Assn 11 Solutions

1) $\begin{bmatrix}9\\-18\\-18\end{bmatrix} \cdot \begin{bmatrix}-2\\-14\\13\end{bmatrix} = \begin{bmatrix}9 & -18 & -18\end{bmatrix}\begin{bmatrix}-2\\-14\\13\end{bmatrix} = -18+14(18)-18(13)$
$= 18(-1+14-13)$
$= 0$

$\boxed{\text{yes, orthogonal}}$

2) $\begin{bmatrix}0 & 4 & 4 & 0\end{bmatrix}\begin{bmatrix}-2\\2\\0\\0\end{bmatrix} = 8 \neq 0 \Rightarrow \boxed{\text{no, not orthogonal}}$

3) $c_1 = \begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix} = (+8+9)\sqrt{2}/2 = 17\sqrt{2}/2$

$c_2 = \begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix} = (-8+9)\sqrt{2}/2 = \sqrt{2}/2$

$\boxed{\vec{v} = \frac{17\sqrt{2}}{2}\begin{bmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix} + \sqrt{2}/2\begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix}}$

4) $c_1 = \begin{bmatrix}3 & 9 & -3\end{bmatrix}\begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\\ 0\end{bmatrix} = 6\sqrt{2}$

$c_2 = \begin{bmatrix}3 & 9 & -3\end{bmatrix}\begin{bmatrix}\sqrt{6}/6\\ -\sqrt{6}/6\\ -\sqrt{6}/3\end{bmatrix} = \sqrt{6}/6(3-9+2(3)) = 0$

$\boxed{\vec{v} = 6\sqrt{2}\begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\\ 0\end{bmatrix} + 3\sqrt{3}\begin{bmatrix}-\sqrt{3}/3\\ \sqrt{3}/3\\ -\sqrt{3}/3\end{bmatrix}}$

$c_3 = \begin{bmatrix}3 & 9 & -3\end{bmatrix}\begin{bmatrix}-\sqrt{3}/3\\ \sqrt{3}/3\\ -\sqrt{3}/3\end{bmatrix} = \sqrt{3}/3(-3+9+3) = 3\sqrt{3}$

5) $\frac{\vec{u}^T\vec{v}}{\|\vec{v}\|^2} = \frac{0}{\|\vec{v}\|^2} = 0 \Rightarrow \boxed{\text{proj. of } \vec{u} \text{ onto span}(\vec{v}) \text{ is } \vec{0}}$

6) $\frac{\vec{u}^T\vec{v}}{\|\vec{v}\|^2} = \frac{-6-50}{4+100} = \frac{-56}{104}$
$= -\frac{7}{13}$

$\boxed{\text{proj. is } -\frac{7}{13}\begin{bmatrix}-2\\0\\-10\end{bmatrix} = \begin{bmatrix}14/13\\0\\70/13\end{bmatrix}}$

7) $c_1 = [-3 \ \ 6 \ \ 2]\begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{2}(3+6) = 9\sqrt{2}/2$

$C_2 = [-3 \ \ 6 \ \ 2]\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} = -2+4-2/3 = 4/3$

proj. is $9\sqrt{2}/2 \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} + 4/3 \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$

$= \begin{bmatrix} -9/2 \\ 9/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 8/9 \\ 8/9 \\ -4/9 \end{bmatrix} = \begin{bmatrix} -65/18 \\ 97/18 \\ -4/9 \end{bmatrix}$

8) $A^TA = \begin{bmatrix} 18 & -18 \\ -18 & 45 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$

$\left[\begin{array}{cc|c} 18 & -18 & 18 \\ -18 & 45 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 27 & 24 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 8/9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 17/9 \\ 0 & 1 & 8/9 \end{array}\right]$

$\hat{x} = \begin{bmatrix} 17/9 \\ 8/9 \end{bmatrix}$

9) $A^TA = \begin{bmatrix} 2 & 3 & -11 \\ 3 & 6 & -21 \\ -11 & -21 & 74 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} -9 \\ -21 \\ 72 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 2 & 3 & -11 & -9 \\ 3 & 6 & -21 & -21 \\ -11 & -21 & 74 & 72 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & -11 & -9 \\ 1 & 3 & -10 & -12 \\ -11 & -21 & 74 & 72 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & -10 & -12 \\ 0 & -3 & 9 & 15 \\ 0 & 12 & -36 & -60 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\cancel{\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -27 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -3 \end{array}\right]} \qquad \cancel{\hat{x} = \begin{bmatrix} -27 \\ -14 \\ -3 \end{bmatrix}}$

$\hat{x}_1 = 3 + \hat{x}_3$
$\hat{x}_2 = -5 + 3\hat{x}_3$
$\hat{x}_3$ free

10)
$$A^TA = \begin{bmatrix} 19 & -4 \\ -4 & 2 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} -79 \\ 12 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 1 & 6 \\ 19 & -4 & -79 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 6 \\ 1 & 5 & -25 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 5 & -25 \\ 0 & 11 & -44 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & -25 \\ 0 & 1 & -4 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & -4 \end{array}\right] \qquad \hat{x} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$$

~~More Diff. Probs~~ True/False

1) F, orthogonal sets are lin. ind.

2) T, as they would form a lin. ind. set of 5 vectors in $\mathbb{R}^4$, and $5 > 4$

3) T, a fact proven in lecture

4) F, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has $\det = -1$

5) F, would imply $n > m$ lin. ind. vectors in $\mathbb{R}^m$

6) T, ~~pro~~ the span remains the same ($\text{span}\{\vec{v}\} = \text{span}\{c\vec{v}\}$ for $c \neq 0$)

7) T, by def'n (or argued in class)

8) T, there is always a minimizer of distance

9) F, the minimizer is unique, but may be mapped onto by multiple points e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

10) T, def'n

More Difficult Problems

1. Pairs where we take one from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and one from $\{\vec{v}_4, \vec{v}_5, \vec{v}_6\}$ are obviously orthogonal. So we just need to check within each group.

$\vec{v}_1^T \vec{v}_2 = -2+2=0$ $\qquad \vec{v}_4^T \vec{v}_5 = 6-6=0$

$\vec{v}_1^T \vec{v}_3 = -5+4+1=0$ $\qquad \vec{v}_4^T \vec{v}_6 = 3-3=0$

$\vec{v}_2^T \vec{v}_3 = -2+2=0$ $\qquad \vec{v}_5 \vec{v}_6 = 2+2-4=0$

Thus, all pairs are orthogonal, and an orthogonal set is given.

2. $A^TA = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\qquad A^T\vec{b} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$ $\qquad$ proj. is $A\hat{x} = \begin{bmatrix} 1/3 + 14/3 + 0 \\ 1/3 + 0 + 5/3 \\ 0 + 14/3 - 5/3 \\ -1/3 + 14/3 + 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

3. $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

dot prod of columns $= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

norm of columns $= \cos^2\theta + \sin^2\theta = 1$

$$R_x\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

also col 1 clearly orthogonal to col 2 & 3
col 2 & 3 orthogonal by calculation for R_θ

norm 1 by previous calculations

$$R_y\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

col 2 orthogonal to cols 1 & 3 clearly
col 1 & 3 orthogonal by R_θ calculation

$$R_z\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Col 3 orthogonal to cols 1 & 2 clearly
Cols 1 & 2 orthogonal by R_θ calculation

norm 1 by previous

(4) $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \text{Col} \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$

Let's call this U

Projecting $\vec{b}$ onto Col U is achieved via the formula!

$$\text{proj}_{\text{Col}\,U}\vec{b} = U(U^TU)^{-1}U^T\vec{b}$$

A $\vec{u}_1, \vec{u}_2$ are orthogonal, $\vec{u}_1 \cdot \vec{u}_2 = 0$ & U^TU is diagonal

$$= U \begin{bmatrix} 1/\|u_1\|^2 & 0 \\ 0 & 1/\|u_2\|^2 \end{bmatrix} \begin{bmatrix} - & \vec{u}_1^{\,T} & - \\ - & \vec{u}_2^{\,T} & - \end{bmatrix} \vec{b}$$

$$= U \begin{bmatrix} - & \left(\frac{1}{\|u_1\|^2}\right)\vec{u}_1^{\,T} & - \\ - & \left(\frac{1}{\|u_2\|^2}\right)\vec{u}_2^{\,T} & - \end{bmatrix} = \boxed{\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} - & \left(\frac{1}{\|u_1\|^2}\right)\vec{u}_1^{\,T} & - \\ - & \left(\frac{1}{\|u_2\|^2}\right)\vec{u}_2^{\,T} & - \end{bmatrix}} \vec{b}$$

Note: $\vec{b}$ maps to $\dfrac{\vec{u}_1 \cdot \vec{b}}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \dfrac{\vec{u}_2 \cdot \vec{b}}{\vec{u}_2 \cdot \vec{u}_2}$

(5) $A^TA\hat{x} = A^T\vec{b}$

For A orthonormal, $A^TA = Id$

$$\boxed{\hat{x} = A^T\vec{b}}$$