Assignment 10 Solutions

Basic Problems

① $$\begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}^{-1} = \frac{1}{-5-(-6)} \begin{bmatrix} -5 & 3 \\ -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -5 & 3 \\ -2 & 1 \end{bmatrix}}$$

② $$\left[\begin{array}{cccccc} 1 & -3 & -2 & 1 & 0 & 0 \\ -3^{+3} & 10^{-9} & 8^{-6} & 0^{+3} & 1 & 0 \\ -2^{+2} & 5^{-6} & 3^{-4} & 0^{+2} & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & -3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & -1^{+1} & -1^{+2} & 2^{+3} & 0^{+1} & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & -3 & -2^{+2} & 1^{+10} & 0^{+2} & 0^{+2} \\ 0 & 1 & 2^{-2} & 3^{-10} & 1^{-2} & 0^{-2} \\ 0 & 0 & 1 & 5 & 1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & -3^{+3} & 0 & 11^{-21} & 2^{-3} & 2^{-6} \\ 0 & 1 & 0 & -7 & -1 & -2 \\ 0 & 0 & 1 & 5 & 1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -10 & -1 & -4 \\ 0 & 1 & 0 & -7 & -1 & -2 \\ 0 & 0 & 1 & 5 & 1 & 1 \end{array}\right]$$

$$\boxed{\begin{bmatrix} -10 & -1 & -4 \\ -7 & -1 & -2 \\ 5 & 1 & 1 \end{bmatrix}}$$

(3)

$$\det(A - \lambda I) = (2-\lambda)(-5-\lambda) + 12$$
$$= -10 - 2\lambda + 5\lambda + \lambda^2 + 12$$
$$= \lambda^2 + 3\lambda + 2$$
$$= (\lambda+2)(\lambda+1)$$

$\lambda = -2, -1$

$$A + I = \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

④ $\begin{bmatrix} -2-\lambda & 0 & 0 \\ 2 & -1-\lambda & -1 \\ 6 & 4 & -5-\lambda \end{bmatrix} \sim \begin{bmatrix} -2-\lambda & 0 & 0 \\ 2(-5-\lambda) & (-1-\lambda)(-5-\lambda) & -(-5-\lambda) \\ 6 & 4 & -5-\lambda \end{bmatrix}$

$$\begin{bmatrix} -2-\lambda & 0 & 0 \\ 2(-5-\lambda)+6 & (-1-\lambda)(-5-\lambda)+4 & 0 \\ 6 & 4 & -5-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \frac{1}{\cancel{-5-\lambda}}(-2-\lambda)\left((-1-\lambda)(-5-\lambda)+4\right)\cancel{(-5-\lambda)}$$

$$= (-2-\lambda)(5+\lambda+5\lambda+\lambda^2+4)$$

$$= -(\lambda+2)(\lambda^2+6\lambda+9)$$

$$= -(\lambda+2)(\lambda+3)^2$$

$$\lambda = -2, -3$$

$$A+2I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 6^{-6} & 4^{-3} & -3^{+3} \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 6 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Not digonalizable. Does not have eigenbasis.

5

$$\lambda(\lambda^2 - 2\lambda - 3) = \lambda(\lambda - 3)(\lambda + 1)$$

$$\lambda = 3, 0, -1$$

$$A - 3I = \begin{bmatrix} -4 & -3 & -6 \\ -8^{+8} & -9^{+6} & -18^{+12} \\ 4^{-4} & 3^{-3} & 6^{-6} \end{bmatrix} \sim \begin{bmatrix} -4 & -3 & -6 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 & -6 \\ -8 & -6 & -18 \\ 4 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 \\ 4^{-4} & 3^{-12} & 9^{-24} \\ 4 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 \\ 0 & -9 & -15 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix}$$

$$A + I = \begin{bmatrix} 0 & -3 & -6 \\ -8 & -5 & -18 \\ 4 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 10 \\ -8^{+8} & -5^{+6} & -18^{+20} \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -3 & -1 \\ -2 & -5 & -2 \\ 1 & 3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

⑥

a. $\|\vec{v}_1\| = \sqrt{49+9} = \sqrt{58}$ $\quad \|\vec{v}_2\| = \sqrt{1+25} = \sqrt{26}$

b. $\langle v_1, v_2 \rangle = 7(-1) + 3(5) = -7 + 15 = 8$

$$\cos^{-1}\left(\frac{8}{\sqrt{58}\cdot\sqrt{26}}\right) \approx 1.36 \text{ rad}$$

c. $\|\vec{v}_1 - \vec{v}_2\| = \left\| \begin{bmatrix} 8 \\ -2 \end{bmatrix} \right\| = \sqrt{64+4} = \sqrt{68}$

d. $\dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{58}}\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ $\quad \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{1}{\sqrt{26}}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

⑦

a. $\|\vec{v}_1\| = \sqrt{4+16+25} = \sqrt{45}$ $\quad \|\vec{v}_2\| = \sqrt{4+4+4} = \sqrt{12}$

b. $\langle \vec{v}_1, \vec{v}_2 \rangle = -2(2) + 4(2) + (-5)(-2) = -4+8+10 = 14$

$$\cos^{-1}\left(\frac{14}{\sqrt{45}\,\sqrt{12}}\right) \approx 0.92 \text{ rad}$$

c. $\|\vec{v}_1 - \vec{v}_2\| = \left\| \begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix} \right\| = \sqrt{16+4+9} = \sqrt{29}$

d. $\dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{45}}\begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix}$ $\quad \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{1}{\sqrt{12}}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix}$

True / False

1. True
2. False $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. True
4. False $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ & $\begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix}$
5. False $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
6. False $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
7. False $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
8. True
9. False $\begin{bmatrix} \underline{1} & 0 \\ \underline{1} & 0 \end{bmatrix}$ $\begin{bmatrix} \underline{1} \\ \underline{1} \end{bmatrix} \in \text{Col}(A)$

   $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \text{Nul}(A)$

More Difficult Problems

① $-\lambda^3 + 5\lambda^2 - 6\lambda = -\lambda(\lambda^2 - 5\lambda + 6)$

$= -\lambda(\lambda - 3)(\lambda - 2)$

$\lambda = 3, 2, 0$

$$A - 3I = \begin{bmatrix} -3 & 3 & -3 \\ -2 & 3 & -4 \\ -2 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 2^{-2} & -3^{+2} & 4^{-2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & +1 & -2 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Nul}(A - 3I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$A - 2I = \begin{bmatrix} -2 & 3 & -3 \\ -2 & 4 & -4 \\ -2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ -2^{+2} & 3^{-4} & -3^{+4} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Nul}(A - 2I) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 0 & 3 & -3 \\ -2 & 6 & -4 \\ -2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ -2^{+2} & 3^{-6} & -1^{+4} \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{ccc} & +3 & -3 \end{array} \\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Nul}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2.

(a) $\lambda = 0 \Rightarrow \dim(\text{Nul}(A)) \geq 1$

$\dim(\text{Nul}(A)) \leq 1$ since deg. in char. poly $= 1$

$\Rightarrow$ $\dim(\text{Nul}(A)) = 1$ and

$\text{rank}(A) + \dim(\text{Nul}(A)) = 5 \Rightarrow$ $\dim(\text{Nul}(A)) = 4$

(b) $\lambda = 2, 0, -1$

$\text{rank}(A - 2I) + \dim(\text{Nul}(A - 2I)) = 5$

$\text{rank}(A - 2I) = 3$

$\Downarrow$

$\dim(\text{Nul}(A - 2I)) = 2$

$\text{rank}(A + I) + \dim(\text{Nul}(A + I)) = 5$

$\text{rank}(A + I) = 3$

$\Downarrow$

$\dim(\text{Nul}(A + I)) = 2$

$\Rightarrow$ A has an eigenbasis $\Rightarrow$ A is diagonalizable

③ $P = [\vec{p}_1 \quad \dots \quad \vec{p}_n]$ $\qquad D = [d_1\vec{e}_1 \quad \dots \quad d_n\vec{e}_n]$

$$AP = [A\vec{p}_1 \quad \dots \quad Ap_n]$$

$$PD = [P(d_1\vec{e}_1) \quad \dots \quad P(d_n\vec{e}_n)] = [d_1 P\vec{e}_1 \quad \dots \quad d_n P\vec{e}_n]$$

$$= [d_1\vec{p}_1 \quad \dots \quad d_n\vec{p}_n]$$

eigenvalues

$$AP = PD \implies A\vec{p}_1 = d_1\vec{p}_1 \text{ and} \dots \text{ and } A\vec{p}_n = d_n\vec{p}_n$$

eigenvector if $\vec{p}_i \neq \vec{0}$

④ A

$$\begin{array}{c} \begin{matrix} -3 & +6 & +3 & -9 \end{matrix} \\ \begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{matrix} \quad & \quad & +1 & -7 \end{matrix} \\ \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

$x_1 = 2x_2 + 4x_4$

x_2 is free

$x_3 = 7x_4$

x_4 is free

$\dim(\mathrm{Nul}(A)) = 2$

5

$$\cos^{-1}\left(\frac{\langle a\vec{u}, b\vec{v}\rangle}{\|a\vec{u}\|\cdot\|b\vec{v}\|}\right) = \cos^{-1}\left(\frac{\cancel{a}\cancel{b}\langle u, v\rangle}{\cancel{a}\|\vec{u}\|\cdot\cancel{b}\|\vec{v}\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle u, v\rangle}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)$$