Assignment 2

CAS CS 132: *Geometric Algorithms*

Due September 18, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Unless otherwise specified, you must show your work. Also make sure to cite your sources per the instructions in the course manual.

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 1.1, 1.2, 1.3.

Basic Problems

1. Determine a pair of row operations that transform the matrix A into the matrix B.

$$A = \begin{bmatrix} -4 & -7 & 4 \\ 3 & 8 & 2 \\ -10 & 1 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 8 & 2 \\ -14 & -6 & -5 \\ -10 & 1 & -9 \end{bmatrix}$$

2. Determine a general form solution for the following linear system by first determining the reduced echelon form of its augmented matrix. If this system has no solutions then write *NO SOLUTION*.

$$\begin{aligned} x_1 - 6x_2 + x_4 &= 2 \\ 2x_1 - 12x_2 + x_3 - 4x_4 &= 7 \\ x_1 - 6x_2 - 2x_3 + 13x_4 &= -4 \end{aligned}$$

3. Determine three particular solutions of a linear system whose augmented matrix is row equivalent to the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & -3 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Compute the following linear combination of vectors.

$$7\begin{bmatrix}2\\-3\\-8\\9\end{bmatrix}+3\begin{bmatrix}2\\-3\\-4\\-3\end{bmatrix}-4\begin{bmatrix}5\\-1\\5\\-7\end{bmatrix}-2\begin{bmatrix}-2\\-9\\-2\\-10\end{bmatrix}$$

5. Determine a vector equation which is equivalent to the following linear system, i.e., that has the same solution set as the following linear system. You do not need to solve the vector equation.

$$\begin{aligned} 8x_1+6x_2-9x_3 &= -5\\ 4x_1+3x_2+9x_3+2x_4 &= -1\\ 4x_1+5x_2 &= 9\\ 3x_2+8x_3-3x_4 &= 2 \end{aligned}$$

6. Determine a vector that is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ but not in $\text{span}\{\mathbf{v}_1\}$ or $\text{span}\{\mathbf{v}_2\}$. Justify your answer.

$$\mathbf{v}_1=\begin{bmatrix}-7\\-1\\0\end{bmatrix}\quad \mathbf{v}_2=\begin{bmatrix}-8\\9\\-7\end{bmatrix}$$

7. Determine a general form solution for the following vector equation. If this equation has no solutions then write *NO SOLUTION*.

$$x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\5\\2\end{bmatrix}+x_3\begin{bmatrix}-8\\21\\10\end{bmatrix}=\begin{bmatrix}16\\-38\\-12\end{bmatrix}$$

8. Determine if the vector $\mathbf{v}_1$ is in the span of the remaining vectors. If it is, determine the corresponding dependence relation, i.e., write $\mathbf{v}_1$ as a linear combination of the remaining vectors.

$$\mathbf{v}_1=\begin{bmatrix}1\\4\\-7\end{bmatrix}\quad \mathbf{v}_2=\begin{bmatrix}0\\1\\-3\end{bmatrix}\quad \mathbf{v}_3=\begin{bmatrix}5\\7\\-1\end{bmatrix}\quad \mathbf{v}_4=\begin{bmatrix}1\\2\\-2\end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. A *consistent* linear system whose augmented matrix is a 207×209 matrix must have infinitely many solutions.

2. A *consistent* linear system whose coefficient matrix is square must have infinitely many solutions.

3. If the rightmost column of the augmented matrix of a linear system is a pivot column, then the system is inconsistent.

4. For any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, there is a vector $\mathbf{u}$ such that $\mathbf{v}_1+\mathbf{v}_2+\mathbf{u}=\mathbf{0}$.

5. For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\mathbf{v}_1 \in \mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3\}$, then $\mathbf{v}_2 \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_3\}$.
6. The span of any two distinct nonzero vectors in $\mathbb{R}^3$ is a plane.
7. For any vector $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, $\mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathsf{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$.

More Difficult Problems

1. Determine every 3×3 matrix in reduced echelon form with at least two pivot positions whose entries are either 0 or 1.
2. Determine a linear system over three variables such that (a, b, c) is a solution exactly when the cubic function $f(x) = ax^3 + bx^2 + c$ intersects the points $(-1, 4)$, $(1, 5)$ and $(2, 10)$. You do not need to solve the linear system.
3. Determine a linear equation whose point set is the span of the following vectors.[1] The linear equation you determine should have relatively prime integer coefficients (i.e., it should not be possible to divide the equation by an integer value and get a new equation with integer coefficients).

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Challenge Problems (Optional)

1. Suppose you're given a system of linear equations with three variables and two equations, and that $(4, 1, 0)$ and $(2, 4, 1)$ are solutions to this system. You are also given that the two equations define distinct planes in $\mathbb{R}^3$. Determine the reduced echelon form of the augmented matrix for this system.
2. There are several ways to define row equivalence. For example, suppose we restrict the replacement rule to only allow operations of the form

$$R_i \leftarrow R_i + R_j$$

   That is, we can only add one row to another, without doing any scaling of that row. We call this an *addition operation*.

   Demonstrate that two matrices A and B are row equivalent if there is a sequence of addition and scaling operations which transform A to B. In particular, addition and scaling operations can simulate replacement and exchange operations.

3. Determine the RREF of the following matrix in terms of x, y, assuming $x \neq 1$.

$$\begin{bmatrix} x^2 & x & 1 & y \\ 1 & 1 & 1 & 1 \\ \left(\frac{x+1}{2}\right)^2 & \frac{x+1}{2} & 1 & \frac{y+1}{2} \end{bmatrix}$$

   Hint: Don't try to row reduce it. Think in terms of polynomial interpolation.

[1] In this problem, we are conflating vectors and points.