Assignment 6

`CAS CS 132`: *Geometric Algorithms*

Due October 16, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 2.2, 2.3, 2.5.

Basic Problems

1. Determine the inverse of the following matrix. If the matrix is not invertible, then explain why.

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Determine the inverse of the following matrix. If the matrix is not invertible, then explain why.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ -2 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \end{bmatrix}$$

3. Determine the inverse of the following matrix. If the matrix is not invertible, then explain why.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 7 & 1 & 0 & 0 \\ 8 & 9 & 10 & 11 & 1 & 0 \\ 12 & 13 & 14 & 15 & 16 & 1 \end{bmatrix}$$

4. Determine the matrix in $\mathbb{R}^{4\times 4}$ that implements the following row operations, in order from top to bottom. That is, determine a matrix A such that AB is the result of applying the following row operations,

from top to bottom, to B.

$$\begin{aligned}
&R_2 \leftarrow R_2 - R_3 \\
&R_1 \leftrightarrow R_4 \\
&R_1 \leftarrow R_1 + 3R_2 \\
&R_3 \leftarrow R_3 - 5R_4 \\
&R_4 \leftarrow -2R_4
\end{aligned}$$

5. Determine the inverse of the following transformation, if it exists. If the following transformation is not invertible, then write *SINGULAR*. Your solution should be in the form of a transformation, as given below.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_3 \\ -3x_1 + x_2 - 4x_3 \\ x_1 + 2x_2 \end{bmatrix}$$

6. Determine an LU factorization of the following matrix.

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

7. Determine an LU factorization of the following matrix.

$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ -2 & 3 & -13 & -1 & 8 \\ 2 & -4 & 18 & 3 & -2 \\ -3 & 5 & -22 & -4 & 3 \end{bmatrix}$$

8. Suppose that A is a matrix in $\mathbb{R}^{3\times 3}$ such that the following sequence of row operations (from top to bottom) transforms A into the identity matrix. Determine the inverse of A. (Hint: Don't determine A first)

$$\begin{aligned}
&R_1 \leftrightarrow R_2 \\
&R_1 \leftarrow R_1 - R_2 \\
&R_3 \leftarrow R_3 - 3R_2 \\
&R_1 \leftarrow R_1 + 3R_2 \\
&R_3 \leftarrow R_3 - 5R_1 \\
&R_1 \leftrightarrow R_2
\end{aligned}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. For any matrices A and B, if there is a unique matrix X such that $AX = B$, then A is invertible.

2. If A and B are invertible, then so is $A + B$.

3. For any matrix $A \in \mathbb{R}^{n \times n}$, if the columns of A^3 span all of $\mathbb{R}^n$, then the columns of A are linearly independent.

4. A square matrix A is called **symmetric** if $A = A^T$. For any matrix $A \in \mathbb{R}^{n \times n}$, the matrix $A + A^T$ is symmetric.

5. If $A \in \mathbb{R}^{n \times n}$ has zeros along its diagonal, then A is not invertible.

6. If $A \in \mathbb{R}^{n \times n}$ has a row of all zeros, then A is not invertible.

7. If $A \in \mathbb{R}^{2 \times 2}$ and A^{-1} has integer entries then the determinant of A is 1.

8. For any square matrices A and B, if $AB = I$, then $AB = BA$.

9. The **Hadamard product** of two matrices is defined as

$$(A \circ B)_{ij} = A_{ij} B_{ij}$$

In other words, A and B are multiplied entry-wise. For any invertible matrices A and B, if $A \circ B$ is invertible, then $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$.

More Difficult Problems

1. Suppose that A and B are invertible matrices such that $AB^T X A^{-1} B = I$ for some matrix X. Determine X in terms of A and B.

2. Let A, B, and C such that $A = A^{-1}$ and $C = C^T$ and

$$A(C^{-1}(AB)^T)^T C$$

is well-defined. Simplify this expression using the algebraic properties of matrix operations.

3. Let A and B be defined as below. Determine an invertible matrix X such that the inverse of $A - AX$ is $X^{-1}B$.

$$A = \begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

4. Compute the following matrix expression. Your answer should be a single matrix with entries given in terms of n. Note that A^{-n} is the same as $(A^n)^{-1}$.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-n}$$

5. Suppose that LU is the LU-factorization of a matrix A, and let B be a matrix such that $BU = I$. Determine the inverse of A in terms of L, U, and B.

6. Let A and B be matrices such that AB is invertible, and let C be the inverse of AB. Demonstrate that A and B are invertible by determining the inverses of A and B. (Note. You can't assume that A and B are invertible)

7. Determine a matrix in $\mathbb{R}^{2\times 2}$ that is equal to its inverse. (Hint. Use the closed-form equation for the inverse of a 2×2 matrix)

8. For what values of k, if any, is the following matrix singular?

$$\begin{bmatrix} 4 & 5 & k \\ k & 0 & 6 \\ 1 & 0 & -2 \end{bmatrix}$$

Challenge Problems (Optional)

1. Determine the reduced echelon form of the matrix

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

   in terms of a, b, c, d. Show your work.

2. Determine two invertible matrices A and B such that $AB^{-1} = -BA^{-1}$.

3. Let A and B be two invertible matrices in $\mathbb{R}^{n\times n}$. Show that if $AB^{-1} = -BA^{-1}$, then the matrix

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

   in $\mathbb{R}^{2n\times 2n}$ is invertible. Also determine the inverse.