Assignment 4

`CAS CS 132`: *Geometric Algorithms*

Due October 2, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual.

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 1.8, 1.9

Basic Problems

1. Determine the domain and the codomain of the following linear transformation T. Also determine a matrix A such that $T(\mathbf{v}) = A(\mathbf{v})$ for all vectors $\mathbf{v}$ in the domain of T.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} -9x_1 + 4x_2 + 5x_3 - 3x_4 + 2x_5 \\ 3x_1 + 8x_2 + 5x_3 - 5x_4 + 7x_5 \\ -x_1 + 5x_2 - x_3 + 4x_4 + 8x_5 \end{bmatrix}$$

2. Determine if the vector $\mathbf{v}$ is in the range of the matrix transformation T given by $\mathbf{x} \mapsto A\mathbf{x}$, where A and $\mathbf{v}$ are defined below. If $\mathbf{v}$ is in the range of T, then determine a vector whose image under T is $\mathbf{v}$. Furthermore, state whether the vector you determined is unique, i.e., determine if there is another vector whose image under T is $\mathbf{v}$. Explain your answer.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ -2 & 2 & -3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 6 \\ 12 \\ -10 \end{bmatrix}$$

3. Let T be a linear transformation with the following input-output behavior.

$$T(\mathbf{v}_1) = \begin{bmatrix} -6 \\ 3 \\ -2 \\ -10 \end{bmatrix} \quad T(\mathbf{v}_2) = \begin{bmatrix} -5 \\ 1 \\ -2 \\ 9 \end{bmatrix} \quad T(\mathbf{v}_3) = \begin{bmatrix} 8 \\ -7 \\ 6 \\ 6 \end{bmatrix}$$

Determine the vector $T(-3\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3)$.

4. Let T be a linear transformation with the following input-output behavior.

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \quad T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

Determine the matrix that implements T, i.e., determine the matrix A such that $T(\mathbf{v}) = A\mathbf{v}$ for all vectors $\mathbf{v}$ in the domain of T.

5. Draw the image of the unit square under the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ where A is defined as below.

$$\begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}$$

6. Suppose that the linear transformation T has the following input-output behavior.

$$T\left(\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T\left(\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad T\left(\begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Determine the image of the following vector under T.

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $T(\mathbf{v}) = A\mathbf{v}$ for all vectors $\mathbf{v}$ in the domain of T, then A is an $m \times n$ matrix.
2. If the columns of A are linearly independent, then the reduced echelon form of A has only 0s and 1s.
3. If A is an $m \times n$ matrix and $m < n$, then the range of $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.
4. If the columns of $A \in \mathbb{R}^{m \times n}$ span all of $\mathbb{R}^m$, then the reduced echelon form of A has only 0s and 1s.
5. The matrix that implements a linear transformation T is unique.

More Difficult Problems

1. Determine the matrix which implements the linear tranformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the x_1x_2-plane.

2. Determine the matrix which implements the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ that repeats the input vector, e.g.,

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

3. Determine the matrix which implements the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the line $y = \tan(\frac{3\pi}{8})x$.

Challenge Problems (Optional)

1. Determine the matrix which implements the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates vectors 120 degrees about

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

   You can choose which direction to rotate.

2. Determine the matrix which implements the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the plane defined by the linear equation $x + y + z = 0$, and then rotates vectors 60 degrees about

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

   You can choose which direction to rotate.

3. Determine the matrix which implements the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the plane defined by the linear equation $x + y + z = 0$.