Assignment 9 Solutions

Basic Probs

(1) $\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ yes, with eigenvalue 5

(2) $\begin{bmatrix} -10 & -3 & -5 \\ 5 & -5 & -3 \\ 5 & 7 & -7 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -30+25 \\ 15+15 \\ 15+35 \end{bmatrix} = \begin{bmatrix} -5 \\ 30 \\ 50 \end{bmatrix}$ no

(3) $A+3I = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ no nontrivial sol'ns $\Rightarrow$ no

(4) $A-4I = \begin{bmatrix} 1 & -6 & 2 \\ 1 & -6 & 2 \\ -1 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

yes; a basis is $\left\{\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$

(5) $A-\lambda I = \begin{bmatrix} -3-\lambda & 2 \\ -10 & 6-\lambda \end{bmatrix} \Rightarrow$ char poly $= (-3-\lambda)(6-\lambda)+20$

$= \lambda^2 - 3\lambda - 18 + 20 = \lambda^2 - 3\lambda + 2$

$= (\lambda-1)(\lambda-2)$

For $\lambda_1 = 1$

$A - I = \begin{bmatrix} -4 & 2 \\ -10 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ basis: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

For $\lambda_2 = 2$

$A - 2I = \begin{bmatrix} -5 & 2 \\ -10 & 4 \end{bmatrix} \sim \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} 2/5 \\ 1 \end{bmatrix}$ basis: $\left\{\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$

(6) Triangular, so eigenvalues are $1, -3, -4$. As all distinct eigenspaces of dim'n 1. For $\lambda_1 = 1$, it's clear that $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ is a basis.

For $\lambda_2 = -3$

$A+3I = \begin{bmatrix} 4 & 16 & -12 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{x} = x_2\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ a basis is $\left\{\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}\right\}$

For $\lambda_3 = -4$

$A+4I = \begin{bmatrix} 5 & 16 & -12 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 20 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_3\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

a basis is $\left\{\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}\right\}$

⑦ As it's triangular $\lambda_1 = 3$, $\lambda_2 = 2$

For $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_3 \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix} \quad \text{a basis: } \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\}$$

For $\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{a basis } \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

⑧
$$\begin{bmatrix} -1 & 1 & -1 \\ 3 & 3 & 3 \\ 1 & -3 & -2 \end{bmatrix} \underbrace{\sim}_{\text{one scaling } R_1 \text{ by } -1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & 0 \\ 0 & -2 & -3 \end{bmatrix} \underbrace{\sim}_{\text{one scaling } R_2 \text{ by } 1/6} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim$$

$$\det(A) = \frac{(-1)^0}{-1/6}(-3) = 18$$

⑨ $\det(A^{-1}) = \frac{1}{\det(A)}$, so let's figure that out

$$\begin{bmatrix} 1 & 3 & 4 \\ -5 & -4 & -3 \\ 2 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 11 & 17 \\ 0 & -6 & -13 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -9 \\ 0 & -6 & -13 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -9 \\ 0 & 0 & 41 \end{bmatrix}$$

only row add'n operations

$$\det(A) = -41 \Rightarrow \det(A^{-1}) = -\frac{1}{41}$$

True/False

① True, the eigenspace for λ is $Nul(A - \lambda I)$

② False, almost all rotation matrices have no eigenvectors, e.g. $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

③ False, again rotation matrix example

④ True, input dimension must equal output dimension

⑤ True, A invertible iff $\dim(Nul\ A) = 0$

6. False, scaling a row scales the determinant
7. False, $\det(A)$ may equal -1
8. False, 3 is simply an upper bound on dimension

More Difficult Problems

1. Find $\vec{x}_0$ in the basis of eigenvectors first.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ 3 & 9 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 18 & -2 & -18 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & -9 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \vec{x}_0 = 5\vec{v}_1 - \vec{v}_2$$

$$A^k\vec{x}_0 = (2^k)5\vec{v}_1 - (1/2)^k\vec{v}_2$$

As $k \to \infty$, the solution grows exponentially and behaves like the term $5(2^k)\vec{v}_1$.

2. $\det R_\theta = \cos^2\theta + \sin^2\theta = \underline{1}$

3. $\det(R_\theta - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta$

$$= \lambda^2 - 2\cos\theta\lambda + \cos^2\theta + \sin^2\theta$$
$$= \lambda^2 - 2(\cos\theta)\lambda + 1$$

$$\lambda = \frac{2(\cos\theta) \pm \sqrt{4\cos^2\theta - 4}}{2} = (\cos\theta) \pm \sqrt{\cos^2\theta - 1}$$

$\leftarrow$ only non-negative when $\cos^2\theta = 1$,
$\Leftrightarrow$ when $\cos\theta = \pm 1$,
$\Leftrightarrow$ when $\theta = 0, \pi$
(or $\theta = \pi k$ for some integer k)

R_θ has real roots only when $\theta = 0$ or π essentially.

When $\theta = 0$, $R_\theta = Id$, and all vectors are eigenvectors with eigenvalue 1. Thus, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a reasonable choice.

When $\theta = \pi$, $R_\theta = -Id$, and again all vectors are eigenvectors with eigenvalue -1. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is again a reasonable choice.

4 $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, $\begin{bmatrix}-3 & 0\\0 & -3\end{bmatrix}$

$\downarrow$ all vectors are eigenvectors with eigenvalue 0

$\searrow$ all vectors are eigenvectors with eigenvalue -3

Essentially, all valid answers must be some scalar multiple of the identity matrix.