Assignment 7

CAS CS 132: *Geometric Algorithms*

Due October 30, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant section for this assignment is: 5.9.

Basic Problems

For each of the following stochastic matrices A, answer the following items:

(a) Draw the state diagram for the corresponding Markov chain. You should label each node with a positive integer for the corresponding column of A (e.g, the node corresponding to the first column should be labeled "1", to the second column labeled "2", and so on).

(b) Determine if A is regular, and write down the smallest k such that A^k has strictly positive values. If it is not regular, justify your answer.

(c) Determine the general form solution of the equation $(A - I)\,\mathbf{x} = \mathbf{0}$. You may use a computer to do this, but you should express your answer in fractions, not decimals.

(d) Determine a steady state vector for A. If the steady state vector is unique, note this. You must do this by hand and show your work.

1.

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.4 & 0.3 & 0 \\ 0 & 0.7 & 0.9 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.4 & 1 & 0.3 \\ 0.4 & 0 & 0.7 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5.

$$A = \begin{bmatrix} 0 & 0 & 0.3 \\ 1 & 0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. **(Challenge)** An *eigenvector* $\mathbf{v}$ of matrix A is a vector such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$. If A is a stochastic matrix, then the all ones vector $\mathbf{1}$ is an eigenvector for A^T.
2. If a stochastic matrix has a unique steady state vector, then the corresponding Markov chain converges to it regardless of starting state.
3. **(Challenge)** A stochastic regular matrix $A \in \mathbb{R}^{n \times n}$ has a smallest integer k such that A^k has strictly positive entries. For such a matrix, $k \leq n$.
4. Every stochastic matrix has at least one steady state vector.
5. If A and B are stochastic matrices, then so is AB.
6. If A and B are stochastic matrices, then so is BA.
7. Every A stochastic matrix is invertible.

More Difficult Problems

1. Suppose that we're trying to predict the preformance of the A_5 Soccer Club based on statistics that we've gathered on their recent games. We've found that:
   (a) After a win, they have a 70% chance of winning, a 15% chance of drawing, and a 15% chance losing their next game.
   (b) After a draw, they have a 20% chance of winning, a 50% chance of drawing, and a 30% chance of losing their next game.
   (c) After a loss, they have a 20% chance of winning, a 70% chance of drawing, and a 10% chance of losing their next game.

If they win their first game, how should we expected them to perform overall? That is, what is the expected percentage of wins, draws, and losses in the long term?

If they lose their first game, what will their overall record tend towards?

Challenge Problems (Optional)

1. Argue that any invertible stochastic matrix A with a stochastic inverse must be a permutation matrix. That is, every column and every row of A has a single 1, with the rest of the entries being 0s. (*Hint:* consider what can be inferred about the entries of a stochastic A^{-1} from its action on $A\mathbf{e}_i$ for any elementary basis vector $\mathbf{e}_i$.)