Assignment 11

CAS CS 132: *Geometric Algorithms*

Due December 4, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 6.2, 6.3, and 6.5.

Basic Problems

1. Determine if the following set of vectors is orthogonal.

$$\left\{ \begin{bmatrix} 9 \\ -18 \\ -18 \end{bmatrix}, \begin{bmatrix} -2 \\ -14 \\ 13 \end{bmatrix} \right\}$$

2. Determine if the following set of vectors is orthogonal.

$$\left\{ \begin{bmatrix} 0 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -6 \end{bmatrix} \right\}$$

3. Express the vector $\mathbf{v}$ in terms of the given orthonormal basis.

$$\mathbf{v} = \begin{bmatrix} -8 \\ 9 \end{bmatrix} \qquad \mathcal{B} = \left[\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right]$$

4. Express the vector $\mathbf{v}$ in terms of the given orthonormal basis.

$$\mathbf{v} = \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix} \qquad \mathcal{B} = \left[\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} \right]$$

5. Determine the projection of $\mathbf{u}$ onto $\text{span}\{\mathbf{v}\}$.

$$\mathbf{u} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

6. Determine the projection of $\mathbf{u}$ onto $\text{span}\{\mathbf{v}\}$.

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ -10 \end{bmatrix}$$

7. Determine the project of $\mathbf{u}$ onto the orthnormal set of vectors V.

$$\mathbf{u} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} \qquad V = \left[\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \right]$$

8. Solve for all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are defined as follows.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 3 & -6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

9. Solve for all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are defined as follows.

$$A = \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -3 \\ -1 & -2 & 7 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -7 \\ -10 \\ 2 \end{bmatrix}$$

10. Solve for all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are defined as follows.

$$A = \begin{bmatrix} -1 & 1 \\ -3 & 1 \\ -3 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 11 \\ 15 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. There can be a linear dependence relationship between vectors in an orthogonal set.
2. In $\mathbb{R}^4$, any set of vectors that has 5 members cannot be an orthogonal set.
3. Orthogonal matrices are invertible.
4. Orthogonal matrices have determinant 1.
5. An $m \times n$ orthonormal matrix may have $m < n$.

6. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

7. For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.

8. Given any matrix-vector equation $A\mathbf{x} = \mathbf{b}$, there always exists a least-squares solution.

9. Given any matrix-vector equation $A\mathbf{x} = \mathbf{b}$, there always exists a unique least-squares solution.

10. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto Col A.

More Difficult Problems

1. Do the following six vectors form an orthogonal set? Argue why or why not. (*Hint:* You should not need to explicitly check every possible pair; you can argue more generally.)

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v_6} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 4 \end{bmatrix}.$$

2. Project $\mathbf{b} = [2\ 5\ 6\ 6]^T$ onto Span$\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$, using the normal equations.

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

3. Show that any 2×2 rotation matrix R_θ is an orthogonal matrix. Show the same for any 3×3 rotation matrices R_x^θ, R_y^θ, and R_z^θ about the x-, y-, and z-axes, respectively. (See Assignment 8 or Graphics Lecture 15 if you need a recall of some of these definitions).

4. Find a matrix that implements orthogonal projection onto Span$\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1}$ and $\mathbf{u_2}$ are orthogonal (as a given, no need to show).

5. Find a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthonormal.

Challenge Problems (Optional)

1. Argue that for an orthogonal matrix A, its row vectors also form an orthonormal set (*Hint:* Use the fact that for any invertible matrix A, we have $AA^{-1} = A^{-1}A = \text{Id}$).

2. Argue that the product UV of two orthogonal matrices U, V is an orthogonal matrix. (*Hint:* As an intermediate step, argue that $(UV)^{-1} = (UV)^T$).