Assignment 12 Solutions

Basic Problems

① Design Matrix

$$X = \begin{bmatrix} 1 & -5 \\ 1 & -2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 6 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 5 & 1 \\ 1 & 25+4+4+36 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 69 \end{bmatrix}$$

$$X^T \vec{y} = \begin{bmatrix} 13 \\ -10-2+8+36 \end{bmatrix} = \begin{bmatrix} 13 \\ 32 \end{bmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} 865/344 \\ 147/344 \end{bmatrix} \approx \begin{bmatrix} 2.51 \\ 0.43 \end{bmatrix}$$

$$y = 2.51 + (0.43)x$$

② Design matrix

$$X = \begin{bmatrix} 1 & -5 & 25 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 6 & 36 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 5 & 1 & 69 \\ 1 & 69 & -125-8+8+216 \\ 69 & \boxed{91} & (25)^2+32+(36)^2 \end{bmatrix}$$

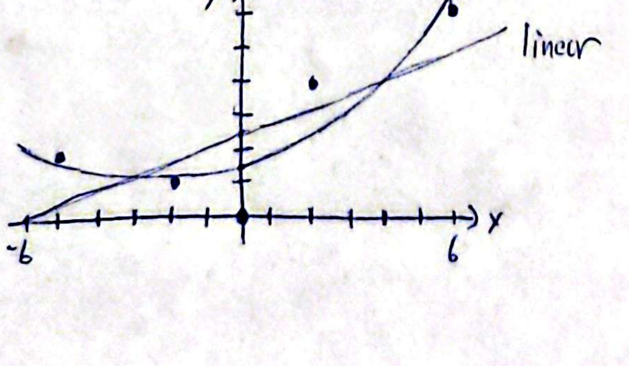
$(25)^2 = 625$

$$\begin{array}{r} 36 \\ \times 36 \\ \hline 216 \\ 108 \\ \hline 1296 \end{array} \qquad \begin{array}{r} 1296 \\ 625 \\ +\ 32 \\ \hline 1953 \end{array}$$

$$= \begin{bmatrix} 5 & 1 & 69 \\ 1 & 69 & 91 \\ 69 & 91 & 1953 \end{bmatrix}$$

$$X^T \vec{y} = \begin{bmatrix} 13 \\ 32 \\ 50+4+16+216 \end{bmatrix} = \begin{bmatrix} 13 \\ 32 \\ 286 \end{bmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T \vec{y} = \frac{1}{157{,}238} \begin{bmatrix} 223{,}500 \\ 52{,}985 \\ 12{,}661 \end{bmatrix} \approx \begin{bmatrix} 1.42 \\ 0.33 \\ 0.08 \end{bmatrix}$$



(3)

$$\begin{bmatrix} 1/32 & -5\sin(5) & 25 & 7 \\ 1/4 & -2\sin(-2) & 4 & 7 \\ 1 & 0 & 0 & 7 \\ 4 & 2\sin(2) & 4 & 7 \\ 64 & 6\sin(6) & 36 & 7 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

design matrix

(4)

$$A - \lambda I = \begin{bmatrix} (3-\lambda) & 1 \\ 1 & (3-\lambda) \end{bmatrix} \quad \det(A-\lambda I) = \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8$$
$$= (\lambda-4)(\lambda-2)$$

$$\lambda_1 = 4, \; A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\lambda_2 = 2, \; A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P^T}$$

(5)

$$A = \begin{bmatrix} 3 & 4.5 & -0.5 \\ 4.5 & 0 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$

(6) $Q(\vec{x}) = Q(x_1, x_2, x_3) = 2x_1^2 + 9x_2^2 + x_3^2 + 6x_2x_3$

(7)

$$\det(A - \lambda I) = (2-\lambda)[(9-\lambda)(1-\lambda) - 9] = (2-\lambda)[\lambda^2 - 10\lambda]$$
$$= -(\lambda-2)(\lambda)(\lambda-10)$$

All eigenvalues nonnegative, so Q is positive semidefinite.

8

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$ eigenvalues: $\lambda_1 = 3,\ \lambda_2 = 2$

singular values: $\sigma_1 = \sqrt{3},\ \sigma_2 = \sqrt{2}$

eigenvectors: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underset{\text{prop. to}}{\vec{u}_3 \propto} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -1\hat{i} + 2\hat{j} - 1\hat{k} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

(cross prod)

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}$$

True/False

1. F, the β_1 parameter is squared (model not linear in β_1)
2. F, $f(x) = \beta_0 + \beta_1 x$ & data points $\{(1,2),(1,3)\}$

   $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow X^TX = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ ~~[illegible]~~ det $= 0$
3. F, ~~[illegible]~~ the set of orthogonally diagonalizable matrices is equal to the set of symmetric matrices
4. F, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
5. T, for diagonalizability there must be an eigenbasis, so each eigenspace must be "full"

⑥ F, $\|x\|^2 = \vec{x}^T \underset{\substack{\uparrow \\ n\times n \text{ identity}}}{I} \vec{x}$

⑦ F, for $\vec{x}=0$ $\vec{x}^T Q \vec{x} = 0$

⑧ F, $\vec{x}^T \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} = x^2 + xy + y^2$

only quadratic terms

⑨ F, if A has ~~an~~ negative eigenvalues, then ~~sing values~~ corresponding singular values ~~[illegible]~~ are equal to their absolute value, e.g.:

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

More Difficult Problems

①

$$X = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & -3 \\ 1 & 1 & 5 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 5 & -3 & 4 \\ -3 & 9 & 11 \\ 4 & 11 & 38 \end{bmatrix} \qquad X^T \vec{z} = \begin{bmatrix} 5 \\ -7 \\ -14 \end{bmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T \vec{z} = \frac{1}{71} \begin{bmatrix} 193 \\ 102 \\ -76 \end{bmatrix} \approx \begin{bmatrix} 1.24 \\ 0.06 \\ -0.26 \end{bmatrix} \qquad f(x,y) = 1.24 + 0.06x - 0.26y$$

② $\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 & 5 \\ 1 & 5-\lambda & 1 \\ 5 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]$

$$- \left[(1-\lambda)-5\right]$$

$$+5\left[1-5(5-\lambda)\right]$$

$$= (1-\lambda)\left[\lambda^2-6\lambda+4\right] \cancel{-(1-\lambda)} + \lambda + 4 + 25\lambda - 120$$

$$= -\lambda^3 + 7\lambda^2 - 10\lambda + 4 + 26\lambda \cancel{+} - 116$$

$$= -\lambda^3 + 7\lambda^2 + 16\lambda - 112$$

$$= -(\lambda-4)(\lambda^2-3\lambda-28) = -(\lambda-4)(\lambda+4)(\lambda-7)$$

$\boxed{-4 \text{ is min}}$ $\boxed{7 \text{ is max}}$

$$A-7I = \begin{bmatrix} -6 & 1 & 5 \\ 1 & -2 & 1 \\ 5 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -11 & 11 \\ 0 & 11 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

argmax $\underset{\text{prop to}}{\propto} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\text{argmax} = \pm \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$A+4I = \begin{bmatrix} 5 & 1 & 5 \\ 1 & 9 & 1 \\ 5 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 9 & 1 \\ 0 & -44 & 0 \\ 0 & -44 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

argmin $\underset{\text{(prop. to)}}{\propto} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\text{argmin} = \pm \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

③ (a) rk A = # of non-zero sing. values = 2

(b) 3rd axis zeroed out, so Col A has basis formed by first two columns of U

$$\left\{ \begin{bmatrix} .40 \\ .37 \\ -.84 \end{bmatrix}, \begin{bmatrix} -.78 \\ -.33 \\ -.52 \end{bmatrix} \right\}$$

Again, as 3rd axis zeroed out, the 3rd column of V is a basis for Nul A (note is 3rd row of V^T)

$$\left\{ \begin{bmatrix} .58 \\ -.58 \\ .58 \end{bmatrix} \right\}$$