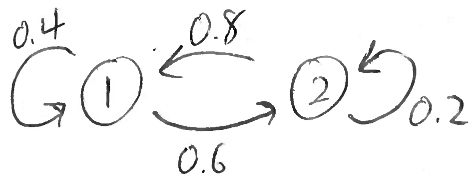
Assn 7 Solution Key

Stochastic Probs:

① (a)



(b) regular for $k=1$

(c) $\left(\begin{array}{cc|c} -0.6 & 0.8 & 0 \\ 0.6 & -0.8 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + R_1} \left(\begin{array}{cc|c} -0.6 & 0.8 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow 0.8x_2 = 0.6x_1 \Rightarrow \boxed{\begin{array}{l} x_1 = \frac{4}{3}x_2 \\ x_2 \text{ free} \end{array}}$

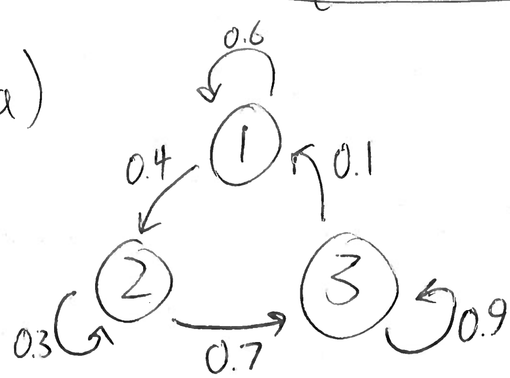
(d) $x_1 + x_2 = 1$ for a prob. vector

$\frac{4}{3}x_2 + x_2 = \frac{7}{3}x_2 = 1 \Rightarrow x_2 = \frac{3}{7}$ & $x_1 = 1 - x_2 = \frac{4}{7}$

$\boxed{\begin{array}{c} \text{steady state vector } \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} \\ \text{unique as } A \text{ regular} \end{array}}$

(also only prob. vector solution to $(A-I)x=0$)

② (a)



(b) regular for $k=2$, possible to go from any i to any j with a length 2 path (or just multiply out)

(c) $\left(\begin{array}{ccc|c} -0.4 & 0 & 0.1 & 0 \\ 0.4 & -0.7 & 0 & 0 \\ 0 & 0.7 & -0.1 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + R_1} \left(\begin{array}{ccc|c} -0.4 & 0 & 0.1 & 0 \\ 0 & -0.7 & 0.1 & 0 \\ 0 & 0.7 & -0.1 & 0 \end{array}\right)$

$\downarrow R_3 \leftarrow R_3 + R_2$

$0.1x_3 = 0.4x_1$
$0.1x_3 = 0.7x_2$ $\Leftarrow \left(\begin{array}{ccc|c} -0.4 & 0 & 0.1 & 0 \\ 0 & -0.7 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

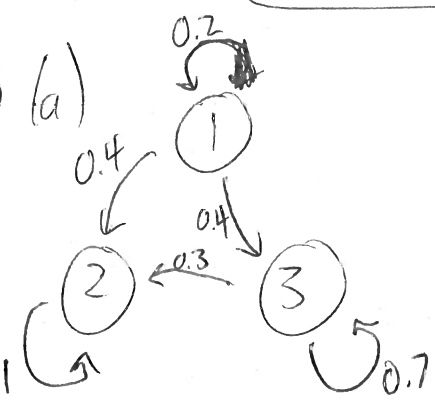
$$\boxed{\begin{array}{l} x_1 = \frac{1}{4}x_3 \\ x_2 = \frac{1}{7}x_3 \\ x_3 \text{ free} \end{array}}$$

(d) $x_1 + x_2 + x_3 = 1 \Rightarrow \frac{1}{4}x_3 + \frac{1}{7}x_3 + x_3 = \frac{7}{28}x_3 + \frac{4}{28}x_3 + \frac{28}{28}x_3 = 1$

$\Rightarrow x_3 = \frac{28}{39}$

$\boxed{\text{steady state is } \begin{bmatrix} \frac{7}{39} \\ \frac{4}{39} \\ \frac{28}{39} \end{bmatrix}, \text{ unique as } A \text{ regular}}$ (also only prob. vector sol'n to $(A-I)x=0$)

③ (a)



(b) Not regular, from state 2 you can never reach other states

(c) $\left(\begin{array}{ccc|c} -0.8 & 0 & 0 & 0 \\ 0.4 & 0 & 0.3 & 0 \\ 0.4 & 0 & -0.3 & 0 \end{array}\right) \xrightarrow[R_3 \leftarrow \frac{1}{2}R_1 + R_3]{R_2 \leftarrow \frac{1}{2}R_1 + R_2} \left(\begin{array}{ccc|c} -0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & -0.3 & 0 \end{array}\right)$

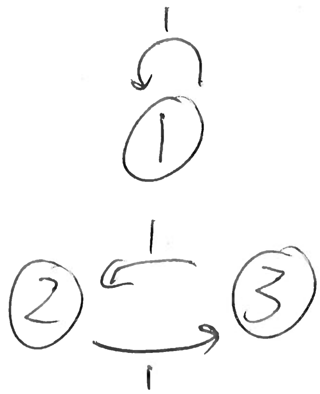
$R_3 \leftarrow R_2 + R_3$

$$\left(\begin{array}{ccc|c} -0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 0.8x_1 = 0 \\ 0.3x_3 = 0 \end{array} \Rightarrow \boxed{\begin{array}{l} x_1 = 0 \\ x_2 \text{ free} \\ x_3 = \cancel{\text{free}}\ 0 \end{array}}$$

(d) $x_1 + x_2 + x_3 = 1 \Rightarrow 0 + 0 + x_3 = 1 \Rightarrow$ steady state vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

unique as only prob. vector sol'n to $(A-I)x = 0$

④ (a)



(b) Not regular as can't get to other states from state 1.

(c) $$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = x_3$$

$$\boxed{\begin{array}{l} x_1 \text{ free} \\ x_2 = x_3 \\ x_3 \text{ free} \end{array}}$$

(d) $x_1 + x_2 + x_3 = 1 \Rightarrow x_1 + x_3 + x_3 = x_1 + 2x_3 = 1$

~~For one choice~~ $x_1 = 1 - 2x_3$

$x_2 = x_3$ are all potential steady state vectors

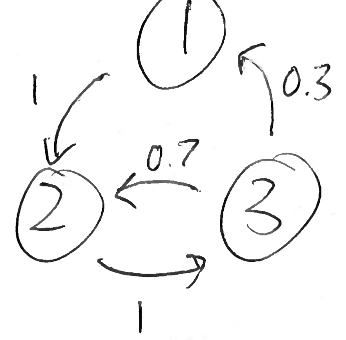
x_3 free

Letting $x_3 = 0$ gives one result: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

not unique as other choices of x_3 give other steady state vectors

(one example: $x_3 = 0.5$ gives $\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$)

(5) (a)



(b)
- k needs to be at least 3 as you need 3 steps to return to state 1 from state 1.
- $k \neq 3$ as no 3-step path from state 1 to state 3
- $k \neq 4$ as no 4-step path from state 1 back to itself
- $k=5$ turns out to work, requires check or computer calculation

(c)
$$\left(\begin{array}{ccc|c} -1 & 0 & 0.3 & 0 \\ 1 & -1 & 0.7 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_1 + R_2} \left(\begin{array}{ccc|c} -1 & 0 & 0.3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\downarrow R_3 \leftarrow R_2 + R_3$$

$$\boxed{\begin{array}{l} x_1 = 0.3x_3 = \frac{3}{10}x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{array}} \Leftarrow \left(\begin{array}{ccc|c} -1 & 0 & 0.3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(d) $x_1 + x_2 + x_3 = 1 \Rightarrow \frac{3}{10}x_3 + x_3 + x_3 = \frac{23}{10}x_3 = 1 \Rightarrow x_3 = \frac{10}{23}$

$$\boxed{\text{steady state vector: } \begin{bmatrix} 3/23 \\ 10/23 \\ 10/23 \end{bmatrix}, \text{ unique as regular}}$$

T/F:

1. True
2. False, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has unique steady state vector $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ but vectors just oscillate & don't converge to it

3. False, see matrix 5 from previous section.
4. True
5. True
6. True
7. False, $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ clearly not invertible as rows identical (& columns)

Word Problem

~~ith result (i+1)th WLD~~ ~~Set~~ Model as a Markov chain

$$\begin{array}{c|ccc} {}_{(i+1)\text{th result}} \backslash {}^{i\text{th result}} & \text{win} & \text{draw} & \text{loss} \\ \hline \text{win} & .7 & .2 & .2 \\ \text{draw} & .15 & .5 & .7 \\ \text{loss} & .15 & .3 & .1 \end{array}$$

$$= A$$

A regular, so will have a unique steady state vector that things converge to.

$$(A-I)x = 0 \xrightarrow{\text{aug}} \left(\begin{array}{ccc|c} -0.3 & 0.2 & 0.2 & 0 \\ 0.15 & -0.5 & 0.7 & 0 \\ 0.15 & 0.3 & -0.9 & 0 \end{array}\right)$$

$R_2 \leftarrow \tfrac{1}{2}R_1 + R_2$

$R_3 \leftarrow \tfrac{1}{2}R_1 + R_3$

$$\left(\begin{array}{ccc|c} -0.3 & 0.2 & 0.2 & 0 \\ 0 & -0.4 & 0.8 & 0 \\ 0 & 0.4 & -0.8 & 0 \end{array}\right) \xrightarrow{R_3 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} -0.3 & 0.2 & 0.2 & 0 \\ 0 & -0.4 & 0.8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 + \frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|c} -0.3 & 0.0 & 0.6 & 0 \\ 0 & -0.4 & 0.8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 0.3x_1 = 0.6x_3 \\ 0.4x_2 = 0.8x_3 \end{array} \Rightarrow \begin{array}{l} x_1 = 2x_3 \\ x_2 = 2x_3 \\ x_3 \text{ free} \end{array}$$

$$x_1 + x_2 + x_3 = 1 \Rightarrow 2x_3 + 2x_3 + x_3 = 5x_3 = 1$$

$$\Rightarrow \begin{array}{l} x_1 = 2/5 \\ x_2 = 2/5 \\ x_3 = 1/5 \end{array}$$

The win/draw/loss percentages will ~~corresp~~ be 40%/40%/20% regardless of starting state. ~~(Winning first gam~~

(Winning/losing their first game corresponds to starting states of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ & $\begin{bmatrix}0\\0\\1\end{bmatrix}$, respectively)