Assn 8 Solutions

Basic Problems

(1) $\begin{bmatrix} 1 & -5 \\ -2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$ $\quad$ $\text{Col } A = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ ← basis $\quad$ $\dim(\text{Col } A) = \text{rk}(A) = 1$

gen. sol'n $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} x_2$ $\quad$ $\text{Nul } A = \text{span}\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right\}$ ← basis $\quad$ $\dim(\text{Nul } A) = 1$

(2) $\text{Col } A = \{0\}$ (0-dim'l so no basis) $\quad$ $\dim(\text{Col } A) = \text{rk } A = 0$

$\text{Nul } A = \mathbb{R}^2$, so $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ works as a basis $\quad$ $\dim(\text{Nul } A) = 2$

(gen. sol'n $\vec{x} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

(3) $\begin{bmatrix} 1 & -4 & 3 & -3 \\ -2 & 8 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\vec{x} = x_2\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad$ basis → $\text{Col } A = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix}\right\}$ $\quad$ $\dim(\text{Col } A) \overset{\text{rk } A}{=} 2$

$\text{Nul } A = \text{span}\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ ← basis $\quad$ $\dim(\text{Nul } A) = 2$

(4) $\begin{bmatrix} 1 & -4 & -3 \\ -3 & 12 & 10 \\ -2 & 8 & 8 \\ -1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $\vec{x} = x_2\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

~~dim~~ $\text{Col } A = \text{span}\left\{\begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 10 \\ 8 \\ 2 \end{bmatrix}\right\}$ ← basis $\quad$ $\dim(\text{Col } A) \overset{\text{rk } A}{=} 2$

$\text{Nul } A = \text{span}\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}\right\}$ ← basis $\quad$ $\dim(\text{Nul } A) = 1$

(5) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -5 \\ 2 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ all lin. ind. so $\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -8 \end{bmatrix}\right\}$ is a basis

(6) $\begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & -4 & -4 \\ 0 & -3 & 18 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & +1 & -6 & -2 \\ 0 & -3 & 18 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

pivots, so a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \right\}$

(7) $\left[\begin{array}{cc|c} 1 & -3 & 8 \\ -1 & 4 & -12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 8 \\ 0 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & -4 \end{array}\right] \Rightarrow [u]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

(8) $\left[\begin{array}{ccc|c} 1 & -1 & -2 & 4 \\ -1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad [u]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}$

True/False

(1) False, many choices of basis are possible for any subspace of dim > 0. E.g. For $\mathbb{R}^2$, we may take $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(2) False, $\dim(\text{Nul } A) + \dim(\text{Col } A) = 4$ and dimensions are non-negative. Thus $\dim(\text{Nul } A) \leq 4$. (Or input dimension is 4, and so Nul A can't be more)

(3) False, $\dim(\text{Nul } A) + \dim(\text{Col } A) = 6$
$\dim(\text{Col } A) \leq 3 \Rightarrow \dim(\text{Nul } A) \geq 3$

(4) True, by IMT, or $\text{rank}(A) = n \Rightarrow \dim(\text{Nul } A) = 0 \Rightarrow A$ invertible

(5) True, $\{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$ is the set of linear comb's of columns of A

(6) False, as small as possible. Any subspace not of dim. 0 has an infinite number of vectors, and you could tack on as many as you want and stay a spanning set.

(7) False, T implemented by a matrix $A \in \mathbb{R}^{3\times 3}$
$\dim(\text{Col } A) = \underbrace{2}_{\text{dim'n of a plane}} \Rightarrow \dim(\text{Nul } A) = \underline{1} \leftarrow$ kernel nontrivial

More Difficult Problems

(1) Closure under addition? $A\vec{u} = \vec{e}_1$, $A\vec{v} = \vec{e}_1$ $\Rightarrow A(\vec{u}+\vec{v}) = 2\vec{e}_1$ NO

Closure under scaling? $A\vec{u} = \vec{e}_1 \Rightarrow A(c\vec{u}) = c\vec{e}_1 \neq \vec{e}_1$ NO
for any $c \neq 1$

Subspace: NO

(2)

$$\begin{bmatrix} 3 & -3 & 0 \\ 0 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & -3 \\ 0 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Big\} \mathbb{R}^2$$

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \right\}$$

anything works, as you're asking what pairs span $\mathbb{R}^2$ in the RREF

(3)

$$\begin{bmatrix} 3 & -3 & -1 & 8 \\ -1 & 1 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 & -4 \\ 0 & 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \end{bmatrix} \right\}$$

(4)

$$\frac{d}{dx}(ax^2+bx+c) = 0x^2 + 2ax + b$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$$

$\text{rk}\, A = 2$ Col A basis $\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Nul A basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(\text{Nul}\, A) = 1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a = 0$
$b = 0$
c free

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Constant functions map to 0 under the derivative!