Assignment 9

CAS CS 132: *Geometric Algorithms*

Due November 13, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 5.1, 5.2, and 5.6.

Basic Problems

1. Determine if $\mathbf{v}$ is an eigenvector of A. If it is, find its corresponding eigenvalue.

$$A = \begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2. Determine if $\mathbf{v}$ is an eigenvector of A. If it is, find its corresponding eigenvalue.

$$A = \begin{bmatrix} -10 & -3 & -5 \\ 5 & -5 & -3 \\ 5 & 7 & -7 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$$

3. Determine if λ is an eigenvalue of A. If it is, find a basis for the corresponding eigenspace.

$$A = \begin{bmatrix} -2 & 2 \\ -1 & -4 \end{bmatrix} \qquad \lambda = -3$$

4. Determine if λ is an eigenvalue of A. If it is, find a basis for the corresponding eigenspace.

$$A = \begin{bmatrix} 5 & -6 & 2 \\ 1 & -2 & 2 \\ -1 & 6 & 2 \end{bmatrix} \qquad \lambda = 4$$

5. For the following matrix, determine all eigenvalues and bases for the corresponding eigenspaces.

$$\begin{bmatrix} -3 & 2 \\ -10 & 6 \end{bmatrix}$$

6. For the following matrix, determine all eigenvalues and bases for the corresponding eigenspaces.

$$\begin{bmatrix} 1 & 16 & -12 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

7. For the following matrix, determine all eigenvalues and bases for the corresponding eigenspaces.

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

8. Calculate the determinant of the following matrix.

$$\begin{bmatrix} -1 & 1 & -1 \\ 3 & 3 & 3 \\ 1 & -3 & -2 \end{bmatrix}$$

9. Calculate $\det(A^{-1})$ where A is given below.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -5 & -4 & -3 \\ 2 & 0 & -5 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. Any eigenspace of A is a null space (potentially of some other matrix).
2. Every matrix has at least one eigenvector.
3. Every matrix $A \in \mathbb{R}^{n \times n}$ has an eigenbasis, i.e., a set of eigenvectors that form a basis for $\mathbb{R}^n$.
4. Only square matrices can have eigenvectors.
5. If 0 is an eigenvalue of A, then A is not invertible.
6. The determinant of a matrix does not change under elementary row operations.
7. If $\det(A^2) = 1$, then $\det(A) = 1$.
8. A matrix with characteristic polynomial $(x+2)^3$ has a single eigenspace with dimension 3.

More Difficult Problems

1. Suppose the eigenvalues of a 3×3 matrix A are $\lambda_1 = 2$, $\lambda_2 = 1/2$, and $\lambda_3 = 1/4$ with corresponding eigenvectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}.$$

   Given a starting state $\mathbf{x_0} = [8, -9, 6]^T$, give a closed form expression for the state after k iterations: $A^k\mathbf{x_0}$. Describe what happens as $k \to \infty$.

2. We learned that a (counterclockwise) rotation by angle θ in the plane can be implemented by the matrix below.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   Calculate $\det(R_\theta)$. (*Hint.* The value does not depend on θ)

3. Determine the characteristic polynomial of R_θ as a function of θ. For what values of θ does R_θ have eigenvalues? When R_θ has eigenvalues, what are bases for the corresponding eigenspaces?

4. Give two examples of 2×2 matrices where every nonzero vector is an eigenvector.

Challenge Problems (Optional)

1. Prove that a pair of eigenvectors $\mathbf{v_1}, \mathbf{v_2}$ with distinct eigenvalues $\lambda_1 \neq \lambda_2$ is linearly independent. (*Hint:* Suppose a linear dependence relation existed between $\mathbf{v_1}$ and $\mathbf{v_2}$, and argue that we reach a contradiction.)

2. Prove that the determinant formula from lecture that uses row reduction is correct. (*Hint:* Consider applying elementary row operations via left multiplication by elementary row matrices, and use the fact that $\det(E_k \ldots E_2E_1A) = \det(E_k) \ldots \det(E_2)\det(E_1)\det A$.)