

# **Gaussian Elimination + Numerics**

**Geometric Algorithms  
Lecture 3**

# Outline

- » Finish our discussion of Gaussian Elimination
- » Think more carefully about number representations, and look at the consequences of floating point representations
- » *If there's time:* Analyze the running time of Gaussian Elimination

# Keywords

forward elimination

back substitution

floating point numbers

IEEE-754

relative error

numpy.isclose

ill-conditioned problems

# Practice Problem

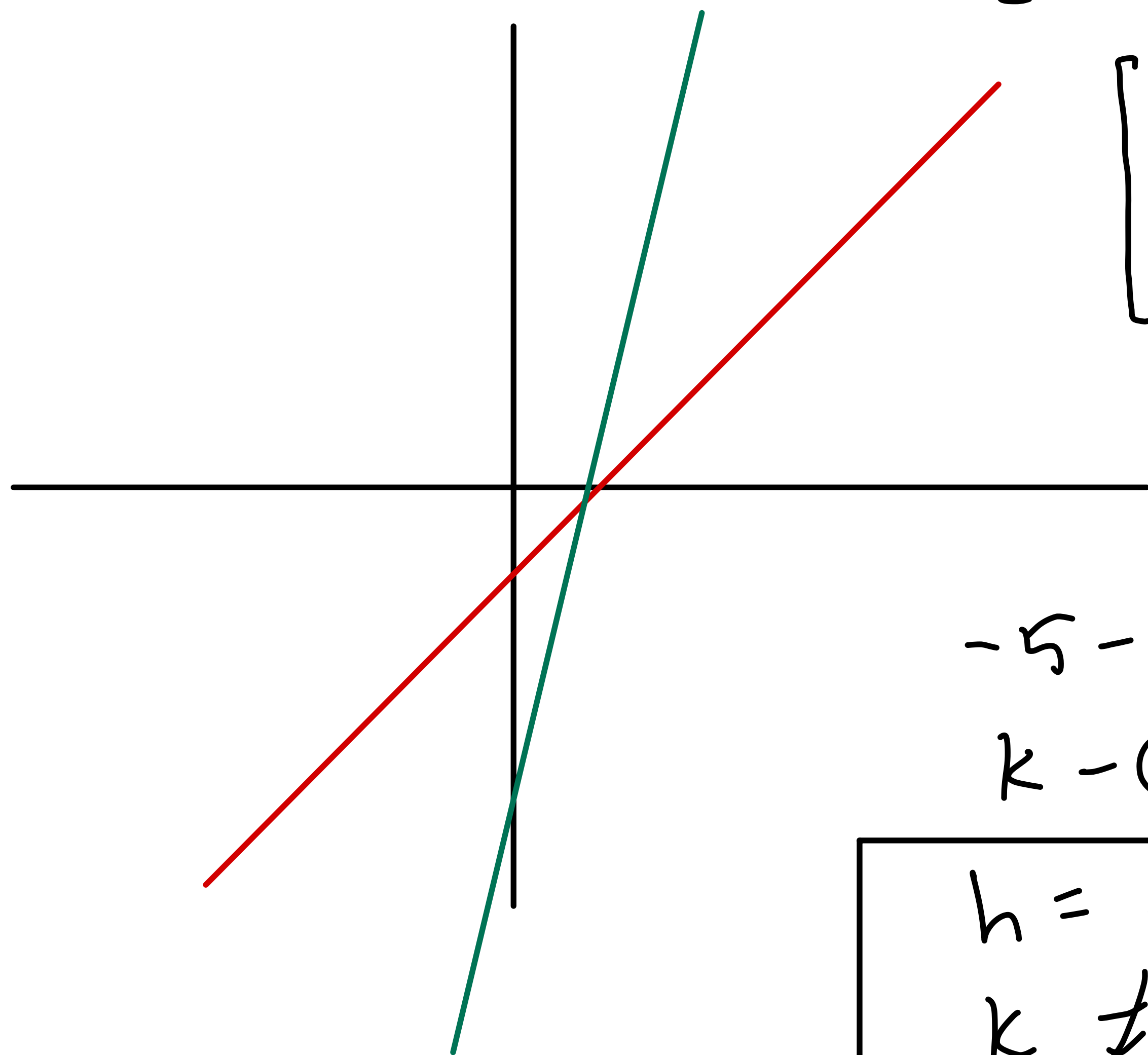
$$h = -2.5$$
$$k \neq 6$$

$$x + hy = 3$$

$$2x - 5y = k$$

*For what values of  $h$  and  $k$  is the above system inconsistent?*

# Solution



$$\begin{bmatrix} 1 & h & 3 \\ 2 & -5 & k \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & h & 3 \\ 0 & -5-2h & k-6 \end{bmatrix}$$

$h = -2.5$

$$x + \boxed{hy} = 3$$

$$2x - 5y = \boxed{k}$$

$k \neq 6$

$$-5 - 2h = 0$$

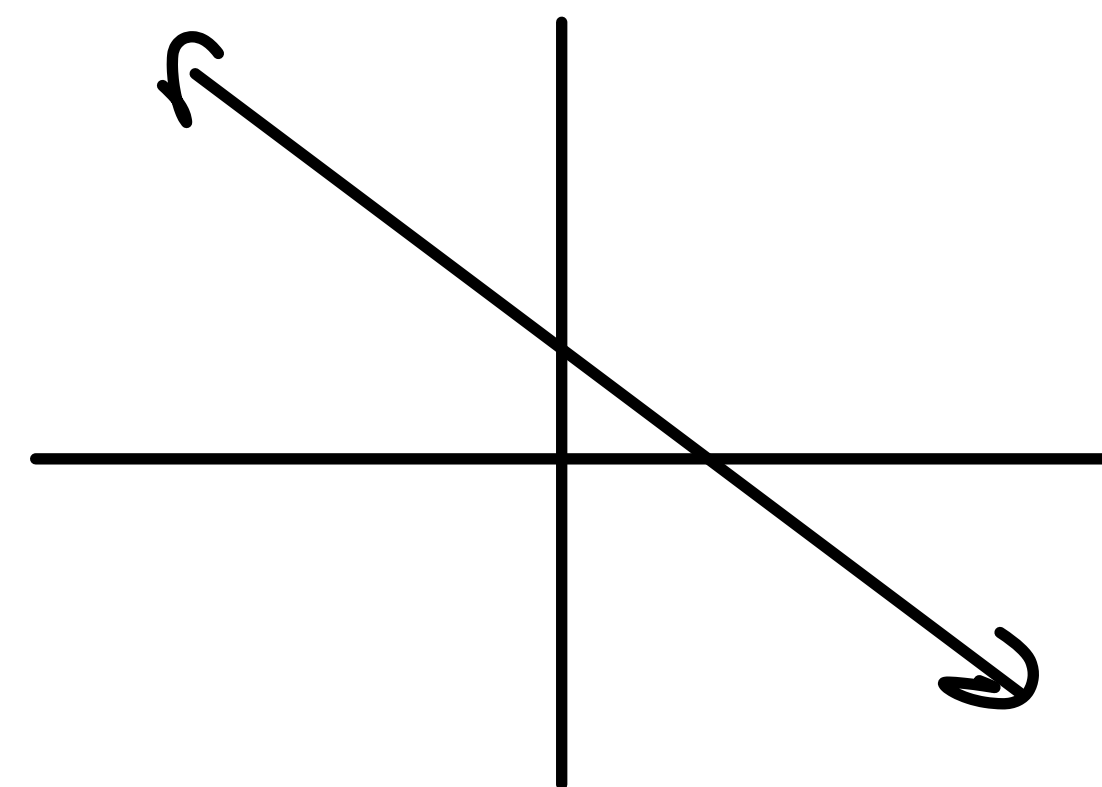
$$k - 6 \neq 0$$

$$h = -2.5$$

$$k \neq 6$$

$$2x + 2y = 2$$

$$x + y = 1$$



# Recap

# Recap: Echelon Form

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare$  = nonzero,  $*$  = anything

# Recap: Echelon Form

next leading entry  
to the right

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all-zero rows at  
the bottom

$\blacksquare$  = nonzero,  $*$  = anything







# **Recap: The Fundamental Points**

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**Point 1.** we can "read off" the solutions of a system of linear equations from its RREF

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**Point 2.** *every* matrix is row equivalent to a unique matrix in reduced echelon form

# Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

$x_3$  is free

# Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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$x_3$  is free

1. For each pivot position  $(i,j)$ , isolate  $x_j$  in the equation in row  $i$

# Recap: General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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1. For each pivot position  $(i,j)$ , isolate  $x_j$  in the equation in row  $i$
2. If  $x_i$  is not in a pivot column then write

$x_i$  is free



# **Recap: Solving a System of Linear Equations**

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3. Read off the solution from the RREF

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# Recap: Echelon Forms Gaussian Elimination

*the goal of back-substitution is to reduce an echelon form matrix to a **reduced** echelon form*

*the goal of Gaussian elimination is to reduce an **augmented** matrix to a **reduced** echelon form*

***reduced echelon forms describe solutions to linear equations***



# Gaussian Elimination

# **At a High Level**

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eliminations + back-substitution

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*we've already done this*

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the algorithm as pseudocode

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eliminations + back-substitution

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**Keep in mind.** How do we turn our intuitions  
into a formal procedure?

# **A Word of Warning**

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The goal is not to understand it entirely, but to get enough intuition to emulate it

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**You should roughly use Gaussian Elimination when solving a system by hand**

demo

# Gaussian Elimination (Specification)

**FUNCTION** GE(A):

# **INPUT:**  $m \times n$  matrix  $A$

# **OUTPUT:** equivalent  $m \times n$  RREF matrix

...

# Gaussian Elimination (High Level)

**FUNCTION** fwd\_elim(A):

# **INPUT:**  $m \times n$  matrix A

# **OUTPUT:** equivalent  $m \times n$  echelon form matrix

...

**FUNCTION** back\_sub(A):

# **INPUT:**  $m \times n$  echelon form matrix A

# **OUTPUT:** equivalent  $m \times n$  RREF matrix

...

**FUNCTION** GE(A):

**RETURN** back\_sub(fwd\_elim(A))

# Elimination Stage

# Elimination Stage (High Level)

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**Input:** matrix  $A$  of size  $m \times n$

**Output:** echelon form of  $A$



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starting at the top left and move down, find a leading entry and eliminate it from latter equations

# Edge cases

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What if the first equation doesn't have the variable  $x_1$ ?

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What if the first equation doesn't have the variable  $x_1$ ?

**Swap rows with an equation that does.**

What if *none* of the equations have the variable  $x_1$ ?

**Find the *leftmost* variable which appears in *any* of the remaining equations.**

# Elimination Stage (Pseudocode)

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**FUNCTION** fwd\_elim(A):



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    FOR [i from 1 to m]: # for each row from top to bottom
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    FOR [i from 1 to m]: # for each row from top to bottom
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        IF [rows i...m are all-zeros]: # if remaining rows are zero
```

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    ELSE:
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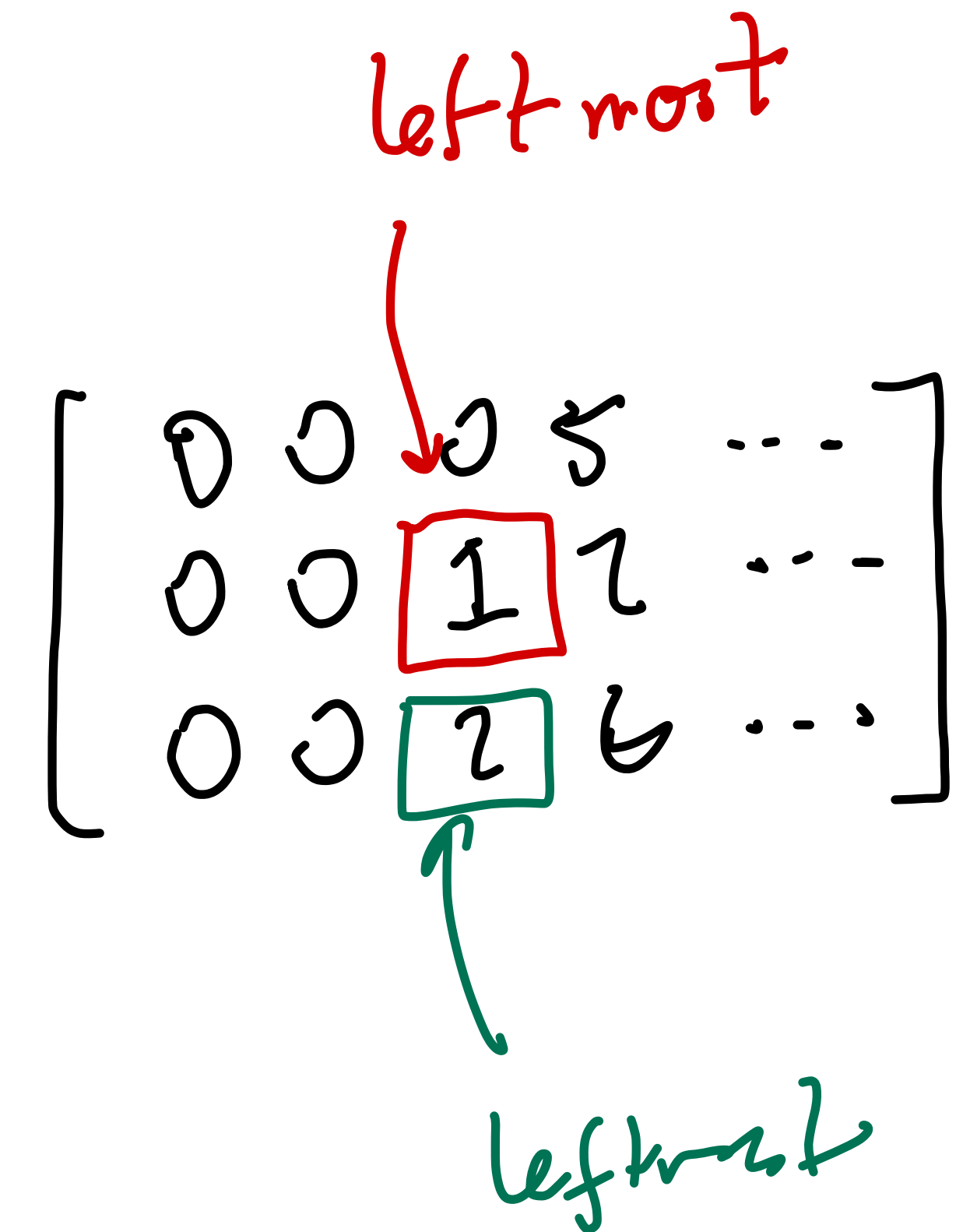
**FOR** [i from 1 to m]: # for each row from top to bottom

**IF** [rows i...m are all-zeros]: # if remaining rows are zero

**RETURN** A

**ELSE:**

(j, k) ← [position of leftmost entry in the rows i...m]



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    [swap row i and row j]

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        (j, k) ← [position of leftmost entry in the rows i...m]

        [swap row i and row j]

**FOR** [l from i + 1 to m]: # for all remaining rows

# Elimination Stage (Pseudocode)

**FUNCTION** fwd\_elim(A):

**FOR** [i from 1 to m]: # for each row from top to bottom

**IF** [rows i...m are all-zeros]: # if remaining rows are zero

**RETURN** A

**ELSE:**

        (j, k) ← [position of leftmost entry in the rows i...m]

        [swap row i and row j]

**FOR** [l from i + 1 to m]: # for all remaining rows

            [zero out A[l, k] using a replacement operation]



# Elimination Stage (Pseudocode)

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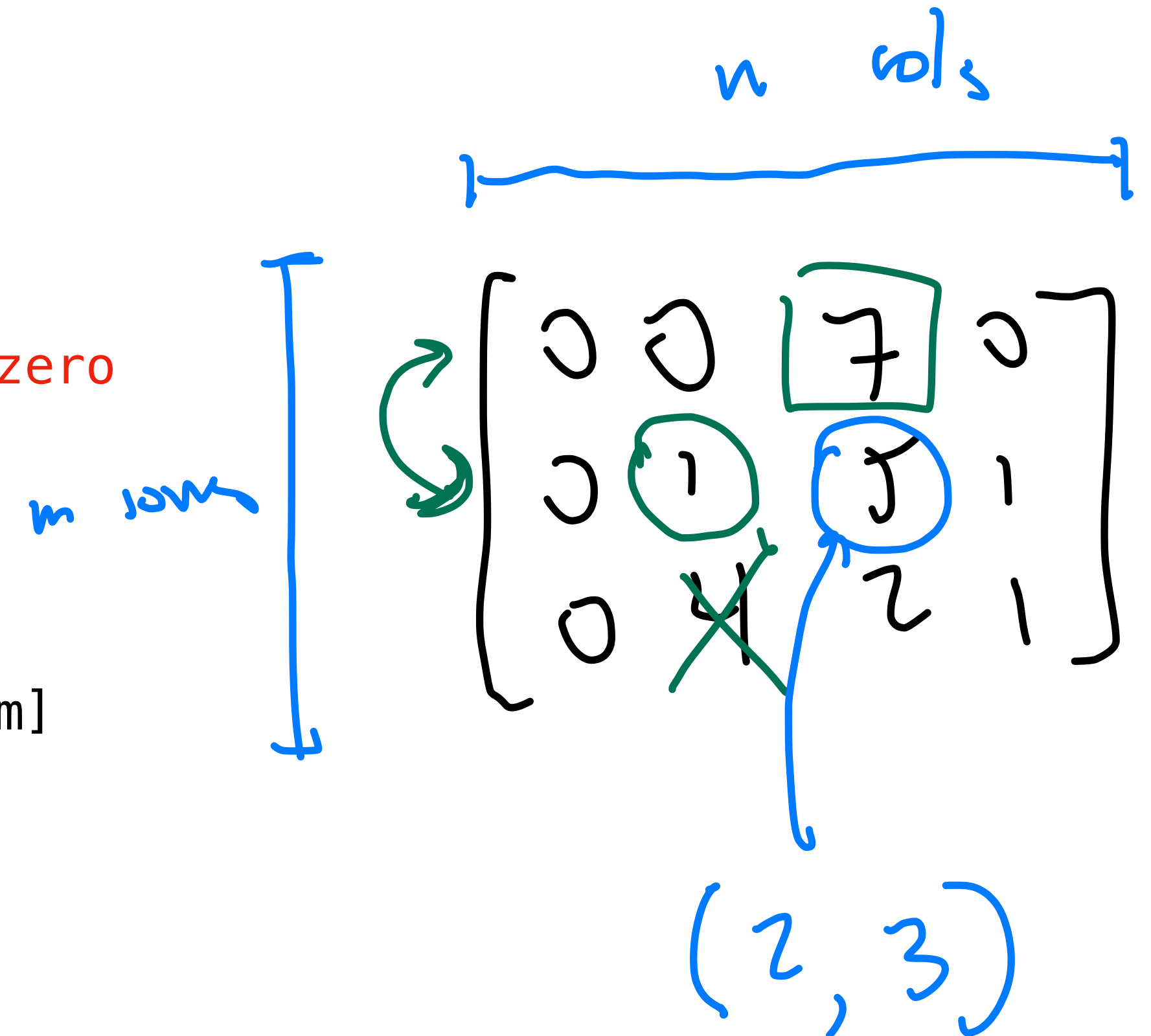
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[swap row i and row j]

**FOR** [l from i + 1 to m]: # for all remaining rows

[zero out A[l, k] using a replacement operation]

**RETURN** A



# Elimination Stage (Example)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

# Elimination Stage (Example)

leftmost  
nonzero  
entry

→

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Swap  $R_1$  and  $R_3$

# Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

# Elimination Stage (Example)

next entry  
to zero

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

# Elimination Stage (Example)

next entry  
to zero

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\cancel{R_2} \leftarrow \cancel{R_2} - \frac{3}{3} R_1$$

# Elimination Stage (Example)

$$\rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$



# Elimination Stage (Example)

leftmost  
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swap  $R_2$  with  $R_2$

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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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# Elimination Stage (Example)

next entry  
to zero

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - \frac{3R_2}{2}$$

# Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Elimination Stage (Example)

leftmost  
nonzero  
entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \curvearrowright$$

# Elimination Stage (Example)

leftmost  
nonzero  
entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

swap  $R_3$  with  $R_3$



# Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Elimination Stage (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

done with elimination stage  
going to back substitution stage

# Back Substitution Stage

# Back Substitution Stage (High Level)

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**Input:** matrix  $A$  of size  $m \times n$  in echelon form

**Output:** reduced echelon form of  $A$

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scale pivot positions and eliminate the variables for that column from the other equations

# Back Substitution (Psuedocode)

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FUNCTION back_sub(A):
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$R_i(A) \leftarrow R_i(A) / A[i, j]$  # divide by leading entry

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$R_i(A) \leftarrow R_i(A) / A[i, j]$  # divide by leading entry

**FOR** [k from 1 to i - 1]: # for the rows above the current one

$R_k(A) \leftarrow R_k(A) - R[k, j] * R_i(A)$

        # zero out R[k, j] above the leading entry

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**RETURN** A

# Gaussian Elimination (Example)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$



# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 / 3$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 / 2$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

next entry  
to zero

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

next entry  
to zero

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$



# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

pivot  
position

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_3 \leftarrow R_3 / 1$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

next entry  
to zero

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# Gaussian Elimination (Example)

next entry  
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

next entry  
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$



# Gaussian Elimination (Example)

next entry  
to zero

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 5R_3$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Gaussian Elimination (Example)

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

done with back substitution phase

# Gaussian Elimination (Example)

$$x_1 = (-24) + 2x_3 - 3x_4$$

$$x_2 = (-7) + 2x_3 - 2x_4$$

$x_3$  is free

$x_4$  is free

$$x_5 = 4$$

# **Recap: Solving a System of Linear Equations**

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Gaussian elimination

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# Numerics

demo

# Significant Figures (Sig Figs)

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*Do you remember sig figs from science class?*

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*We run into a similar problem with decimal numbers  
in programs*

# Number Representations



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Your computer is a collection of fixed size registers

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Each register holds a sequence of bits

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The Goal. represent numbers so they fit in those registers

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this is, of course, ~~a lie~~ an abstraction

# Number Representations

[illegible]

# Number Representations

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

**Question.** How do we slice up our fixed sequence to represent numbers?

# Number Representations

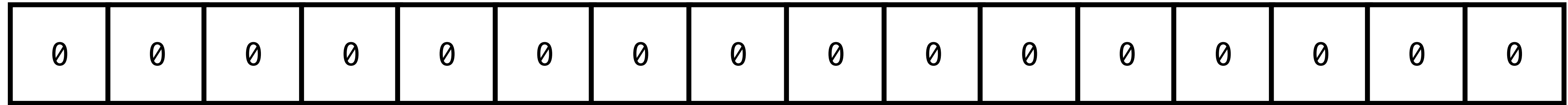
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

**Question.** How do we slice up our fixed sequence to represent numbers?

things to consider:

- » simple idea (easy to understand)
- » maximize coverage (not too redundant)
- » simple numeric operations (easy to use)

# Unsigned Integers



value

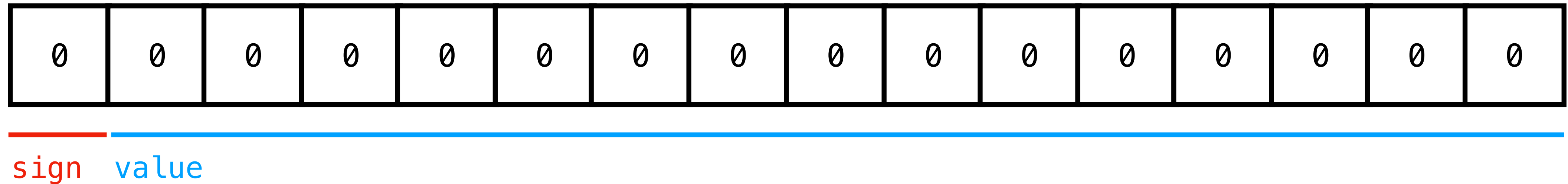
binary value (we should know this by now)

e.g. **1**000**1**0**1**0 represents

$$1(2^7) + 0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$$



# Signed Integers



sign bit + binary value

e.g. **1**000**1010** represents

$$\text{−1} \times (0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0))$$

# **Floating-Point Numbers (Some Figures)**

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floats in python use *64 bits*

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*We can't represent everything. We'll have to choose and then round*

**Question.** Which ones should we represent?

# Floating-Point Numbers (An Idea)

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Integers work because they are **discrete and evenly spaced**



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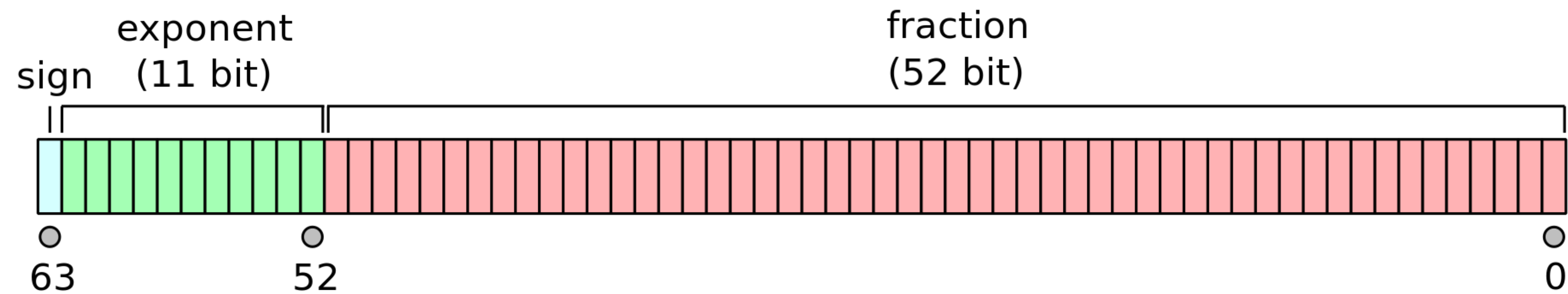
i.e., represent

..., -0.001, 0, 0.00~~1~~1, 0.002, 0.003, 0.004, ...

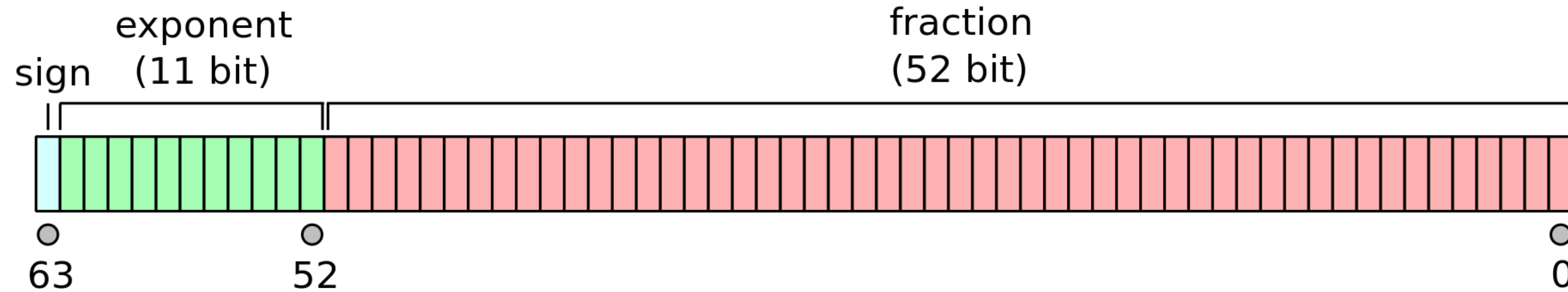
# Question

*Discuss the advantages and disadvantages of this approach*

# Floating-Point Numbers (IEEE-754)

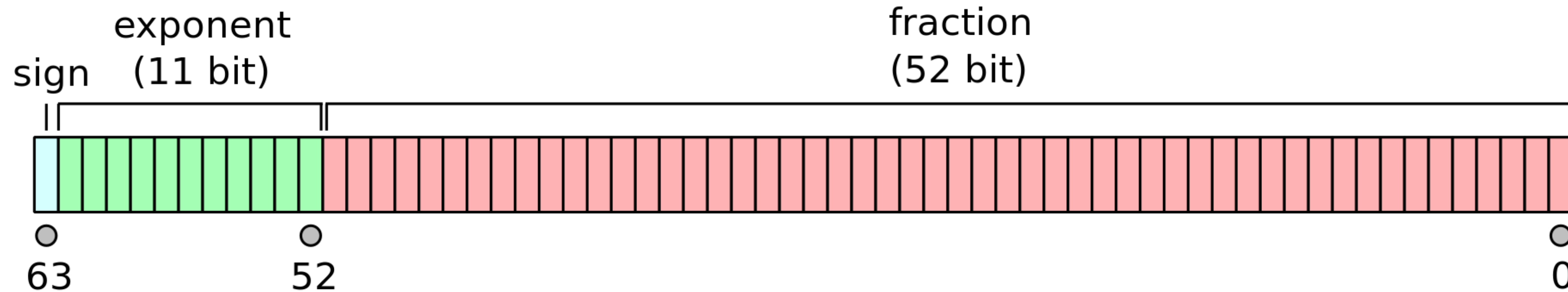


# Floating-Point Numbers (IEEE-754)



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This is like scientific notation, but binary:

$$(-1)^{\text{sign}} \times \left( 1 + \frac{\text{fraction}}{2^{52}} \right) \times 2^{\text{exponent} - (2^{10} - 1)}$$

It's an accepted standard, not perfect, but it works well

# Question

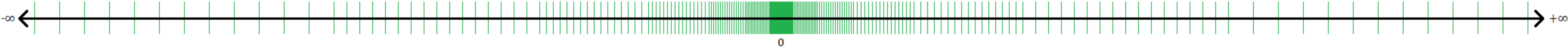
$$(-1)^{\text{sign}} \times \left( 1 + \frac{\text{fraction}}{2^{52}} \right) \times 2^{\text{exponent} - (2^{10} - 1)}$$

*Any ideas why this is better/worse?*

*And why not have a sign bit for the exponent?*

# Step Size

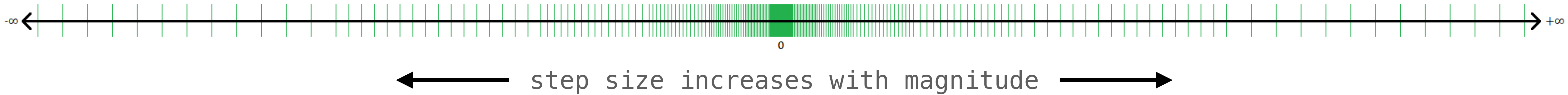
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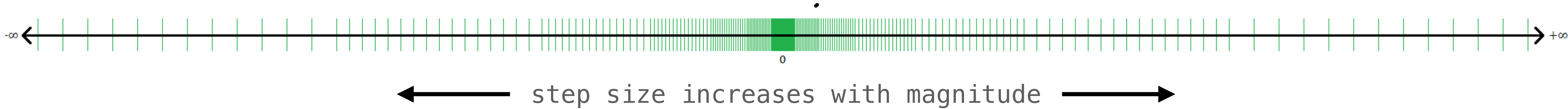
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**Definition.** step size is the space between two floating-point representations

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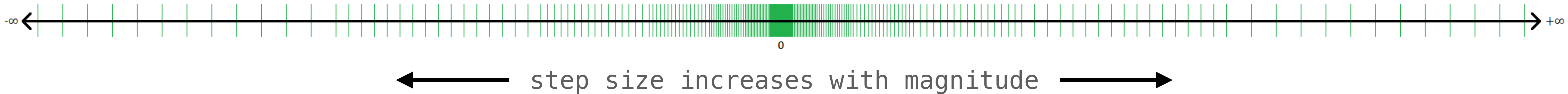
for fixed exponent  $n$  two numbers are at least

$$0.00\dots001 \times 2^n = 2^{-52} \times 2^n$$

away (why?)

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away (why?)

Step size doubles for each exponent

# Things to Keep in Mind

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operations on floating point numbers attempt to give you the closest to the actual value, though there will be errors

we can assume when we write down a number like '0.3' we get the closest IEEE-754 value

# Relative Error

**Observation.**  $\pm 0.001$  is *tiny* error for  $10^{20}$  but *massive* for  $10^{-20}$



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**Relative Error.**

$$\text{err}_{\text{rel}} = \frac{\text{err}}{\text{val}}$$

IEEE-754 keeps relative error small

# Relative Error (Calculation)

$$(-1)^{\text{sign}} \times \left( 1 + \frac{\text{fraction}}{2^{52}} \right) \times 2^{\text{exponent} - (2^{10} - 1)}$$

*(fix an exponent  $n$ )*

# Relative Error (Calculation) $(-1)^{\text{sign}} \times \left(1 + \frac{\text{fraction}}{2^{52}}\right) \times 2^{\text{exponent} - (2^{10} - 1)}$

*(fix an exponent  $n$ )*

error is determined by step-size

$$\text{err} \leq 2^{-52} \times 2^n$$

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*(fix an exponent  $n$ )*

the smallest number we can represent at least  
 $1.0 \times 2^n$

$$\text{val} \geq 1.0 \times 2^n$$

(why do we care about a lower bound on val?)

# Relative Error (Calculation)

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$\approx 16$  digits of accuracy

Not bad, but also not great

# demo

(example from the notes)

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What do we do about it?



# Best Practices

1. don't compare floating points for equality
2. be aware of ill-conditioned problems
3. be aware of small differences

# Principle 1: Closeness

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*When doing floating-point calculations in a program, define an error margin and use that for equality checking*

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**In Practice.**

Replace  
with

`x == y`  
`numpy.isclose(x, y)`

demo

# **Principle 2: Ill-Conditioned Problems**

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*Make sure your problem is not sensitive to small errors.*

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*Make sure your problem is not sensitive to small errors.*

**In Practice.** for example, don't divide by numbers much smaller than your error tolerance



demo

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**In Practice.** Don't expect  $a - b$  to be small when  $a$  and  $b$  are "close" but very large.

demo

# One Last Note: Special Numbers

`0` (we can't already represent 0?)

`nan` stands for not a number, .e.g, `sqrt(-2)`

`inf` symbolic infinity, behaves as expected

# **Extra Topic: Analyzing the Algorithm**

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$2n$  vs.  $n$  is very different  
when  $n \sim 10^{20}$

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A function  $f(n)$  is ***asymptotically equivalent*** to  $g(n)$  if

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for polynomials, they are equivalent to their dominant term

# Dominant Terms

The **dominant term** of a polynomial is the monomial with the highest degree

$$\lim_{i \rightarrow \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$  dominates the function even though the coefficient for  $x^2$  is so large



# Parameters

$n$  : number of variables

$m$  : number of equations (we will assume  $m = n$ )

$n + 1$  : number of rows in the augmented matrix

# The Cost of a Row Operation

$$R_i \leftarrow R_i + aR_j$$

$n + 1$  multiplications for the scaling

$n + 1$  additions for the row additions

Tally:  $2(n + 1)$  FLOPS

# Cost of First Iteration of Elimination

$$R_2 \leftarrow R_2 + a_2 R_1$$

$$R_3 \leftarrow R_3 + a_3 R_1$$

$$\vdots$$

$$R_n \leftarrow R_n + a_n R_1$$

Repeated row operations for each row except the first

Tally:  $\approx 2n(n+1)$  FLOPS

# Rough Cost of Elimination

repeating this last process at most  $n$  times  
gives us a dominant term  $2n^3$

we can give a better estimation...

Tally:  $\approx 2n^2(n + 1)$  FLOPS

# Cost of Elimination

0	■	*	*	*	*	*	*	*	*
0	0	0	■	*	*	*	*	*	*
0	0	0	0	*	*	*	*	*	*
0	0	0	0	*	*	*	*	*	*
0	0	0	0	*	*	*	*	*	*
0	0	0	0	0	0	0	0	0	0

At iteration  $i$ , we're only interested in rows after  $i$

And to the right of column  $i$

# Cost of Elimination

Iteration 1:  $2n(n+1)$

Iteration 2:  $2(n-1)n$

Iteration 3:  $2(n-2)(n-1)$

$\vdots$

+

---

$$\sum_{k=1}^n 2k(k+1) \approx \frac{2n(n+1)(2n+1)}{6} \sim (2/3)n^3$$

Tally:  $\sim (2/3)n^3$  FLOPS

# Cost of Back Substitution

(Let's assume no free variables)

for each pivot, we only need to:

- >> zero out a position in 1 row (0 FLOPS)

- >> add a value to the last row (1 FLOP)

**at most 1 FLOP per row per pivot  $\sim n^2$**

Tally:  $\sim (2/3)n^3$  FLOPS

# Cost of Gaussian Elimination

Tally:  $\sim (2/3)n^3$  FLOPS

(dominated by elimination)



# Summary

floating point numbers are **represented** in your computer

Floating point operations are *not* exact, and this can have unintended consequences

we get **16 digits** of accuracy