Assignment 5

`CAS CS 132`: *Geometric Algorithms*

Due October 9, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 2.1, 2.2.

Basic Problems

1. Determine if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, onto, both, or neither, where A is defined as below. Explain your answer.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ -1 & -3 & -3 \end{bmatrix}$$

2. Determine if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, onto, both, or neither, where A is defined as below. Explain your answer.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & -2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

3. Determine if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, onto, both, or neither, where A is defined as below. Explain your answer.

$$\begin{bmatrix} 1 & 2 & -1 & 6 & 9 \\ 2 & 5 & -5 & 13 & 22 \\ -3 & -4 & -3 & -16 & -19 \end{bmatrix}$$

4. Draw the image of the unit square under the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ where A is defined as below. In addition, determine the other matrix which transforms the unit square to the same set of

points.

$$\begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}$$

5. Compute the result of the expression $2A - 4B - 3C$ where the matrices in the expression are defined as follows.

$$A = \begin{bmatrix} 6 & 1 \\ -1 & -3 \\ 8 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -5 \\ -6 & -2 \\ -3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} -4 & 0 \\ -8 & 5 \\ -4 & -1 \end{bmatrix}$$

6. Compute the matrix multiplication AB where A and B are given below. If it's not possible to multiply A with B, then explain why.

$$A = \begin{bmatrix} -2 & 6 \\ -2 & -7 \\ -3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 8 & 4 \\ -10 & -4 & 9 \end{bmatrix}$$

7. Compute the matrix multiplication AB where A and B are given below. If it's not possible to multiply A with B, then explain why.

$$A = \begin{bmatrix} 8 & 6 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 5 \end{bmatrix}$$

8. Compute the matrix multiplication AB where A and B are given below. If it's not possible to multiply A with B, then explain why.

$$A = \begin{bmatrix} -3 \\ 6 \\ 8 \\ 5 \end{bmatrix} \quad B = \begin{bmatrix} -9 & -7 & 0 & -8 \end{bmatrix}$$

9. Compute the matrix multiplication AB where A and B are given below. If it's not possible to multiply A with B, then explain why.

$$A = \begin{bmatrix} 7 & -10 \\ -6 & 6 \\ 1 & 0 \\ -4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -8 & 9 \\ -10 & 0 \\ -9 & 1 \\ -4 & -3 \end{bmatrix}$$

10. Let A be matrix such that A^{-1} is defined as below. Use this to determine the solution to the matrix

equations of the form $A\mathbf{x} = \mathbf{b}_i$, where each $\mathbf{b}_i$ is defined below.

$$A^{-1} = \begin{bmatrix} -10 & -10 & -1 \\ -2 & -4 & -3 \\ -8 & -6 & 1 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} -5 \\ 6 \\ -1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ -3 \\ -10 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix}$$

11. Determine the inverse of the following matrix. If the matrix is not invertible, then explain why.

$$\begin{bmatrix} -8 & -2 \\ -2 & -3 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. If $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto, then A is a square matrix.
2. If $\mathbf{x} \mapsto A\mathbf{x}$ is neither one-to-one or onto, and the reduced echelon form of A only has 0s and 1s, then one of the columns of A is $\mathbf{0}$.
3. For any matrix $A \in \mathbb{R}^{m \times n}$ where $m > n$, it is not possible for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one.
4. For all matrices A and B in $\mathbb{R}^{m \times n}$ and C in $\mathbb{R}^{n \times m}$, we have $(A + B + C^T)^T = C + B^T + A^T$.
5. For all matrices A and B such that AB is defined, we have $AB \neq BA$.
6. If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$ in the span of the columns of A, then A is invertible.
7. For any matrix A and vector $\mathbf{v}$, if $\mathbf{v}^T A$ is defined, then A is single column.
8. For any matrices A and B, if $A^{-1} = B^{-1}$, then $A = B$.

More Difficult Problems

1. Determine the values of k, if any, such that $AB = BA$, where A and B are defined below.

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$$

2. Let A be as defined below. Is it possible to write the inverse of A as a power of A? If so, determine the smallest positive integer n such that $A^n = A^{-1}$.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

3. Let A be as defined below. Is it possible to write the inverse of A as a power of A? If so, determine the smallest positive integer n such that $A^n = A^{-1}$.

$$A = \begin{bmatrix} \cos\frac{\pi}{9} & -\sin\frac{\pi}{9} & 0 \\ \sin\frac{\pi}{9} & \cos\frac{\pi}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Challenge Problems (Optional)

1. Determine *three* matrices in $\mathbb{R}^{2\times 2}$ that satisfy the following equation. In particular, you must demonstrate that each matrix in your solution satisfies the equation.

$$X^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$