

# Linear Equations

**Geometric Algorithms**  
**Lecture 1**

# Outline

- » Give a few motivating examples for the study of linear systems
- » Formally define linear systems
- » Solve some systems of linear equations

# Keywords

Systems of linear equations

Solutions

Coefficient matrix

Augmented matrix

Elimination and Back-substitution

Replacement, interchange, scaling

Row Equivalence

(In)consistency

# Motivation

# Lines (Slope-Intercept Form)

$$y = mx + b$$

# Lines (Slope-Intercept Form)

$$y = \textcolor{red}{mx} + \textcolor{blue}{b}$$

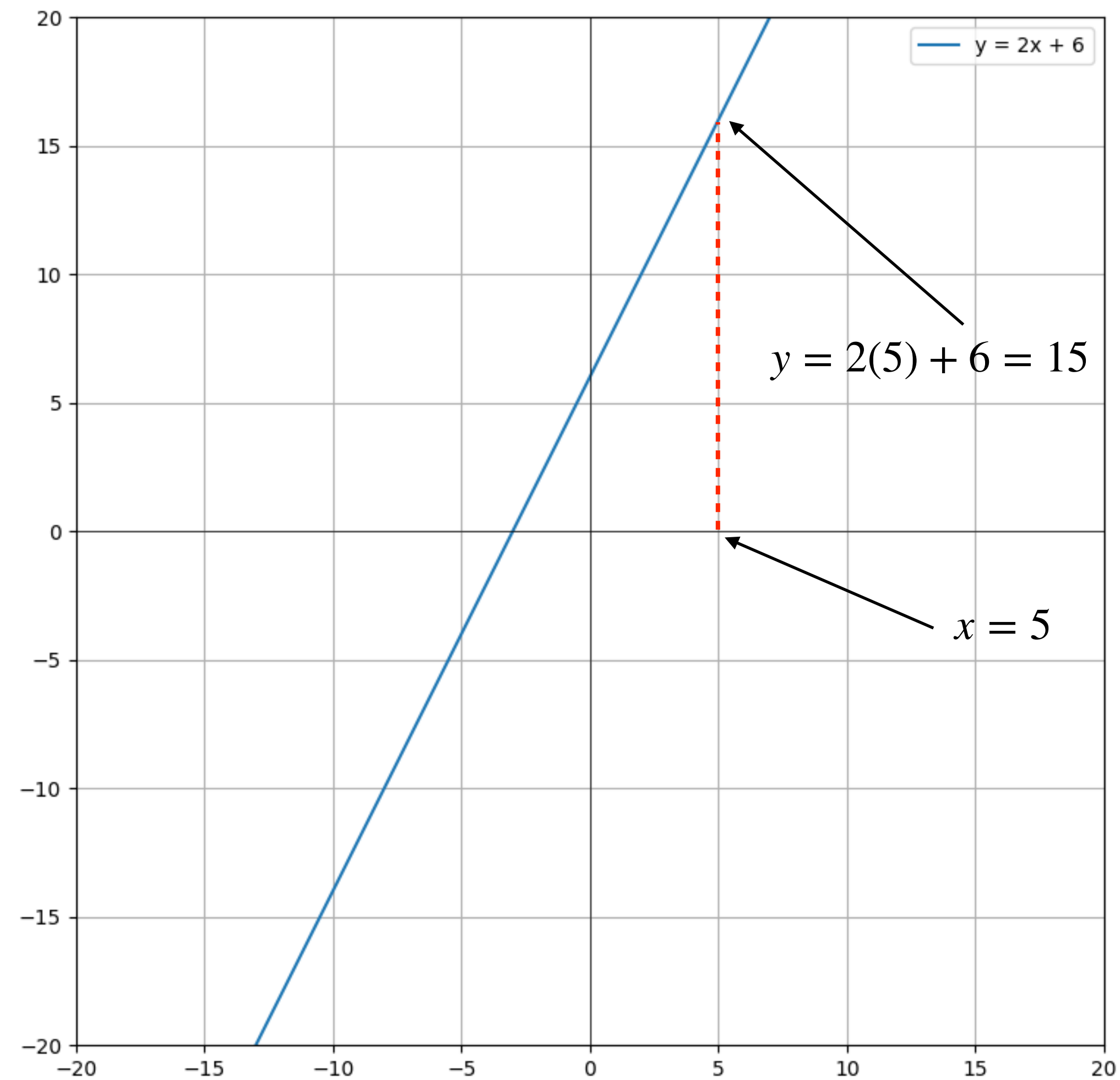
slope                      y-intercept

# Lines (Slope-Intercept Form)

$$y = \underbrace{mx}_{\text{slope}} + \underbrace{b}_{\text{y-intercept}}$$

Given a value of  $x$ , I can compute a value of  $y$

# Lines (Graph)





# Lines (General Form)

$$ax + by = c$$

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$$ax + by = c$$

x-intercept:  $-\frac{c}{a}$

# Lines (General Form)

$$ax + by = c$$

x-intercept:  $\frac{c}{a}$

y-intercept:  $\frac{c}{b}$

# Lines (General Form)

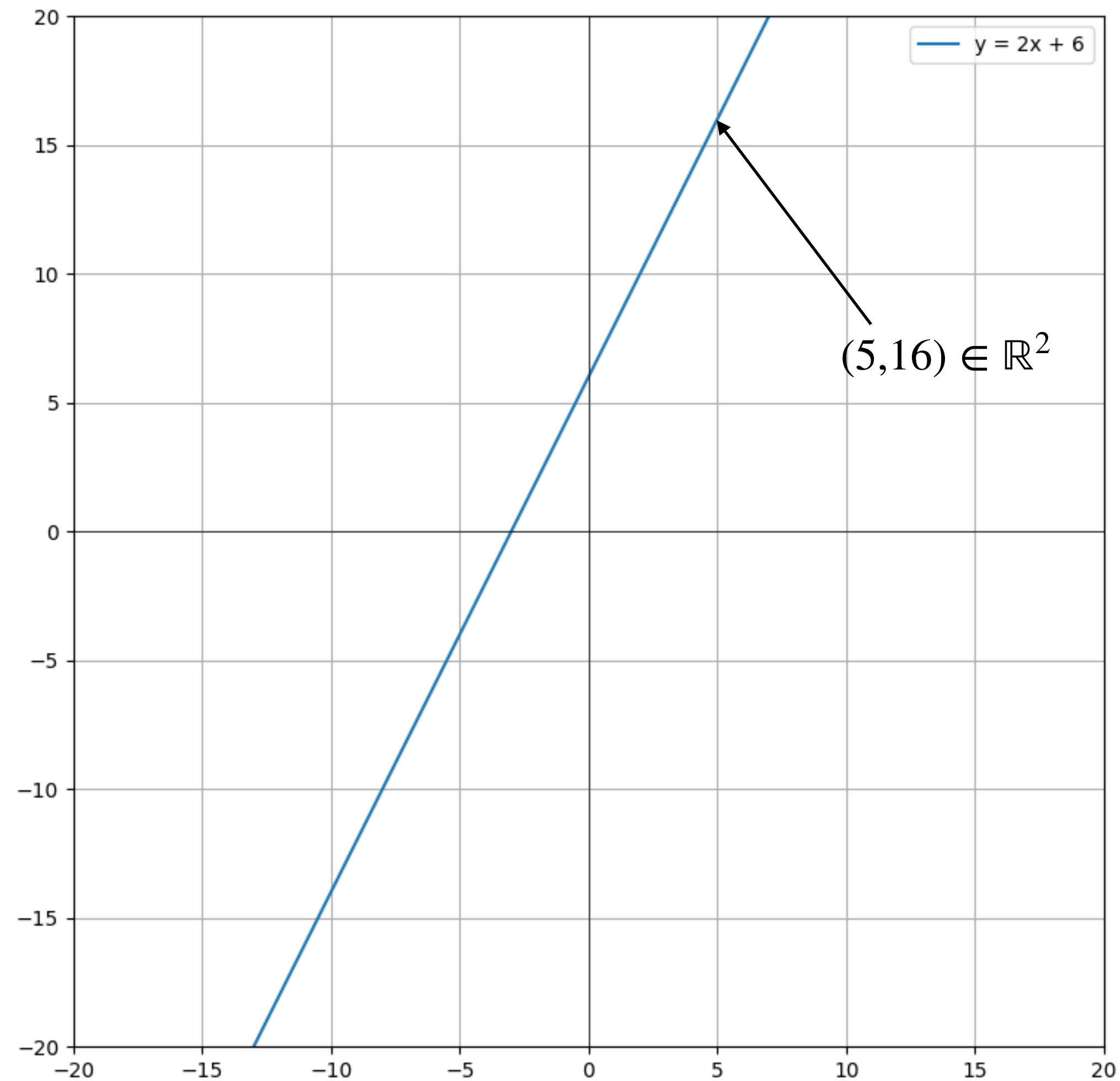
$$ax + by = c$$

$$\text{x-intercept: } \frac{c}{a}$$

$$\text{y-intercept: } \frac{c}{b}$$

What values of  $x$  and  $y$  make the equality hold?

# Lines (Graph)



$$\{(x, y) : (-2)x + y = 6\}$$

# Lines

slope-int  $\rightarrow$  general

$$(-m)x + y = b$$

general  $\rightarrow$  slope-int

$$y = \left( \frac{-a}{b} \right) x + \frac{c}{b}$$

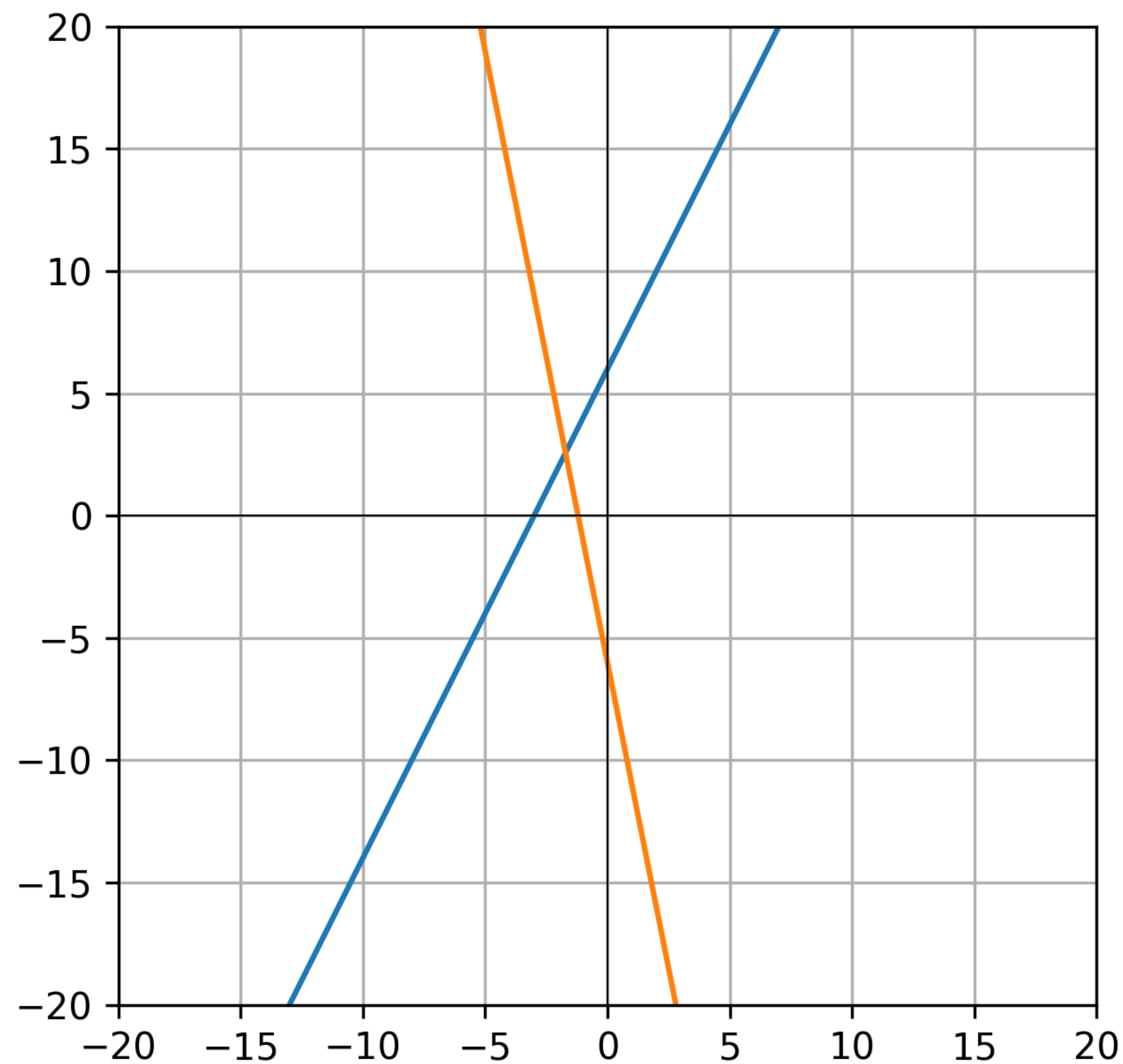
# Line Intersection

$$y = m_1x + b_1$$

$$y = m_2x + b_2$$

**Question.** Given two lines, where do they intersect?

# Line Intersection (Graph)





# Line Intersection (Alternative)

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

**Question.** Given two (general form) lines, what values of  $x$  and  $y$  satisfy ***both*** equations?

# Line Intersection (Alternative)

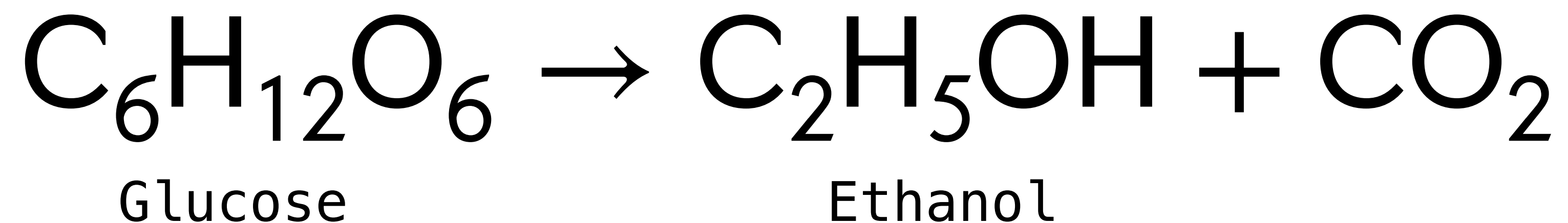
$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

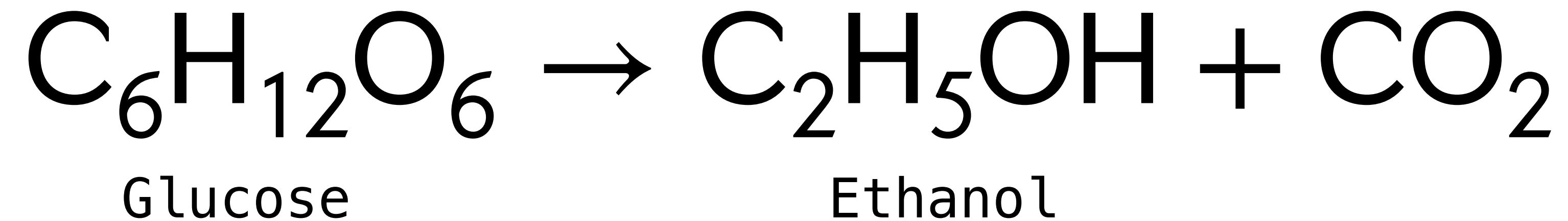
**Question.** Given two (general form) lines, what values of  $x$  and  $y$  satisfy ***both*** equations?

This is the same question

# Example: Balancing Chemical Equations



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We want to know how much ethanol is produced by fermentation (for science)

# Example: Balancing Chemical Equations



We want to know how much ethanol is produced by fermentation (for science)

The **number of atoms** has to be *preserved* on each side of the equation

# Balancing Chemical Equations



# Balancing Chemical Equations



$$6\alpha = 2\beta + \gamma \quad (\text{C})$$

$$12\alpha = 6\beta \quad (\text{H})$$

$$6\alpha = \beta + 2\gamma \quad (\text{O})$$

# Balancing Chemical Equations



$$6\alpha - 2\beta - \gamma = 0 \quad (\text{C})$$

$$12\alpha - 6\beta = 0 \quad (\text{H})$$

$$6\alpha - \beta - 2\gamma = 0 \quad (\text{O})$$



# Formal Definitions

# Linear Equations

**Definition.** A *linear equation* in variables  $x_1, x_2, \dots, x_n$  is an equation which can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where  $a_1, a_2, \dots, a_n, b$  are real numbers ( $\mathbb{R}$ )

# Linear Equations

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$$\text{coefficients}$$
$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

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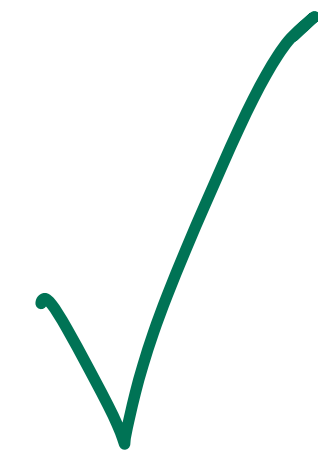
$$\begin{array}{c} \text{coefficients} \qquad \qquad \qquad \text{unknowns} \\ a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \end{array}$$

where  $a_1, a_2, \dots, a_n, b$  are real numbers ( $\mathbb{R}$ )

# Examples

$$2x + 3y - 4z = 7$$

$$2x_1 + 3x_2 - 4x_3 = 7$$

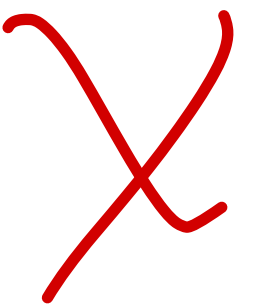


$$2x_1 = 3x_2 - 4x_3$$

$$x_1 x_3 = 4$$

$$x_1^2 + x_2 = 4$$

$$\log(x_2) + 3x_3 = 7$$



# Linear Equations (Point sets)

Linear equations describe *point sets*:

$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1s_1 + a_2s_2 + \dots + a_ns_n = b\}$$

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Linear equations describe *point sets*:

$$a_1 x_1 + \dots + a_n x_n = b$$

$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \dots + a_n s_n = b\}$$

The collections of numbers such that the equation holds

# Examples

$$2x + 4y - 3z = 0$$

$$(0, 0, 0)$$

$$(1, 1, 2)$$

$$(3, 0, 2)$$

$$2(3) + 4(0) - 3(2) \\ 6 - 6 = 0 \quad \checkmark$$



# Linear Equations (Geometrically)

If a 2D linear equation is a *line* then a 3D linear equation is...

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*Not a line...*

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# Linear Equations (Geometrically)

If a 2D linear equation is a *line* then a 3D linear equation is...

*A plane(!)*

# Example 1

$$0x + 0y + z = 5$$

This equation describes the solution set

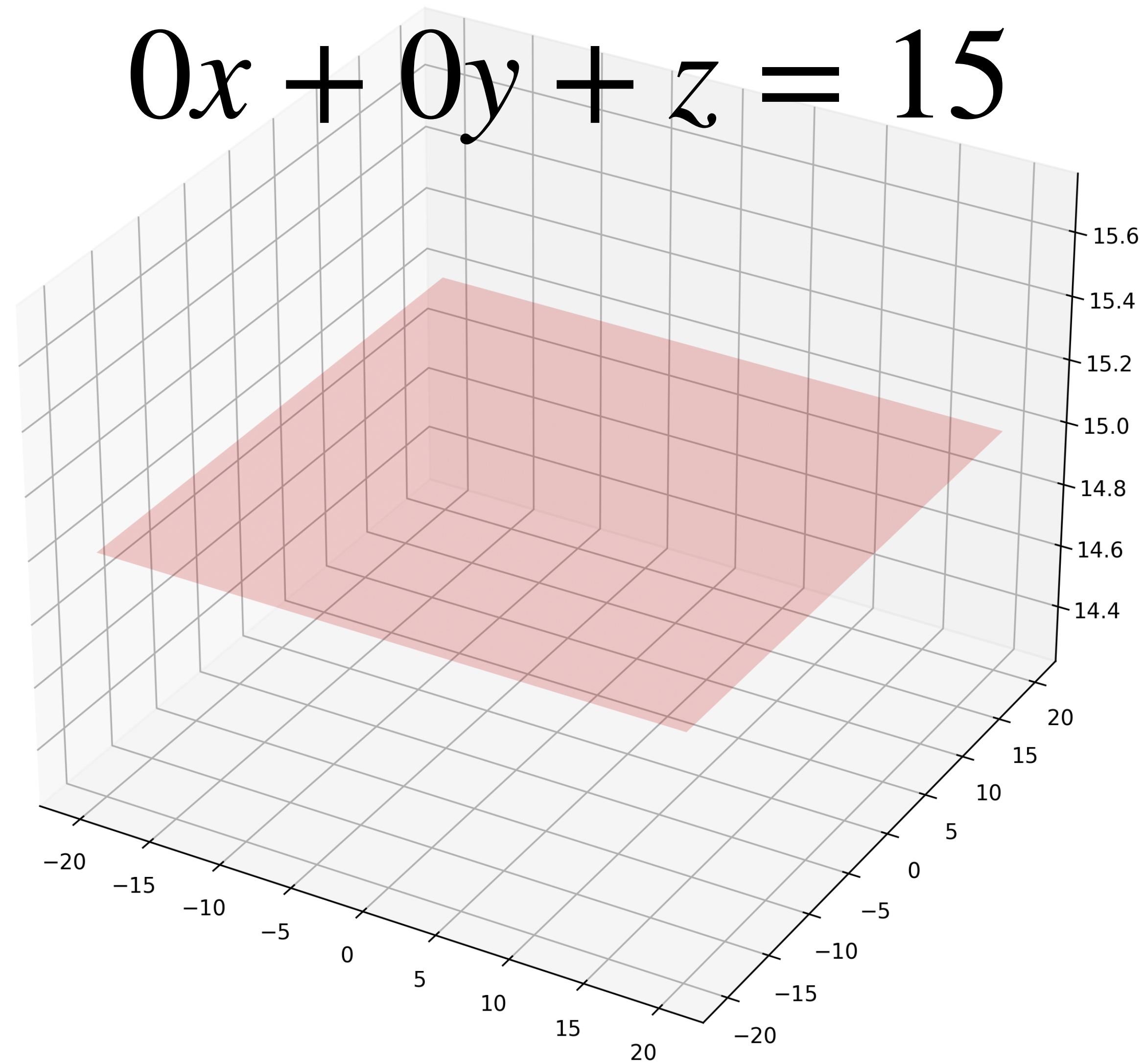
$$\{(x, y, z) : z = 5\}$$

$$(1, 2, 5)$$
$$(2, 2, 5)$$

so  $x$  and  $y$  can be whatever we want

# Example 1

$$0x + 0y + z = 15$$



## Example 2

$$-x + 0y + z = 5$$

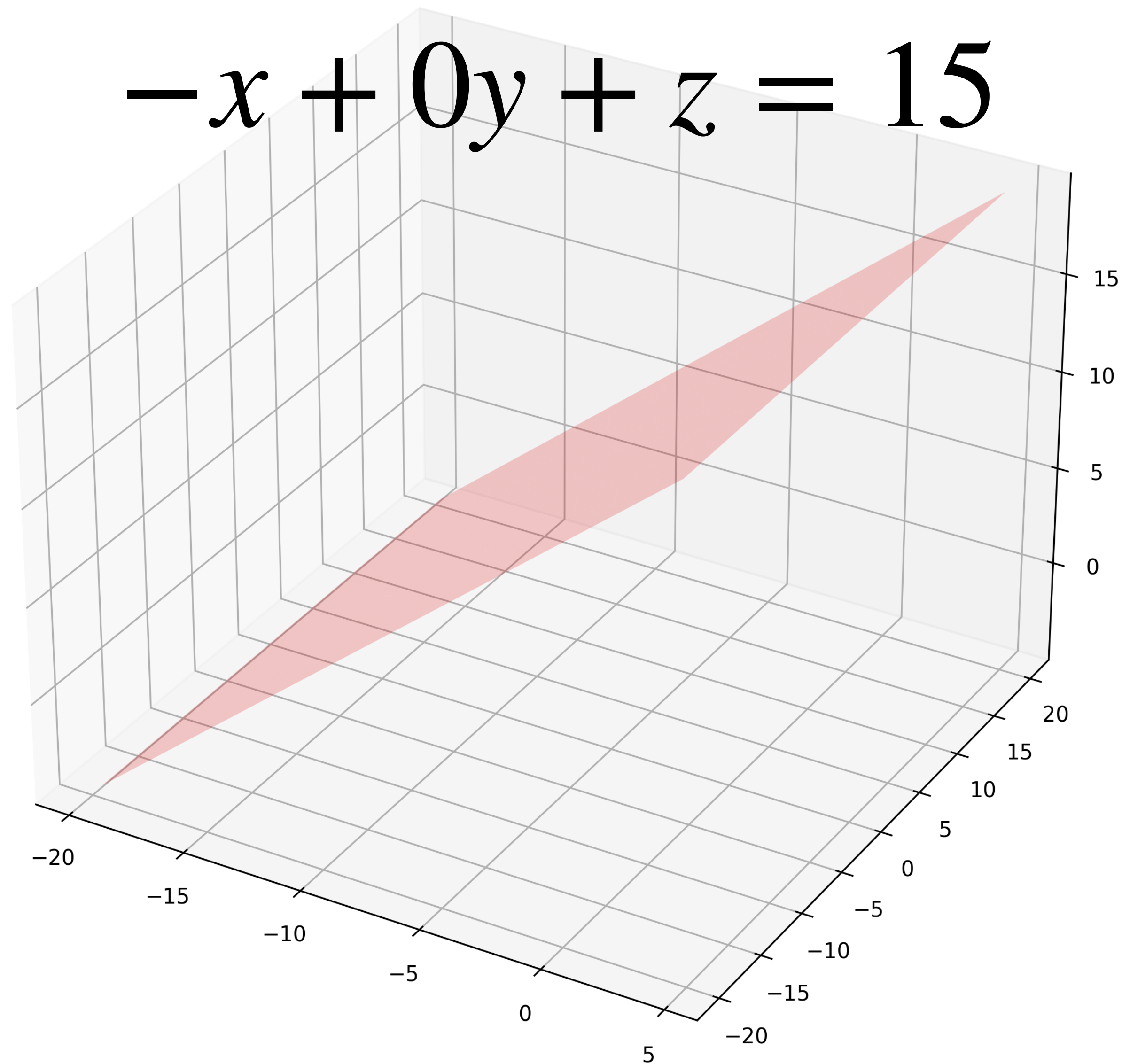
This equation describes the point set

$$\{(x, y, z) : z = x + 5\}$$

so  $y$  can be whatever we want

# Example 2

$$-x + 0y + z = 15$$





## Example 3

$$-x + -y + z = 5$$

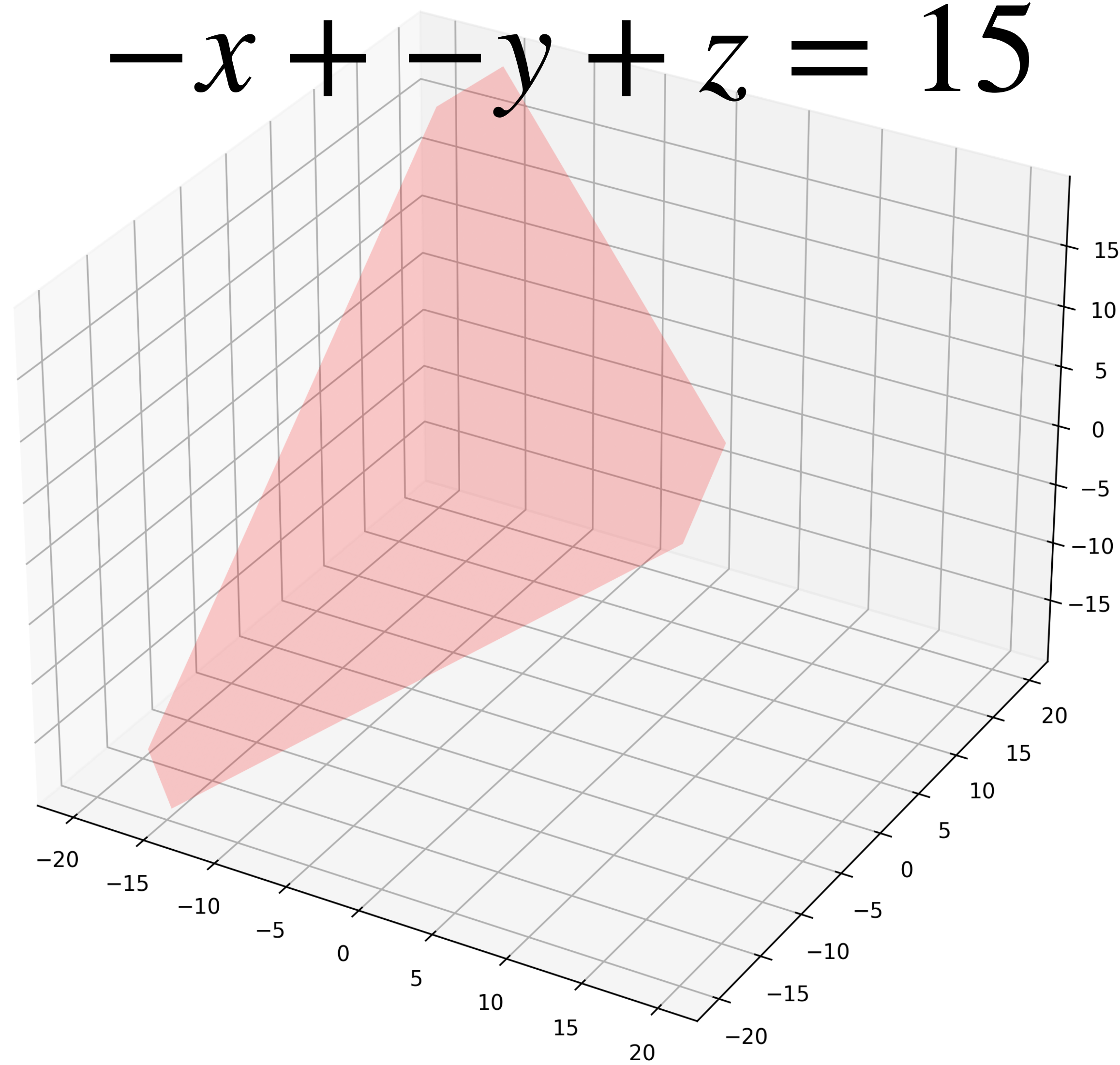
This equation describes the solution set

$$\{(x, y, z) : z = x + y + 5\}$$

so all variables depend on each other

# Example 3

$$-x + -y + z = 15$$



# XYZ-intercepts

$$ax + by + cz = d$$

Just like with lines, we can define

$$\text{x-intercept: } \frac{d}{a} \quad \text{y-intercept: } \frac{d}{b} \quad \text{z-intercept: } \frac{d}{c}$$

These three points define the plane

# Question

*I just lied*

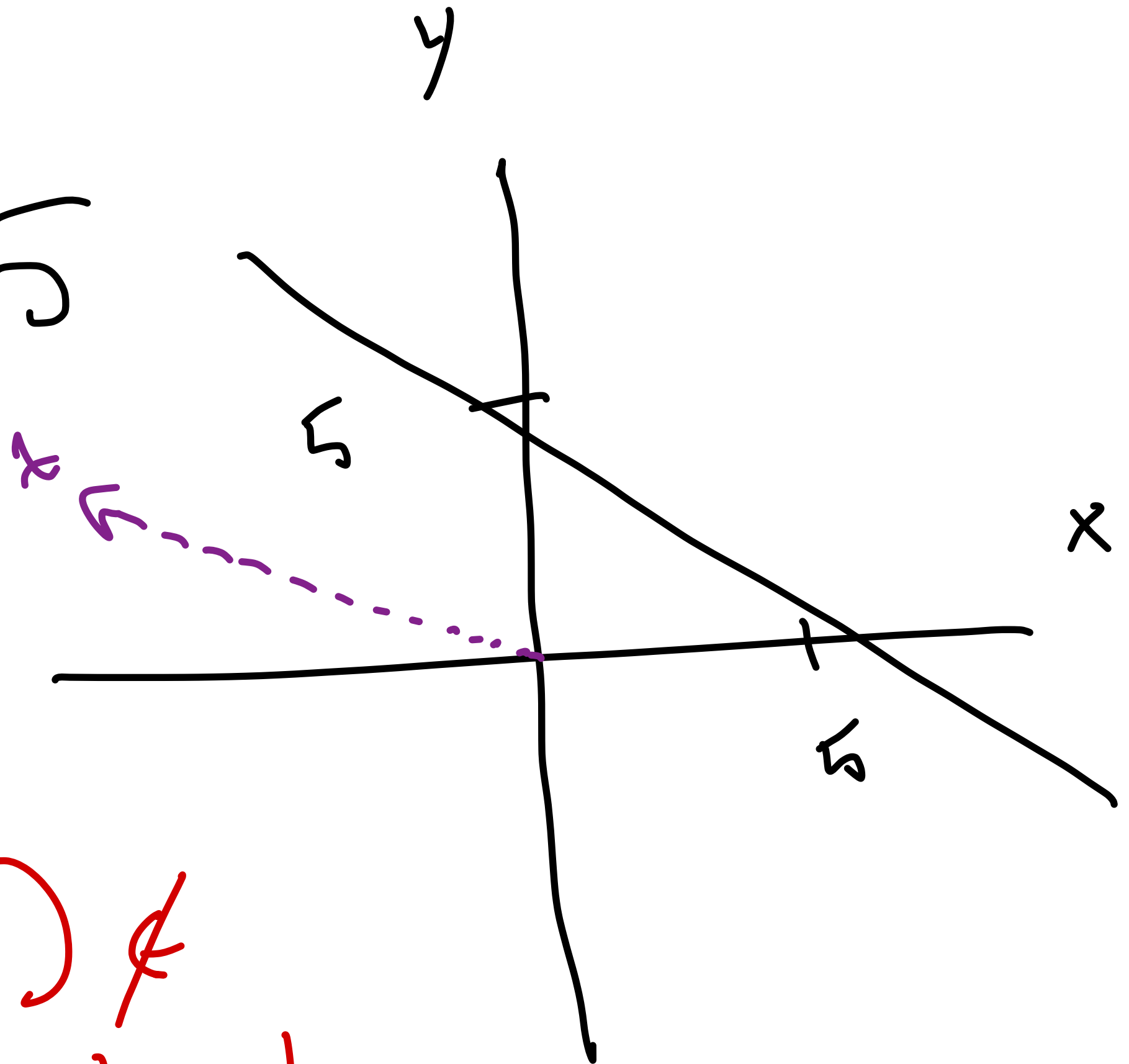
*Give an example of a linear equation that defines a plane with an  $x$ -intercept and  $y$ -intercept but no  $z$ -intercept*

# Answer

$z$ -intersept :  $(0, 0, n)$

$$x + y + 0z = 5$$

$$x + y = 5$$



$$(2.5, 2.5, 7)$$

$$(0, 5, 8)$$

$(0, 0, 8) \notin$   
point-set

# Hyperplanes

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After three dimensions, we can't visualize planes

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The point set of a linear equation is called a **hyperplane**



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After three dimensions, we can't visualize planes

The point set of a linear equation is called a **hyperplane**

Theme of the course: Hyperplanes "behave" like 3D planes in many respects

# Systems of Linear Equations

**Definition.** A *system of linear equations* is just a collection of linear equations over the same variables

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**Definition.** A *system of linear equations* is just a collection of linear equations over the same variables

**Definition.** A *solution* to a system is a point that satisfies all its equations *simultaneously*

# Example

linear system:

$$x + 2y = 1$$

$$-x - y - z = -1$$

$$2x + 6y - z = 1$$

solution:  $(3, -1, -1)$

$$3 + 2(-1) = 1 \quad \checkmark$$

$$-3 + 1 + 1 = -1 \quad \checkmark$$

$$2(\cancel{3}) + \cancel{6}(-1) + (+1) = 1 \quad \checkmark$$

# System of Linear equations (General-form)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

# System of Linear equations (General-form)

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Does a system have a solution?

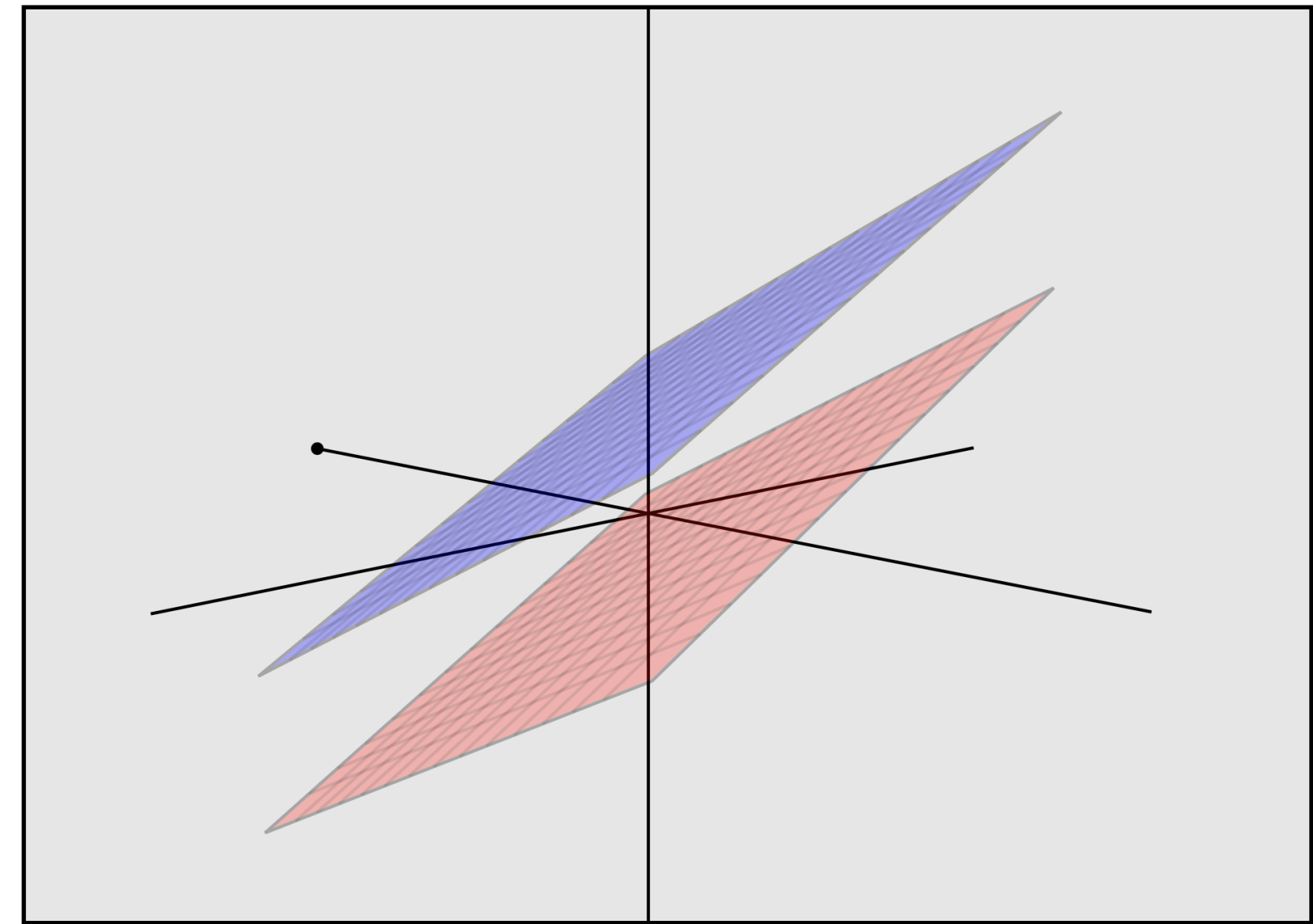
How many solutions are there?

What are its solutions?

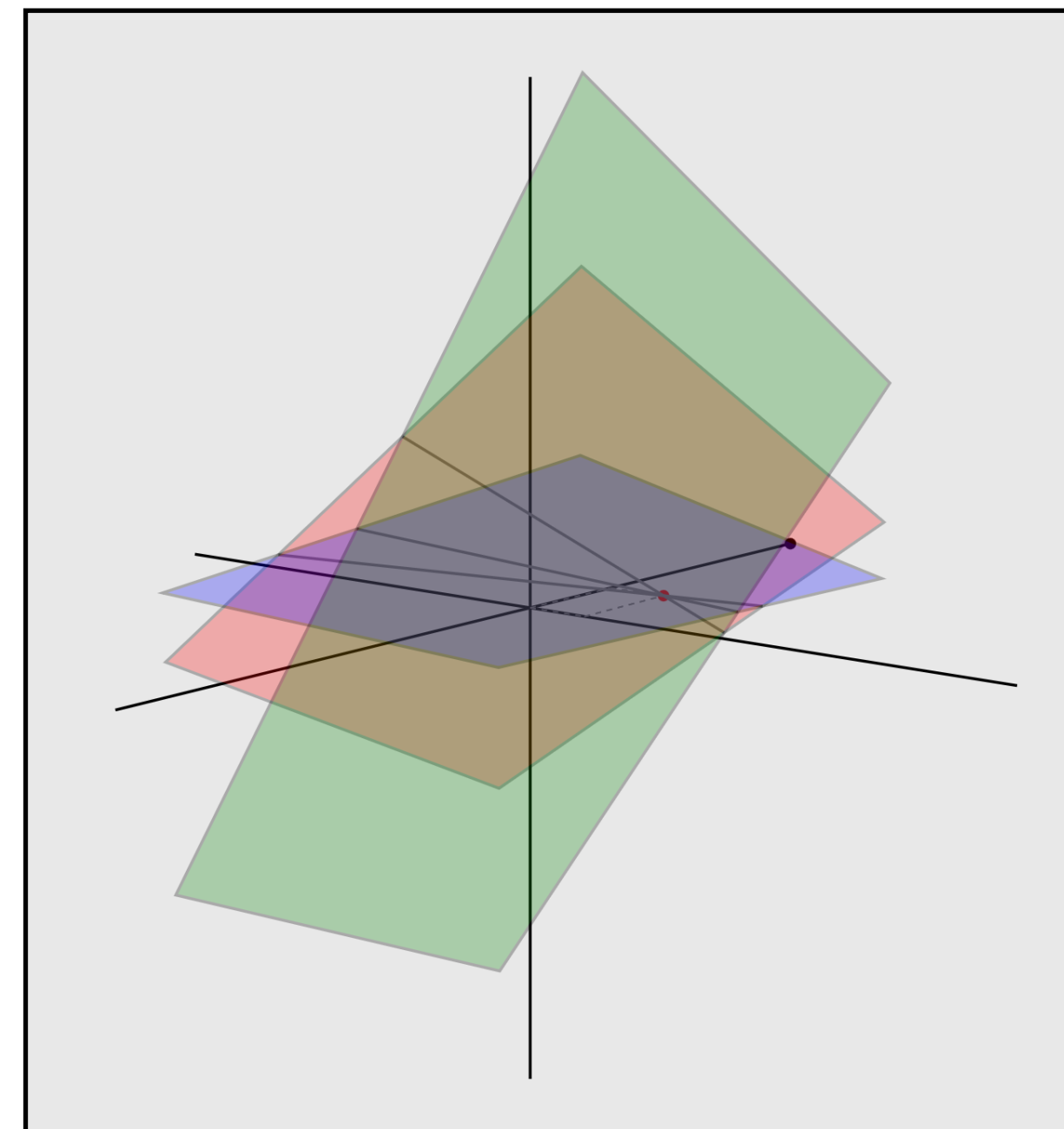
# Consistency

**Definition.** A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has no solutions



**inconsistent**

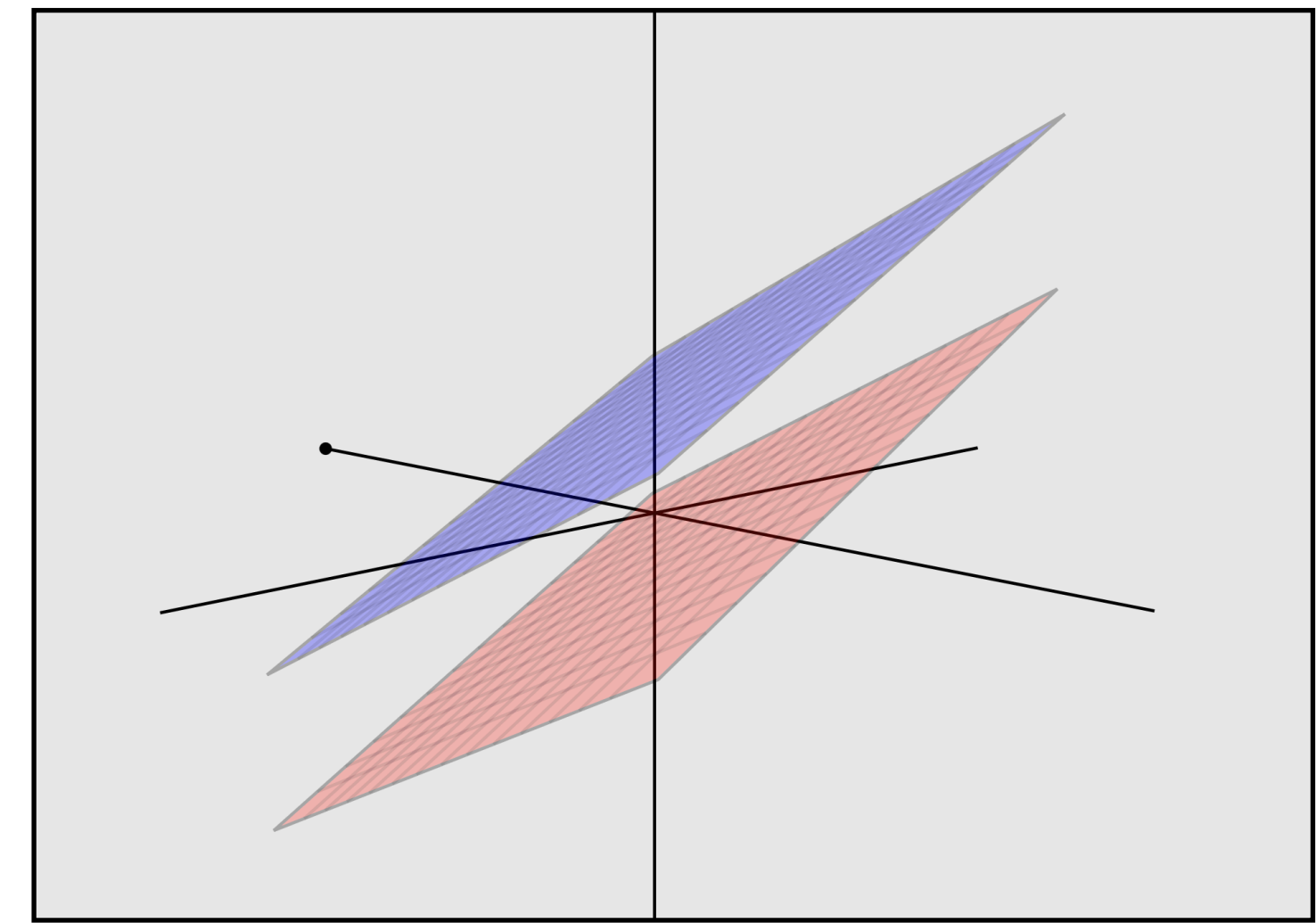


**consistent**

# Example

$$2x + y - z = 5$$

$$2x + y - z = 7$$





# Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

# Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

# Matrix Representations

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \approx \begin{array}{l} 2x + 3y = -8 \\ y = 2 \\ 2y = 0 \end{array}$$

# Matrix Representations

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \approx \begin{array}{l} 2x + 3y = -8 \\ y = 2 \\ 2y = 0 \end{array}$$

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We'll write down linear systems as **matrices**, which are just 2D grids of numbers with *fixed* width and height

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Writing down the unknowns is *tedious* (and more difficult to input into a computer)

We'll write down linear systems as **matrices**, which are just 2D grids of numbers with *fixed* width and height

a matrix is just a representation

# Matrix Representations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

# Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$



# Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

# Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coefficient matrix

# Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \quad (\text{C})$$

$$12\alpha - 6\beta = 0 \quad (\text{H})$$

$$6\alpha - \beta - 2\gamma = 0 \quad (\text{O})$$

# Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

# Solving Linear Systems

# Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

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**The Approach**

# Solving Systems with Two Variables

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## The Approach

Solve for  $x$  in terms of  $y$  in EQ1



# Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

Substitute result for  $x$  in EQ2 and solve for  $y$

# Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

Substitute result for  $x$  in EQ2 and solve for  $y$

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

## The Approach

**Solve for  $x$  in terms of  $y$  in EQ1**

Substitute result for  $x$  in EQ2 and solve for  $y$

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

## The Approach

**Solve for  $x$  in terms of  $y$  in EQ1**

Substitute result for  $x$  in EQ2 and solve for  $y$

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

**Substitute result for  $x$  in EQ2 and solve for  $y$**

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

**Substitute result for  $x$  in EQ2 and solve for  $y$**

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

**Substitute result for  $x$  in EQ2 and solve for  $y$**

Substitute result for  $y$  in EQ1 and solve for  $x$

# Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$y = -2$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

**Substitute result for  $x$  in EQ2 and solve for  $y$**

Substitute result for  $y$  in EQ1 and solve for  $x$



# Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$

$$y = -2$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

Substitute result for  $x$  in EQ2 and solve for  $y$

**Substitute result for  $y$  in EQ1 and solve for  $x$**

# Solving Systems with Two Variables

$$x = 3 - 3$$

$$y = -2$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

Substitute result for  $x$  in EQ2 and solve for  $y$

**Substitute result for  $y$  in EQ1 and solve for  $x$**

# Solving Systems with Two Variables

$$x = 0$$

$$y = -2$$

## The Approach

Solve for  $x$  in terms of  $y$  in EQ1

Substitute result for  $x$  in EQ2 and solve for  $y$

**Substitute result for  $y$  in EQ1 and solve for  $x$**

another perspective...

# Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

## The Approach

*Eliminate  $x$  from the EQ2 and solve for  $y$*

*Eliminate  $y$  from EQ1 and solve for  $x$*

Let's work through it

$$2x + 3y = -6$$

$$4x - 5y = 10$$

$$\begin{array}{rcl} \overset{4}{\cancel{2}}x + \overset{6}{\cancel{3}}y & = & \overset{-12}{\cancel{-6}} \\ \overset{-4x}{4}x - \overset{-6y}{5}y & = & \overset{+12}{10} \end{array}$$

↓

$$2x + 3y = -6$$

$$\cancel{-11y} = \cancel{22}^{-2}$$

↓

$$2x + \overset{-3y}{3}y = \overset{+6}{-6}$$

$$\overset{3}{3}y = \overset{-6}{-2}$$



$$x = 0$$

$$y = -2$$

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

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## The Approach

Eliminate  $x$  from the EQ2 and EQ3

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

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## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Elimination

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

Back-Substitution

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

## The Approach

**Eliminate  $x$  from the EQ2 and EQ3**

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

## The Approach

**Eliminate  $x$  from the EQ2 and EQ3**

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17y + 3z = -34$$

## The Approach

**Eliminate  $x$  from the EQ2 and EQ3**

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1



# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

**Eliminate  $y$  from EQ3**

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

**Eliminate  $y$  from EQ3**

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$68z - 34 - 3z = 26$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

**Eliminate  $y$  from EQ3**

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$71z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

**Eliminate  $y$  from EQ3**

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y + 0 = 5$$

$$2y - 8(0) = -4$$

$$z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

**Eliminate  $z$  from EQ2 and EQ1**

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2y = 5$$

$$2y = -4$$

$$z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

**Eliminate  $z$  from EQ2 and EQ1**

Eliminate  $y$  from EQ1

# Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

**Eliminate  $y$  from EQ1**

# Solving Systems with Three Variables

$$x = 1$$

$$y = -2$$

$$z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Eliminate  $z$  from EQ2 and EQ1

**Eliminate  $y$  from EQ1**



# Solving Systems with Three Variables

$$x = 1$$

$$y = -2$$

$$z = 0$$

## The Approach

Eliminate  $x$  from the EQ2 and EQ3

Eliminate  $y$  from EQ3

Elimination

Eliminate  $z$  from EQ2 and EQ1

Eliminate  $y$  from EQ1

Back-Substitution

# Verifying the Solution

Practice Ex.

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

# Solving Systems as Matrices

How does this look with matrices?

**Observation.** Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

# Solving Systems as Matrices

How does this look with matrices?

**Observation.** Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Can we represent these intermediate steps as operations on matrices?

**Let's look back at this...**

$$2x + 3y = -6$$

$$4x - 5y = 10$$

$$\begin{array}{r} 2x + 3y = -6 \\ 4x - 5y = 10 \end{array}$$

*(Note: In the original image, the second equation is annotated with purple ink:  $-4x$  above the 4,  $-6y$  above the  $-5y$ , and  $+12$  above the 10.)*

$$\begin{array}{r} 2x + 3y = -6 \\ 0x - 11y = 22 \end{array}$$

⋮

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

↓  $R_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

⋮

# Elementary Row Operations

scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows


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scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows

These operations don't change the solutions

# Scaling Example

$$\begin{array}{l} 2x + 3y = -6 \\ 4x - 5y = 10 \end{array}$$

$$R_1 \leftarrow 2R_1$$


$$\begin{array}{l} 4x + 6y = -12 \\ 4x - 5y = 10 \end{array}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$



# Replacement Example

$$\begin{array}{l} 2x + 3y = -6 \\ 4x - 5y = 10 \end{array}$$

$$R_2 \leftarrow R_2 + R_1$$



$$\begin{array}{l} 2x + 3y = -6 \\ 6x - 2y = 4 \end{array}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

# Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned}$$

$$R_1 \leftrightarrow R_2$$



$$\begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

# Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

# Example: Row Reductions

$$\begin{array}{ccc} & R_2 \leftarrow R_2 - 2R_1 & \\ \begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix} \\ & R_2 \leftarrow R_2 / (-11) & \\ & \longrightarrow & \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix} \end{array}$$

# Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$R_2 \leftarrow R_2 / (-11)$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

# Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$



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$$R_2 \leftarrow R_2 / (-11)$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 / 2$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = -2$$

# Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / (-11)$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_1 \leftarrow R_1 / 2$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

# Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / (-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_1 \leftarrow R_1 / 2$$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$



# Row Equivalence

**Definition.** Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

# Row Equivalence

**Definition.** Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

# Question

*How do we know when we're done? What is the "target" matrix?*

We'll get to that next time...

# Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve