Assignment 6

CAS CS 132

Basic Problems

(1)

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ -3 & 4 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

②

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ -2^{+2} & -1^{+2} & -1 & 0^{-2} & 0^{+2} & 1 & 0 & 0 \\ 1^{-1} & 1^{-1} & 0 & 0^{+1} & 0^{-1} & 0 & 1 & 0 \\ -2^{+2} & -2^{+2} & 0 & 1^{-2} & 0^{+2} & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

3 pivots, 4 columns / rows, not invertible by the invertible matrix theorem (IMT)

③ Note: This problem was harder than expected.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 \\ -28 & 22 & -7 & 1 & 0 & 0 \\ 268 & -211 & 67 & -11 & 1 & 0 \\ -3924 & 3089 & -981 & 161 & -161 \end{bmatrix}$$

④

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3R_2} \begin{bmatrix} 0 & 3 & -3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 5R_4} \begin{bmatrix} 0 & 3 & -3 & 1 \\ 0 & 1 & -1 & 0 \\ -5 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \leftarrow -2R_4}$$

$$\begin{bmatrix} 0 & 3 & -3 & 1 \\ 0 & 1 & -1 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

(5)

$$\vec{x} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix} \vec{x}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -3^{+3} & 1 & -4^{+3} & 0^{+3} & 1 & 0 \\ 1^{-1} & 2 & 0^{-1} & 0^{-1} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 2^{-2} & -1^{+2} & -1^{-6} & 0^{-2} & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1^{-1} & 1^{+7} & 0^{+2} & 0^{-1} \\ 0 & 1 & -1^{+1} & 3^{-7} & 1^{-2} & 0^{+1} \\ 0 & 0 & 1 & -7 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & 8 & 2 & -1 \\ 0 & 1 & 0 & -4 & -1 & 1 \\ 0 & 0 & 1 & -7 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 8 & 2 & -1 \\ -4 & -1 & 1 \\ -7 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 8x_1 + 2x_2 - x_3 \\ -4x_1 - x_2 + x_3 \\ -7x_1 - 2x_2 + x_3 \end{bmatrix}}$$

(6)

$$\begin{bmatrix} 1 & -2 \\ -2^{+2} & 5^{-4} \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

7

$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ -2^{+2} & 3^{-2} & -13^{+8} & -1 & 8^{-10} \\ 2^{-2} & -4^{+2} & 18^{-8} & 3 & -2^{+10} \\ -3^{+3} & 5^{-3} & -22^{+12} & -4 & 3^{-15} \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 + 3R_1]{\substack{R_2 \leftarrow R_2 + 2R_1 \\ R_3 \leftarrow R_3 - 2R_1}}$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ 0 & 1 & -5 & -1 & -2 \\ 0 & -2^{+2} & 10^{-10} & 3^{-2} & 8^{-4} \\ 0 & 2^{-2} & -10^{+10} & -4^{+2} & -12^{+4} \end{bmatrix} \xrightarrow{\substack{R_3 \leftarrow R_3 + 2R_2 \\ R_4 \leftarrow R_3 - 2R_2}}$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ 0 & 1 & -5 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2^{+2} & -8^{+8} \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + 2R_3}$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ 0 & 1 & -5 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ -3 & 2 & -2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ 0 & 1 & -5 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \leftarrow R_3 - 3R_2} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 3R_2} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 5R_1}$

$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ -13 & -5 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -13 & -5 & 1 \end{bmatrix}}$

True / False

(1) False $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(2) False $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(3) True

(4) True

(5) False $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(6) True

(7) False $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(8) True

(9) False

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad (A \circ B)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} \circ B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

More Difficult Problems

(1) $AB^T X A^{-1} B = I \Rightarrow$

$B^T X A^{-1} B = A^{-1} \Rightarrow$

$X A^{-1} B = (B^T)^{-1} A^{-1} \Rightarrow$

$X A^{-1} = (B^{-1})^T A^{-1} B^{-1} \Rightarrow$

$\boxed{X = (B^{-1})^T A^{-1} B^{-1} A}$

(2) $A(C^{-1}(AB)^T)^T C =$

$A(C^{-1} B^T A^T)^T C =$

$A(C^{-1} B^T A^T)^T C^T =$

$A(C C^{-1} B^T A^T)^T =$

$A((AB)^T)^T = AAB =$

$\boxed{B}$

(3)

$$(A - AX)^{-1} = X^{-1}B \Rightarrow$$

$$X = B(A - AX) \Rightarrow$$

$$X + BAX = BA \Rightarrow$$

$$(I + BA)X = BA \Rightarrow$$

$$X = BA(I + BA)^{-1}$$

$$BA = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 19 & -15 \\ -14 & 11 \end{bmatrix}$$

$$I + BA = \begin{bmatrix} 20 & -15 \\ -14 & 12 \end{bmatrix}$$

$$\det(I + BA) = 20(12) - 14(15)$$

$$= 240 - 210$$

$$= 30$$

$$(I + BA)^{-1} = \frac{1}{30}\begin{bmatrix} 12 & 15 \\ 14 & 20 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/2 \\ 7/15 & 2/3 \end{bmatrix}$$

$$BA(I + BA)^{-1} = \frac{1}{30}\begin{bmatrix} 19 & -15 \\ -14 & 11 \end{bmatrix}\begin{bmatrix} 12 & 15 \\ 14 & 20 \end{bmatrix} = \boxed{\begin{bmatrix} 3/5 & -1/2 \\ -7/15 & 1/3 \end{bmatrix}}$$

(done with calculator)

④

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3n & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3n & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3n & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 6n^2 & 2n \\ 0 & 1 & 0 \\ 0 & 3n & 1 \end{bmatrix}}$$

⑤

$A = LU \qquad BU = I$

$\Downarrow \qquad\qquad \Downarrow$

$L^{-1}A = U \qquad B(L^{-1}A) = I$

$$\boxed{A^{-1} = BL^{-1}}$$

⑥ $C = (AB)^{-1} \Rightarrow CAB = I \Rightarrow \boxed{CA = B^{-1}}$

$\Rightarrow ABC = I \Rightarrow \boxed{A^{-1} = BC}$

(7) I

Note. We forgot to say a matrix that is not I or $-I$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$c = \frac{-c}{ad-bc} \Rightarrow ad-bc = -1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix}^{-1} = \frac{1}{-9+8}\begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix}$$

(8) $\boxed{k = -3}$

This makes the second and third rows scalar multiples of each other. If $k \neq -3$, then the matrix A^T has linearly independent columns. (Note there are other ways of reasoning about this.)