Assignment 10

`CAS CS 132`: *Geometric Algorithms*

Due November 20, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 4.4, 5.3, and 6.1.

Basic Problems

1. Determine the change-of-basis matrix for the following basis.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \end{bmatrix} \right\}$$

2. Determine the change-of-basis matrix for the following basis.

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 10 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \\ 3 \end{bmatrix} \right\}$$

3. Determine a diagonalization of the following matrix, if it exists. You can leave the rightmost factor in the form P^{-1}, i.e., you don't have to compute the inverse of P.

$$\begin{bmatrix} 2 & -6 \\ 2 & -5 \end{bmatrix}$$

4. Determine a diagonalization of the following matrix, if it exists. You can leave the rightmost factor in the form P^{-1}, i.e., you don't have to compute the inverse of P.

$$\begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & -1 \\ 6 & 4 & -5 \end{bmatrix}$$

5. Determine a diagonalization of the following matrix, if it exists, given that its characteristic polynomial is $\lambda^3 - 2\lambda^2 - 3\lambda$. You can leave the rightmost factor in the form P^{-1}, i.e., you don't have to

compute the inverse of P.

$$\begin{bmatrix} -1 & -3 & -6 \\ -8 & -6 & -18 \\ 4 & 3 & 9 \end{bmatrix}$$

6. Determine the following: (a) the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$, (b) the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, (c) the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$, (d) the unit-length normalizations of $\mathbf{v}_1$ and $\mathbf{v}_2$. You may need to use a calculator for some of these values, but you must simplify the expression as much as you can before giving the approximate result to a couple decimal places.

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

7. Determine the following: (a) the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$, (b) the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, (c) the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$, (d) the unit-length normalizations of $\mathbf{v}_1$ and $\mathbf{v}_2$. You may need to use a calculator for some of these values, but you must simplify the expression as much as you can before giving the approximate result to a couple decimal places.

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. If $\mathcal{B}$ is the standard basis for $\mathbb{R}^n$, then for any $\mathbf{x} \in \mathbb{R}^n$ we have that $[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$
2. All square matrices are diagonalizable.
3. Similar matrices have the same eigenvalues.
4. Similar matrices have the same eigenvectors.
5. If a matrix does not have n distinct eigenvalues, then it is not diagonalizable.
6. If a matrix is diagonalizable, then it is invertible.
7. A diagonalization of a matrix A if it exists, is unique.
8. If $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
9. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

More Difficult Problems

1. For the following matrix, its characteristic polynomial is $-\lambda^3 + 5\lambda^2 - 6\lambda$. Find the eigenvalues and bases for the eigenspaces.

$$\begin{bmatrix} 0 & 3 & -3 \\ -2 & 6 & -4 \\ -2 & 3 & -1 \end{bmatrix}.$$

2. A is a 5×5 matrix with characteristic polynomial $(\lambda - 2)^2(\lambda + 1)^2\lambda$ and rank $A - 2I = \text{rank}A + I = 3$. (a) What can you conclude about rank, A? (b) Can you conclude that A is diagonalizable? Please justify your answer, either way.

3. Suppose that $AP = PD$ for a square matrix A, diagonal matrix D, and arbitrary $m \times n$ matrix P (not necessarily invertible). Show that nonzero columns of P are eigenvectors of A and find their corresponding eigenvalues in terms of entries of D.

4. Write out a matrix equation to figure out the set of vectors that are orthogonal to both vectors listed below.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 3 \\ -6 \\ -2 \\ 2 \end{bmatrix}.$$

   Now solve this equation, and find the general solution. What is the dimensionality of the subspace of vectors orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$?

5. Prove that the angle between two vectors does not depend on their length. In other words, show that for any pair of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the angle between them is equal to the angle between $a\mathbf{u}, b\mathbf{v}$ for any $a, b \neq 0$.

Challenge Problems (Optional)

1. A Jordan block is a square matrix of the form:

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix},$$

   where λ denotes an eigenvalue, and there are 1's on the superdiagonal. (a) Calculate the characteristic polynomial and the dimension of the eigenspace for λ. (b) Now, assume you are given a characteristic polynomial $(\lambda - 3)^3(\lambda + 2)^2$. Using Jordan blocks of varying sizes, construct six matrices that achieve this characteristic polynomial and achieve all possible combinations of eigenspace dimensions for $\lambda = 3$ (1,2, or 3) and for $\lambda = 2$ (1 or 2).