Assignment 3

CAS CS 132: *Geometric Algorithms*

Due September 25, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual.

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 1.4, 1.7.

Basic Problems

1. Compute the matrix-vector multiplication $A\mathbf{v}$ where A and $\mathbf{v}$ are given below. If it's not possible to multiply A with $\mathbf{v}$, then explain why.

$$A = \begin{bmatrix} -10 & 6 & 2 & 8 \\ -1 & 3 & 4 & 5 \\ 0 & -2 & 0 & -9 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 4 \\ -5 \\ 3 \\ 1 \end{bmatrix}$$

2. Compute the matrix-vector multiplication $A\mathbf{v}$ where A and $\mathbf{v}$ are given below. If it's not possible to multiply A with $\mathbf{v}$, then explain why.

$$A = \begin{bmatrix} 6 & 1 & -8 & -3 \\ 5 & 0 & -9 & -4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

3. Determine a general form solution for the matrix equation $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given below. If this equation has no solutions then write *NO SOLUTION*.

$$A = \begin{bmatrix} 1 & -3 & 1 & -9 \\ 1 & -2 & 0 & -5 \\ -3 & 8 & -1 & 19 \\ -2 & 4 & 0 & 10 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ -18 \\ -14 \end{bmatrix}$$

4. Determine if the columns of the following matrix span all of $\mathbb{R}^2$. Show your work and justify your answer.

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

5. Determine if the columns of the following matrix span all of $\mathbb{R}^4$. Show your work and justify your answer.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ -1 & -1 & 2 & 9 \\ -1 & -2 & 1 & 1 \\ 0 & 2 & 6 & 36 \end{bmatrix}$$

6. Determine if the columns of the following matrix span all of $\mathbb{R}^5$. Show your work and justify your answer.

$$\begin{bmatrix} 8 & 3 & 8 \\ 8 & -4 & 4 \\ -2 & 1 & -5 \\ -5 & -7 & -10 \\ -8 & 9 & 1 \end{bmatrix}$$

7. Determine if the following vectors are linearly dependent. If they are write a dependence relation, i.e., determine linear combination of the given vectors which sums to $\mathbf{0}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

8. Determine if the following vectors are linearly dependent. If they are write a dependence relation, i.e., determine linear combination of the given vectors which sums to $\mathbf{0}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 7 \\ -2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -6 \\ 15 \\ -3 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. For $A \in \mathbb{R}^{m \times n}$ where $m > n$, it is not possible for the columns of A to span $\mathbb{R}^m$.

2. For $A \in \mathbb{R}^{m \times n}$, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then it is not possible for the columns of A to span $\mathbb{R}^m$.

3. For $A \in \mathbb{R}^{m \times n}$, if $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the columns of A span $\mathbb{R}^m$.

4. For $A \in \mathbb{R}^{m \times n}$, if $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $\{\mathbf{v} : A\mathbf{v} = \mathbf{0}\} = \mathbb{R}^n$.

5. For matrices A and B in $\mathbb{R}^{m \times n}$, let $[A\ B] \in \mathbb{R}^{m \times 2n}$ be the matrix obtained by hortizontally stacking A and B. If $A\mathbf{x} = \mathbf{b}$ is consistent and $B\mathbf{x} = \mathbf{b}$ is consistent, then so is $[A\ B]\mathbf{x} = \mathbf{b}$.

6. For matrices A and B in $\mathbb{R}^{m \times n}$, if $[A\ B]\mathbf{x} = \mathbf{b}$ is consistent then $A\mathbf{x} = \mathbf{b}$ is consistent or $B\mathbf{x} = \mathbf{b}$ is consistent.

7. For $A \in \mathbb{R}^{m \times n}$, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then A does not have a pivot position in every column.

8. For $A \in \mathbb{R}^{m \times n}$, if the columns of A are linearly dependent, then they do not span $\mathbb{R}^m$.

More Difficult Problems

1. Determine all values of h for which the following set of vectors is linearly dependent. Show your work.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 9 \\ -3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -2 \\ h \\ -6 \end{bmatrix}$$

2. Determine three *nonzero* vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that

   - $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and
   - $\mathbf{v}_1$ **cannot** be written as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

Challenge Problems (Optional)

1. Determine all values of h for which the following set of vectors is linearly dependent. Show your work.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ h \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 13 \\ h \\ -1 \end{bmatrix}$$