Orthogonal Sets

Geometric Algorithms
Lecture 22

CAS CS 132

Practice Problem

$$\mathscr{B} = \left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right)$$

Determine $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathscr{B}}$

Answer

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathscr{B}}$$

$$\mathscr{B} = \left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right)$$

Objectives

1. Recap analytic geometry in R^n
2. Try to understand why it is useful to work with orthogonal vectors
3. Get a sense of how to compute orthogonal vectors
4. Start to connect orthogonality to matrices and linear transformations

Keywords

orthogonal

orthogonal set

orthogonal basis

orthogonal projection

orthogonal component

orthonormal

orthonormal set

orthonormal basis

orthonormal matrix

orthogonal matrix

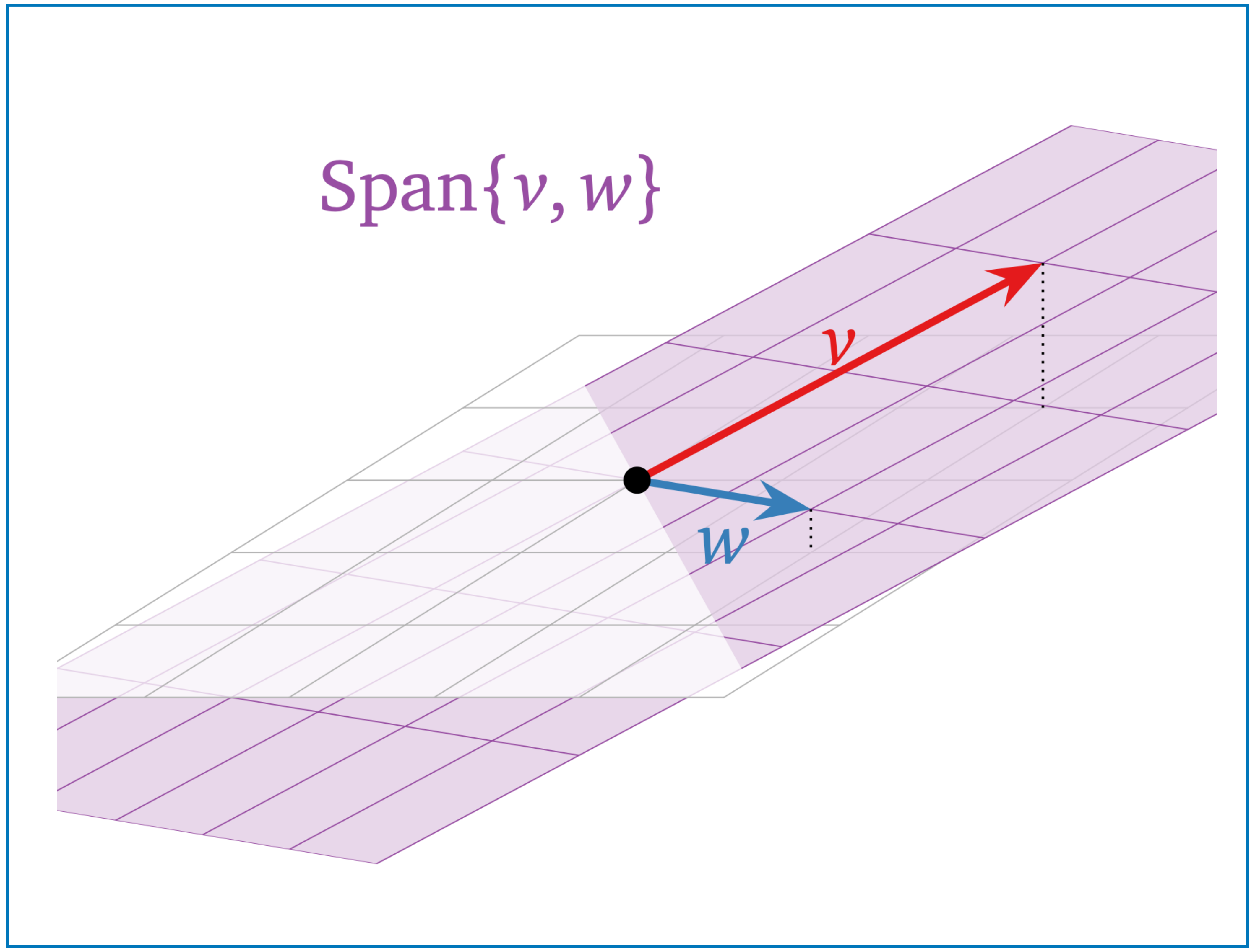
Recap: Analytic Geometry

Recall: The First Key Idea

Angles make sense in *any* dimension

Any pair of vectors in $\mathbb{R}^n$ span a (2D) plane

(We could formalize this via change of bases)



Recall: The Second Key Idea

All of the basic concepts of analytic geometry can be defined *in terms of inner products*

Spaces with inner products (like $\mathbb{R}^n$) are places where you can do analytic geometry

Recall: Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Inner Products

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Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

a.k.a. dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Norms and Inner Products

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

The norm of a vector is the square root of the inner product with itself.

Recall: Norms and Inner Products

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$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

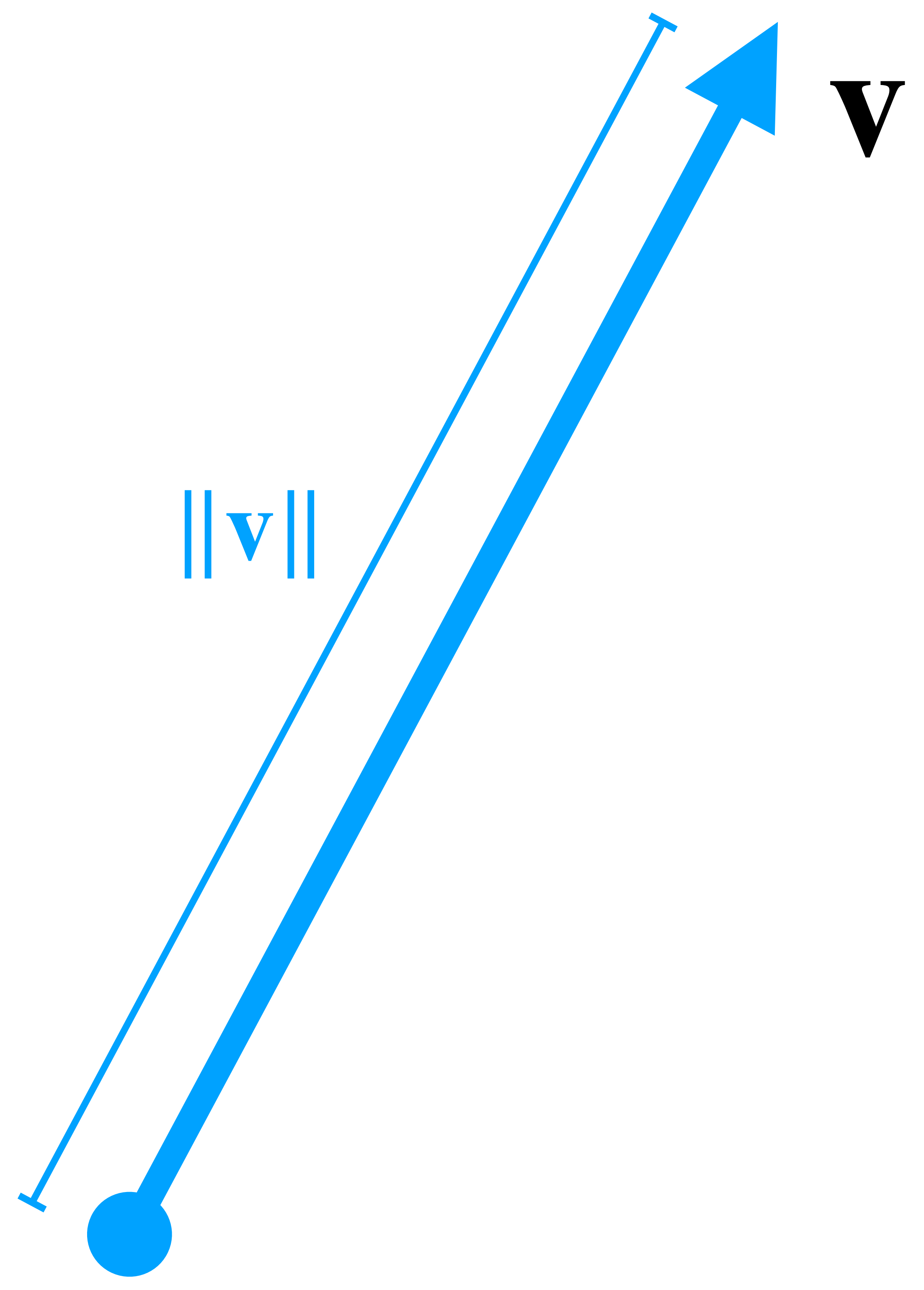
The norm of a vector is the square root of the inner product with itself.

It's important that $\mathbf{v}^T\mathbf{v}$ is nonnegative.

Recall: Norms and Length

Norms give us a notion of <u>length</u>.

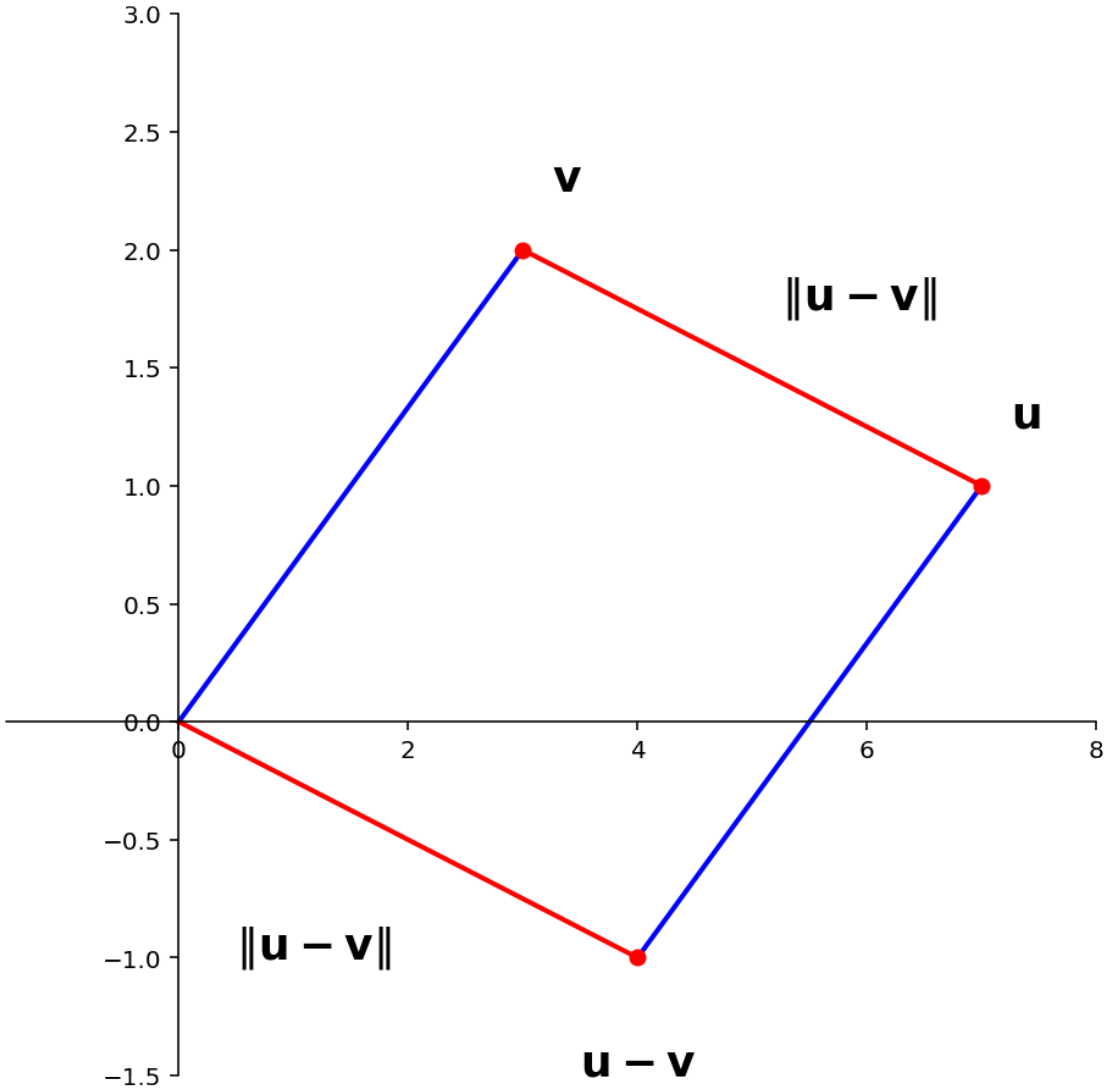
In $\mathbb{R}^2$ and $\mathbb{R}^3$ this is our existing notion of length.



Recall: Distance

If we know how to calculate lengths of vectors, we know how to calculate distances.

Recall: Distance (Pictorially)



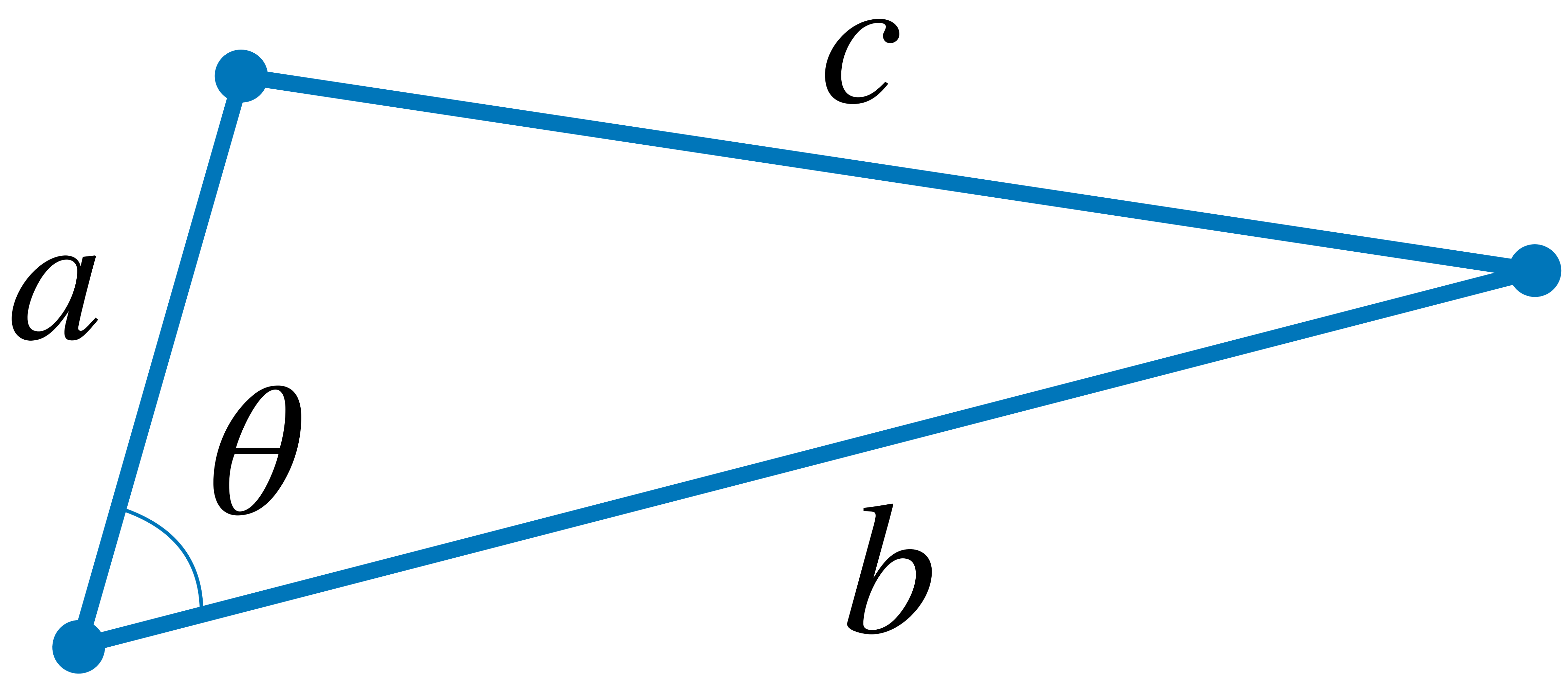
Recall: Distance (Algebraically)

Definition. The distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is given by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g., $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

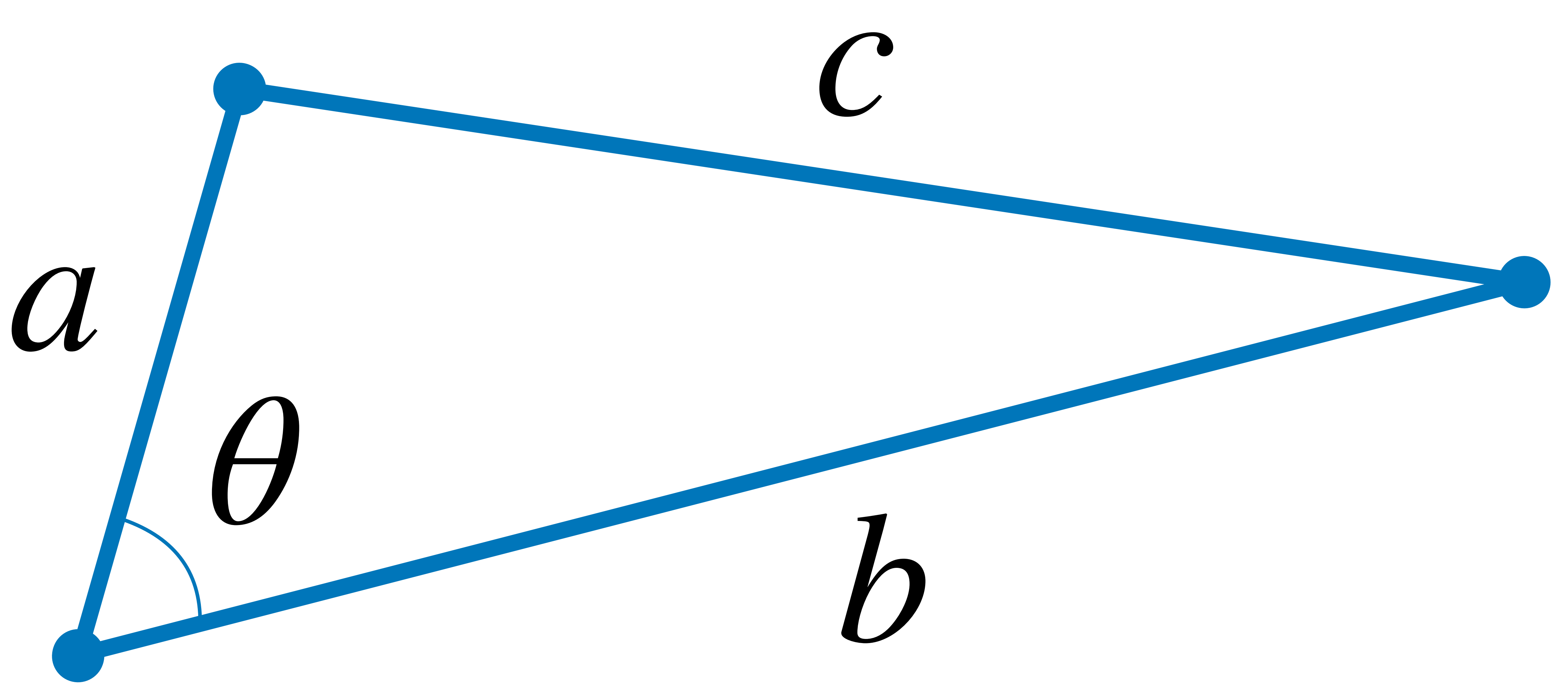
Recall: Law of Cosines



Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Recall: Law of Cosines



Theorem.

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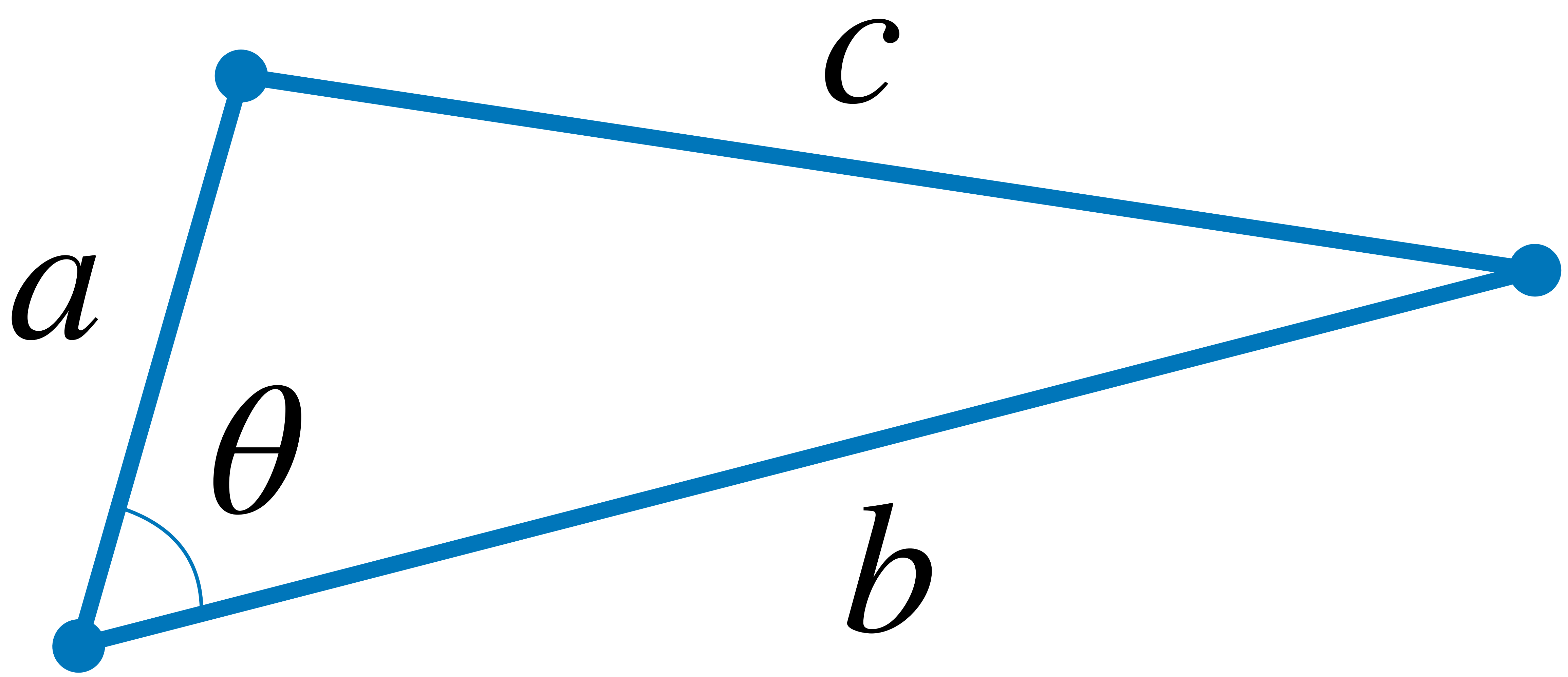
Generalized the Pythagorean Theorem

Recall: Law of Cosines



Theorem.

0 exactly when $\theta = 90^\circ$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Generalized the Pythagorean Theorem

Recall: Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

$$\cos\theta = \left\langle \frac{\mathbf{u}}{||\mathbf{u}||}, \frac{\mathbf{v}}{||\mathbf{v}||} \right\rangle$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations

Recall: Orthogonality

Definition. Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Orthogonal and perpendicular are the same thing.

With inner product we can...

- Given a vector we can determine its length
- Given two points (vectors) we can determine the distance between them
- Given two vectors we can determine the angle between them

Orthogonal Sets

Orthogonal Sets

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ of vectors from R^n is an **orthogonal set** if every pair of distinct vectors is orthogonal: if $i \neq j$ then

$$\langle u_i, u_j \rangle = 0$$

Each vector is pairwise/mutually perpendicular

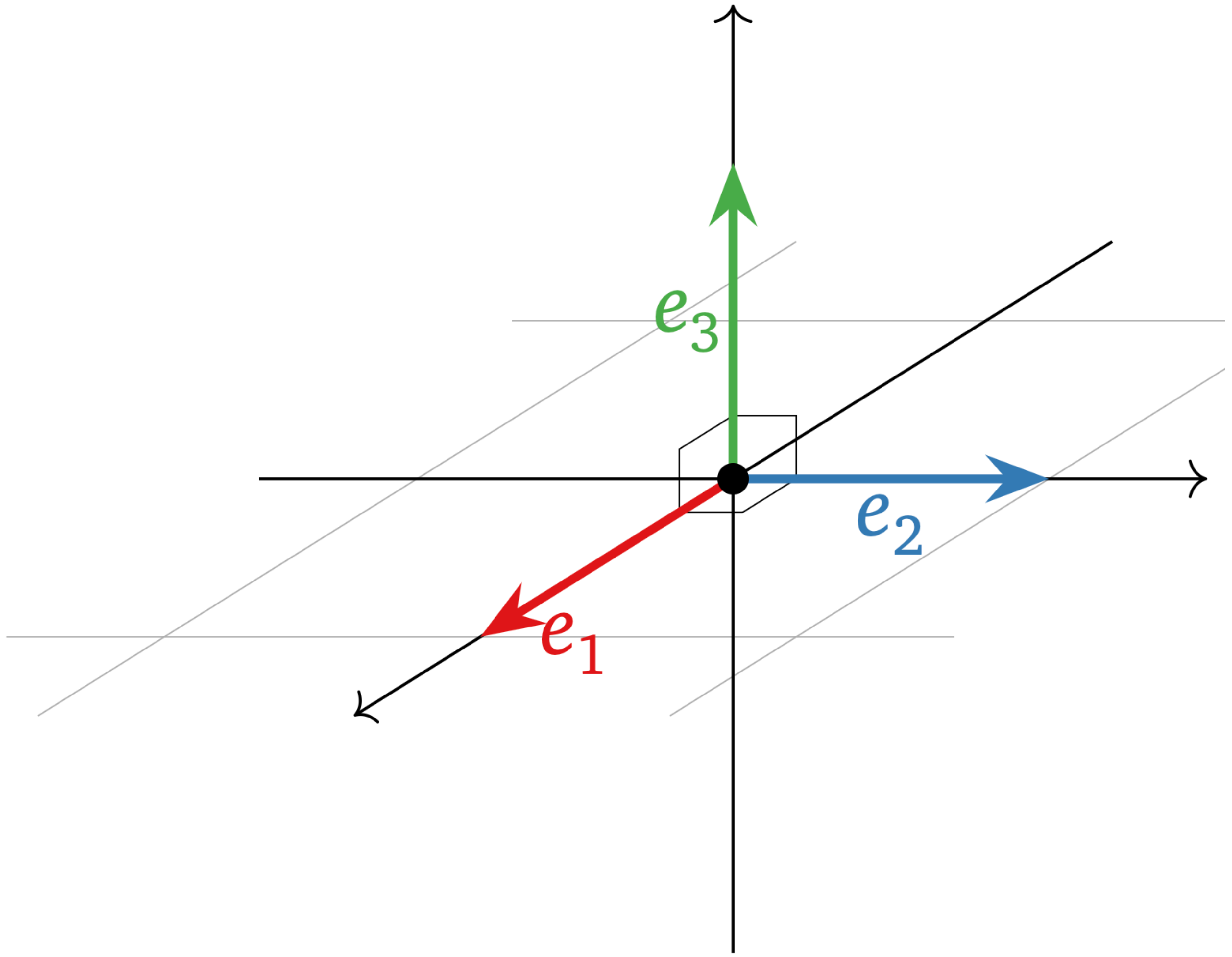
Example

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

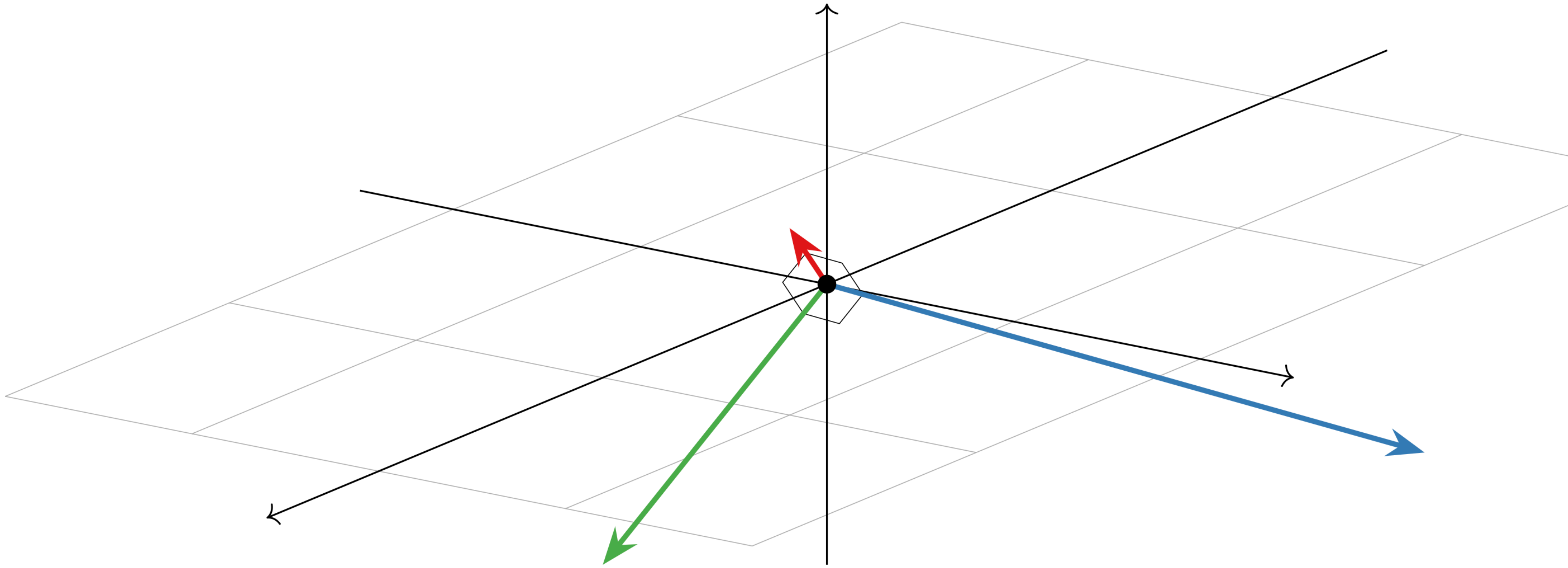
`Verify:`

What do orthogonal sets look like?

The Picture



the standard basis forms a
"centered" orthogonal set



an orthogonal set is like
the standard basis *after*
some rotations and scalings

https://textbooks.math.gatech.edu/ila/orthogonal-sets.html

Orthogonal Sets and Independence

Theorem. If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set of *nonzero* vectors from R^n, then it is linearly independent

Verify:

The Takeaway

If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set, then it is a **basis** for $span\{u_1, u_2, \ldots, u_k\}$

Orthogonal Basis

Definition. An **orthogonal basis** for a subspace W of R^n is a basis for W which is also an orthogonal set.

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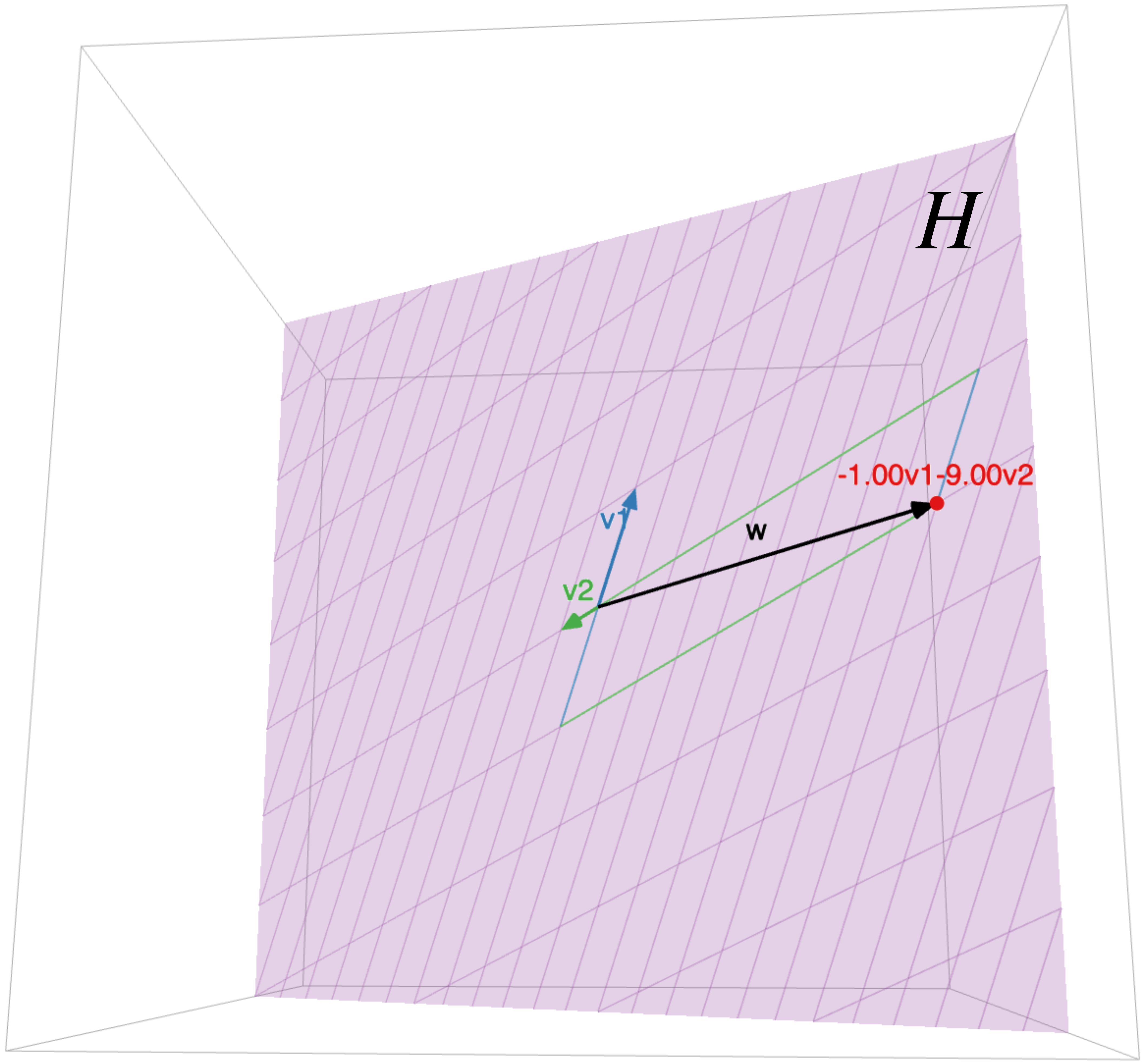

v_1 and v_2 form a basis of H

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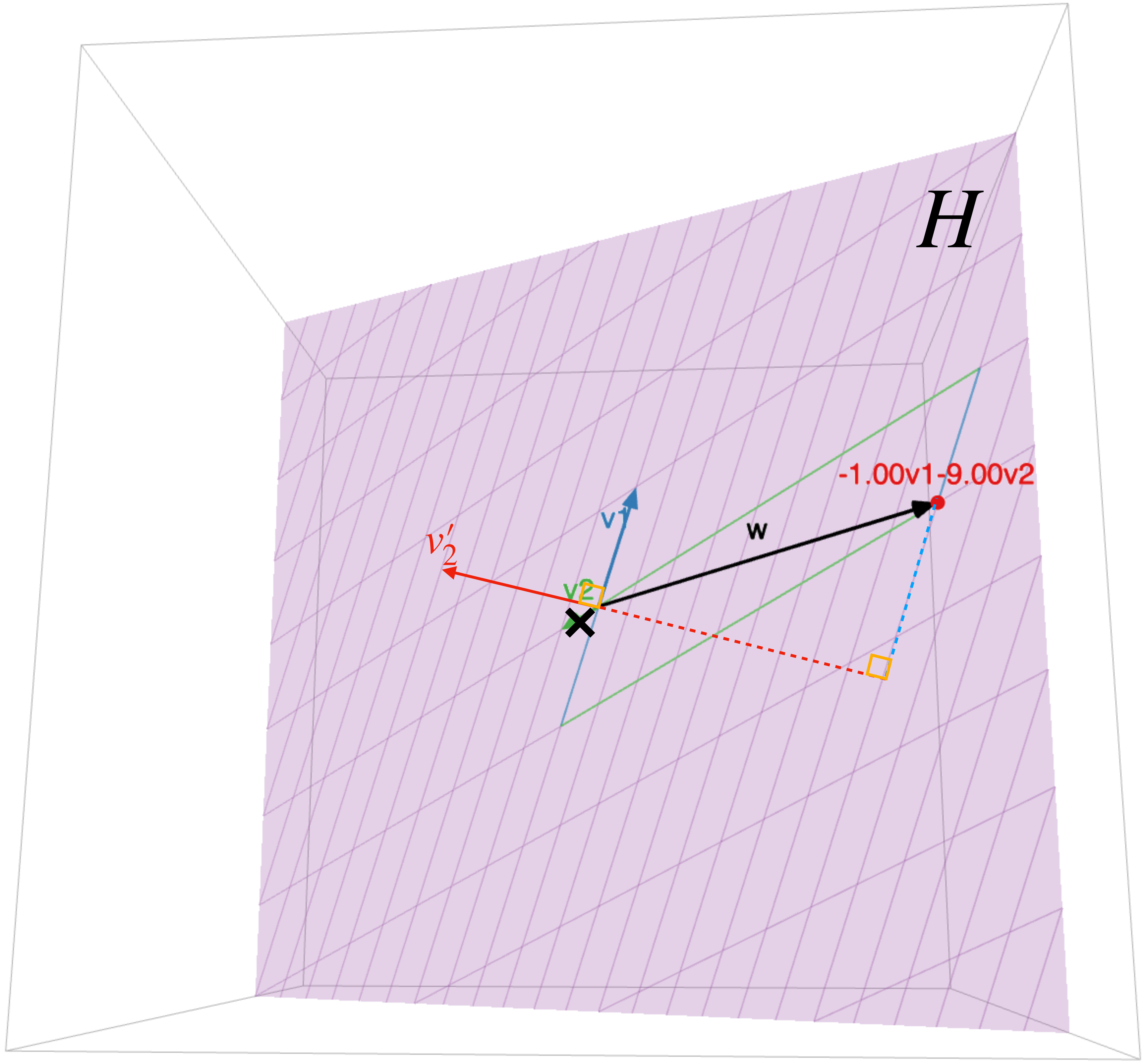

v_1 and v_2 form a basis of H

v_1 and v_2' form an **orthogonal** basis of H

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What's nice about an orthogonal basis?

Recall: How To: Bases

Recall: How To: Bases

Question. Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

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Solution. Solve the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_p\mathbf{u}_p = \mathbf{w}$$

by Gaussian elimination, matrix inversion, etc.

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by Gaussian elimination, matrix inversion, etc.

This takes work

Orthogonal Bases and Linear Combinations

Theorem. For an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, if $\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ then for $j = 1,\ldots,p$

$$c_j = \frac{\mathbf{y}^T\mathbf{u}_j}{\mathbf{u}_j^T\mathbf{u}_j}$$

Verify:

How To: Orthogonal Bases

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Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

How To: Orthogonal Bases

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Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

Much easier to compute.

Question

Express $[6 \;\; 1 \;\; (-8)]^T$ *as a linear combination of vectors in* $\{u_1, u_2, u_3\}$ *where*

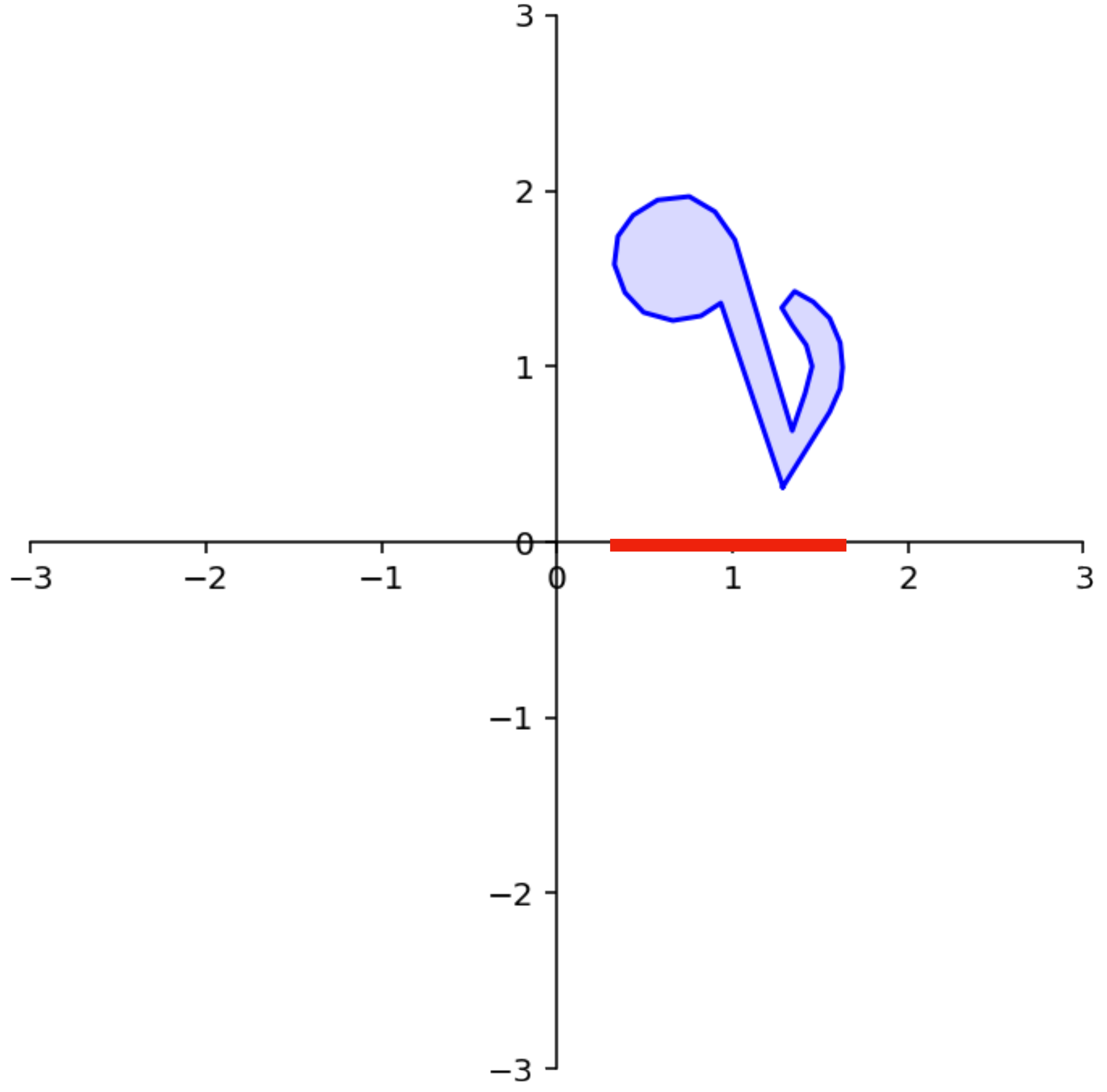
$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Answer: $u_1 - 2u_2 - 2u_3$

Orthogonal Projection

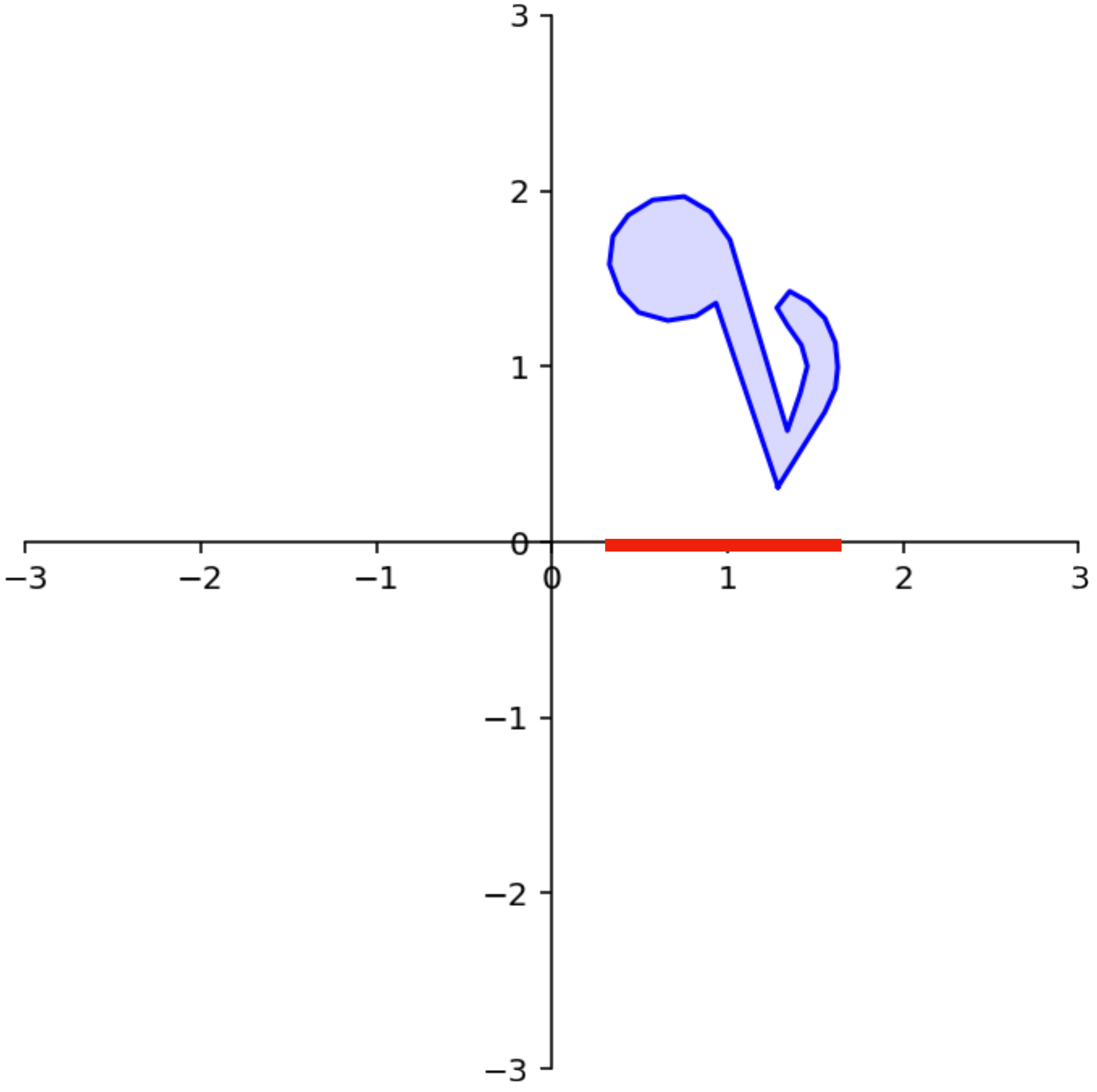
Why does that formula in the last example work?

Recall: Projection onto the x-axis



Recall: Projection onto the x-axis

We've seen simple projections in R^2



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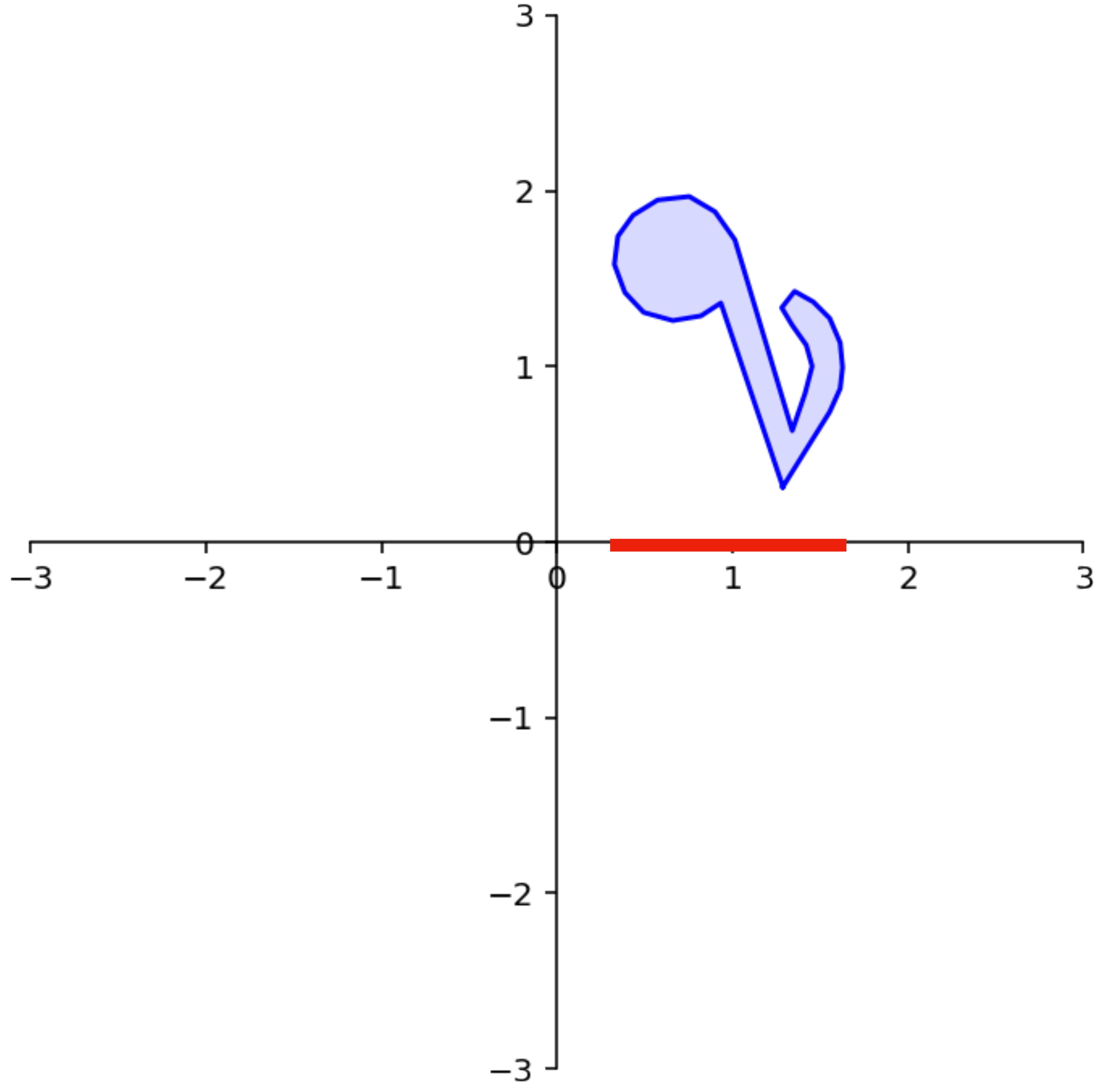

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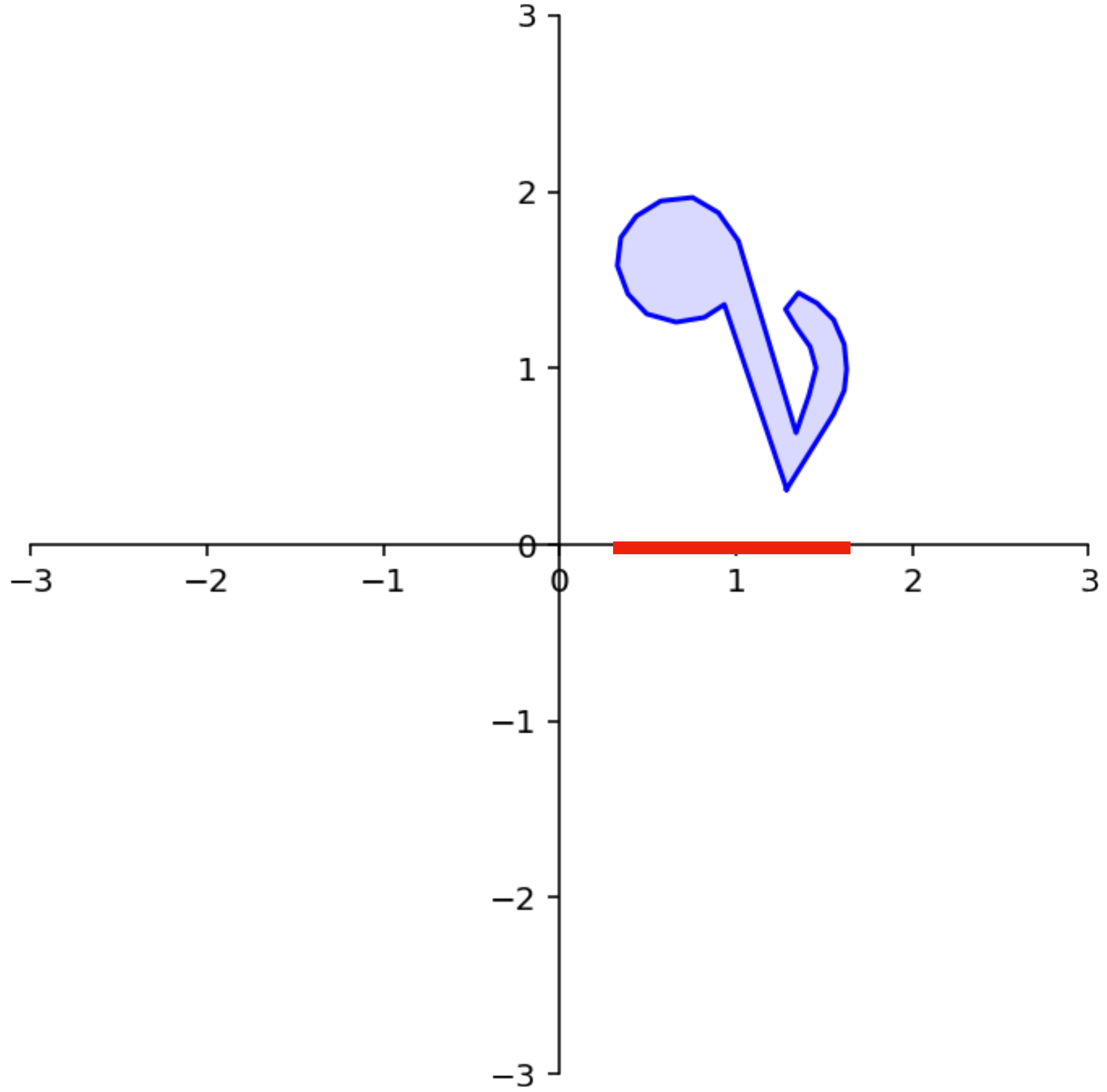
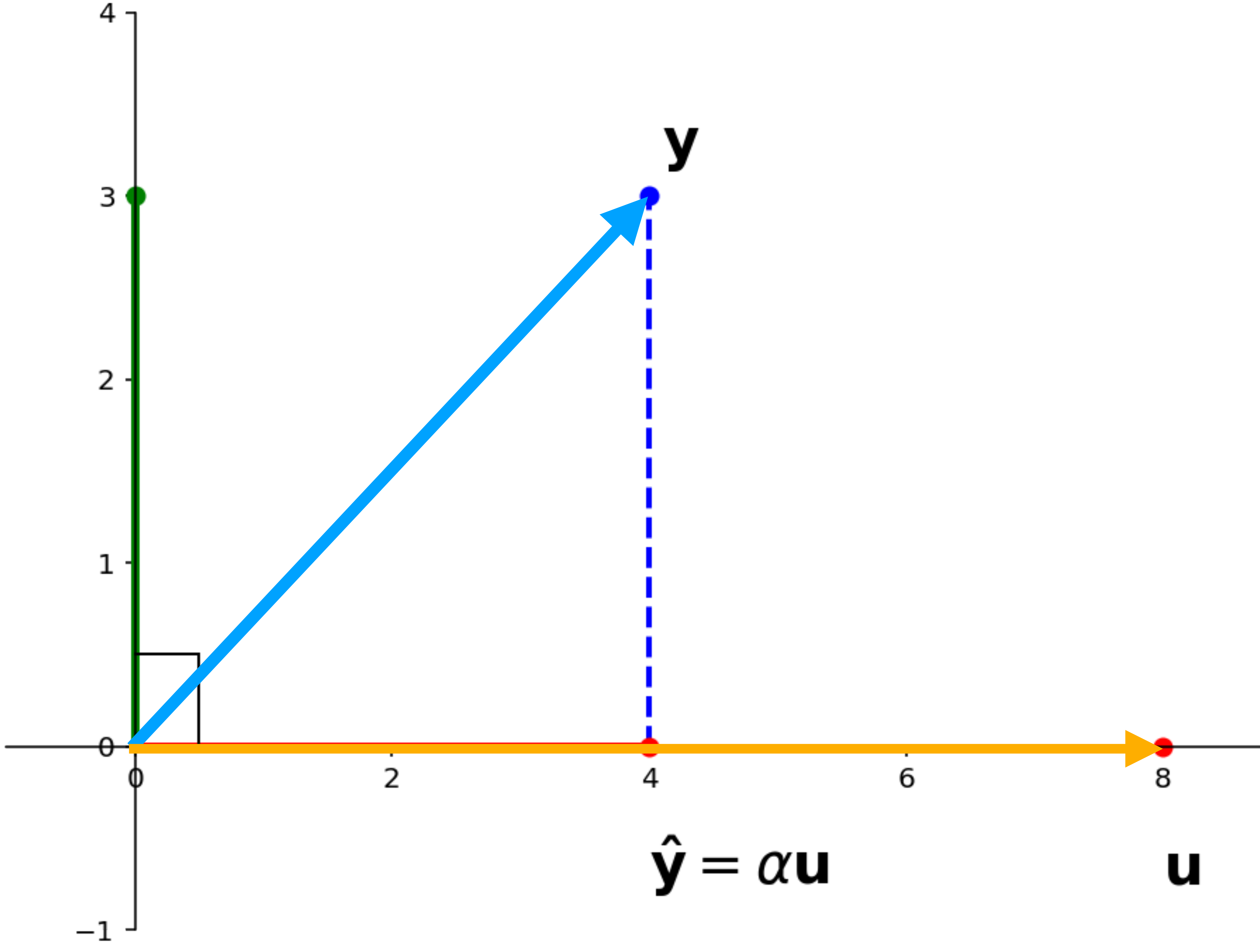
What we really did was a kind of projection onto the basis vectors

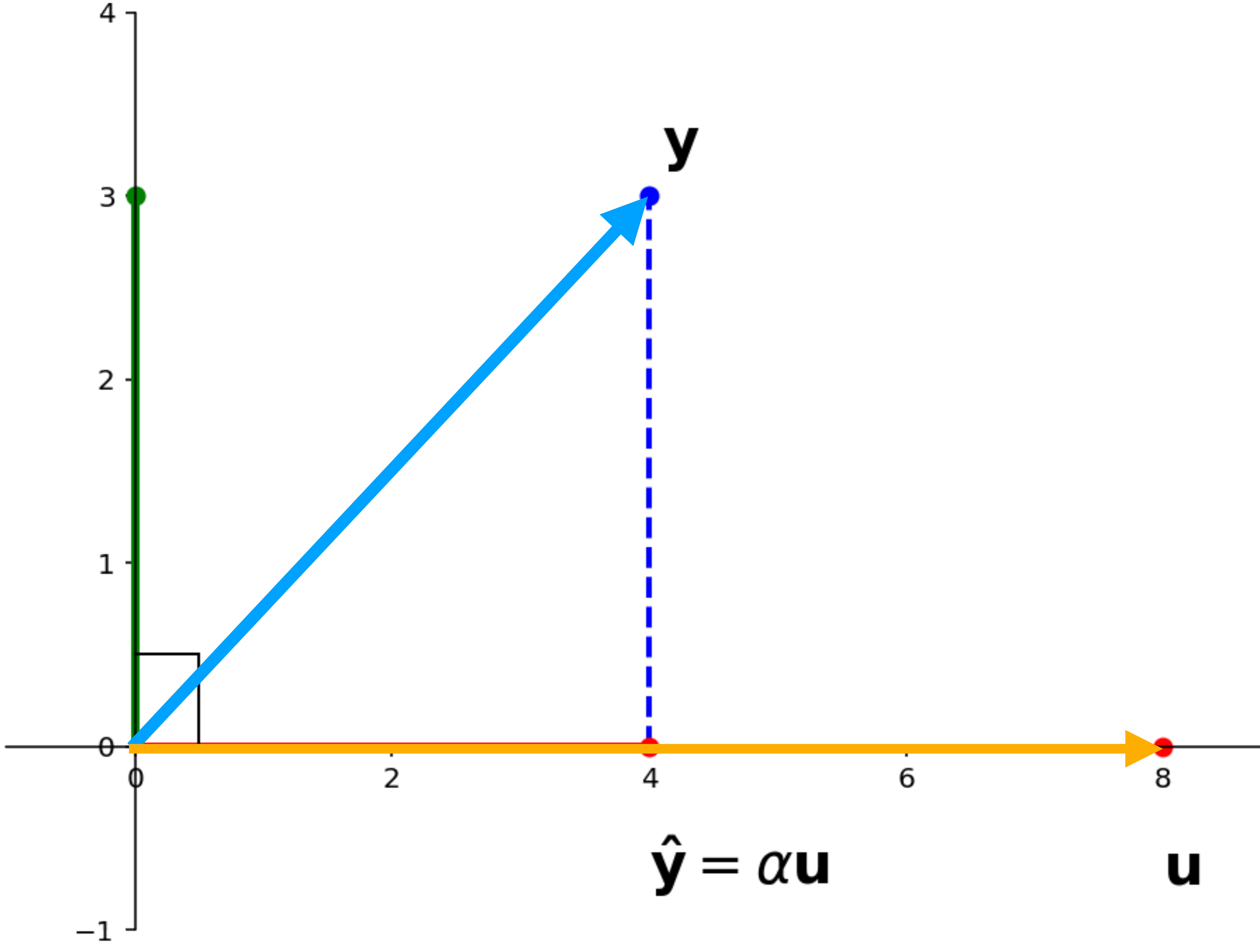


Orthogonal Projection



Orthogonal Projection

Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that



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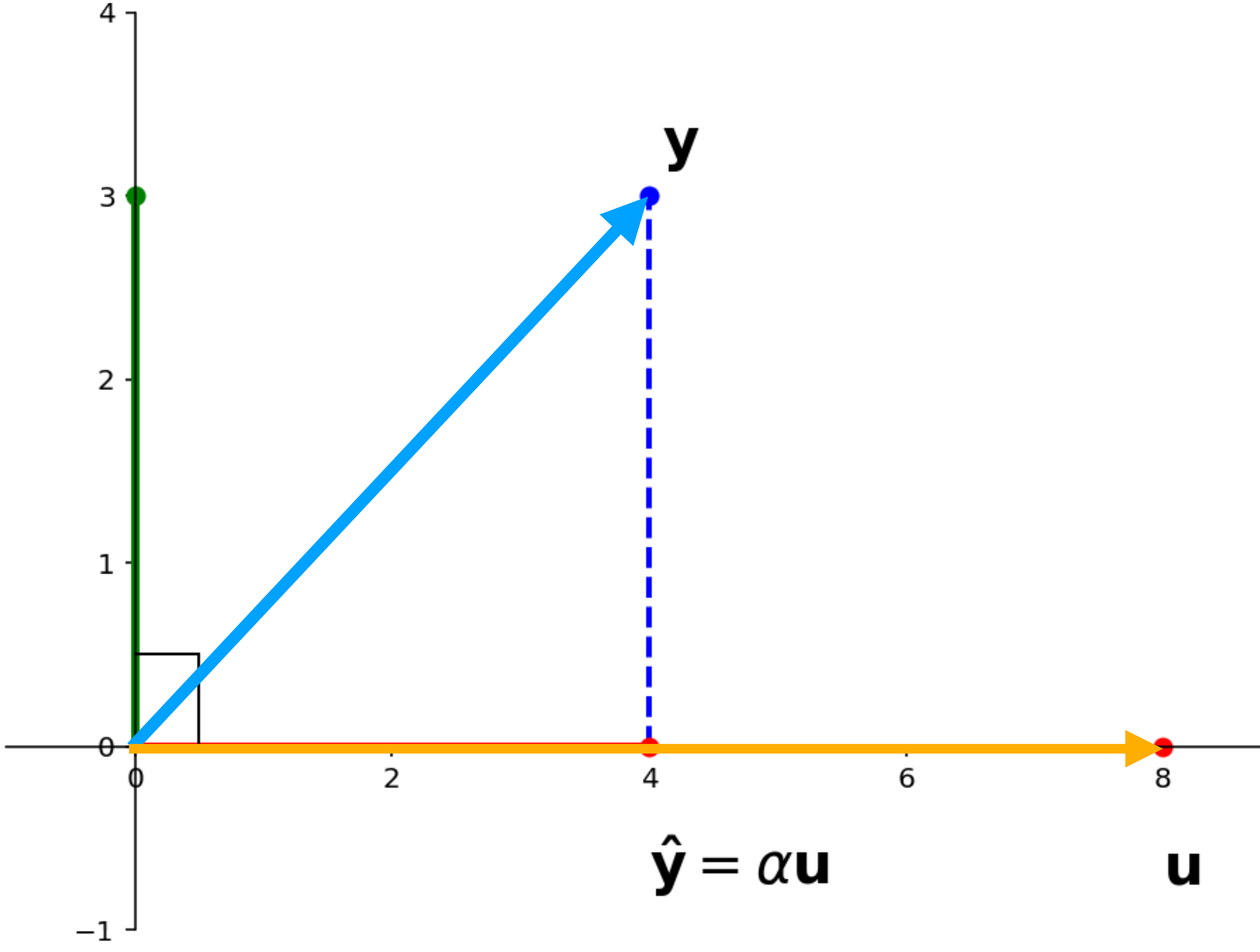
» $\mathbf{z}$ is orthogonal to $\mathbf{u}$ (i.e., $\mathbf{z} \cdot \mathbf{u} = 0$)



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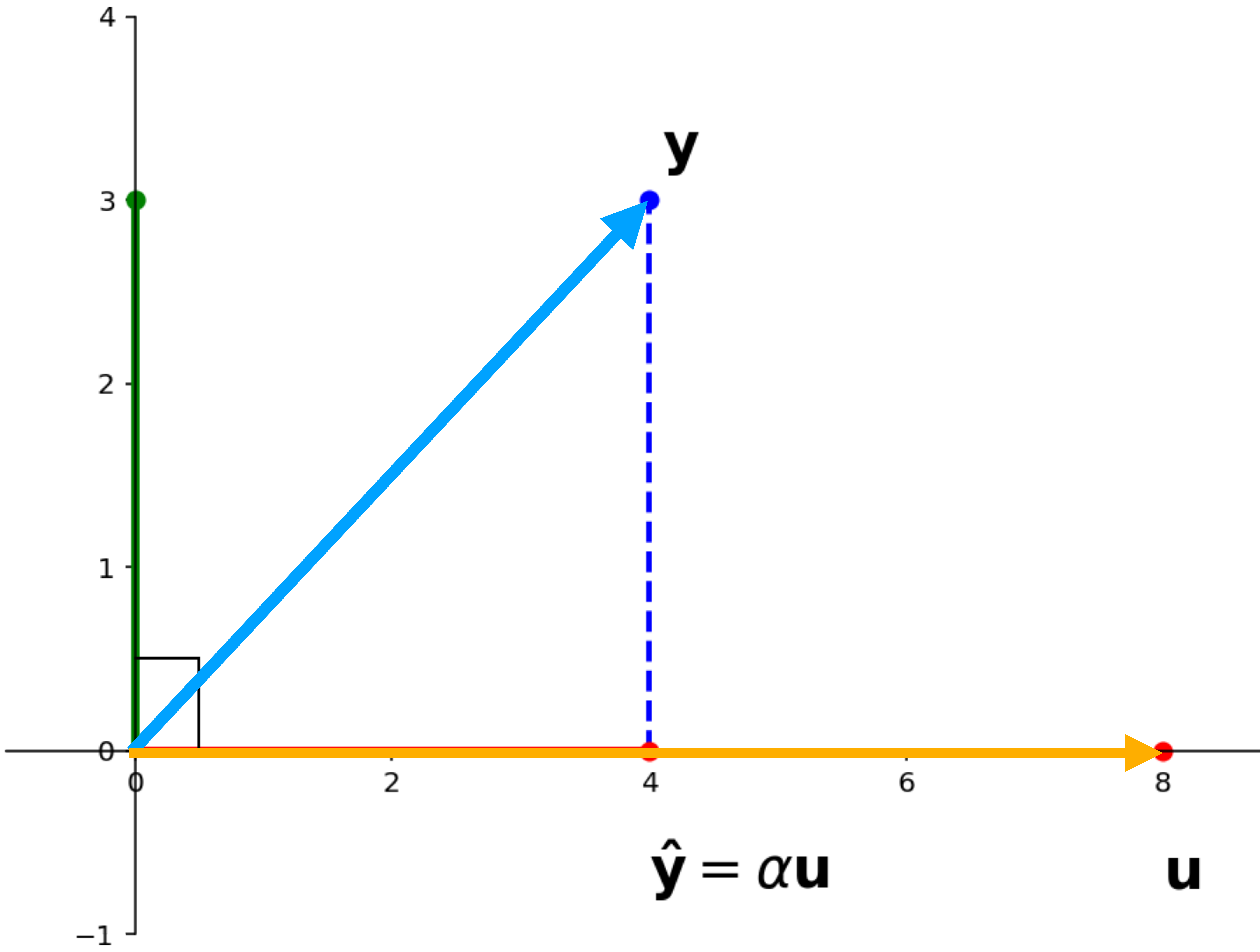
» $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$



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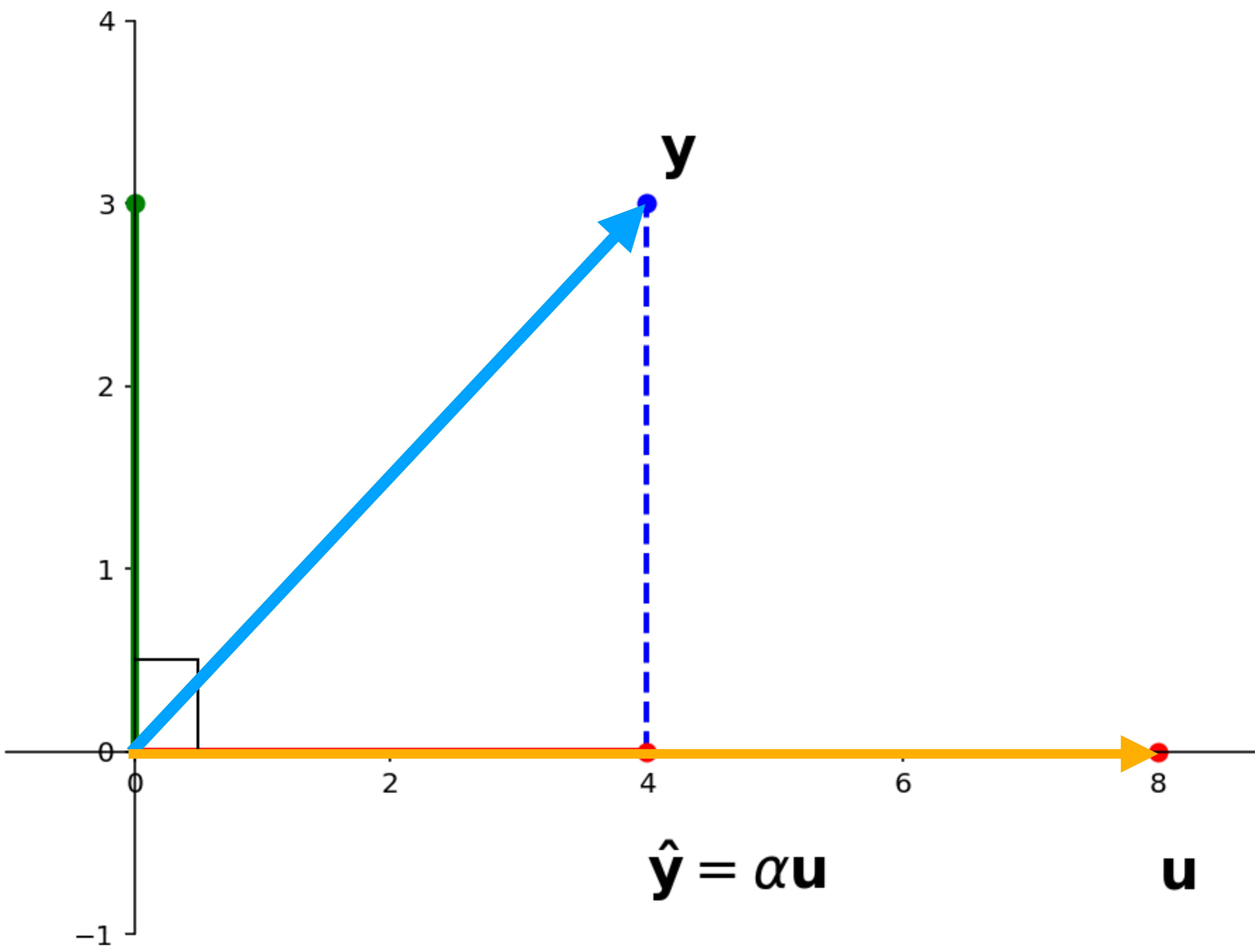
» $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$

» $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$



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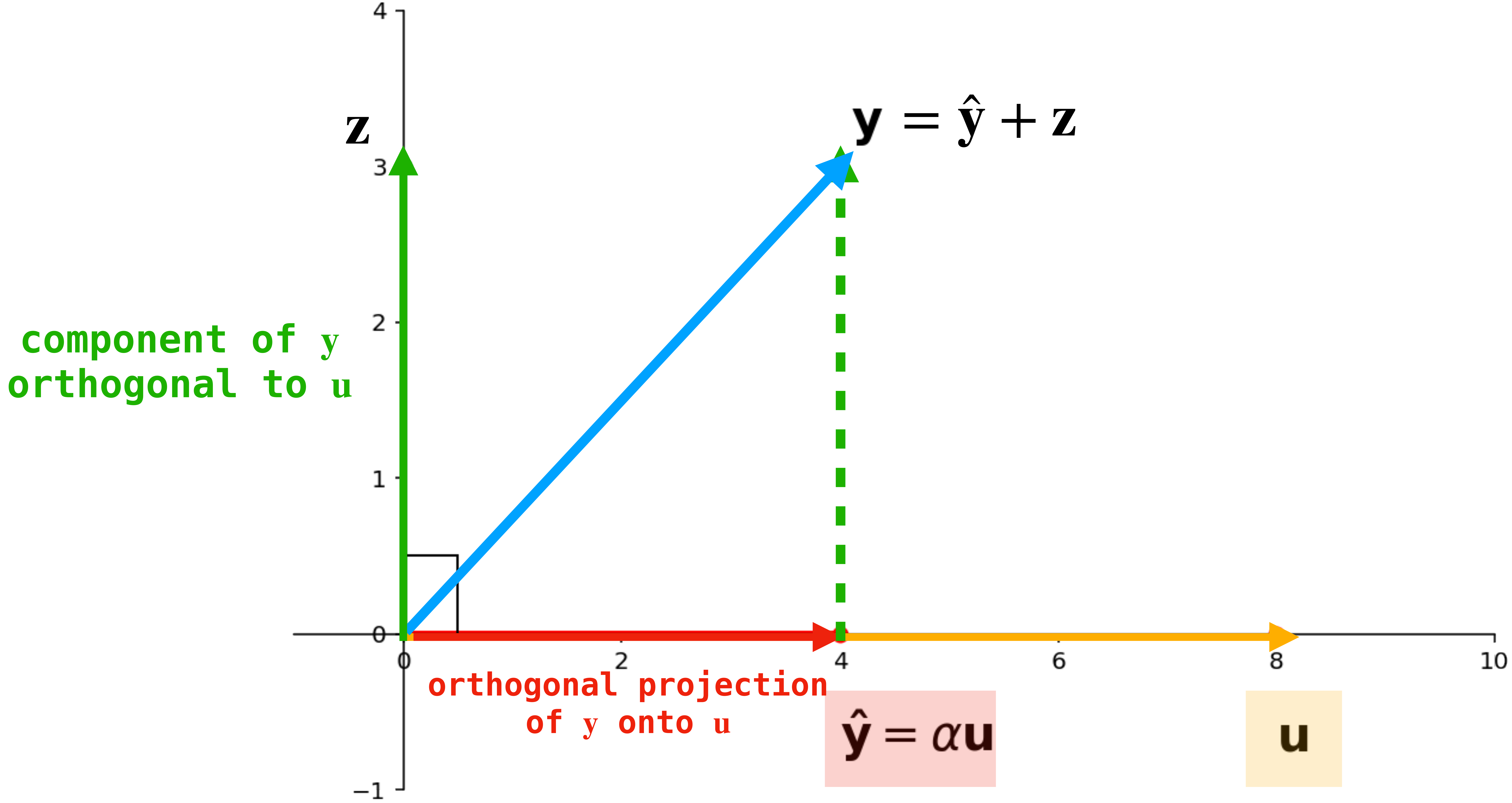

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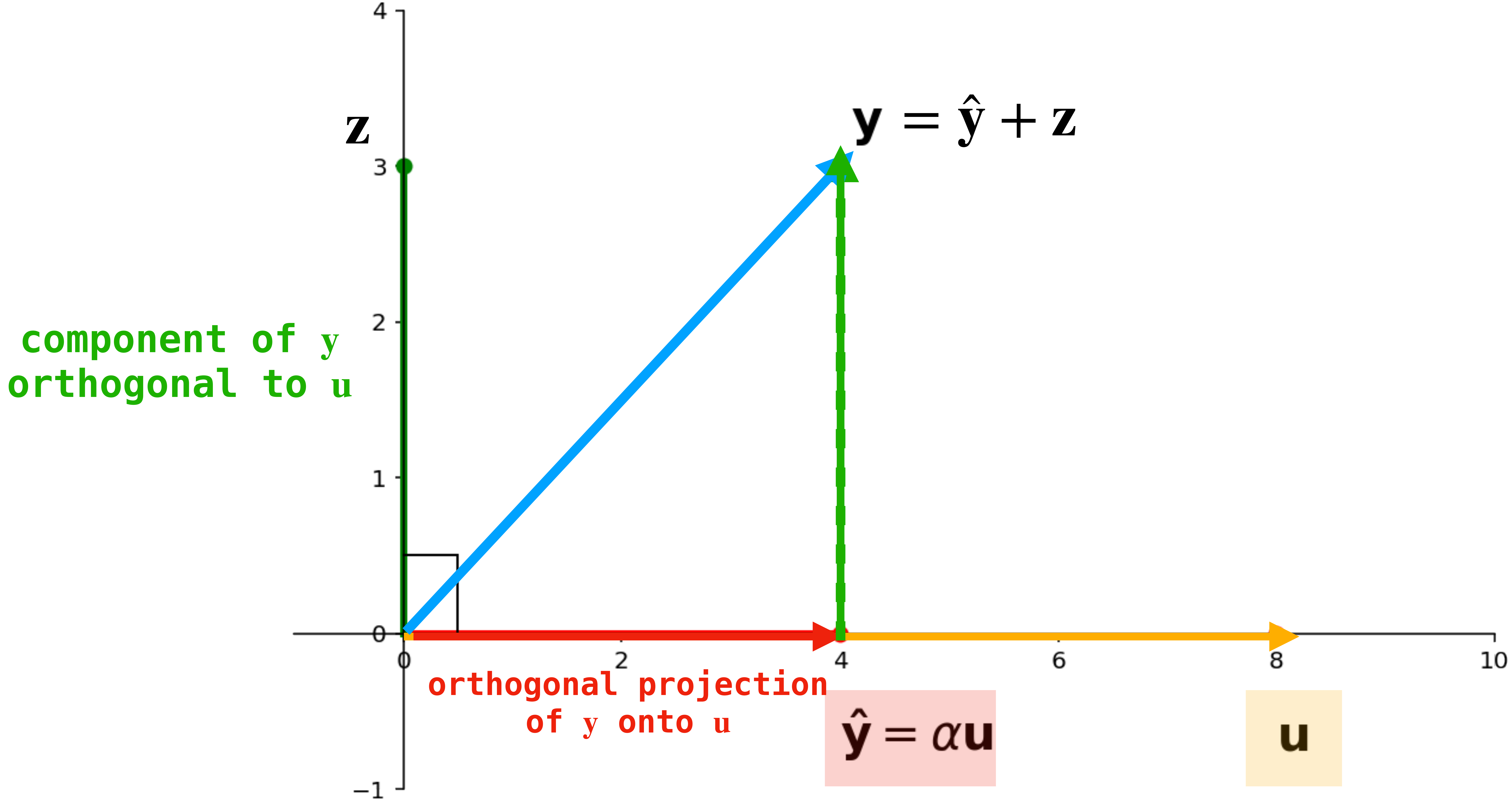
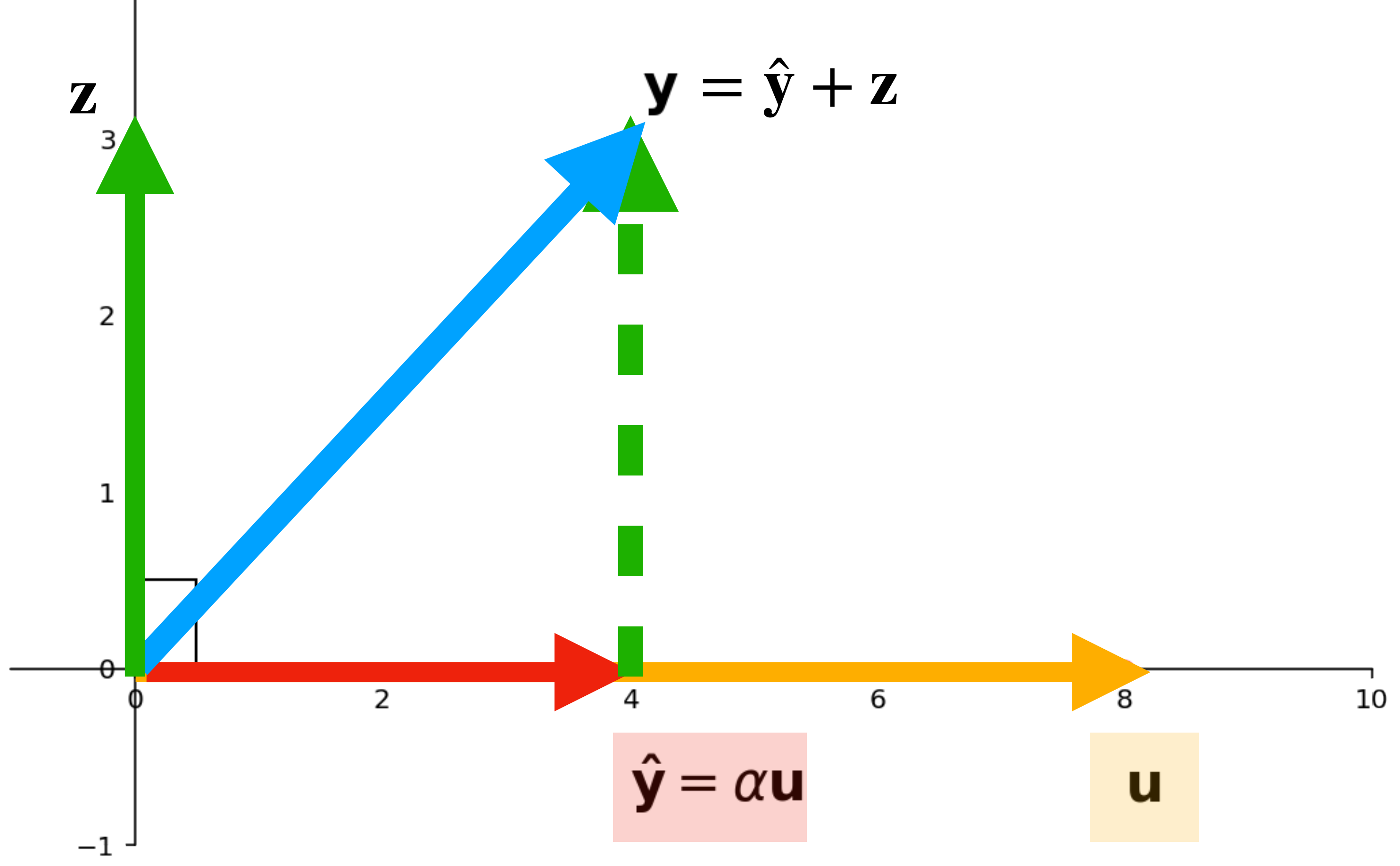


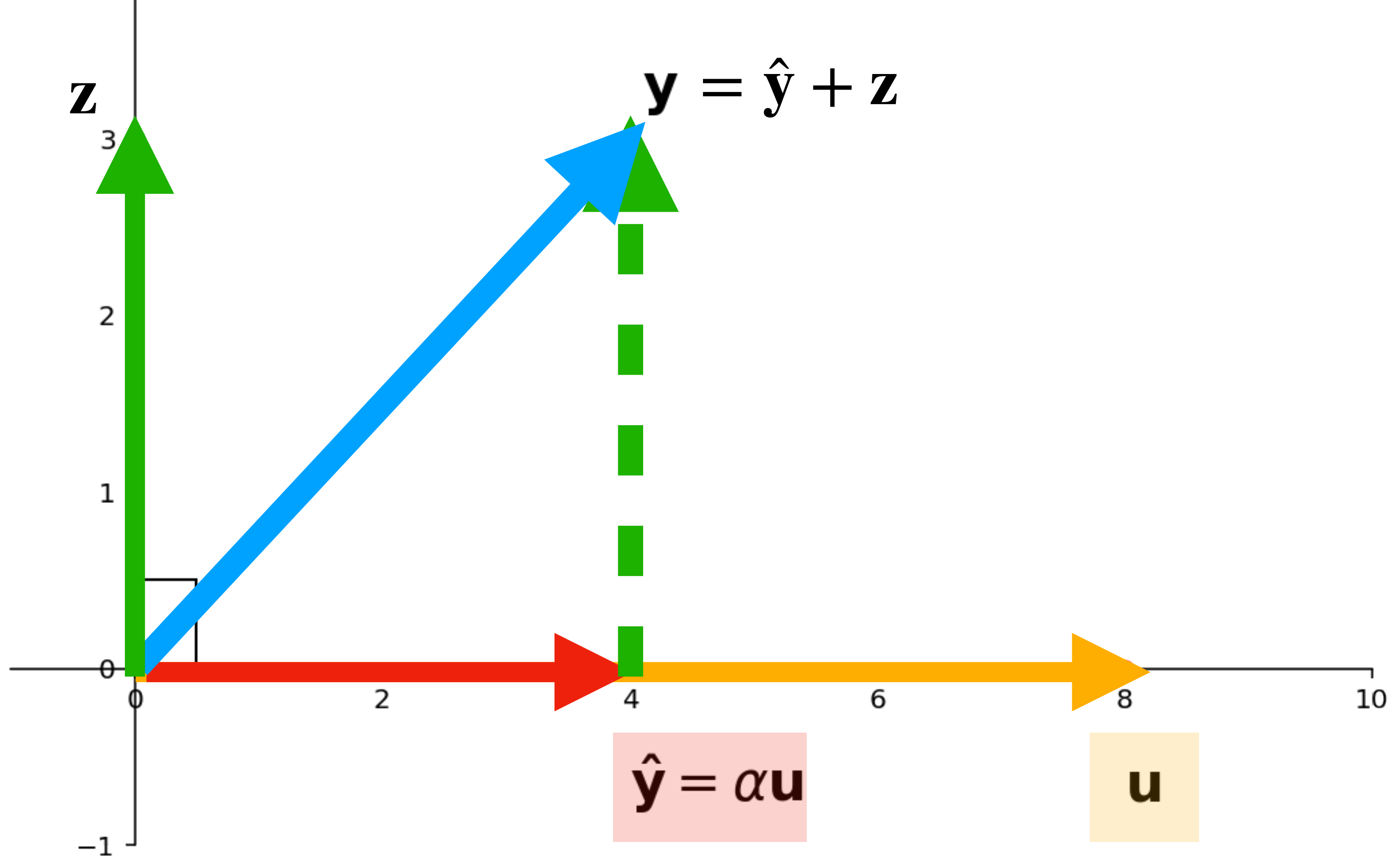
How do we find the orthogonal projection and orthogonal component?

What we know



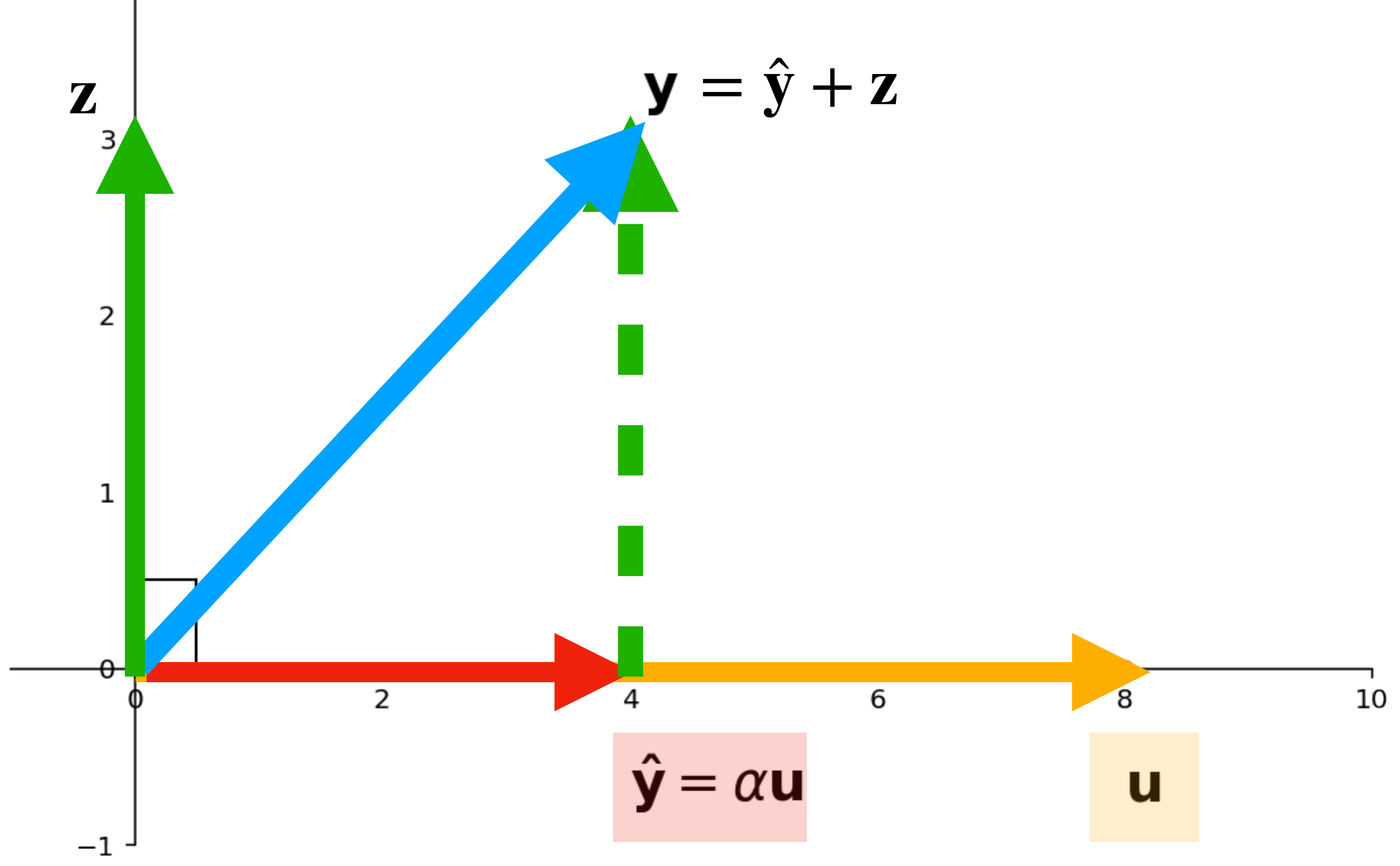
$\mathbf{z}$
$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$
3
2
1
0
−1
0
2
4
6
8
10
$\hat{\mathbf{y}} = \alpha\mathbf{u}$
$\mathbf{u}$

What we know



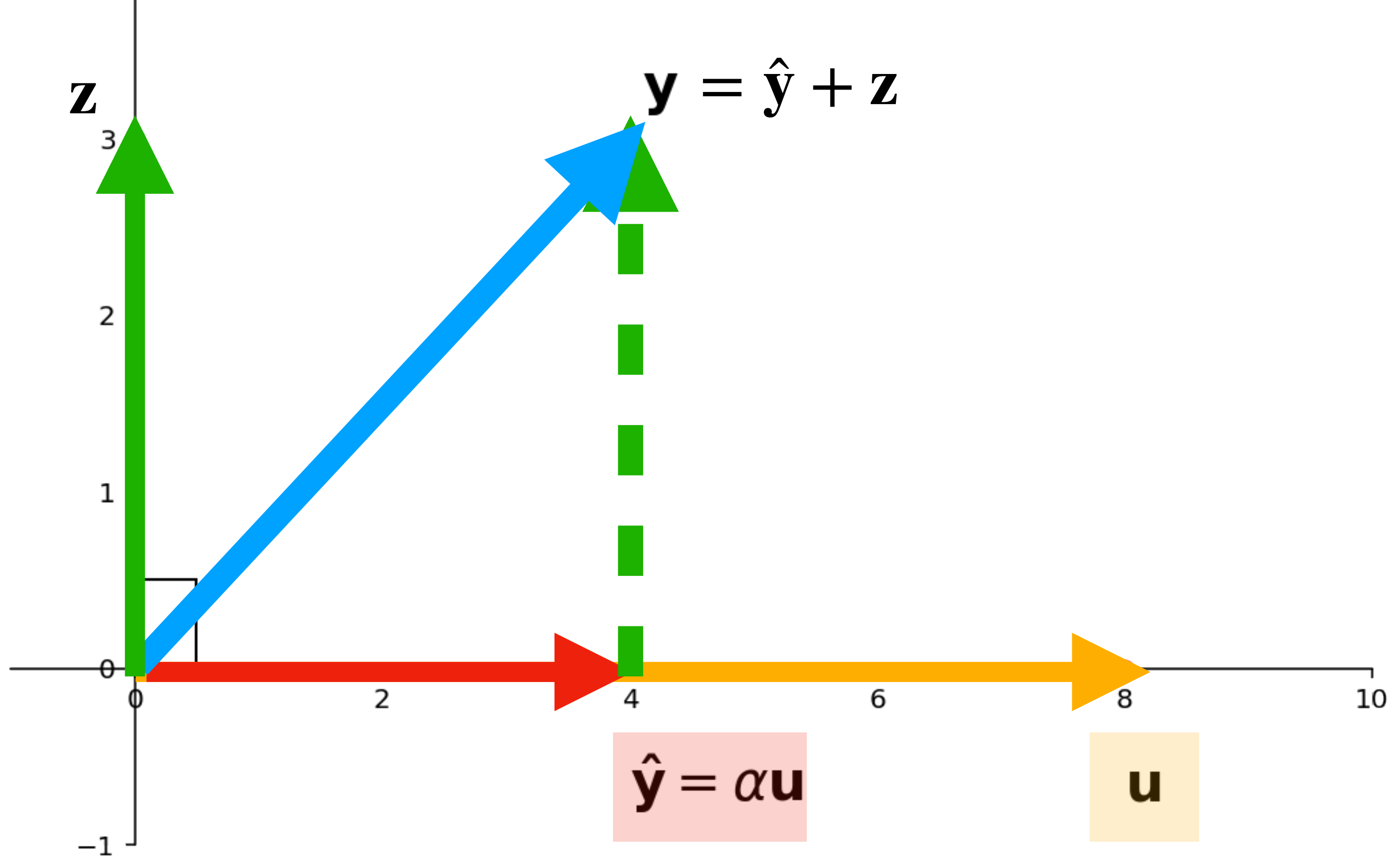
- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)

What we know



- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)

What we know



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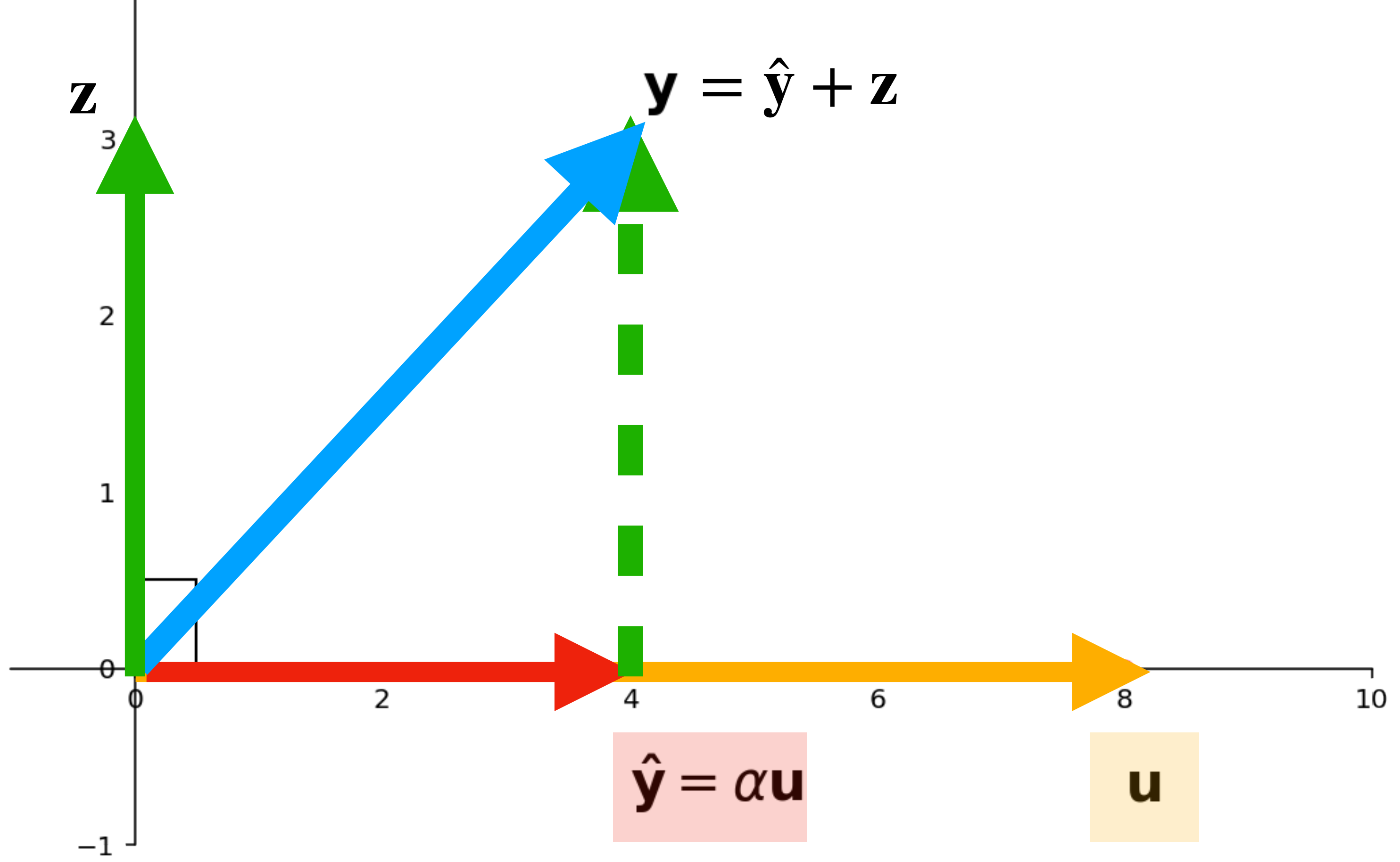
- $\langle \mathbf{z}, \mathbf{u} \rangle = 0$ (since $\mathbf{z}$ is orthogonal with $\mathbf{u}$)

What we know



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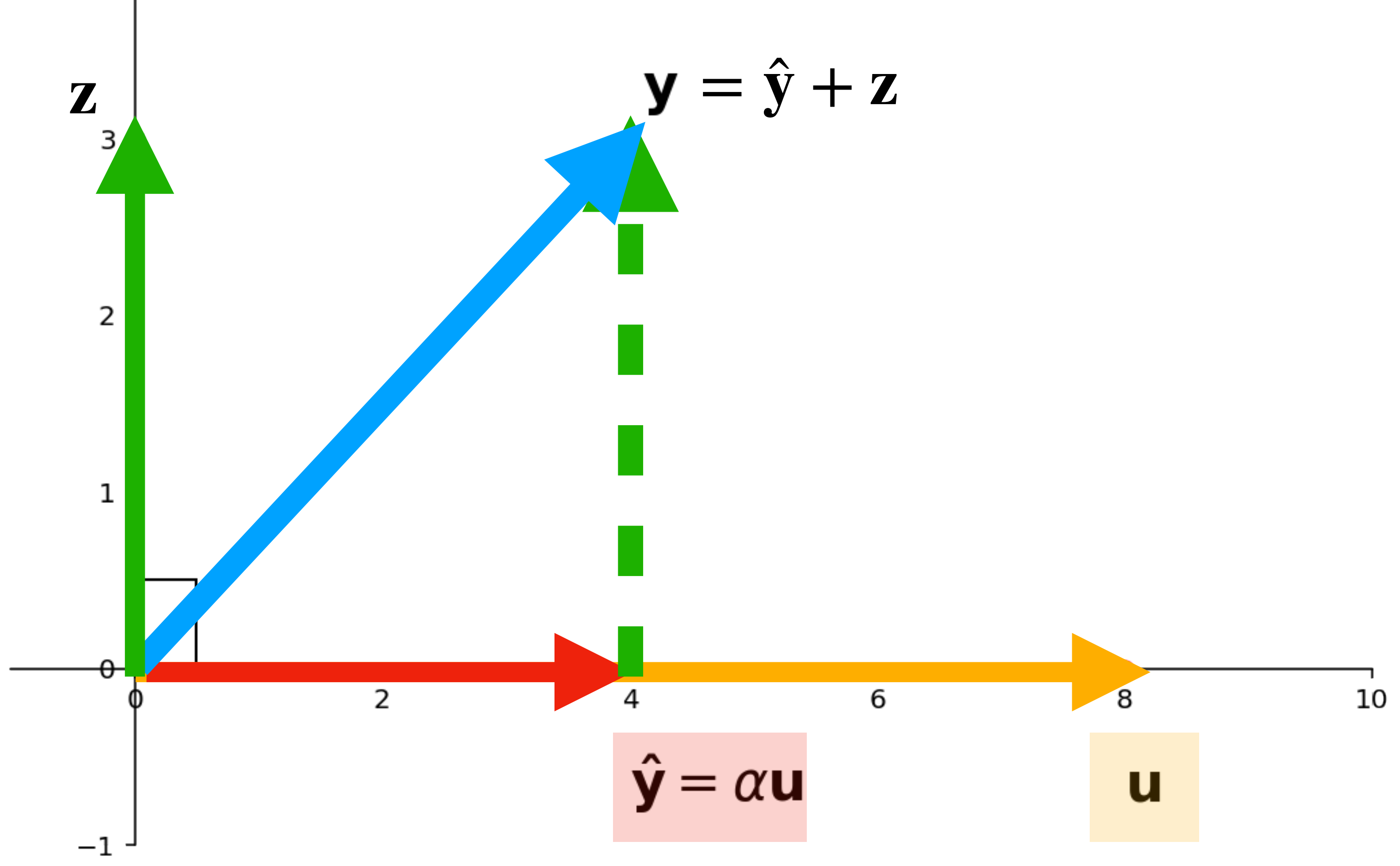
Therefore:

$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

What we know



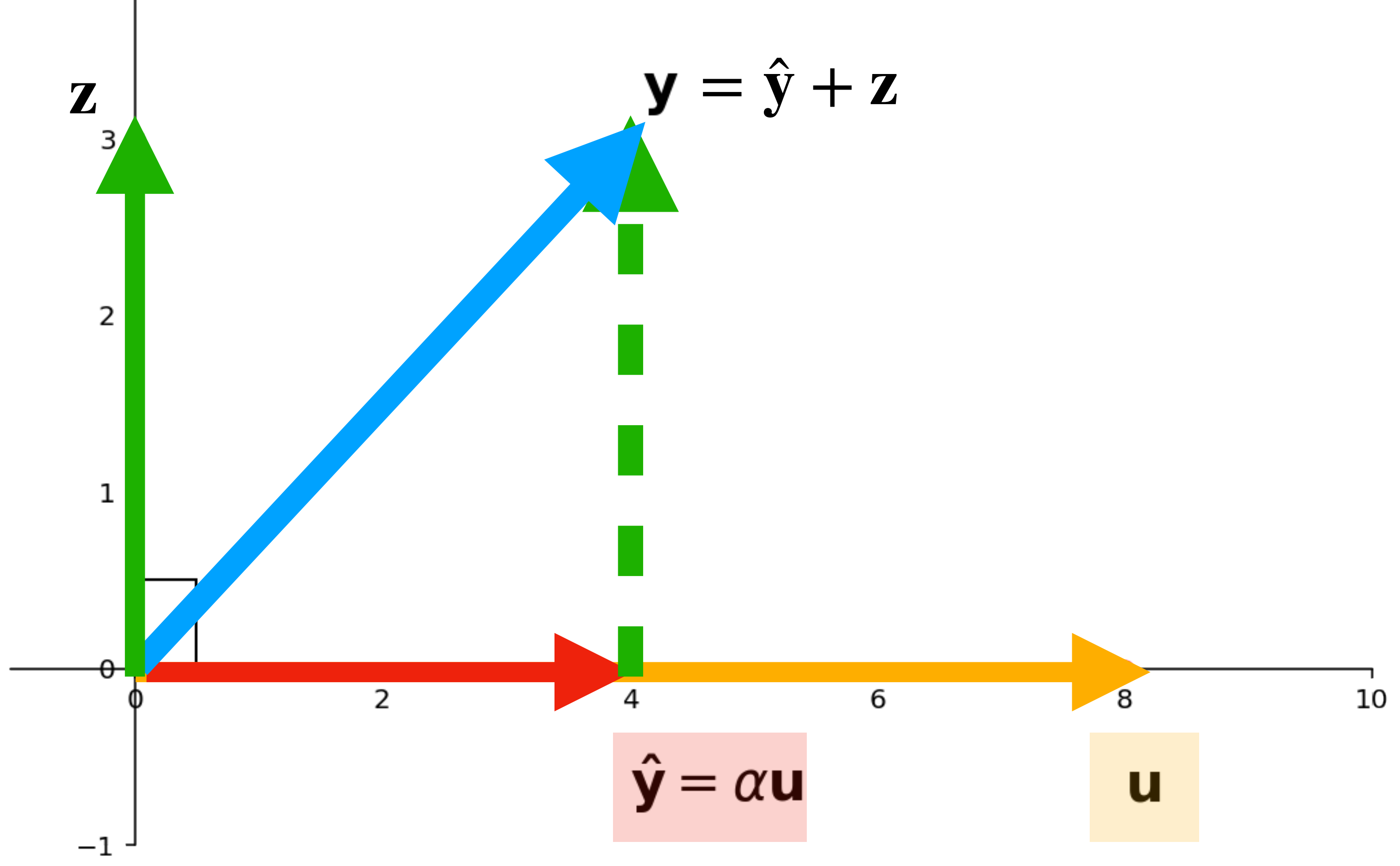
- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)
- $\langle \mathbf{z}, \mathbf{u} \rangle = 0$ (since $\mathbf{z}$ is orthogonal with $\mathbf{u}$)

Therefore:

$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

Once we have α, we can compute both $\hat{\mathbf{y}}$ and $\mathbf{z}$

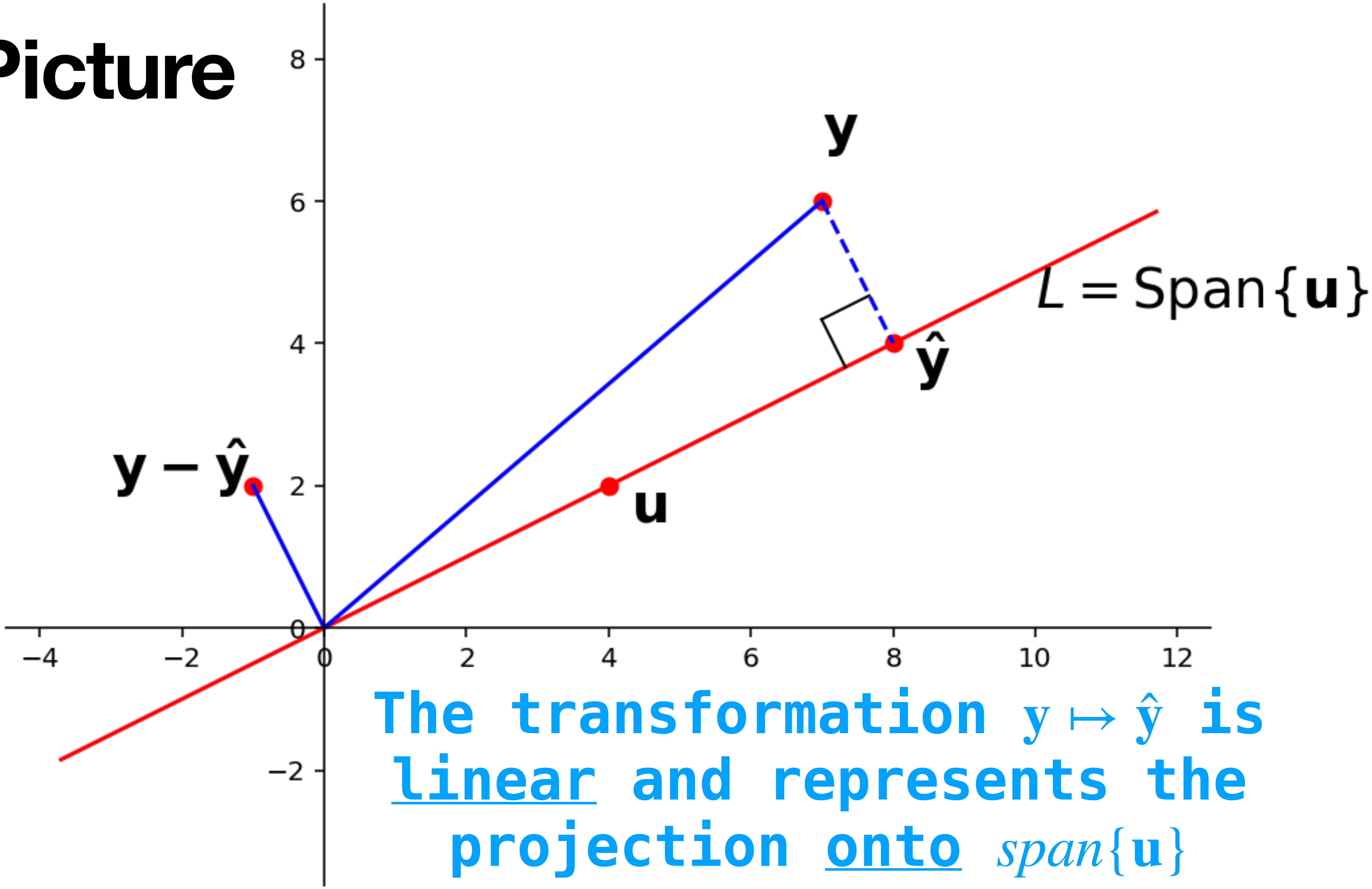
Step 1: Finding α



$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

Let's solve for α, $\hat{\mathbf{y}}$ and $\mathbf{z}$:

The Picture



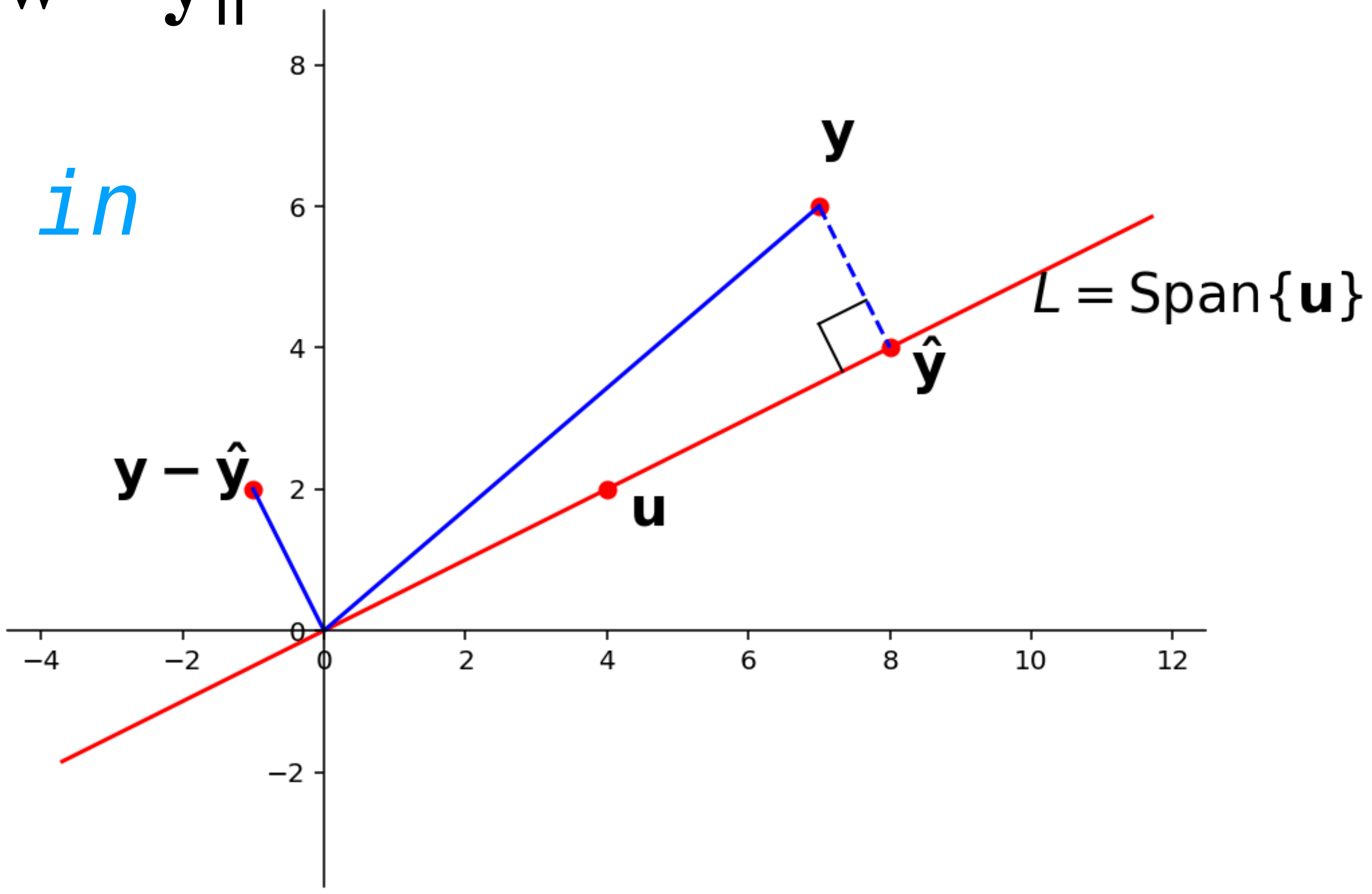
The transformation $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is <u>linear</u> and represents the projection <u>onto</u> $span\{\mathbf{u}\}$

$\hat{\mathbf{y}}$ and Distance

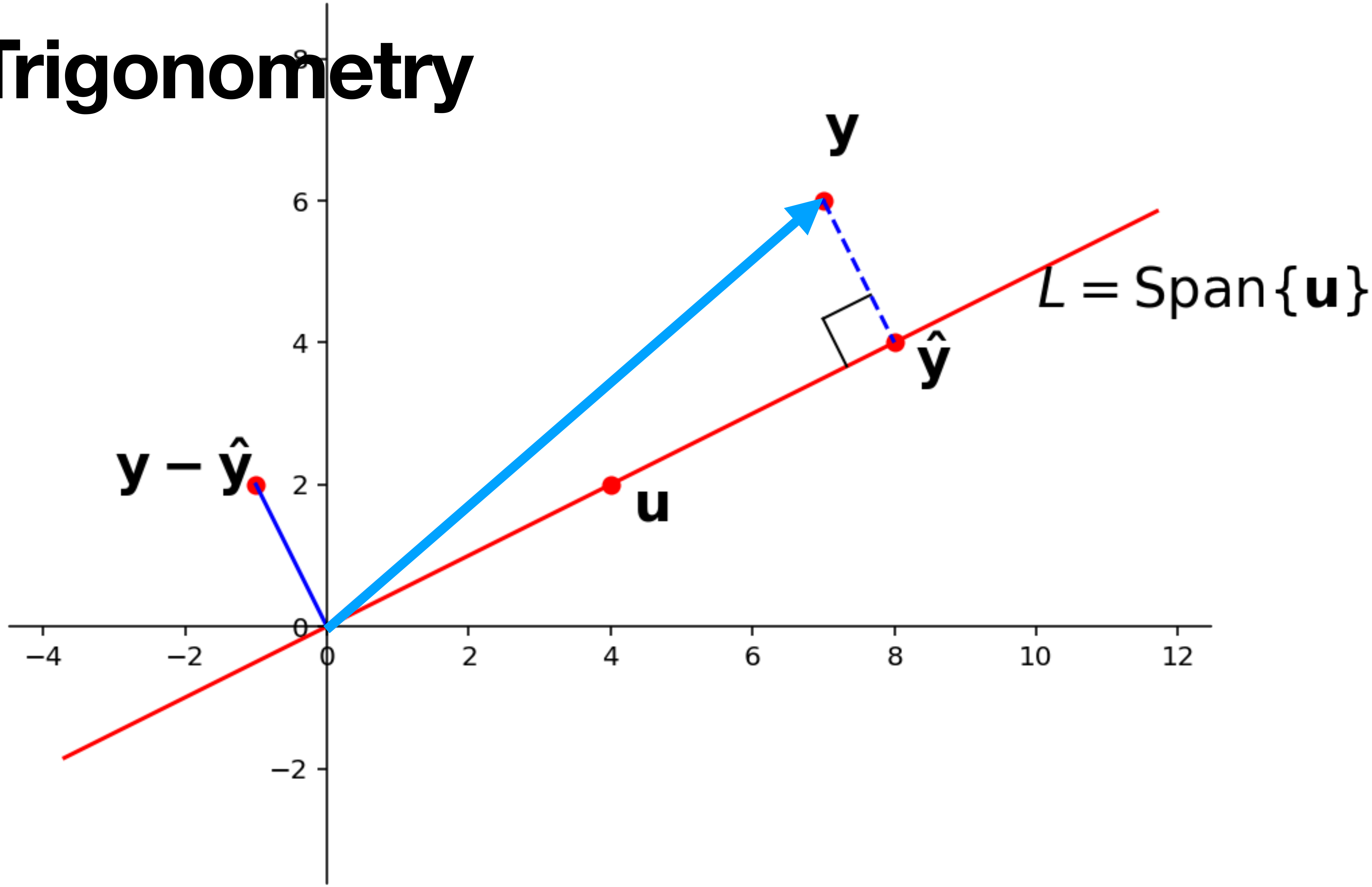
Theorem. $\|\hat{\mathbf{y}} - \mathbf{y}\| = \min_{\mathbf{w} \in span\{\mathbf{u}\}} \|\mathbf{w} - \mathbf{y}\|$

$\hat{\mathbf{y}}$ is the closest vector in $span\{\mathbf{u}\}$ to $\mathbf{y}$.

"Proof" by inspection:



The Trigonometry



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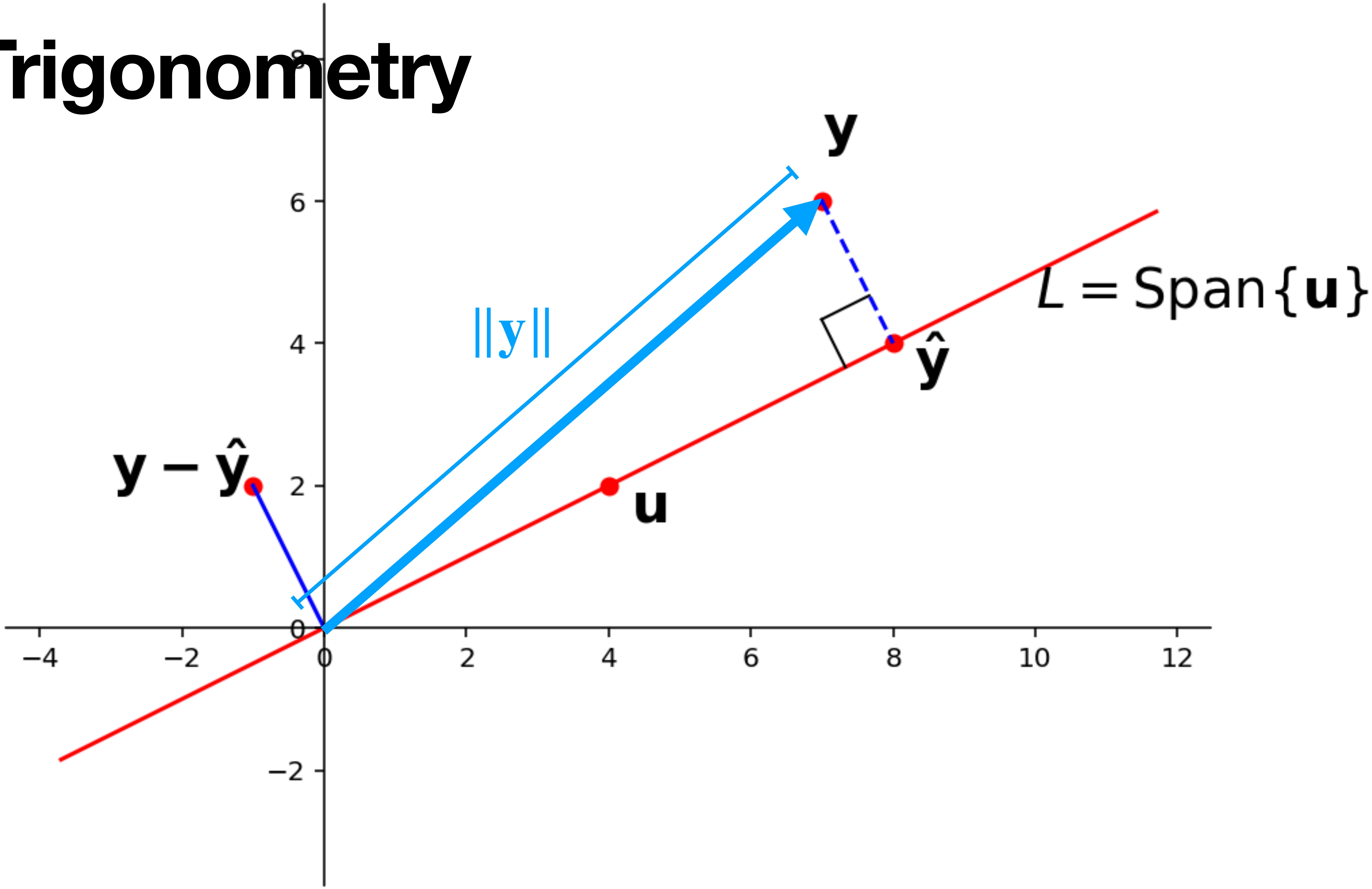
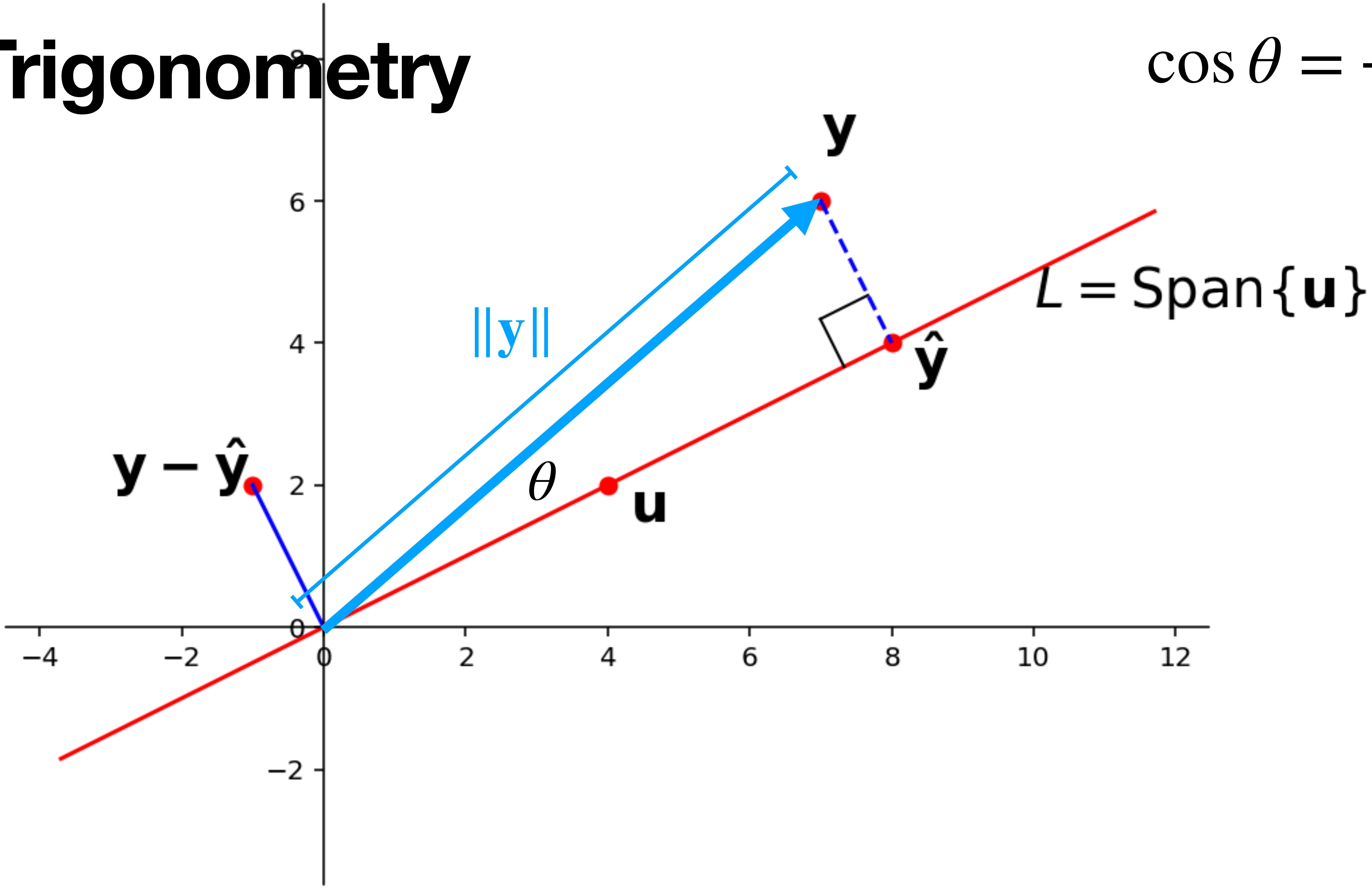


The Trigonometry

$$\cos\theta = \frac{\|\hat{\mathbf{y}}\|}{\|\mathbf{y}\|}$$



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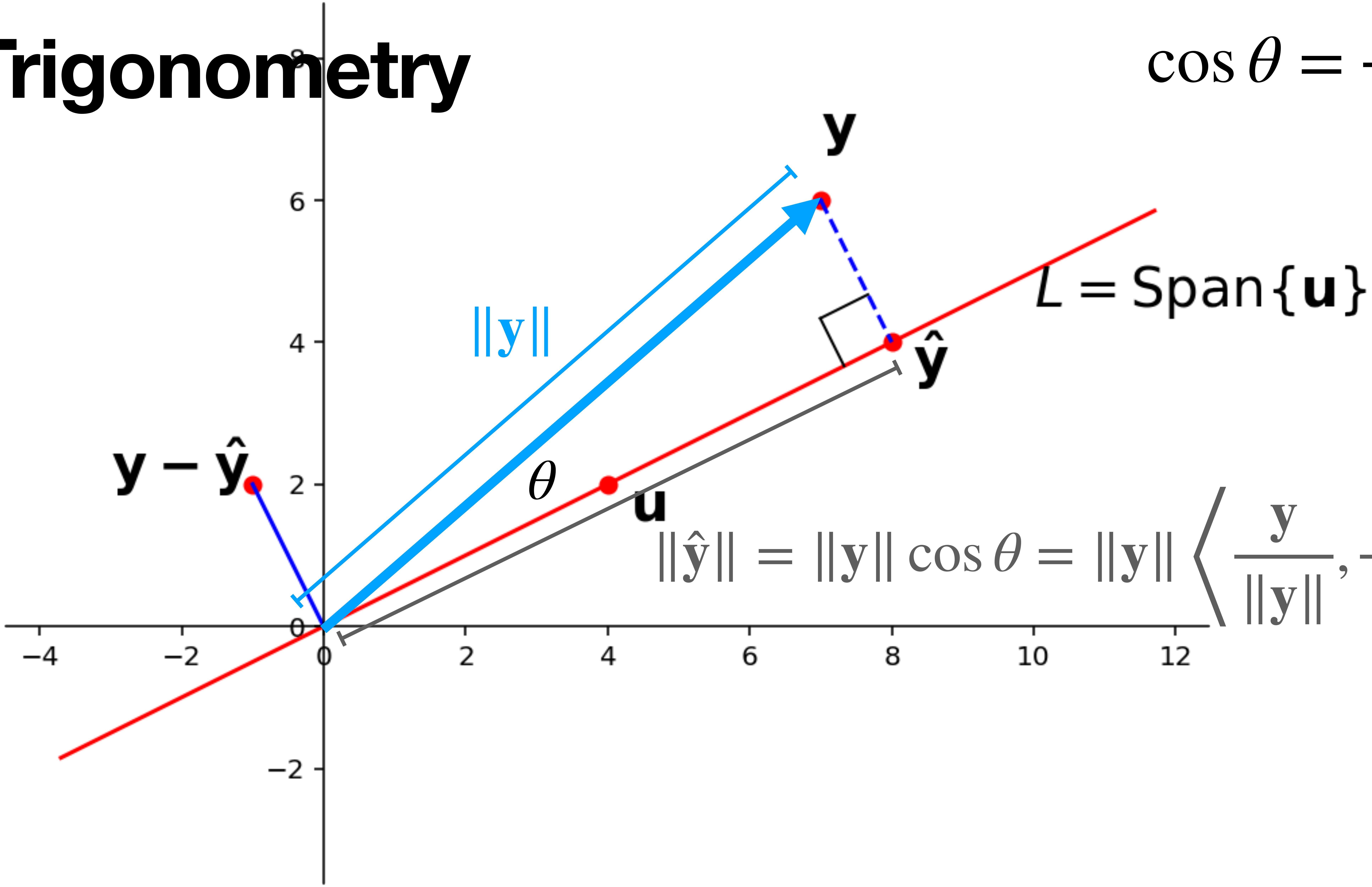

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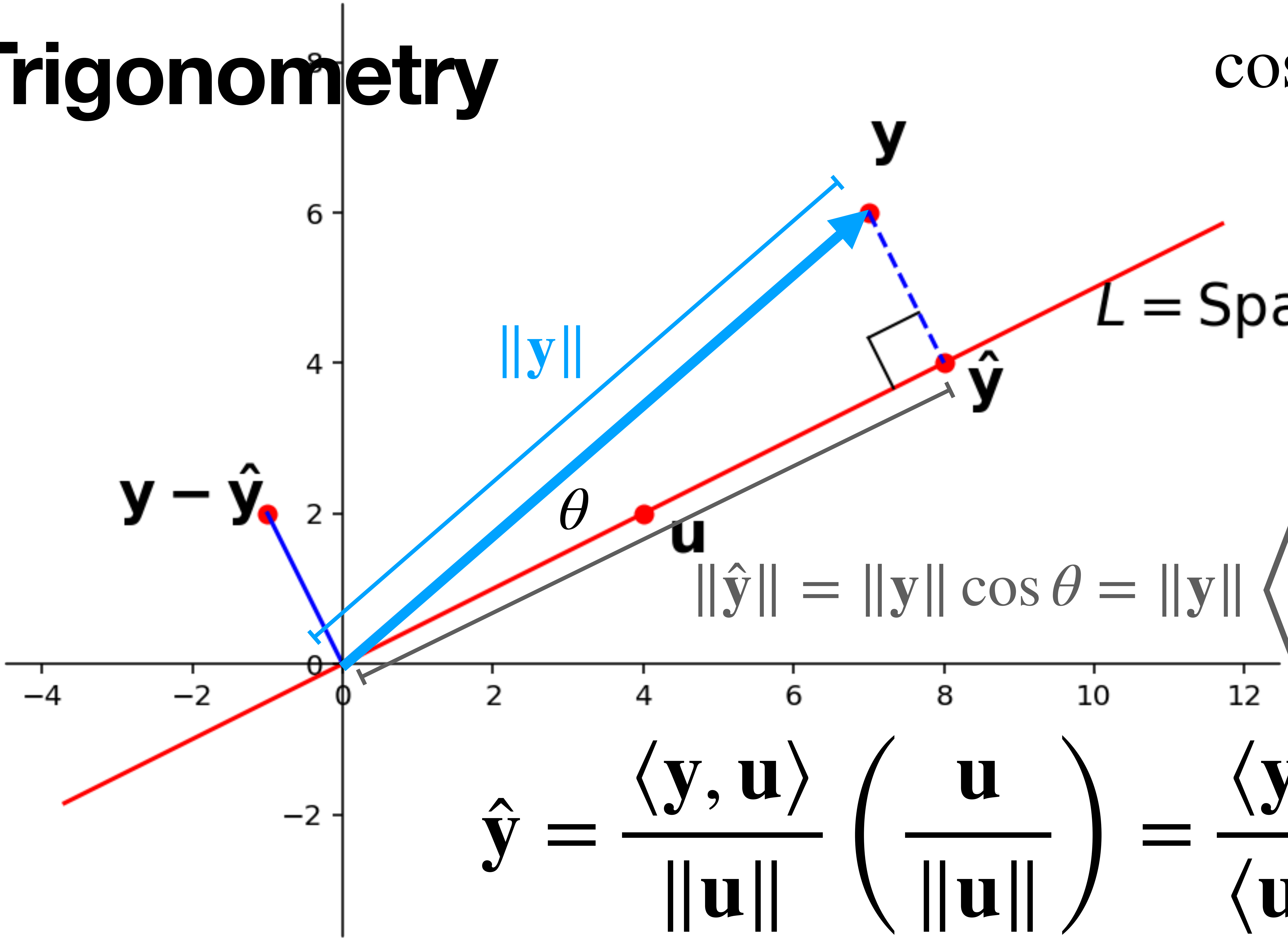

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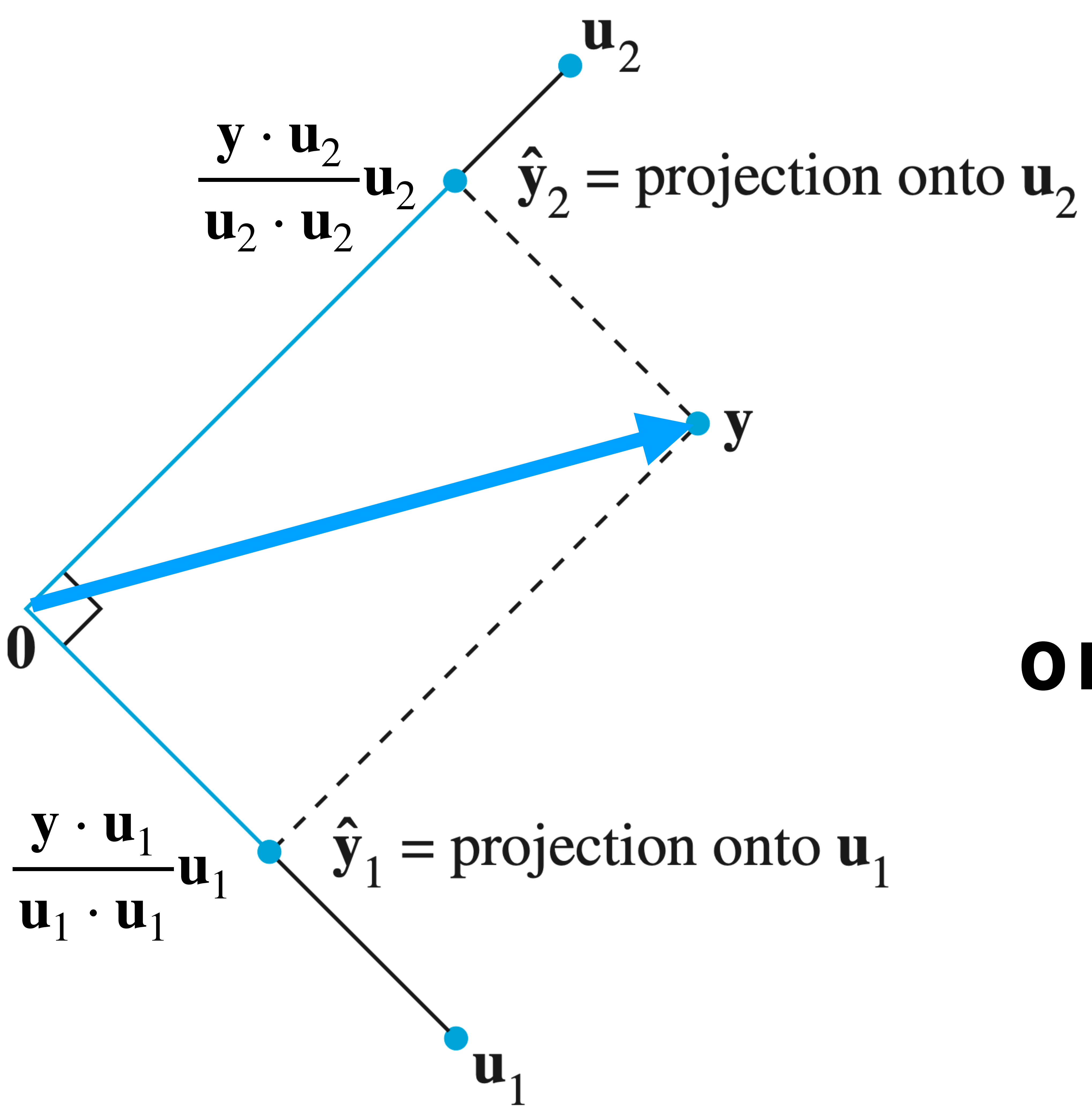
$$\hat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u} \rangle}{\|\mathbf{u}\|}\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\langle \mathbf{y}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

Orthogonal Projections and Orthogonal Bases



Each component of $\mathbf{y}$ written in terms of an *orthogonal* basis is an **orthogonal projection onto to a basis vector**

Linear Algebra and its Applications, Lay, Lay, McDonald

How To:

Question. Find the projection of $\mathbf{y}$ onto the span of $\mathbf{u}$

Solution. Calculate $\alpha = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$, then the solution is $\alpha\mathbf{u}$

Question

$$\alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

Find the matrix which implements orthogonal projection onto the span of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Answer

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

Orthonormal Sets

Orthogonal sets would be easier to work with if every vector was a <u>unit</u> vector

Orthonormality

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Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of <u>unit</u> vectors

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Orthonormal Matrices

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This is incredibly confusing, but we'll try to be consistent and clear

Orthonormal Matrices and Transposition

Theorem. For an $m \times n$ orthonormal matrix U

$$U^T U = I_n$$

Verify:

Inverses of Orthogonal Matrices

Theorem. If an $n \times n$ matrix U is orthogonal (square orthonormal) then it is invertible and

$$U^{-1} = U^T$$

Verify:

Orthonormal Matrices and Inner Products

Theorem. For a $m \times n$ orthonormal matrix U, and any vectors x and y in R^n

$$\langle Ux, Uy \rangle = \langle x, y \rangle$$

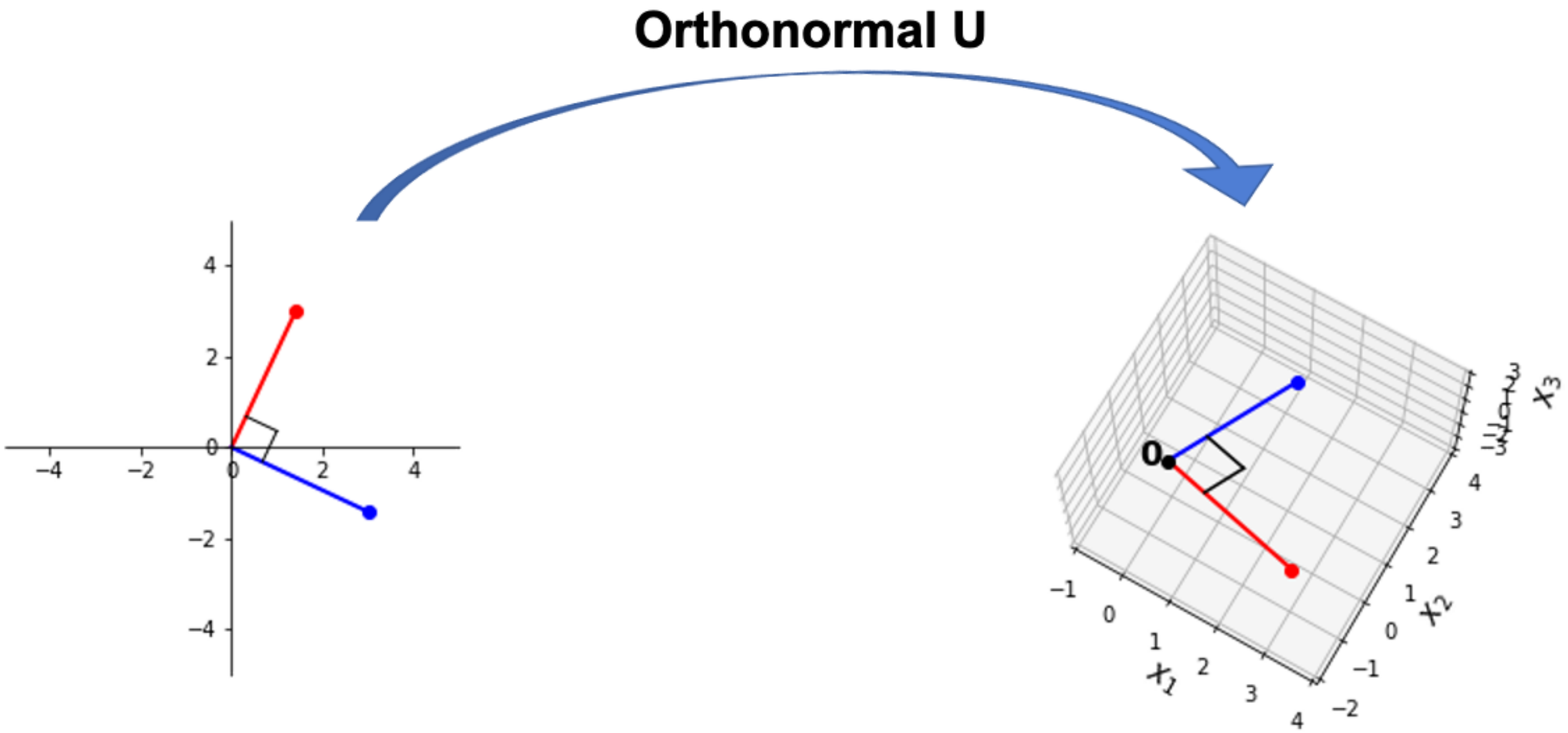
Orthonormal matrices preserve inner products

Verify:

Length, Angle, Orthogonality Preservation

Since lengths and angles are defined in terms of inner products, they are also preserved by orthonormal matrices:

The Picture



Example

$$U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \qquad x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

Question (Conceptual)

Suppose A is an $m \times n$ matrix with orthogonal but ***not*** *orthonormal columns. What is $A^T A$?*

Answer

If $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$ then $A^T A$ is a diagonal matrix D where

$$D_{ii} = \|\mathbf{a}_i\|^2$$

Summary

Orthogonal sets allow for simpler calculations of coordinates

Finding these coordinates is a really about find the orthogonal projections onto each vector in the orthogonal set

We can apply these ideas to matrices and describe a class of very well behaved transformations via orthonormal matrices