Assignment 12

CAS CS 132: *Geometric Algorithms*

Due December 10, 2025 by 8:00PM

Your solutions should be submitted via Gradescope as a single pdf. They must be exceptionally neat (or typed) and the final answer in your solution to each problem must be abundantly clear, e.g., surrounded in a box. Also make sure to cite your sources per the instructions in the course manual. **In all cases, you must show your work and explain your answer.**

These problems are based (sometimes heavily) on problems that come from *Linear Algebra and it's Applications* by David C. Lay, Steven R. Lay, and Judi J. McDonald. As a reminder, we recommend this textbook as a source for supplementary exercises. The relevant sections for this assignment are: 6.6, 7.1-7.4.

Basic Problems

1. Determine the least squares linear fit to the following data points. Sketch the graphs to verify your answer. You should set up the required matrix equations by hand, but you can use a computer to determine the coefficients of your model.

$$\{(-5,2),(-2,1),(0,0),(2,4),(6,6)\}$$

2. Determine the least squares quadratic fit to the data points from the previous problem. Sketch the graphs to verify your answer. You should set up the required matrix equations by hand, but you can use a computer to determine the coefficients of your model.

3. Determine the design matrix for the following function model using the data points from the previous problem.

$$f(x) = \beta_1 \log_2(x) + \beta_2 x \sin(x) + \beta_3 x^2 + 7\beta_4$$

4. Determine an orthogonal diagonalization of the following matrix.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

5. For the following quadratic form $Q(\mathbf{x})$, determine the symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

$$Q([x_1\ x_2\ x_3]^T) = 3x_1^2 + 4x_1x_2 + 5x_2x_1 - x_1x_3$$

6. For the following matrix A, determine the quadratic form $Q(\mathbf{x})$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

7. What is the type of the quadratic form from the prior question?

8. Calculate the SVD for the following matrix.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, give a counterexample when possible.

1. Given a model function $f(x) = \beta_0 + \beta_1^2 x$, we may minimize for least squares error by using the least squares solution to a matrix-vector equation.

2. If X denotes the design matrix for a least squares regression problem, then $X^T X$ is always invertible.

3. There are orthogonally diagonalizable matrices that are not symmetric.

4. An $n \times n$ symmetric matrix must have n distinct eigenvalues.

5. For a symmetric matrix, the dimension of each eigenspace is equal to the algebraic multiplicity of the corresponding eigenvalue.

6. $||\mathbf{x}||^2$ is not a quadratic form.

7. A positive definite quadratic form $Q(\mathbf{x})$ satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

8. $\mathbf{x}^T A\mathbf{x}$ defines a quadratic form only if A is symmetric.

9. An orthogonal diagonalization of a symmetric matrix $A = PDP^T$ is also a singular value decomposition of A.

More Difficult Problems

1. Consider the following multivariate function model: $f(x, y) = \beta_0 + \beta_1 x + \beta_2 y$, and solve for the β_0, β_1, $\beta_2 \in \mathbb{R}$ that minimize least squares error for the following data points:

$$\{(-2, 0, 0), (0, 0, 3), (0, 2, 0), (-2, -3, 3), (1, 5, -1)\}.$$

   You may use a computer only to do matrix inverses and matrix vector multiplications, but show all work otherwise. This includes the matrix-vector equation that you solve in the least squares sense (and thus the design matrix).

2. Minimize and maximize, **and** find the argmin and argmax (so four answers needed) of $\mathbf{x}^T A\mathbf{x}$ subject to the unit norm constraint $||\mathbf{x}|| = 1$, where:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}.$$

To help you in this task, we note that 4 is an eigenvalue of A.

3. Suppose the factorization below is an SVD of a matrix A with the entries in U and V rounded to two decimal places.

$$A = \begin{bmatrix} .40 & -.78 & .47 \\ .37 & -.33 & -.87 \\ -.84 & -.52 & -.16 \end{bmatrix} \begin{bmatrix} 7.10 & 0 & 0 \\ 0 & 4.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} .30 & -.51 & -.81 \\ .76 & .64 & -.12 \\ .58 & -.58 & 58 \end{bmatrix}$$

Please answer the following: (a) What is the rank of A? (b) Using no calculations, write down a basis for Col A and Nul A.