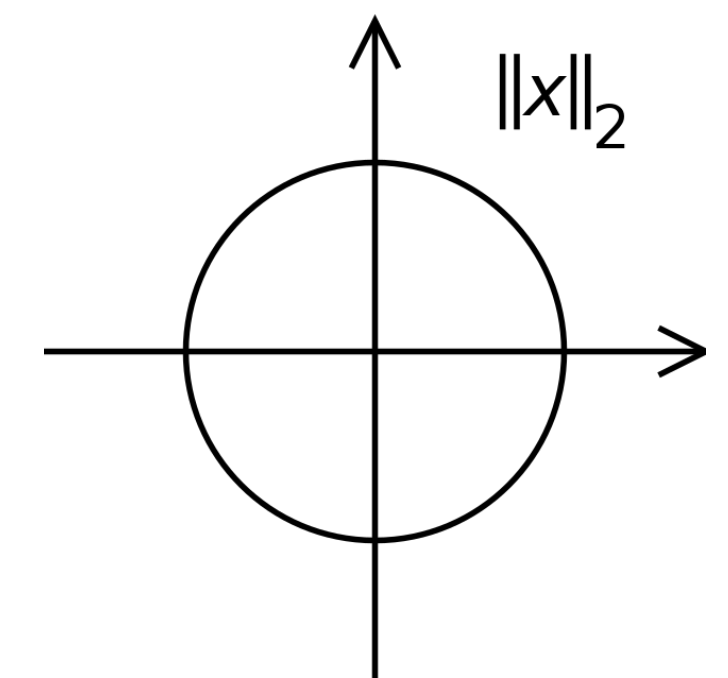
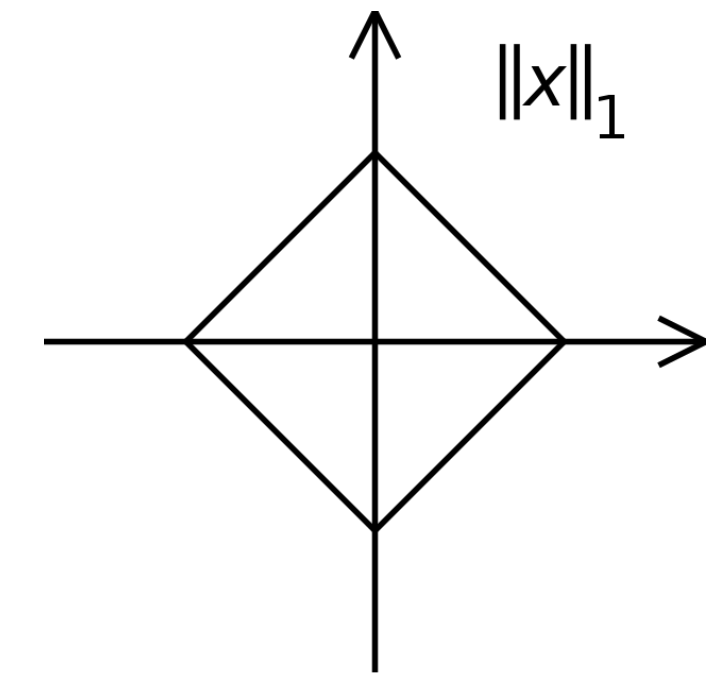
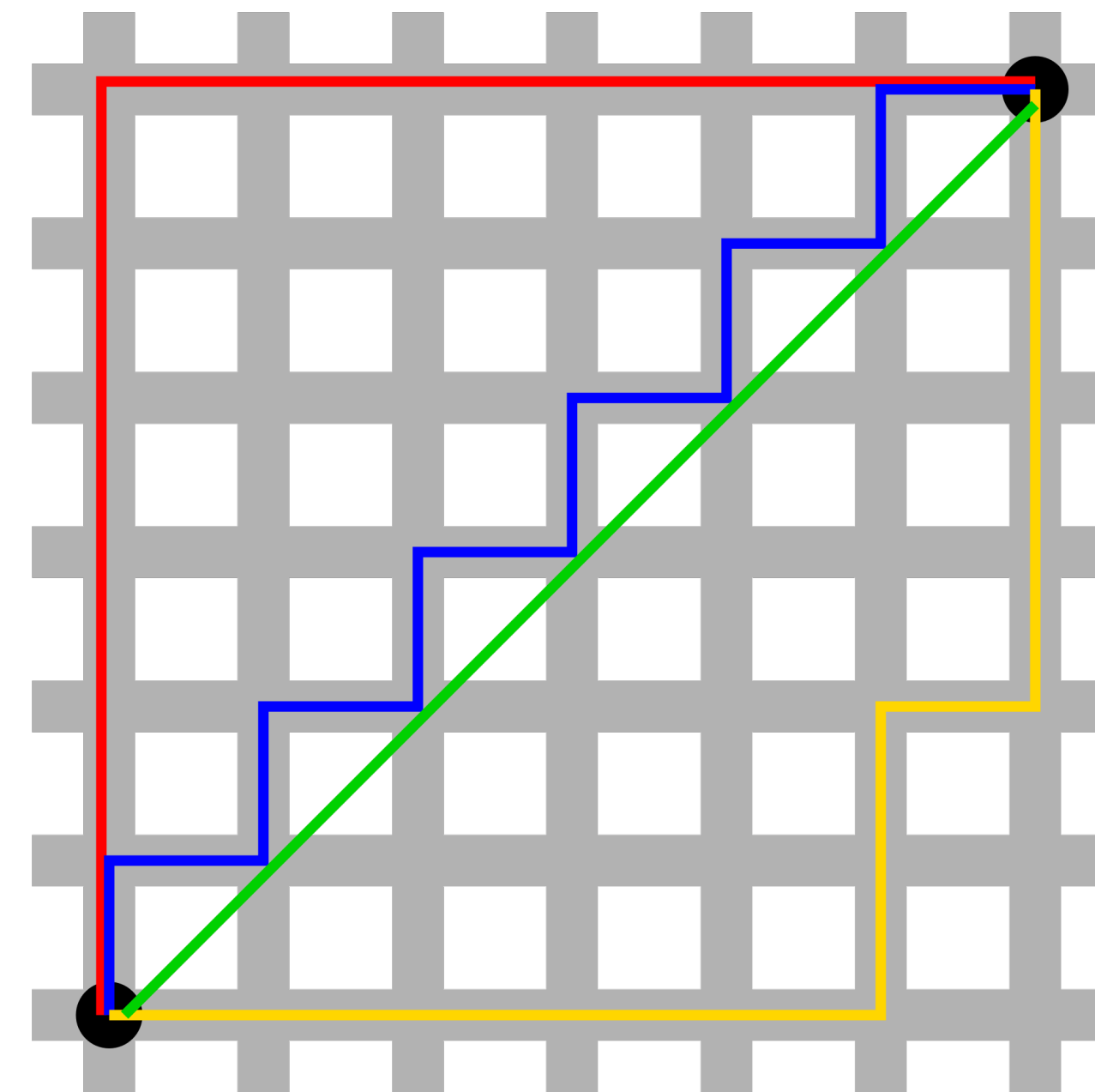
**Definition.** The unit  $n$ -sphere is the collection of all unit vectors in  $\mathbb{R}^n$ .

Vector norms allow us to talk about spheres in higher dimensions.

*A sphere is a collection of points equidistant from a center point.*

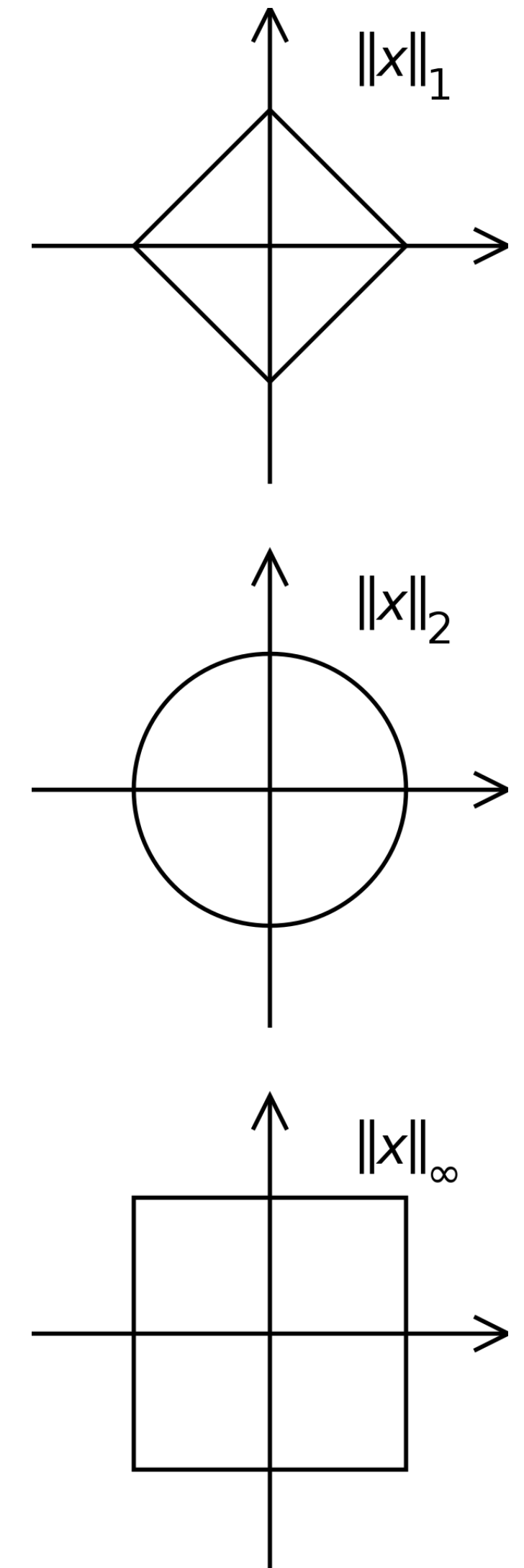
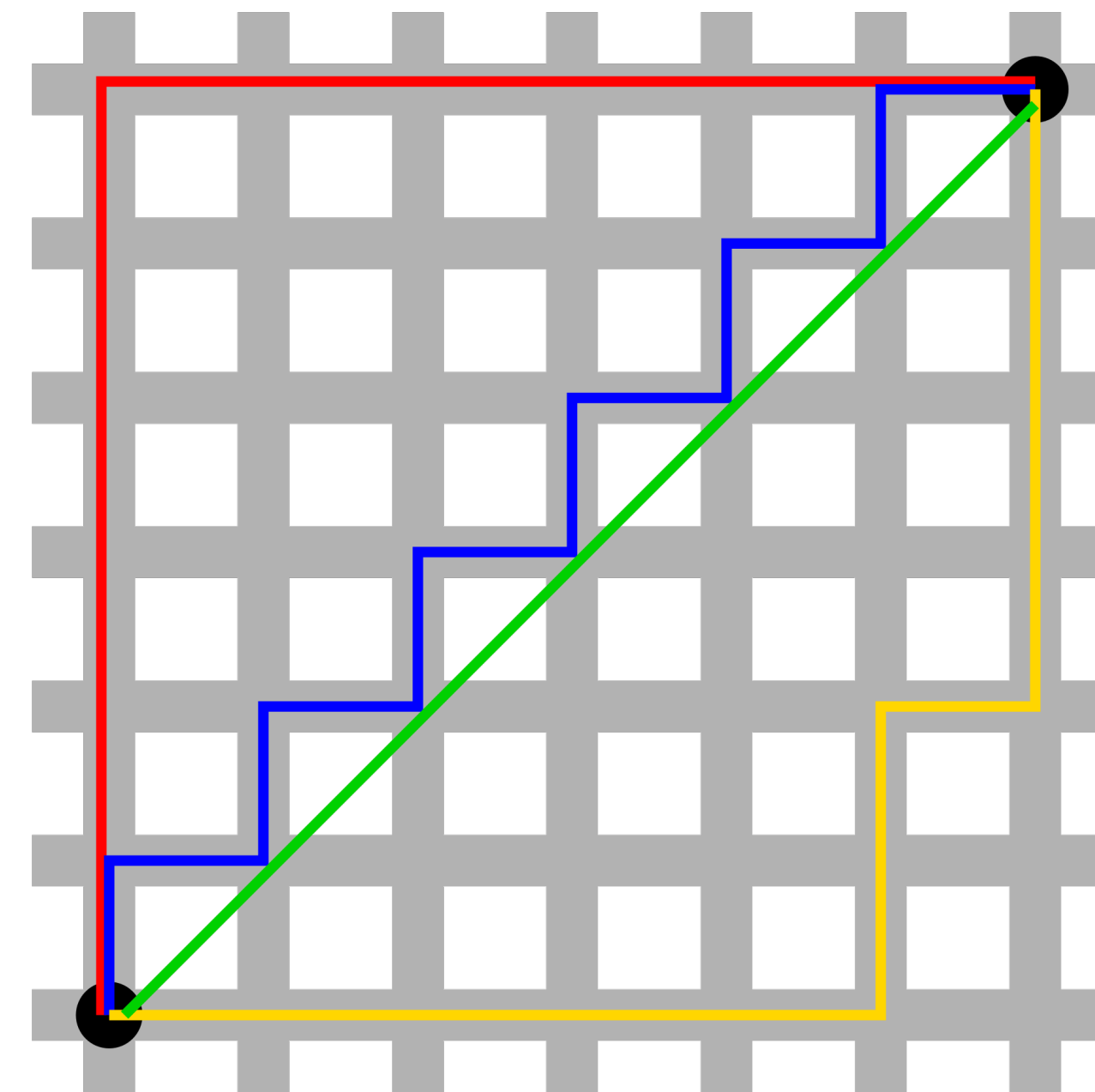


# An Aside: Other Notions of Distance



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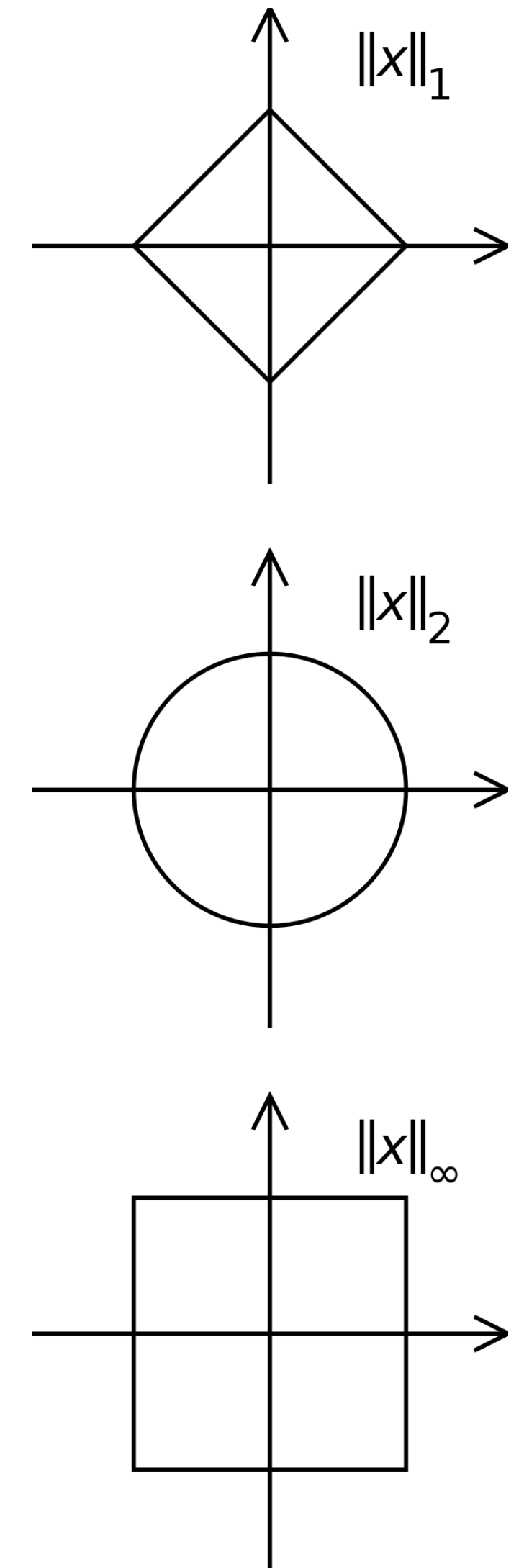
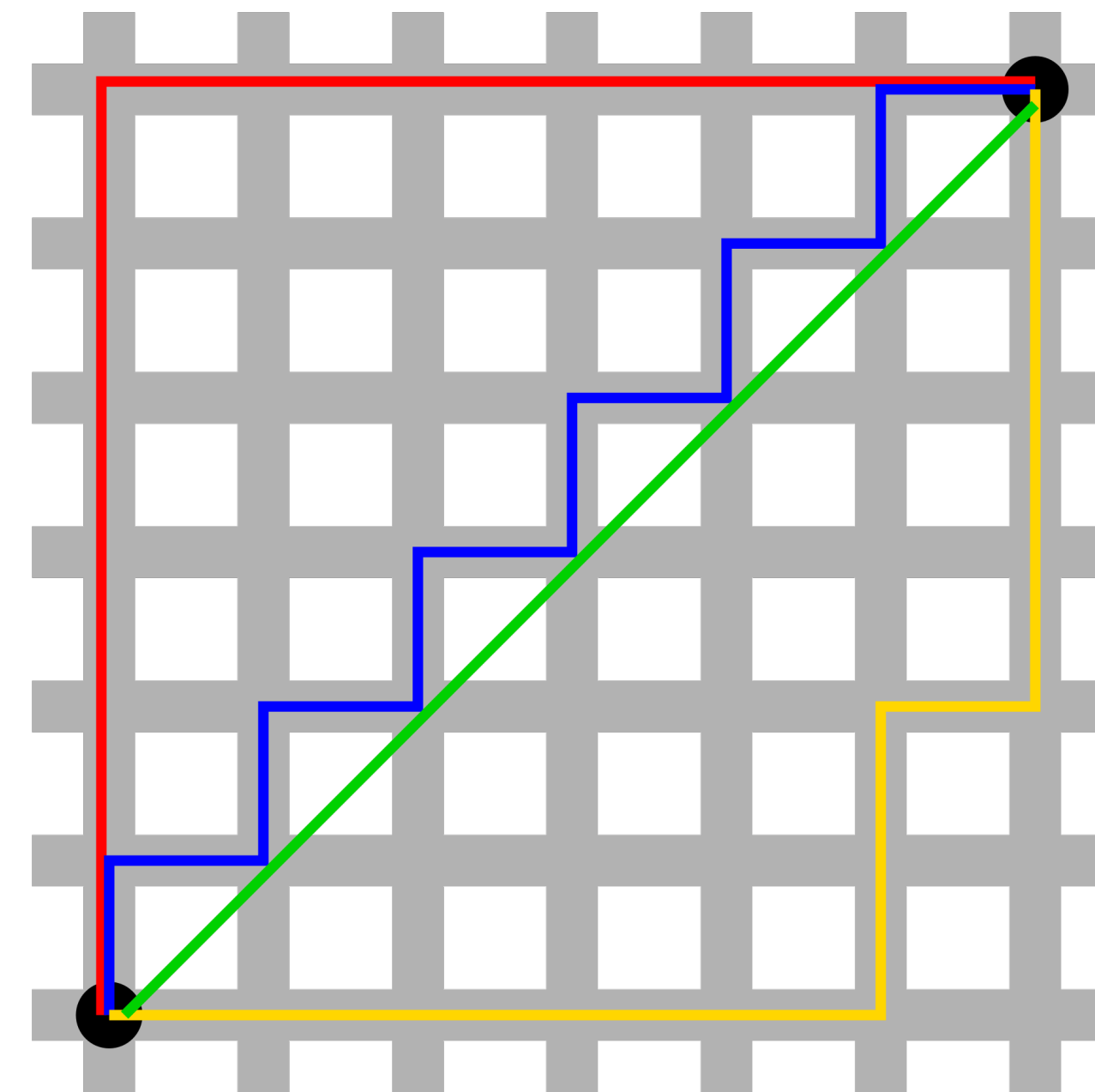
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# An Aside: Other Notions of Distance

*Why are we talking about norms and inner products so generally?*

***Because there are other inner products and norms.***

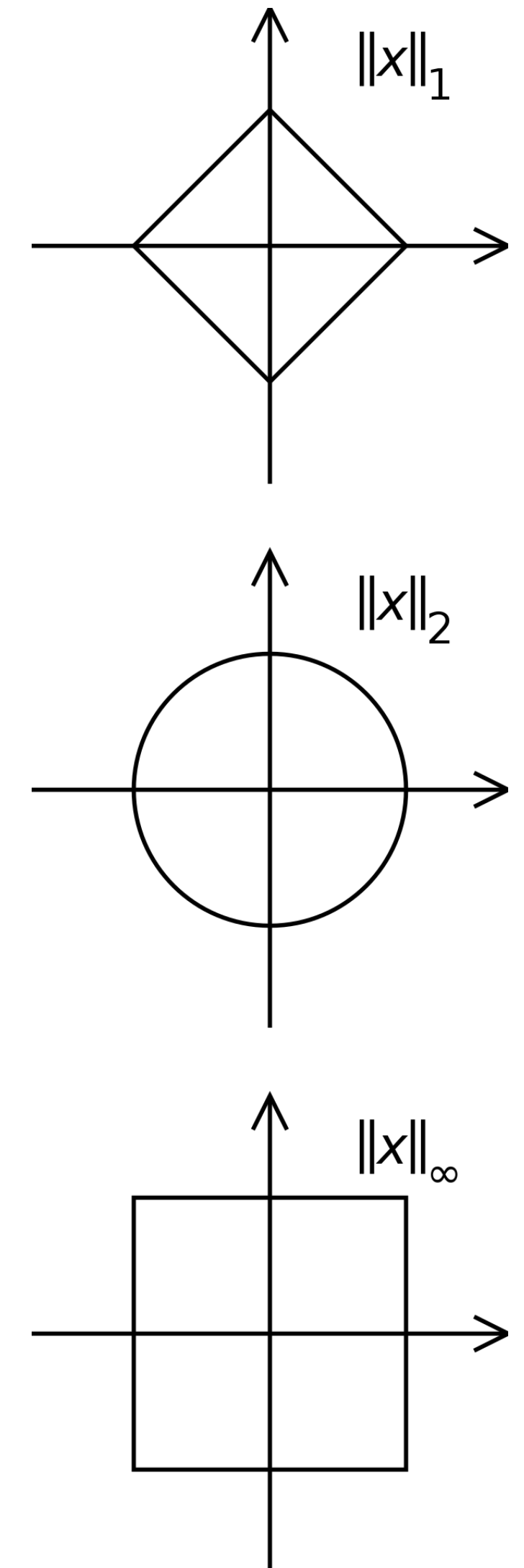
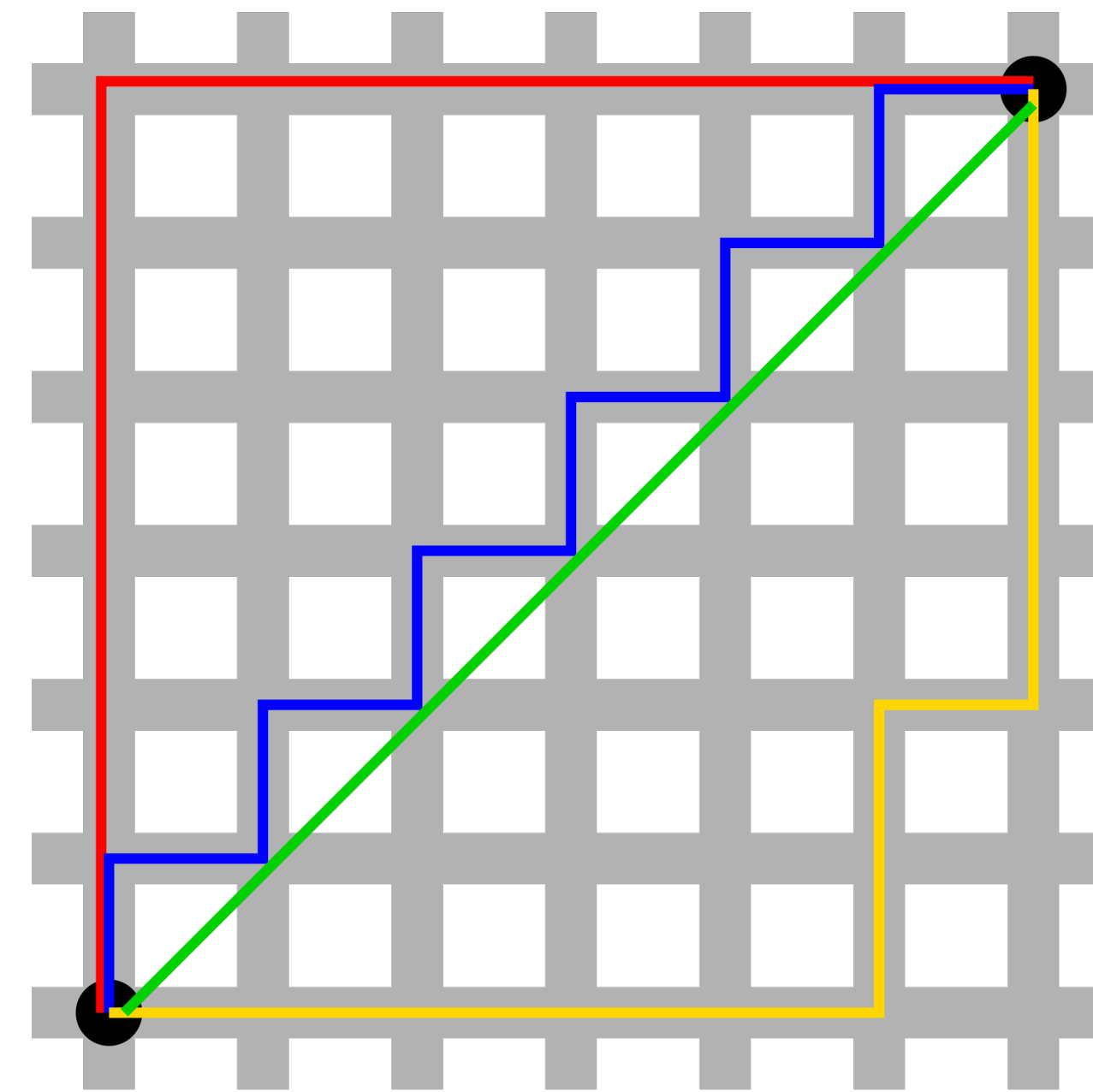


# An Aside: Other Notions of Distance

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e.g., Manhattan distance

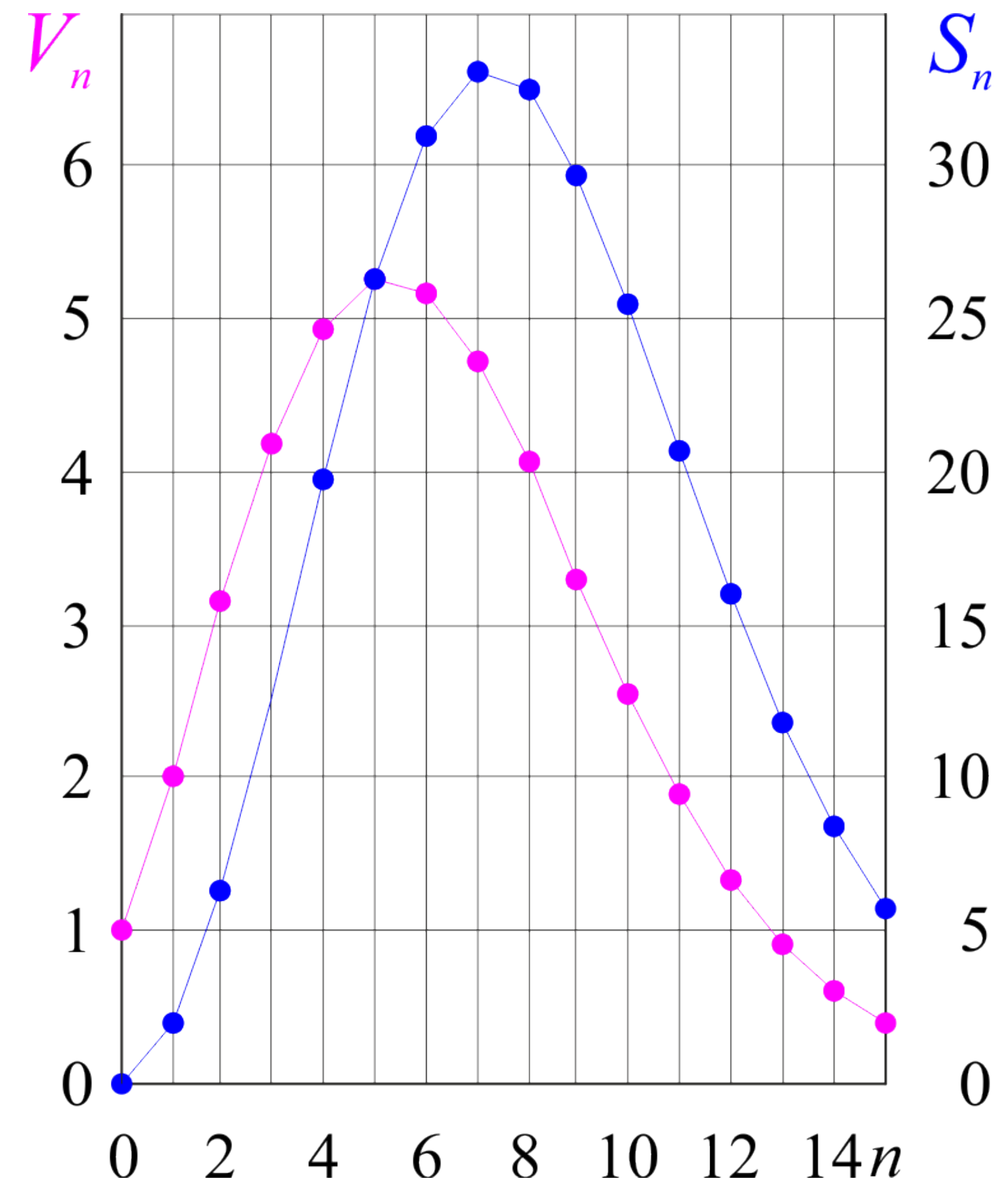


# Another Aside: Surface Area and Volume

With a bit of calculus, we can calculate the surface area and volume of the unit  $n$ -sphere.

And the result is bizarre...

*the infinite dimensional unit sphere has no volume or surface area...*



moving on . . .

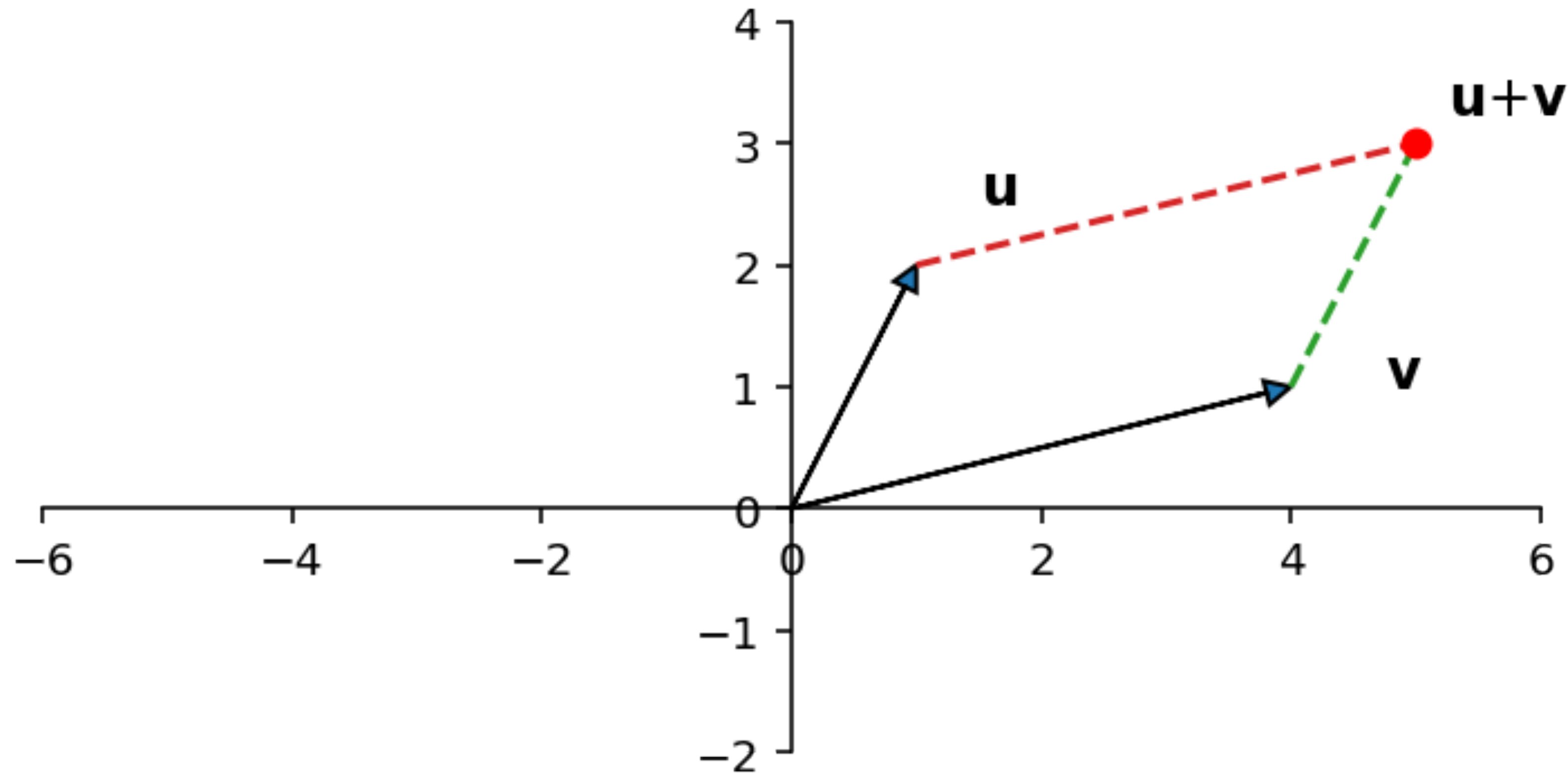
# Distance

If we know how to calculate lengths of vectors, we know how to calculate distances.



# Recall: Vector Addition

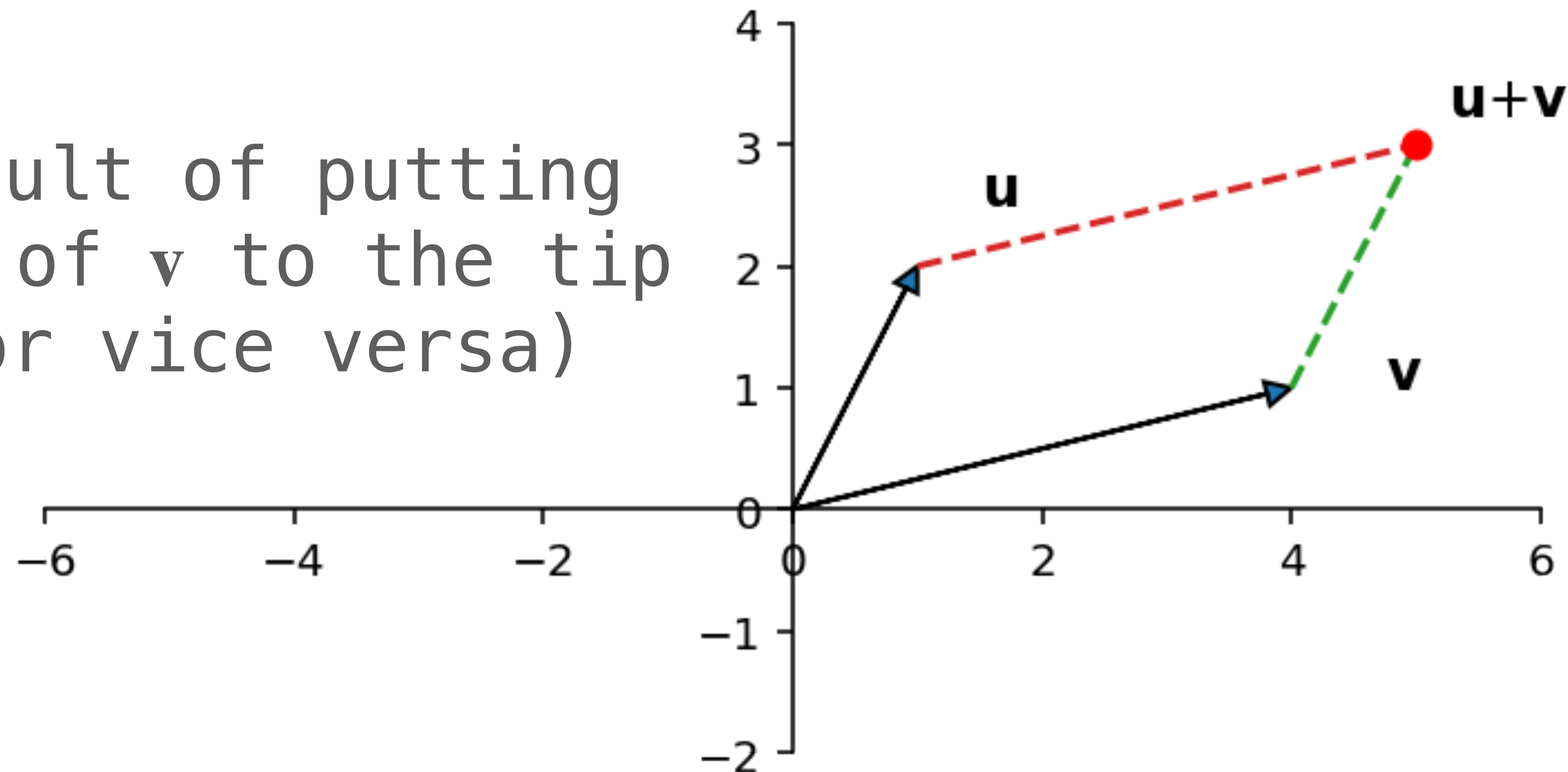
*tip-to-tail rule:*



# Recall: Vector Addition

*tip-to-tail rule:*

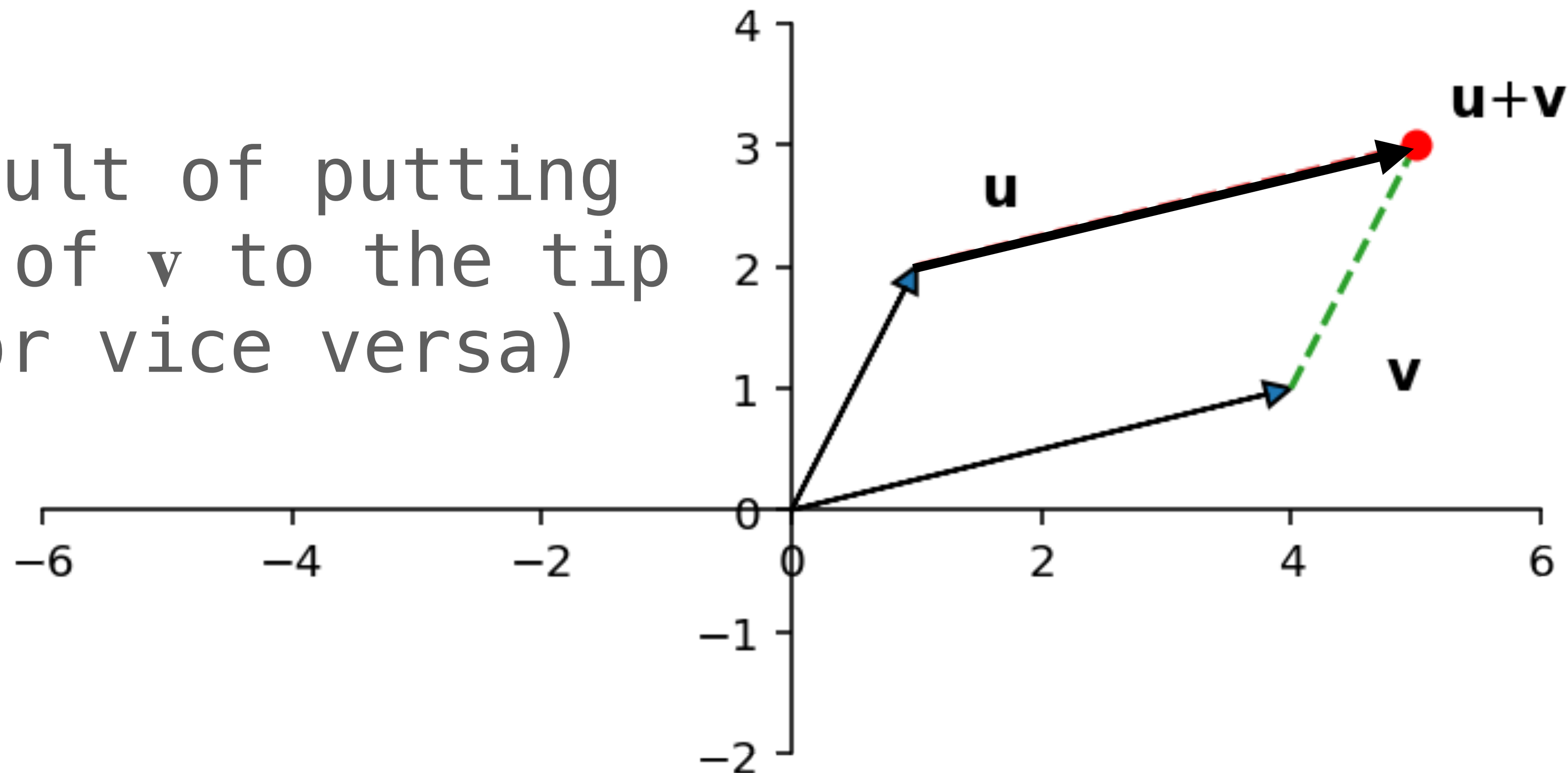
$\mathbf{u} + \mathbf{v}$  result of putting the tail of  $\mathbf{v}$  to the tip of  $\mathbf{u}$  (or vice versa)



# Recall: Vector Addition

*tip-to-tail rule:*

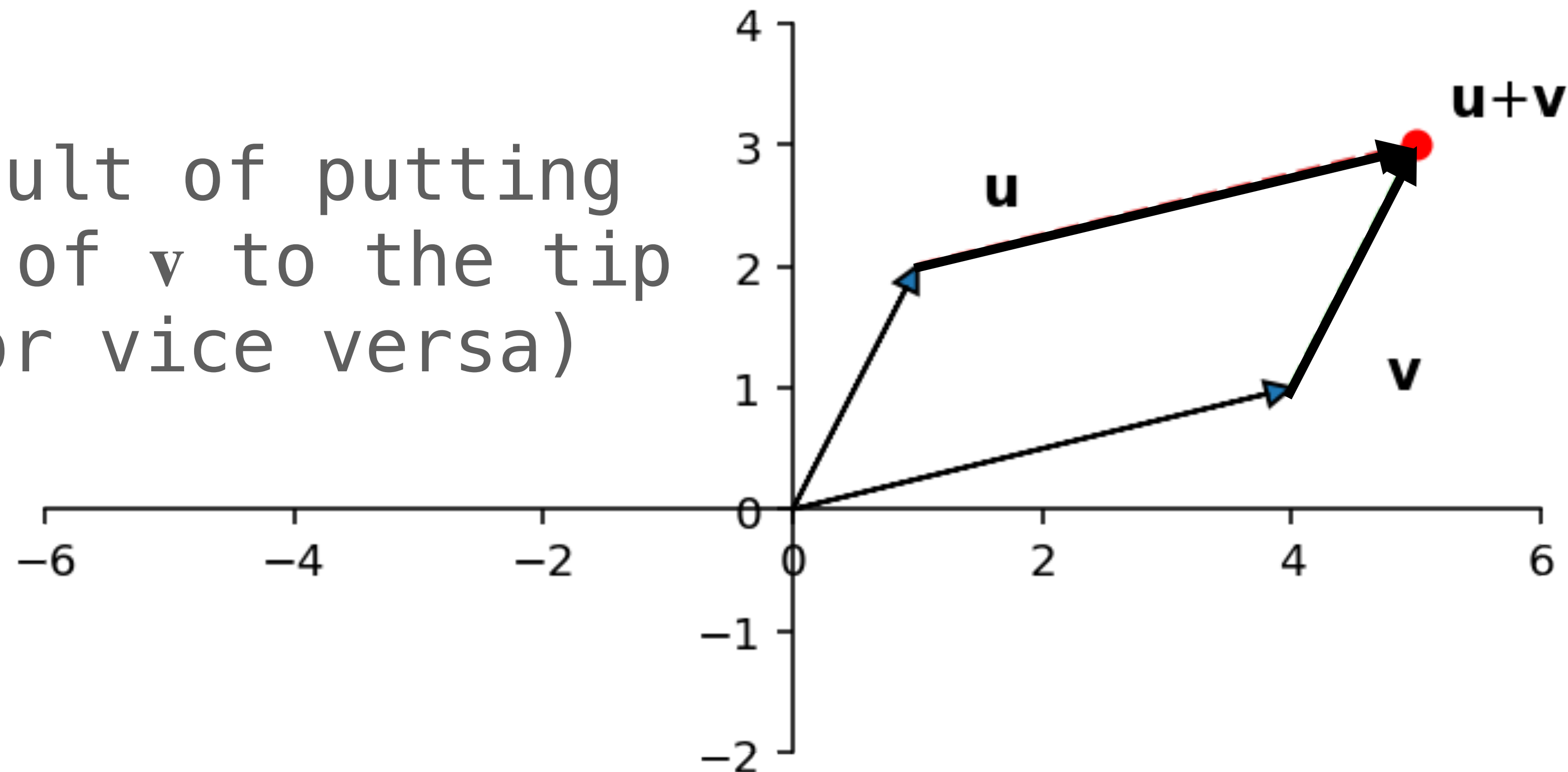
$u + v$  result of putting  
the tail of  $v$  to the tip  
of  $u$  (or vice versa)



# Recall: Vector Addition

*tip-to-tail rule:*

$u + v$  result of putting  
the tail of  $v$  to the tip  
of  $u$  (or vice versa)

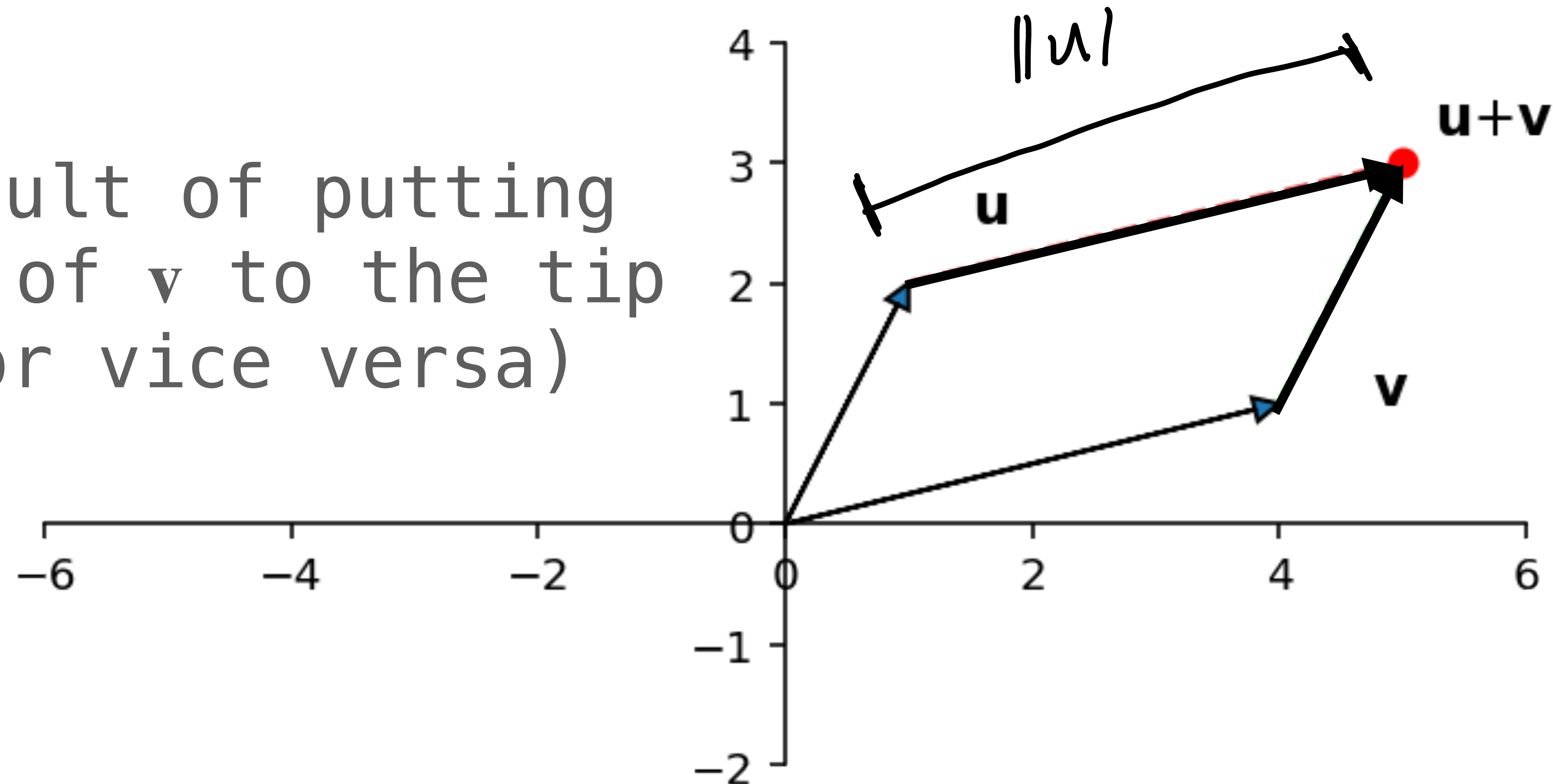


# Recall: Vector Addition

The distance between  $u$  and  $u+v$  is the length of  $v$

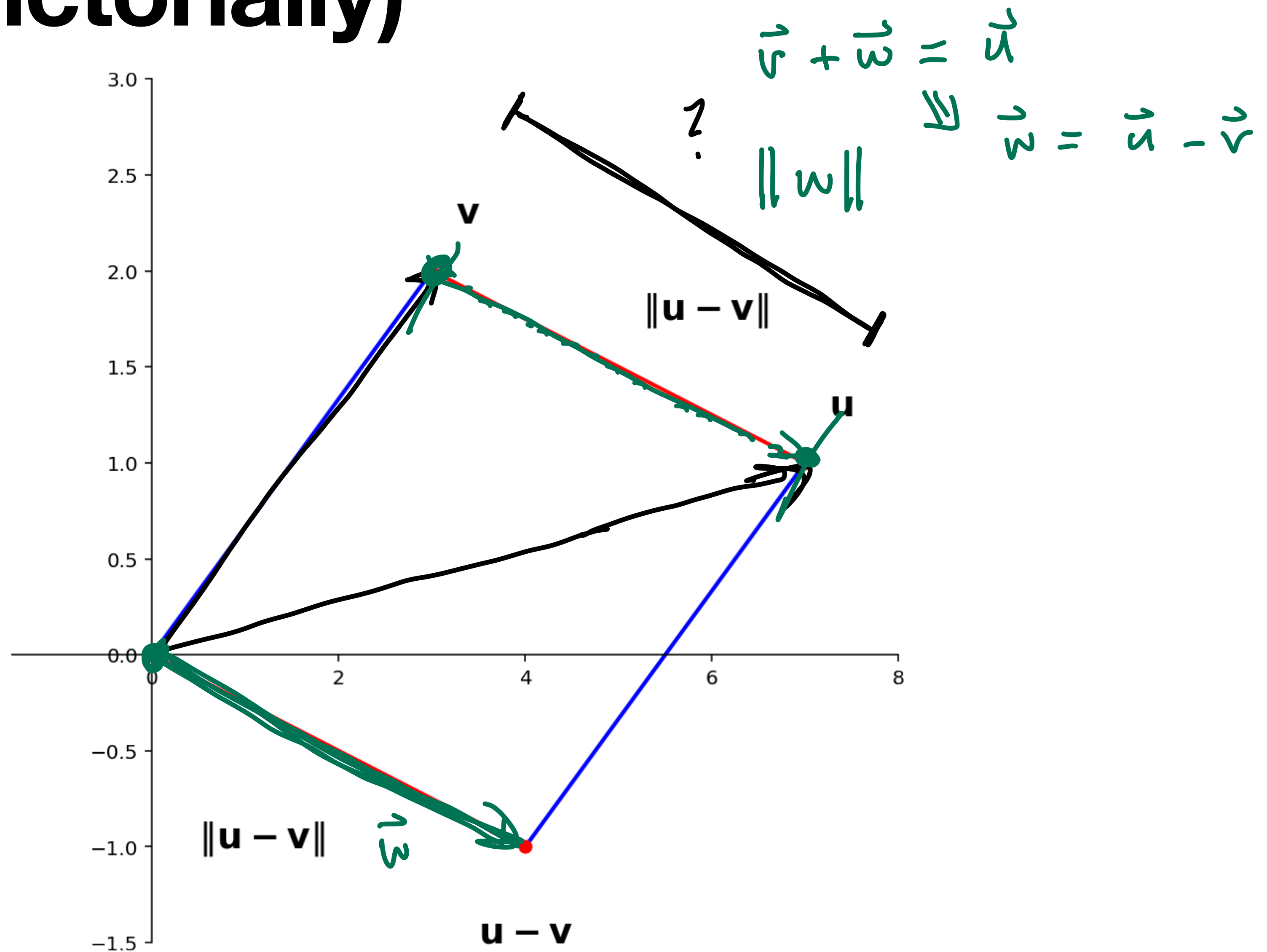
*tip-to-tail rule:*

$u+v$  result of putting the tail of  $v$  to the tip of  $u$  (or vice versa)



# Distance (Pictorially)

$$\cancel{\vec{v}} + (\vec{u} - \cancel{\vec{v}}) = \vec{u}$$



# Distance (Algebraically)

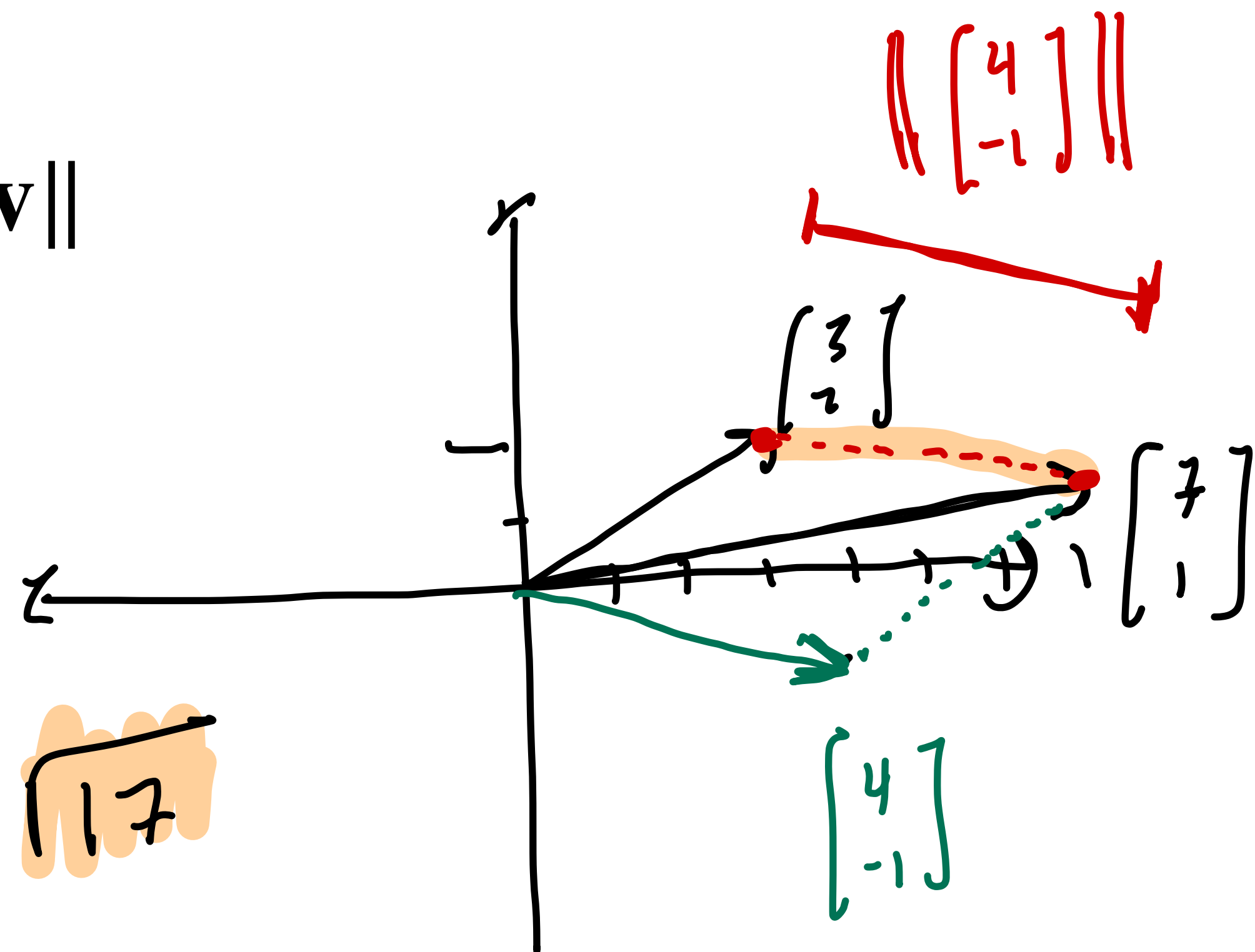
$$\begin{aligned} \left\| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\| &= \sqrt{\left\langle \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\rangle} = \sqrt{4^2 + (-1)^2} \\ &= \sqrt{16 + 1} \\ &= \sqrt{17} \end{aligned}$$

**Definition.** The distance between two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  is given by

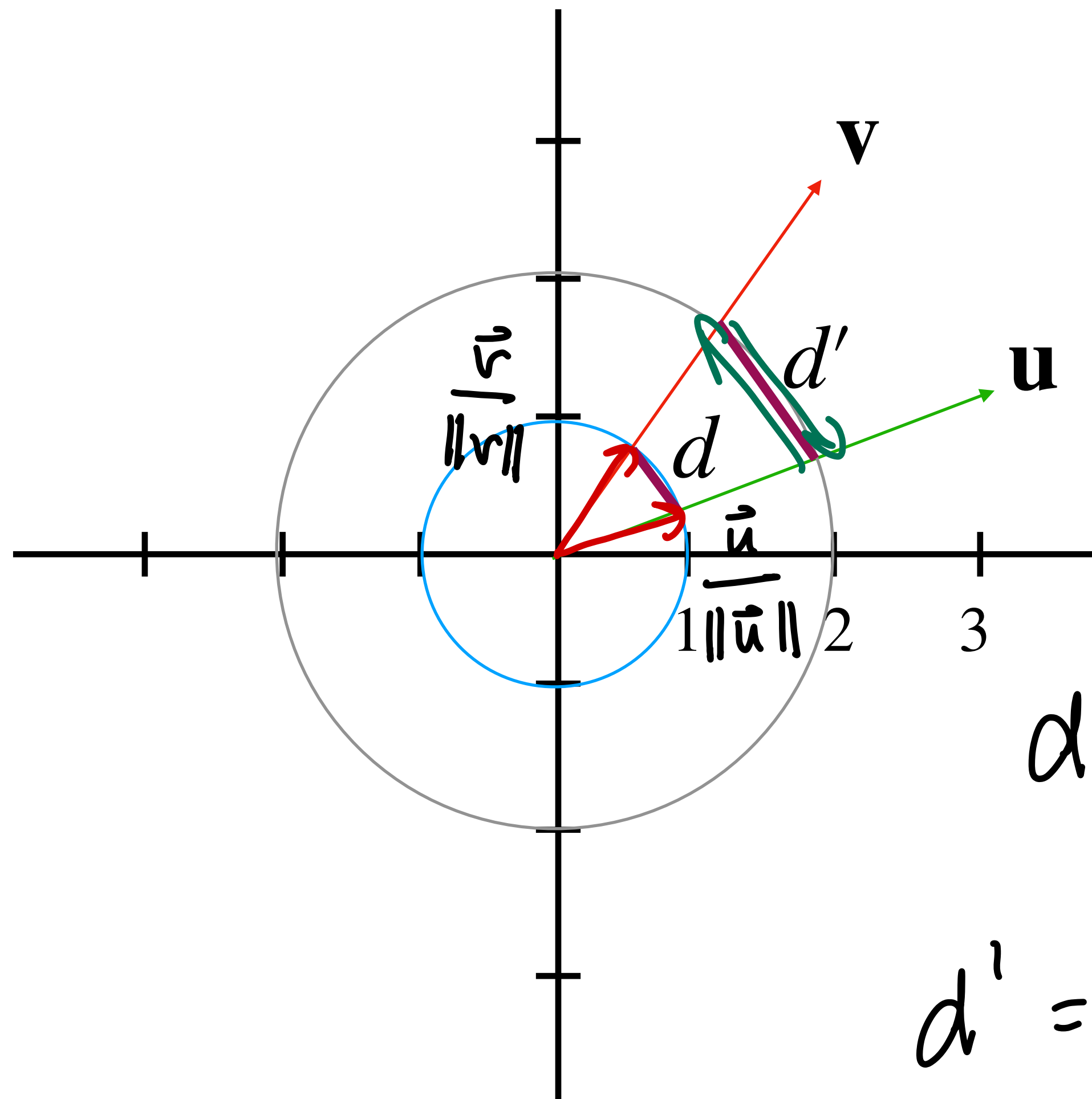
$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g.,  $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \left\| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\| = \sqrt{17}$$



# Question



$$d = \left\| \frac{v}{\|v\|} - \frac{u}{\|u\|} \right\|$$

$$d' = \left\| \frac{2v}{\|v\|} - \frac{2u}{\|u\|} \right\| = 2d$$

Find an expression for the distance  $d$ .

**Challenge.** Find an expression for  $d'$ .

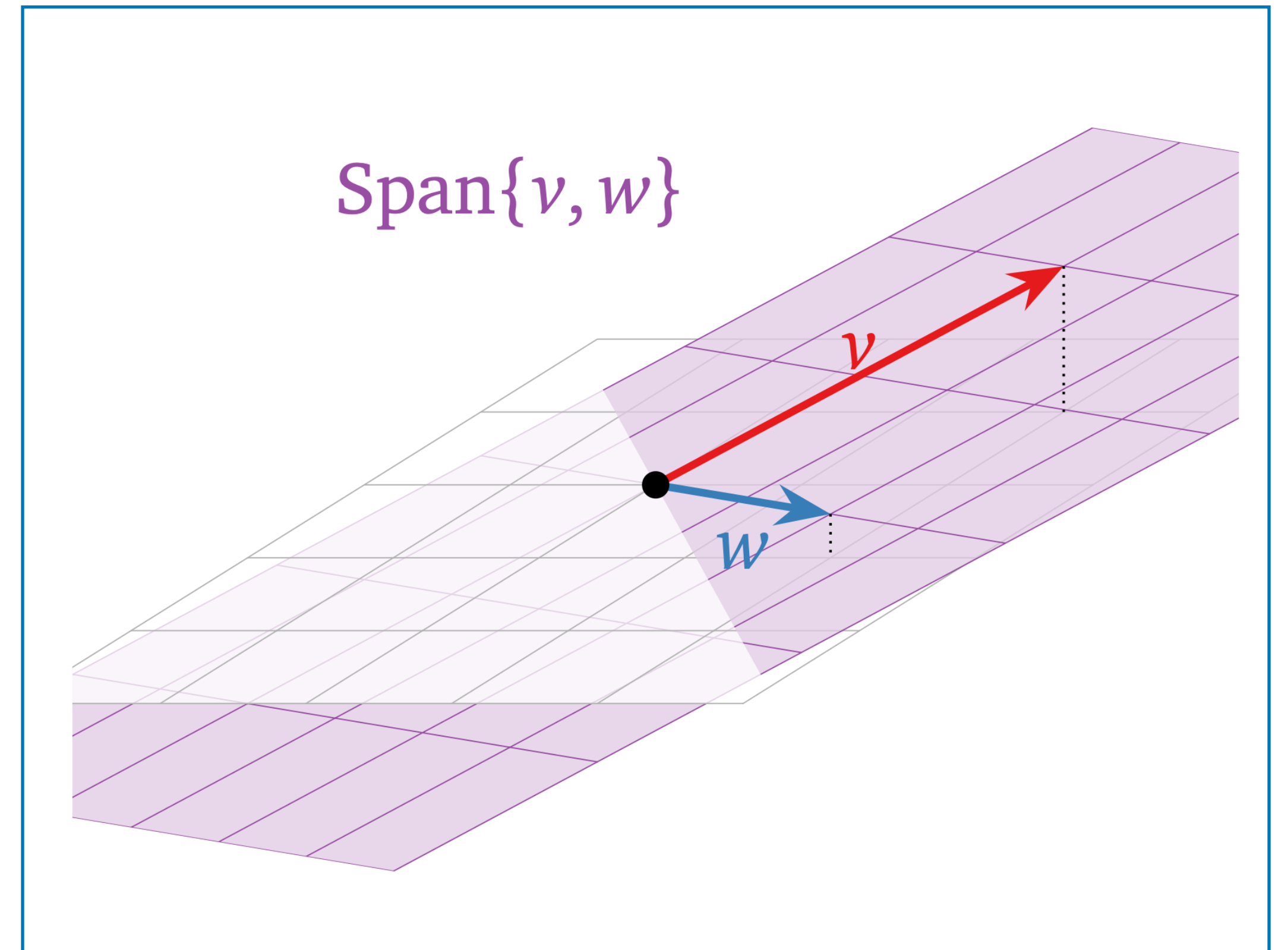


**Answer**

# Angles

# Again, Angles still make sense

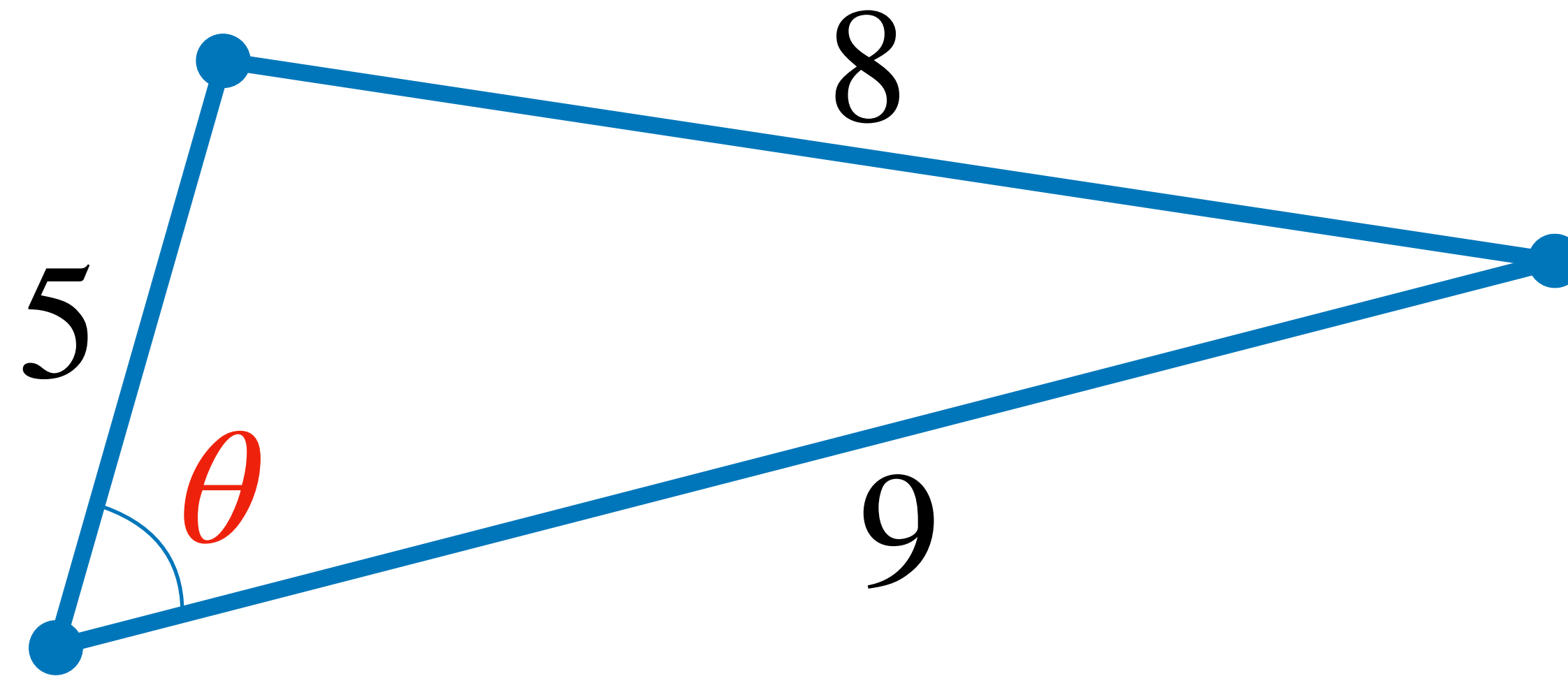
Any pair of vectors in  $\mathbb{R}^n$   
span a (2D) plane.



# Fundamental Question

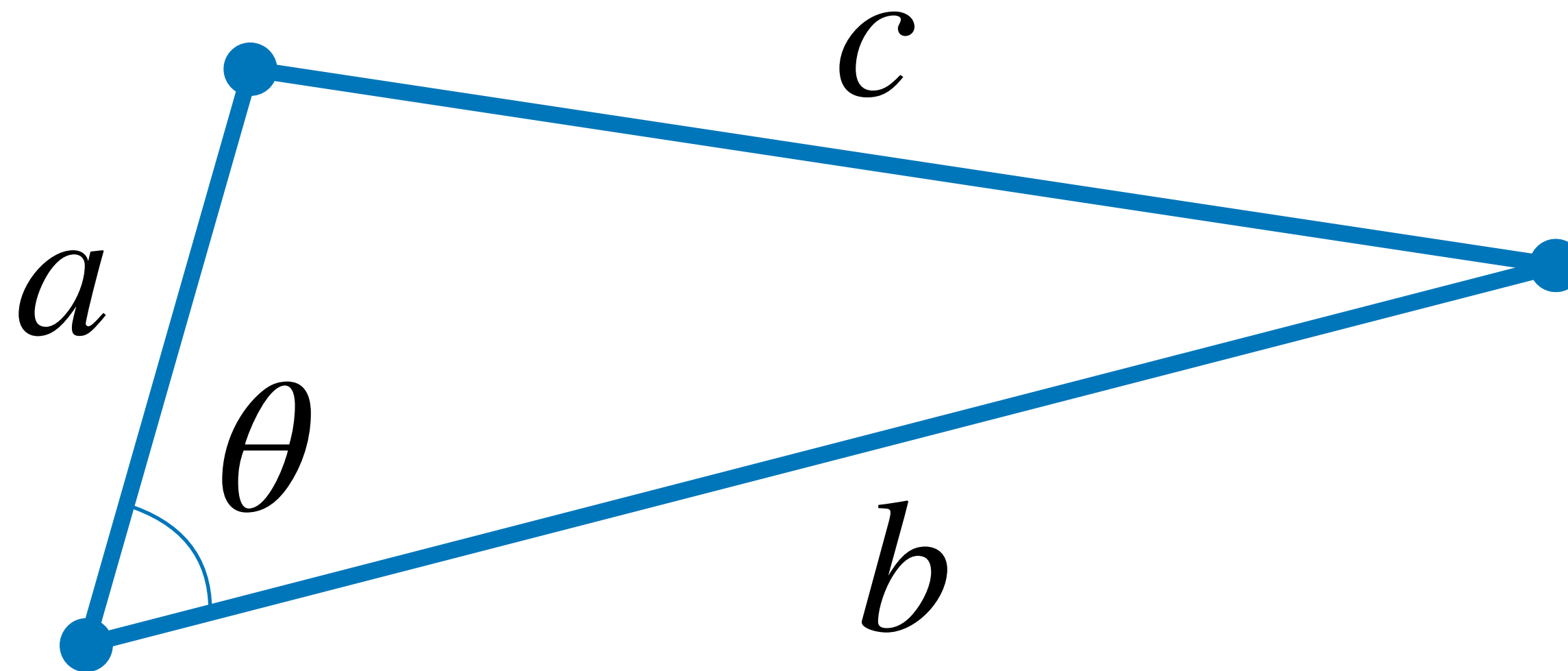
How do we determine the angle  
between any two vectors?

# Recall: A Potentially Familiar Example



*What is the value of  $\theta$ ?*

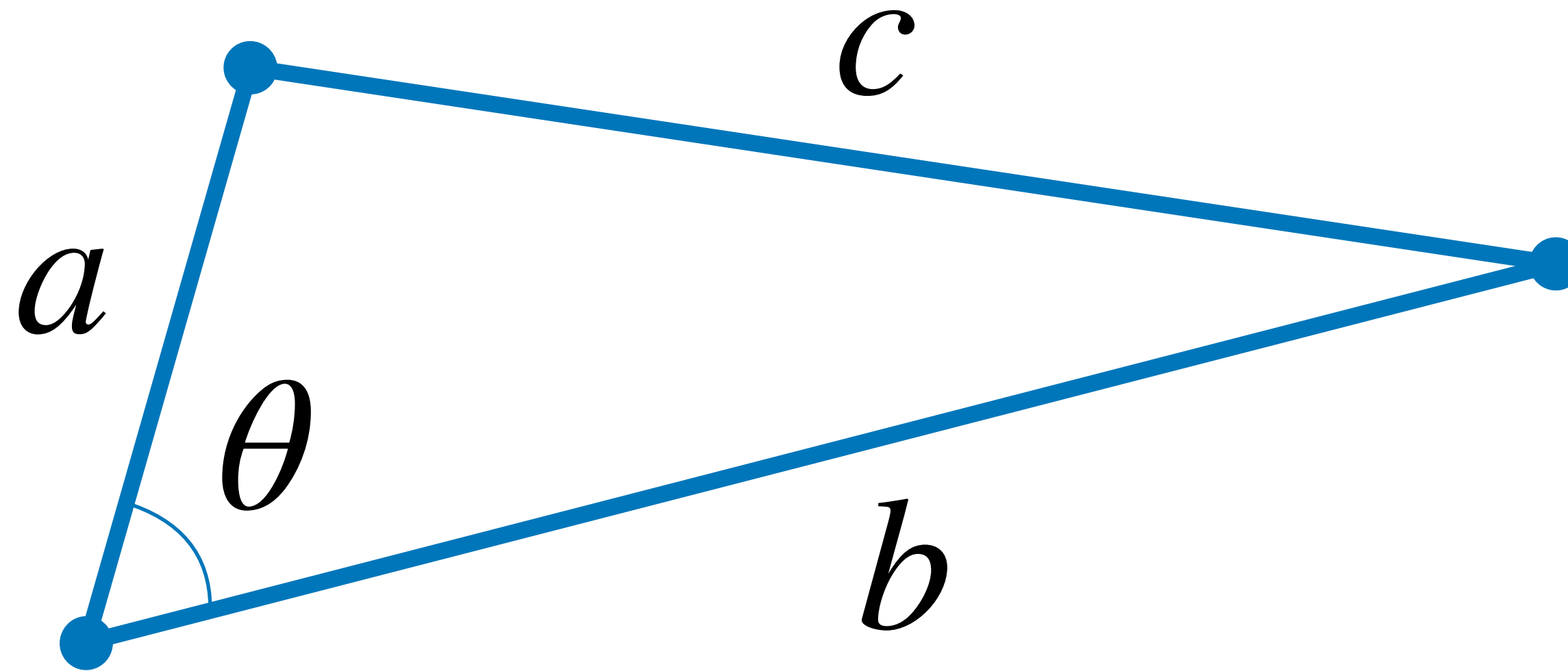
# Law of Cosines



**Theorem.**

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

# Law of Cosines

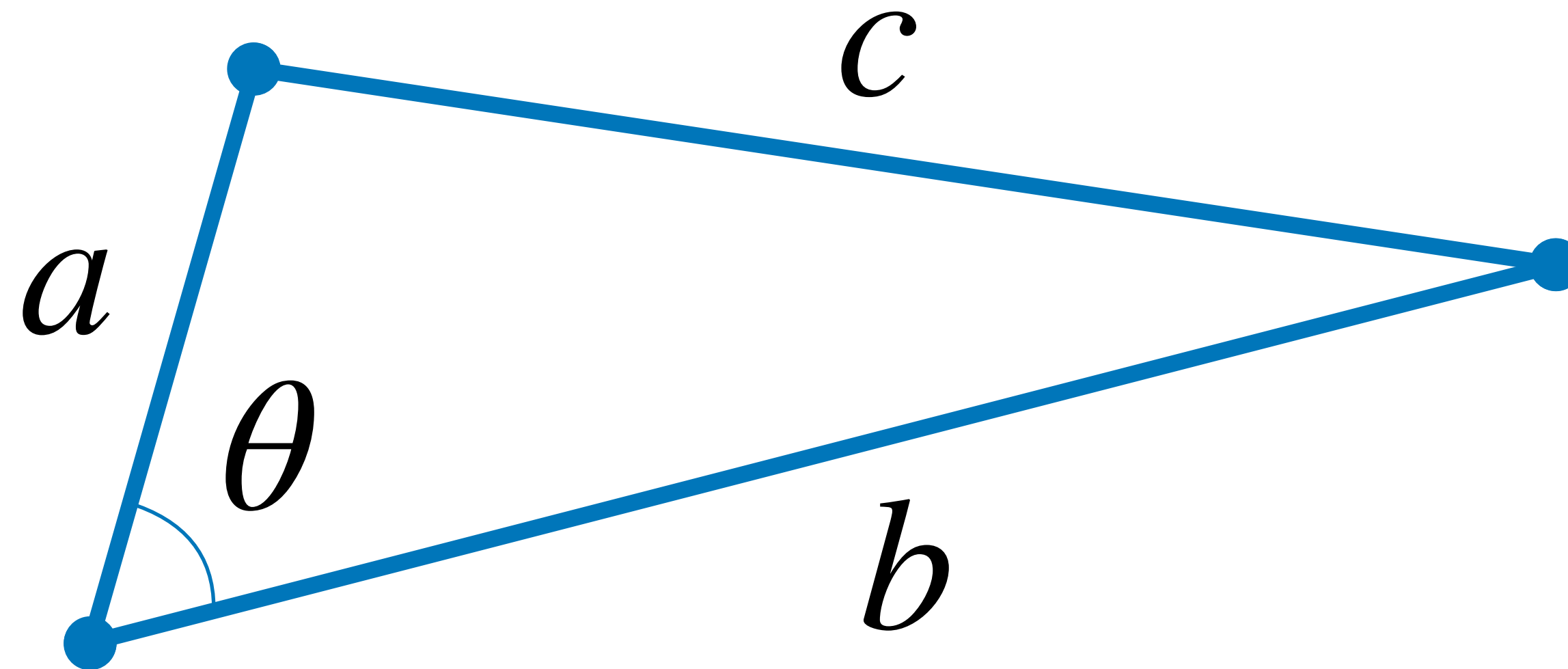


**Theorem.**

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

**Generalized the Pythagorean Theorem**

# Law of Cosines



**Theorem.**

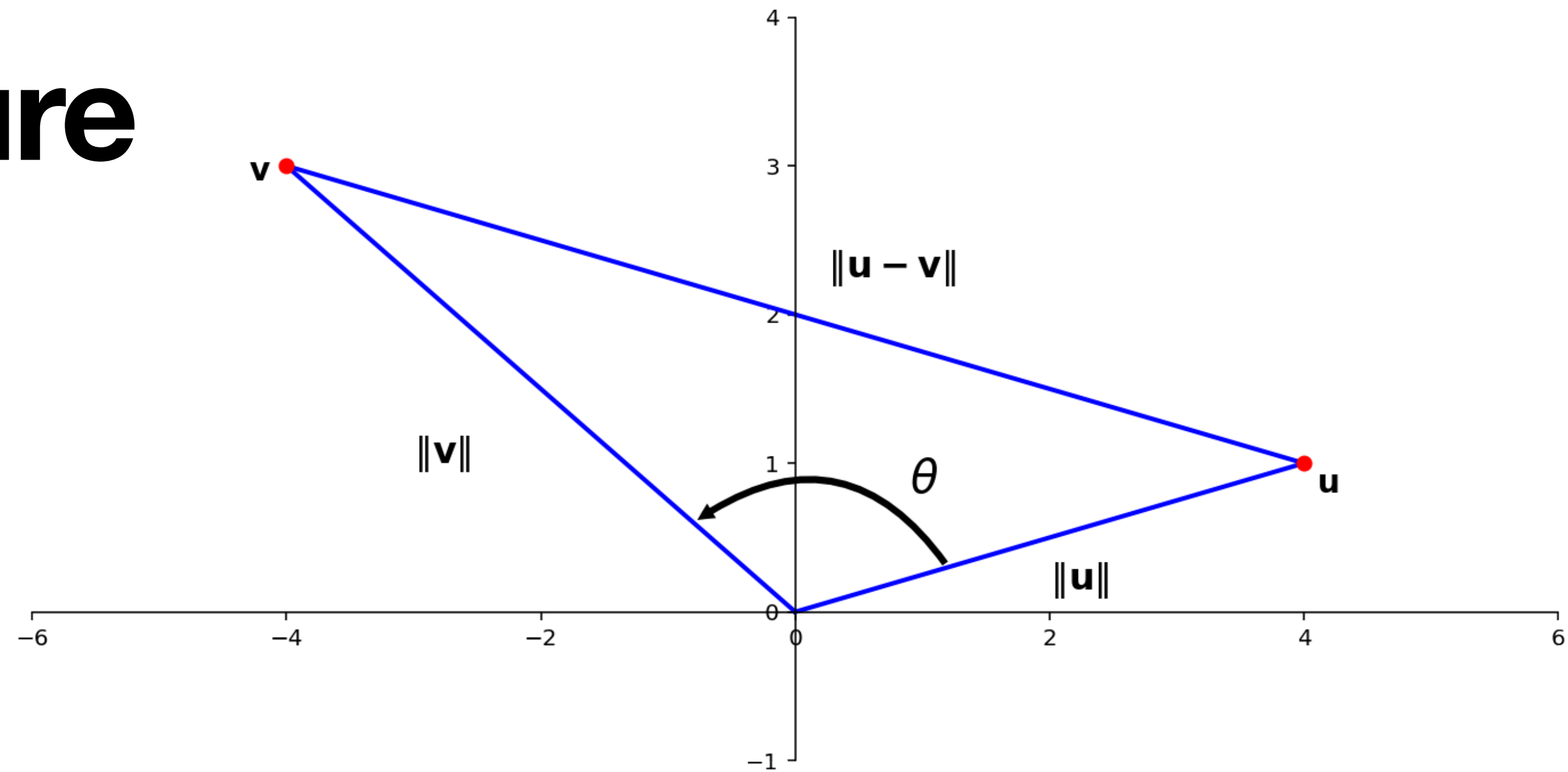
$\theta$  exactly when  $\theta = 90^\circ$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

**Generalized the Pythagorean Theorem**



# The Picture



In more "vector"-y terms:

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

# Isolating $\theta$

$$\cos \theta = \frac{c^2 - a^2 - b^2}{2ab}$$

$$\theta = \cos^{-1} \left( \frac{c^2 - a^2 - b^2}{2ab} \right)$$

We might remember these equations...

# Isolating $\theta$

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta$$

Let's isolate  $\theta$  in this equation:  $\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$

$$\|u - v\|^2 - \|u\|^2 - \|v\|^2 = -2\|u\|\|v\| \cos \theta$$

$$\cos \theta = \frac{\|u - v\|^2 - \|u\|^2 - \|v\|^2}{-2\|u\|\|v\|}$$

$$= \frac{\langle u - v, u - v \rangle - \langle u, u \rangle - \langle v, v \rangle}{-2\|u\|\|v\|} = \frac{\cancel{\langle u, u \rangle} - 2\langle u, v \rangle + \cancel{\langle v, v \rangle} - \cancel{\langle u, u \rangle} - \cancel{\langle v, v \rangle}}{-2\|u\|\|v\|}$$

# Cosines and Unit Vectors

**Theorem.** For vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  with an angle  $\theta$  between them,

$$\cos \theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$

*The cosine of the angle between two vectors is the inner product of their  $\ell^2$  normalizations.*

# How To: Angles

**Question.** Find the angle between the two vectors  $\mathbf{u}$  and  $\mathbf{v}$ .

**Solution.** Compute  $\cos^{-1} \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$  (with a calculator).

# Example

*Find the angle between the vectors*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -7 \\ -2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 8 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

# Example: Step 1

Compute  $\|\mathbf{u}\|$  and  $\|\mathbf{v}\|$ .

$$\|\mathbf{u}\| = \sqrt{1^2 + 3^2 + (-7)^2 + (-2)^2} = 7.93$$

$$\|\mathbf{v}\| = \sqrt{8^2 + (-2)^2 + 4^2 + 6^2} = 10.95$$

# Example: Step 2

Normalize the vectors.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{bmatrix} 0.13 \\ 0.38 \\ -0.88 \\ -0.25 \end{bmatrix}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 0.73 \\ -0.18 \\ 0.36 \\ 0.54 \end{bmatrix}$$



## Example: Step 3

Find their inner product.

$$\left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle = (0.13 \cdot 0.73) + (0.38 \cdot -0.18) + (-0.88 \cdot 0.36) + (-0.25 \cdot 0.54)$$
$$= -0.44$$

# Example: Step 4

Compute the angle.

$$\theta = \cos^{-1}(-0.44) = 116^\circ$$

# Orthogonality (Perpendicularity)

# A Simpler Fundamental Question

How do we determine if angle  
between any two vectors is  $90^\circ$ ?

# Orthogonality

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**Definition (Informal).** Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  are orthogonal if the angle between them is  $90^\circ$ .

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**Orthogonal and perpendicular are the same thing.**

**But it doesn't connect back to inner products.**

**(and it's difficult to compute with)**

# Recall: Cosines and Unit Vectors

**Theorem.** For vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  with an angle  $\theta$  between them,

$$\cos \theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$

*The cosine of the angle between two vectors is the inner product of their  $\ell^2$  normalizations.*

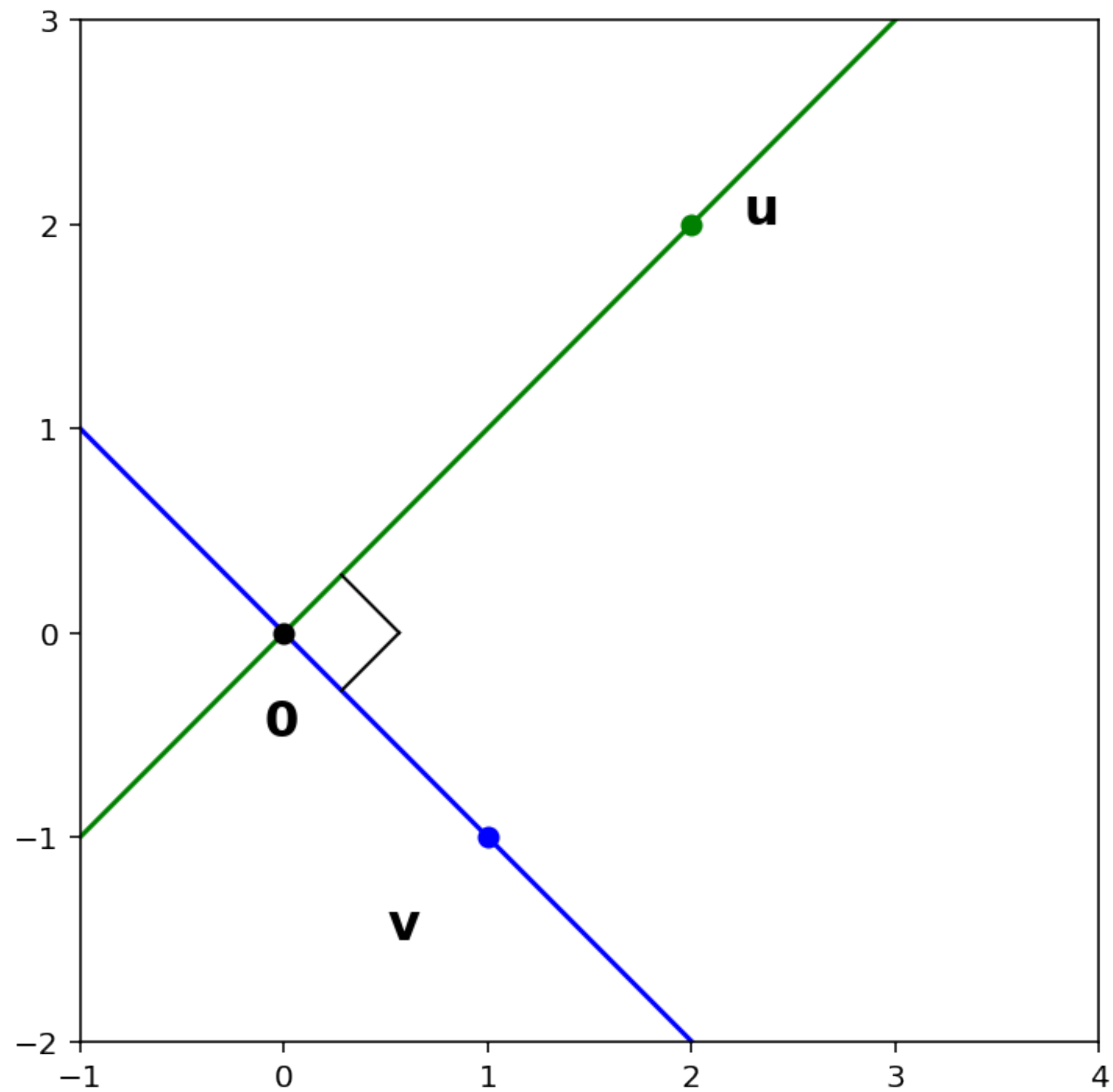
# Orthogonality

**Definition (Actual).** Vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

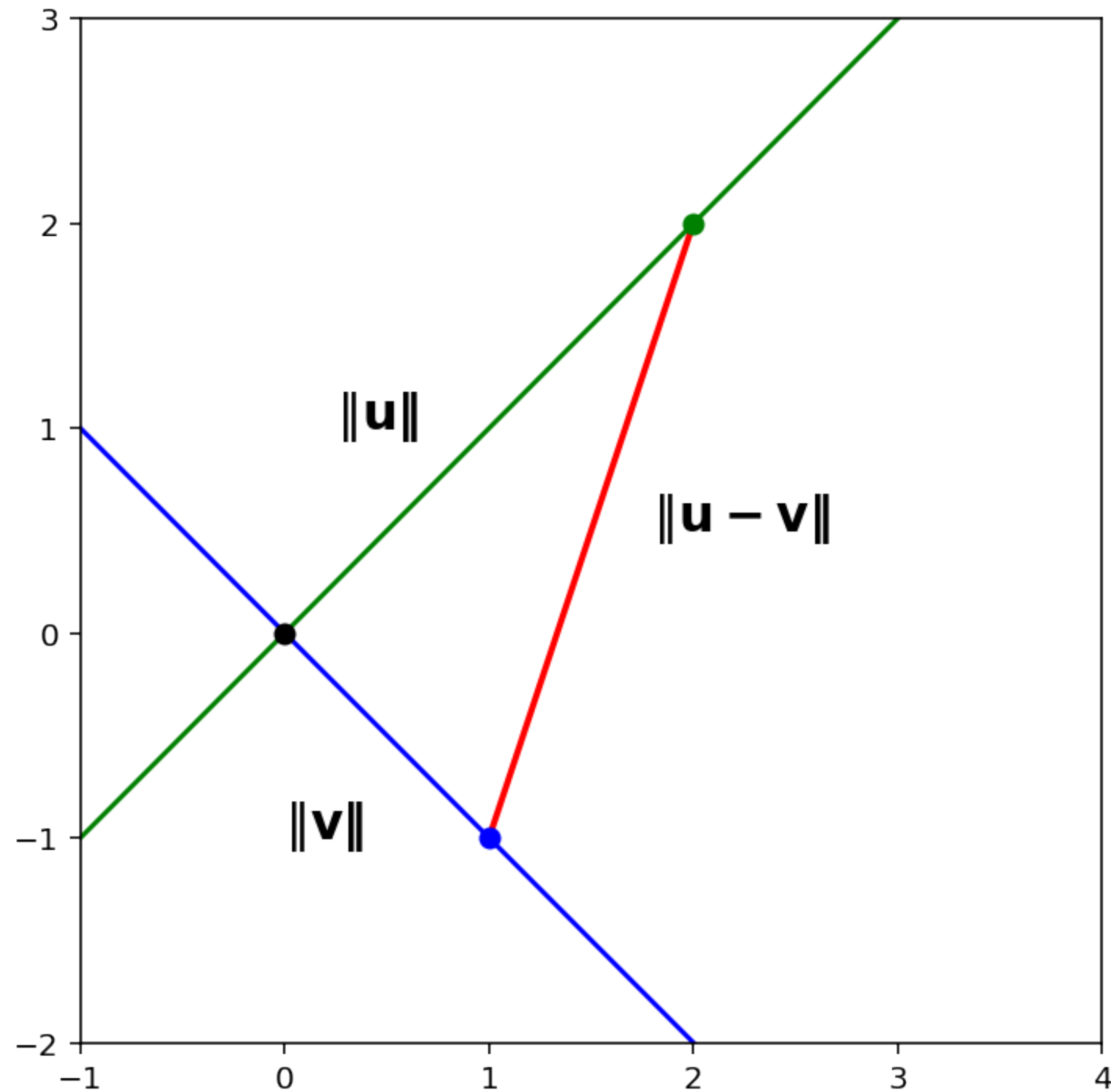
*This definition gives an easy computational way to determine orthogonality.*

Example.

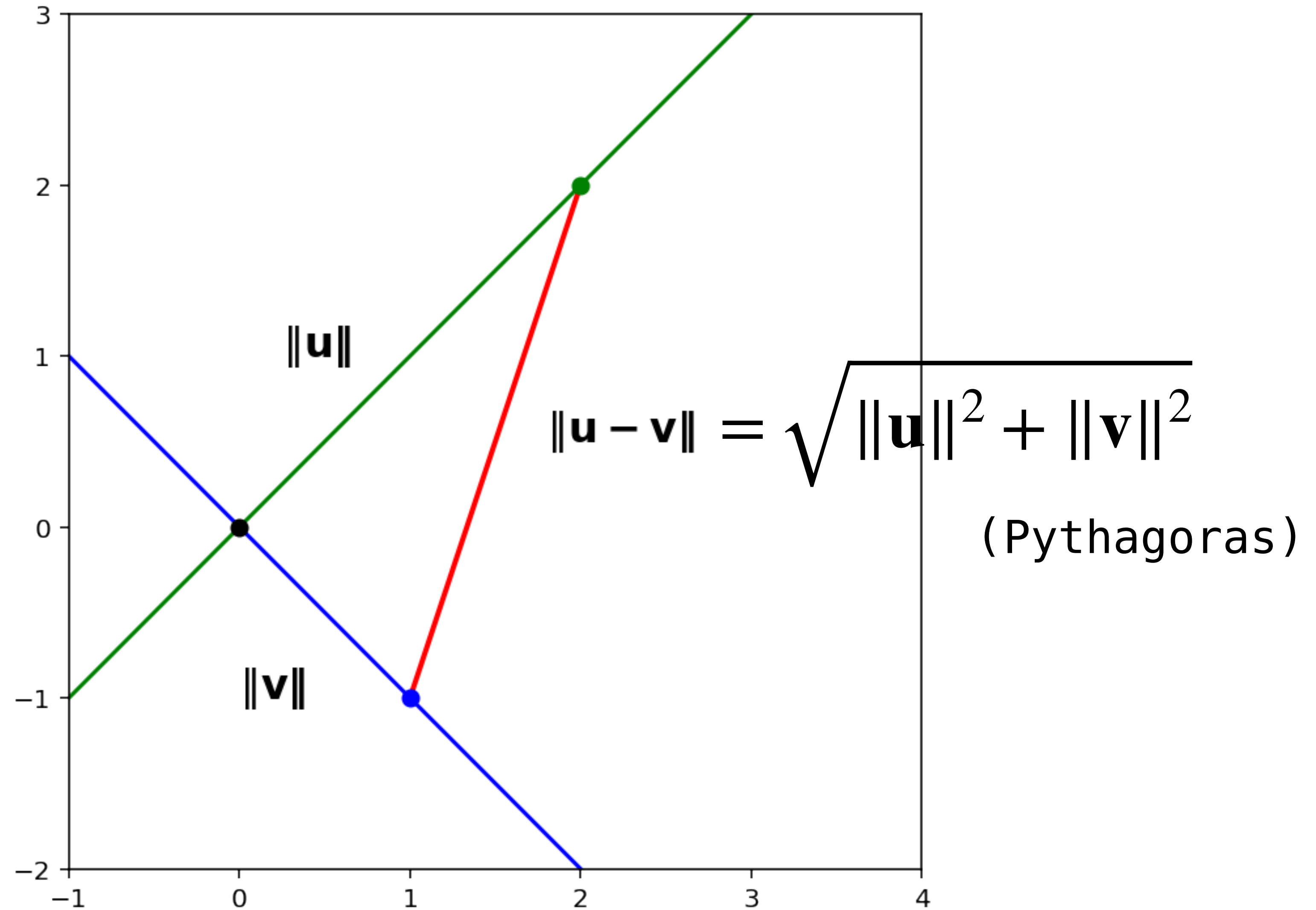
# Derivation by Picture



# Derivation by Picture



# Derivation by Picture



# Derivation by Algebra

$\mathbf{u}$  and  $\mathbf{v}$  are orthogonal exactly when

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

Let's simplify this a bit:



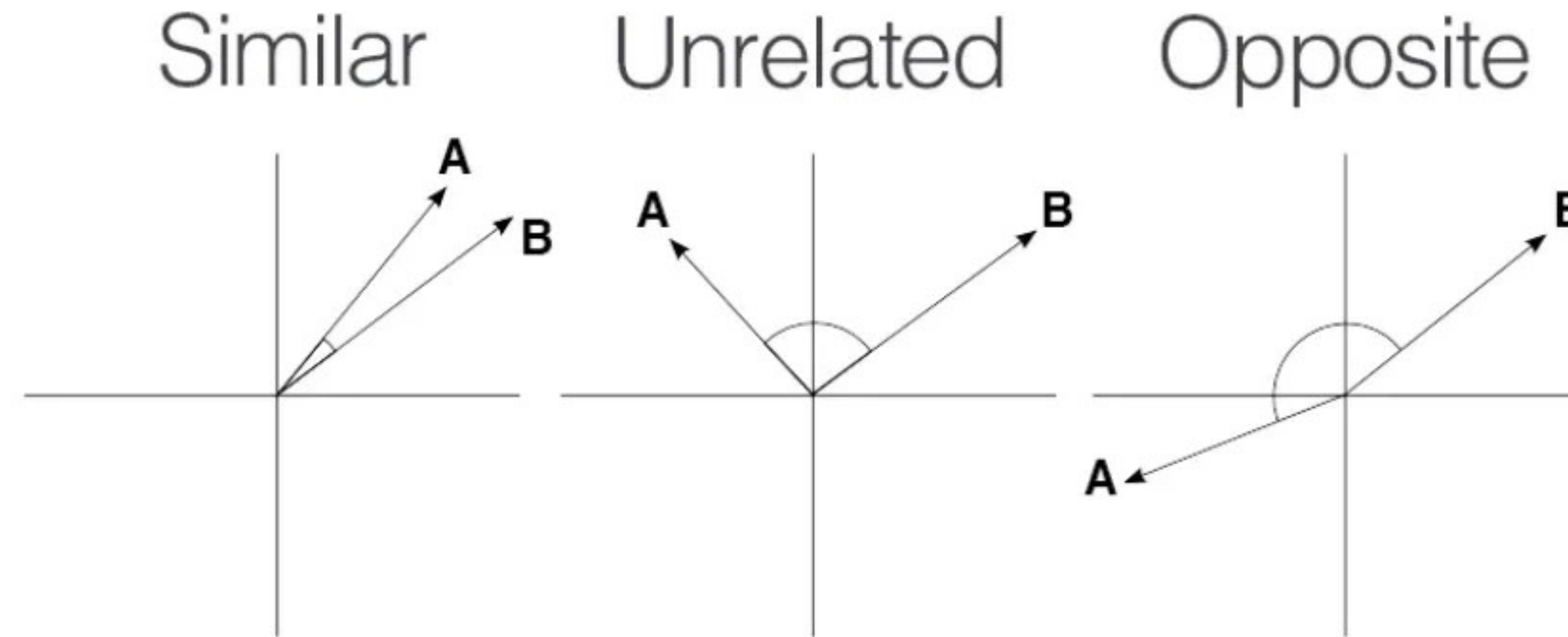
# How To: Orthogonality

**Question.** Determine if  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular.

**Solution.** Determine if  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ . If yes, then they are perpendicular. If no, then they are not.

# Application: Cosine Similarity

# High Level



Data points are very big vectors.

Similar vectors "point in nearly the same direction."

# Example: Netflix Users

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix} \quad \text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix} \quad \text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{l} \text{comedy} \\ \text{drama} \\ \text{horror} \\ \text{romance} \end{array}$$

A Netflix user might be represented as a vectors whose  $i$ th entry is the number of movies they've watched in a particular genre.

**Who are more likely to share similar interests in movies?**

# Cosine Similarity

**Definition.** The **cosine similarity** of two vectors is the cosine of the angle between them.

*If its close to 0, then two Netflix users watch very different movies.*

*If its close to 1, then two Netflix users watch very similar movies.*

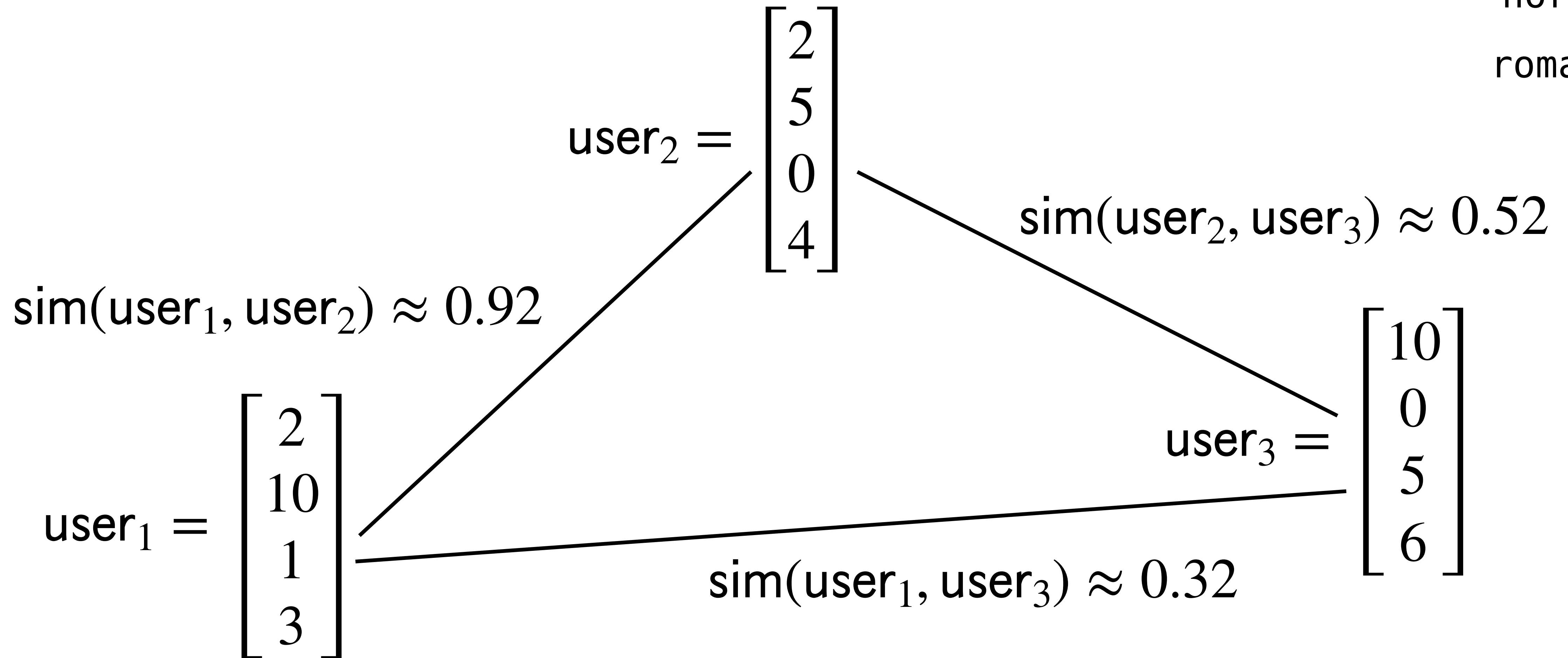
# Example: Netflix Users

comedy

drama

horror

romance



# Other Examples

- ***Document similarity***
  - Documents  $\mapsto$  word count vectors
  - Similar documents should use similar words
- ***Word2Vec***
  - Words  $\mapsto$  vector *somehow*
  - This underlies modern natural language processing (NLP)

# Summary

We can talk about distances and angles in  $\mathbb{R}^n$ .

Every basic geometric concept connects to inner products.

Once we can talk about distances and angles we can talk about similarity.