Vector Equations

Geometric Algorithms
Lecture 4

CAS CS 132

Practice Problem

Suppose that A is a 322×245 augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that A can have?

Answer

243

$A \in \mathbb{R}^{10 \times 5}$ 4, 5

$\min(10, 5)$

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

$A \in \mathbb{R}^{10\times 5}$

A is aug. matrix 4 5

of consistent system, 4

with inf. many sol. 3

Answer

$$A \in \mathbb{R}^{3 \times 4}$$

(3: rows, 4: cols)

$$A \sim \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 1 - 4x_3$$

$$x_2 = 2 - 5x_3$$

x_3 is free

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = 4$$

Outline

» Formally define vectors

» Discuss vector operations and vector algebra

» Draw the connection between vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Motivation (An Aside)

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all n

Changing Perspective

1000 - 1 =
999

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100...000 - 000...001 = 011...111$$

show that this holds for all n

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

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show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

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maybe I'm reaching...

Big Data (More Practical Motivation)

A piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

Maybe if they are **close together** in a geometric sense

A Note on Algebra

$$\mathbf{v} = \mathbf{w}$$

$$\mathbf{v} + \mathbf{w}$$

$$a\mathbf{v}$$

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In programming an *interface* is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

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Doing abstract algebra is like implementing an interface

We're defining an new thing called a *column vector*

We need to define what **equality** and **adding** and **multiplying by a number** means for column vectors

$$\mathbf{v} = \mathbf{w}$$

$$\mathbf{v} + \mathbf{w}$$

$$a\mathbf{v}$$

Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

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it's common to conflate points and vectors

Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$$\underbrace{\overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n}}_{m} \in \mathbb{R}^{m \times n}$$

$$\underbrace{\overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1}}_{4} \in \mathbb{R}^{4 \times 1} = \mathbb{R}^{4}$$

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$$m \; \overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n} \qquad\qquad 4 \; \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1}$$

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

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$$m \left\{ \overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n} \right.$$

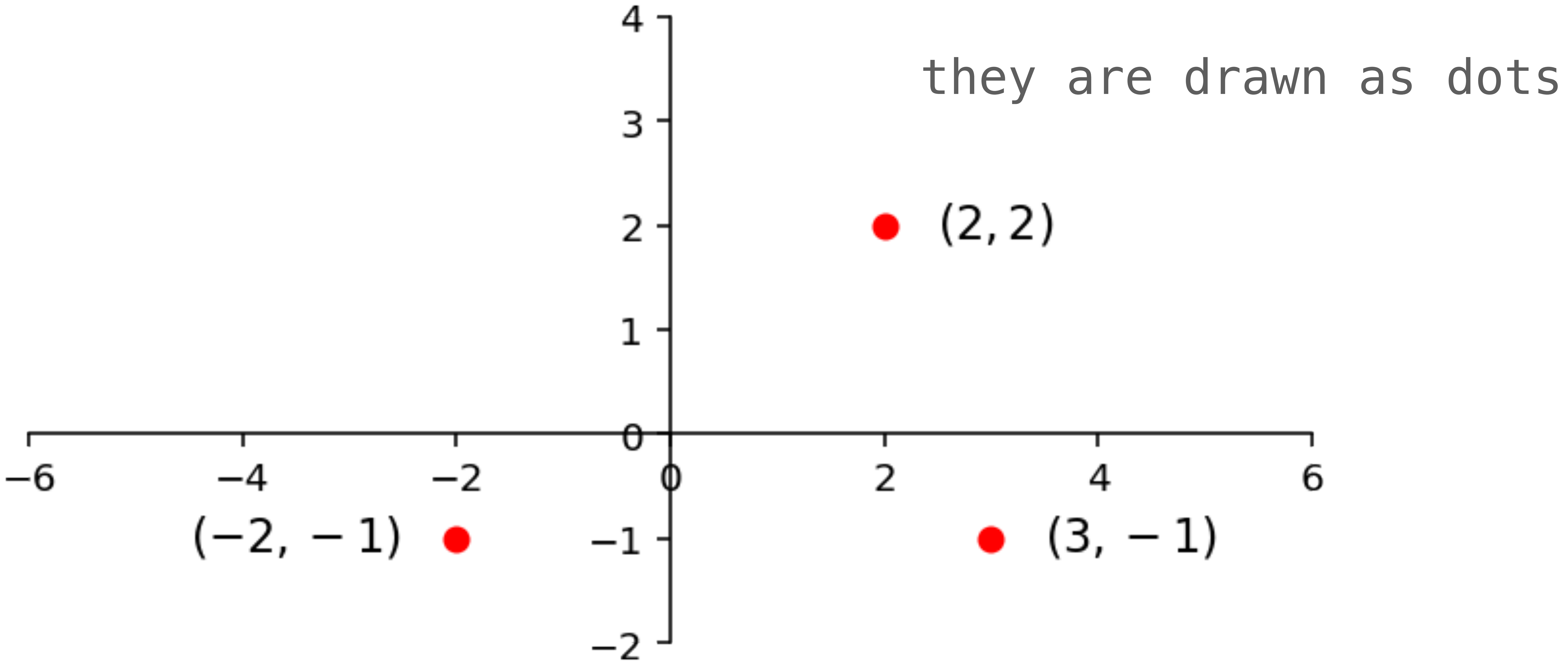
$$4 \left\{ \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1} \right.$$

the number of rows of a vectors is called its **dimension**

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

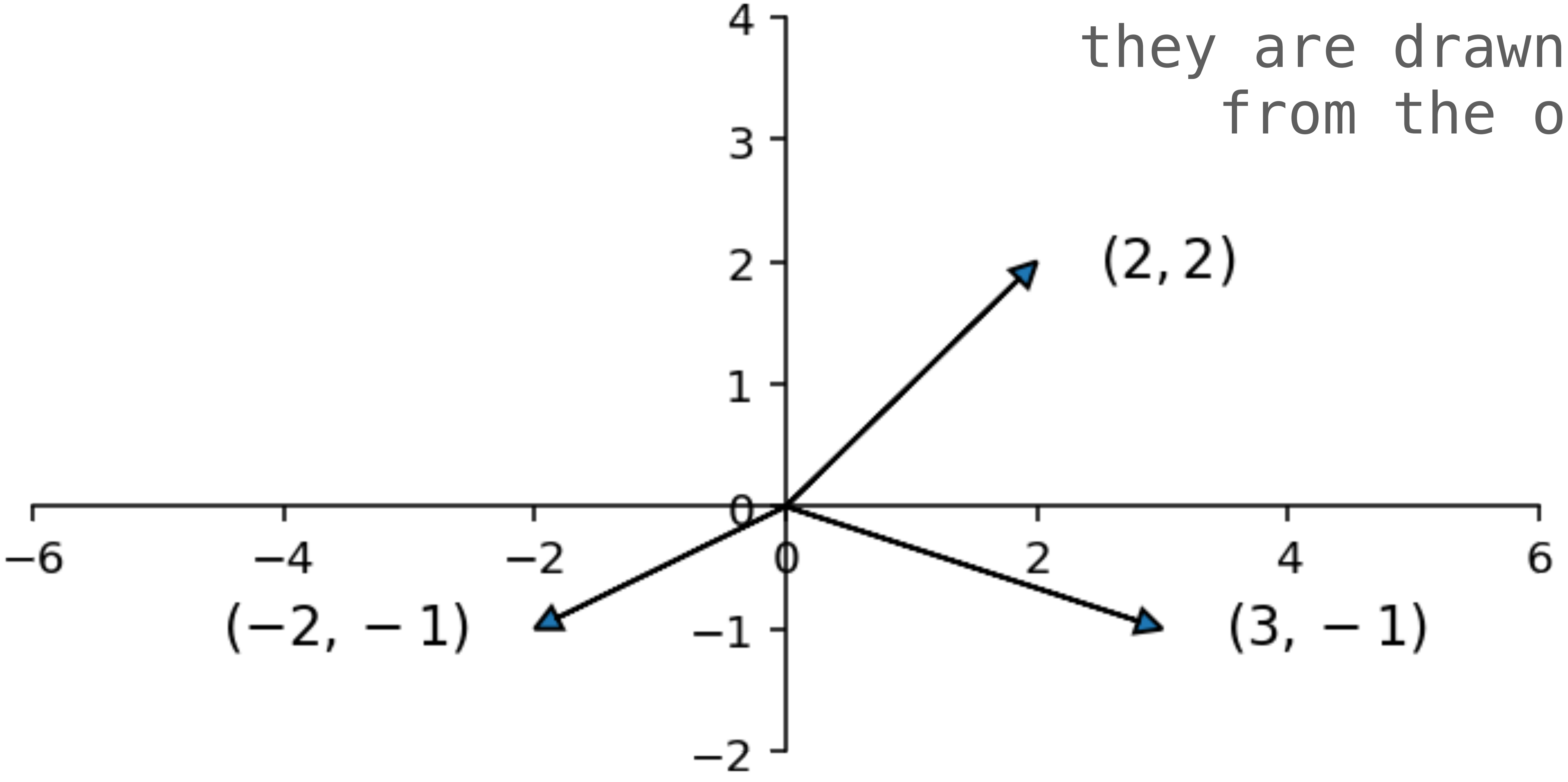
Examples

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



they are drawn as arrow
from the origin

vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation (Looking ahead)

we will often write $[a_1\ a_2\ \ldots\ a_n]^T$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^{n \times 1} \qquad [a_1\ \ a_2 \ldots\ a_n] \in \mathbb{R}^{1 \times n}$$

!!IMPORTANT!!

$(a_1, a_2, \ldots, a_n)$ is not the same as $[a_1\ \ a_2\ \ \ldots\ \ a_n]$

Vector Operations

Vector "Interface"

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equality	what does it mean for two vectors to be equal?

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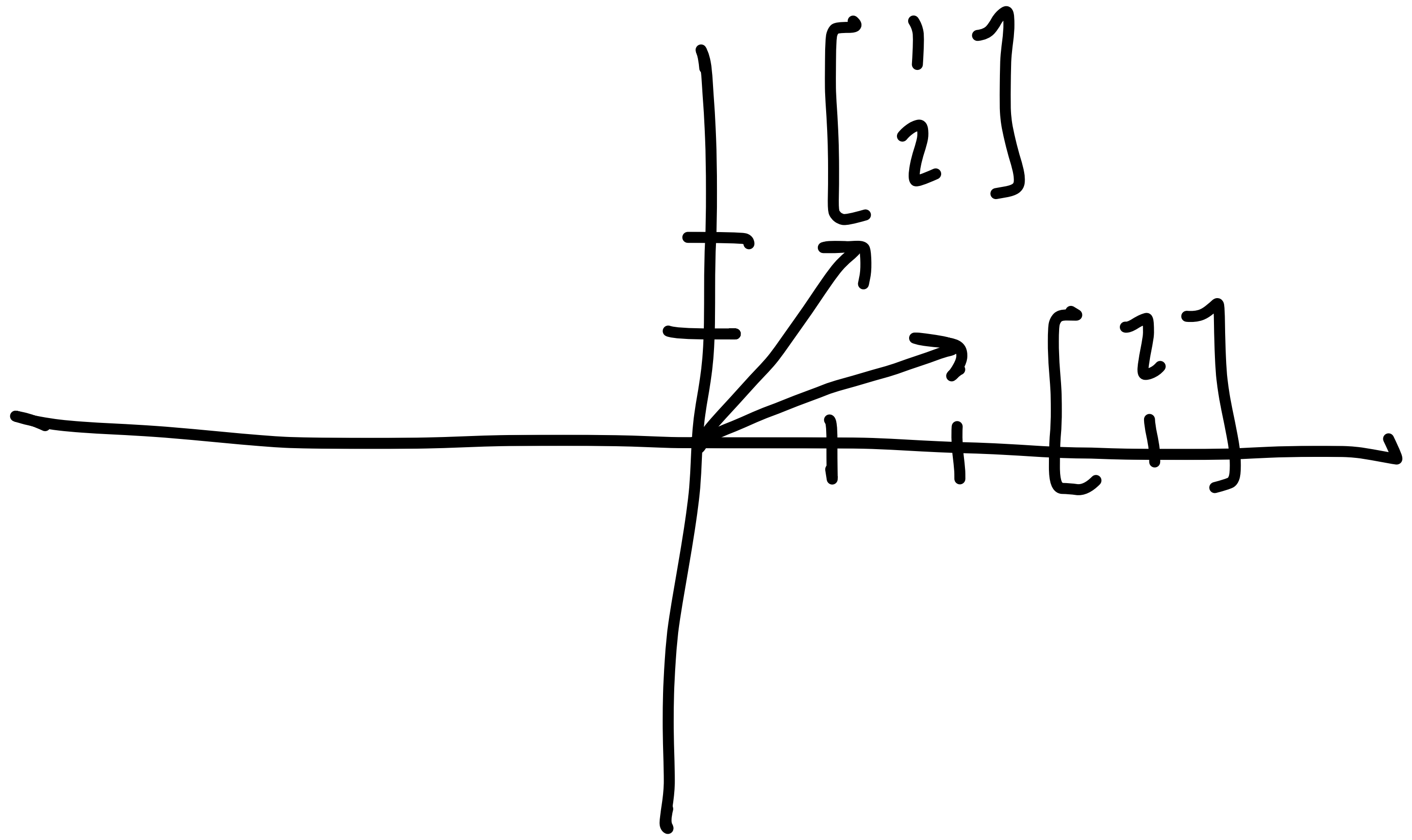
Vector Equality



two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!

ORDER MATTERS

$$\begin{bmatrix}1\\2\end{bmatrix} \neq \begin{bmatrix}2\\1\end{bmatrix}$$

Vector Equality

$$\begin{bmatrix}1\\2\end{bmatrix} \neq \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$$

is the same as

$$\begin{gathered} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{gathered}$$

Examples

$$\left[\frac{1}{2}\right] = \left(\frac{1}{2}\right)$$

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Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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!!IMPORTANT!!

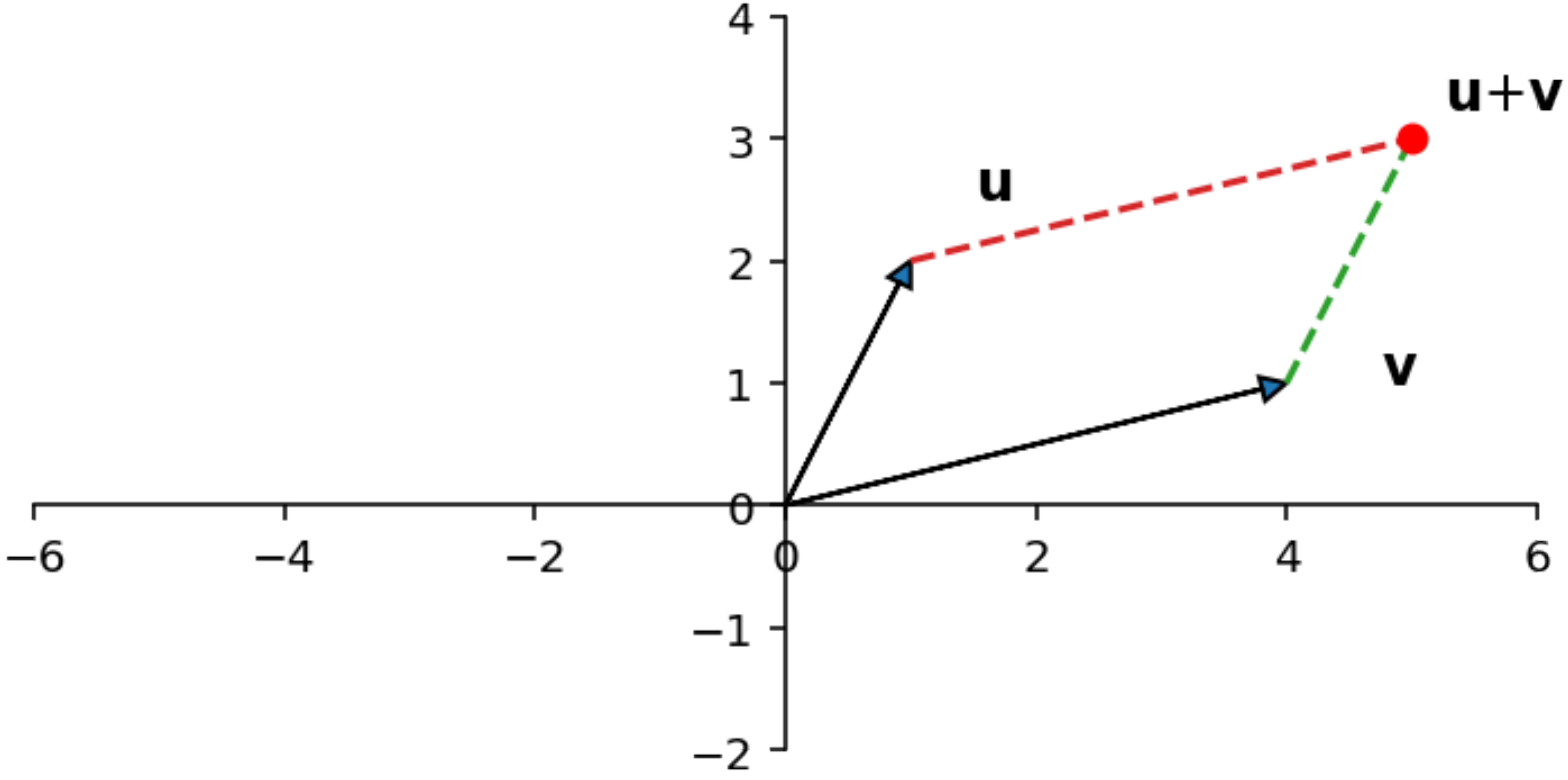
WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Examples

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Vector Addition (Geometrically)

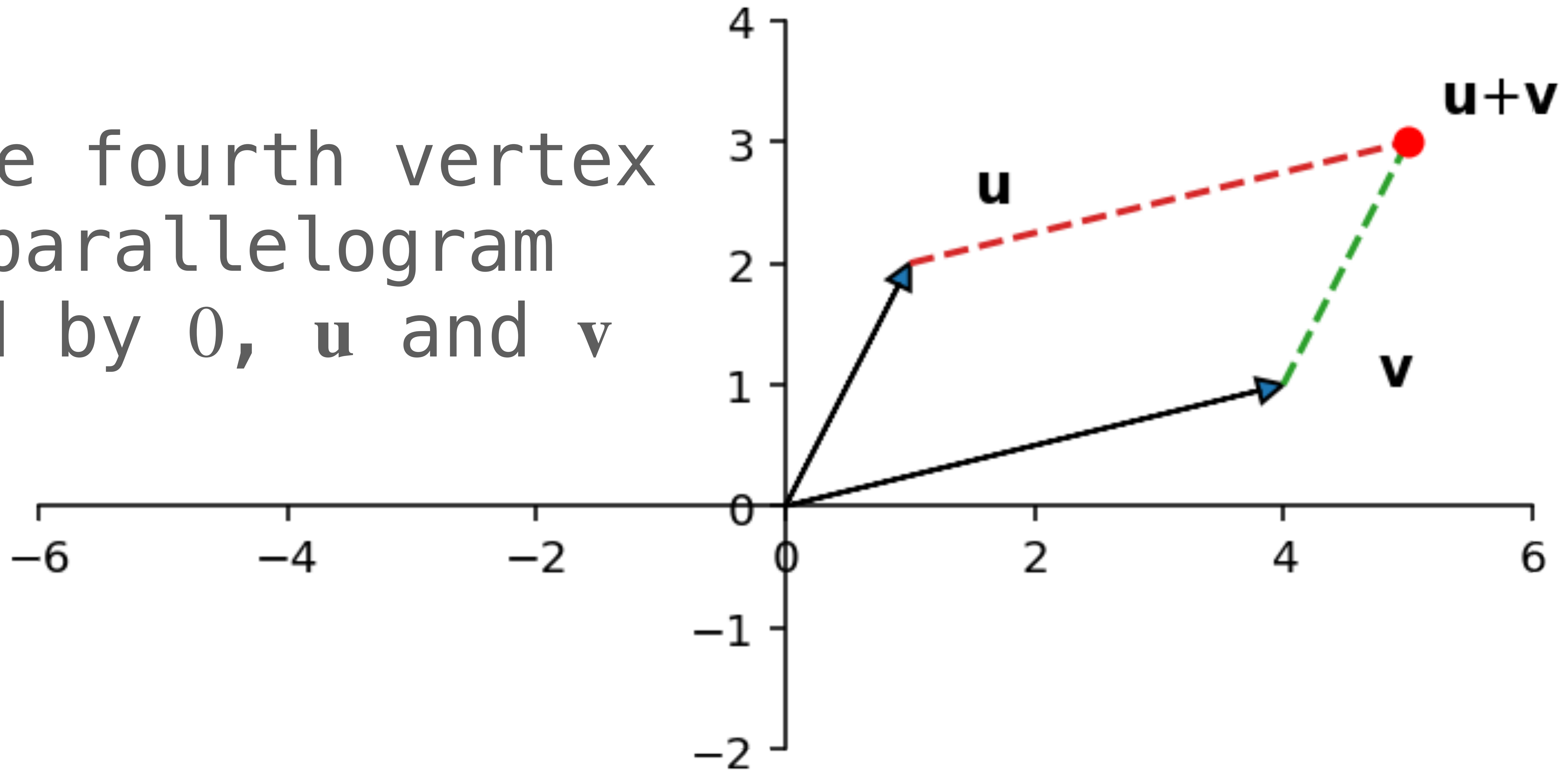
in $\mathbb{R}^2$ it's called the *parallelogram rule*



Vector Addition (Geometrically)

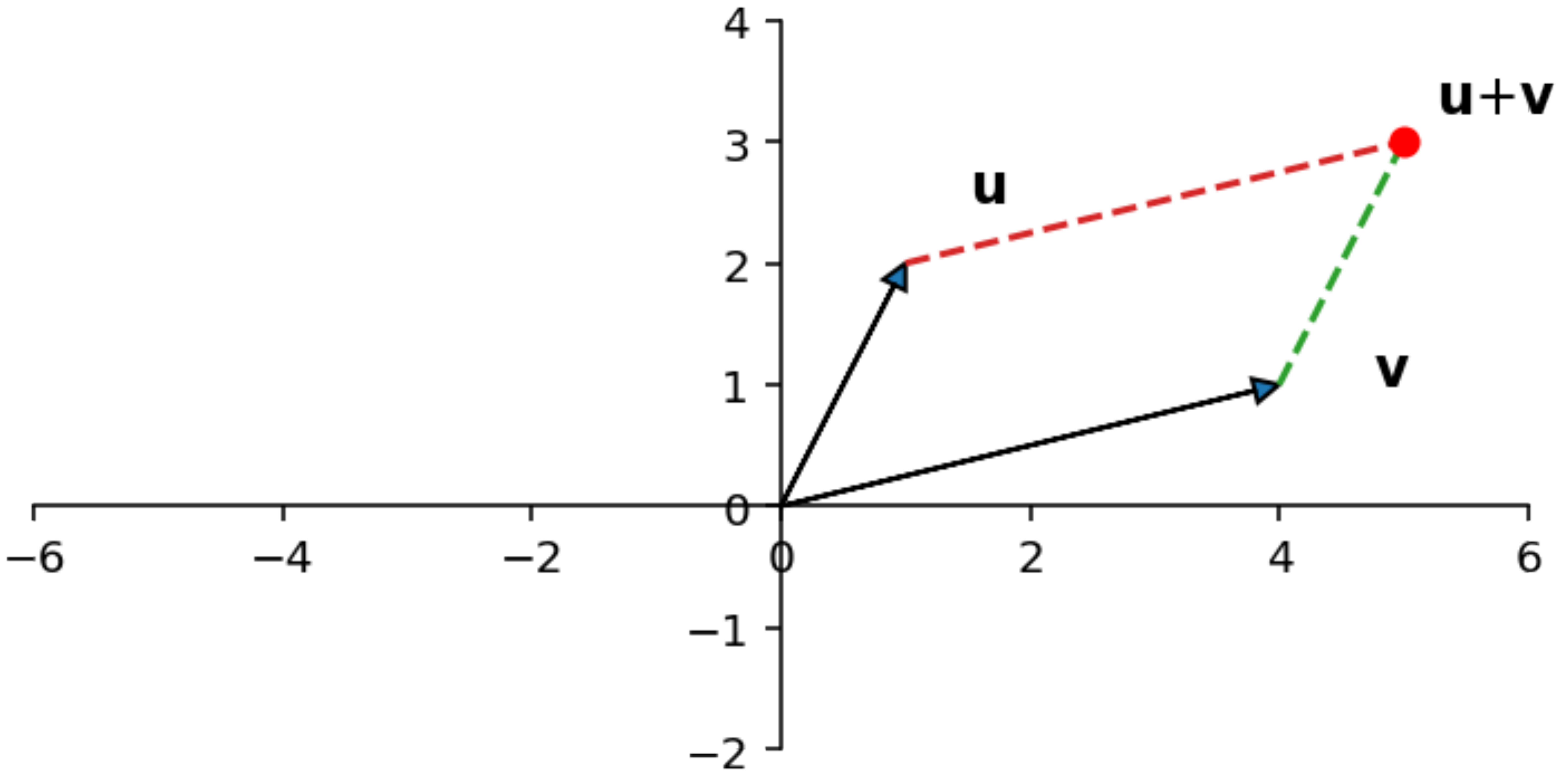
in $\mathbb{R}^2$ it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

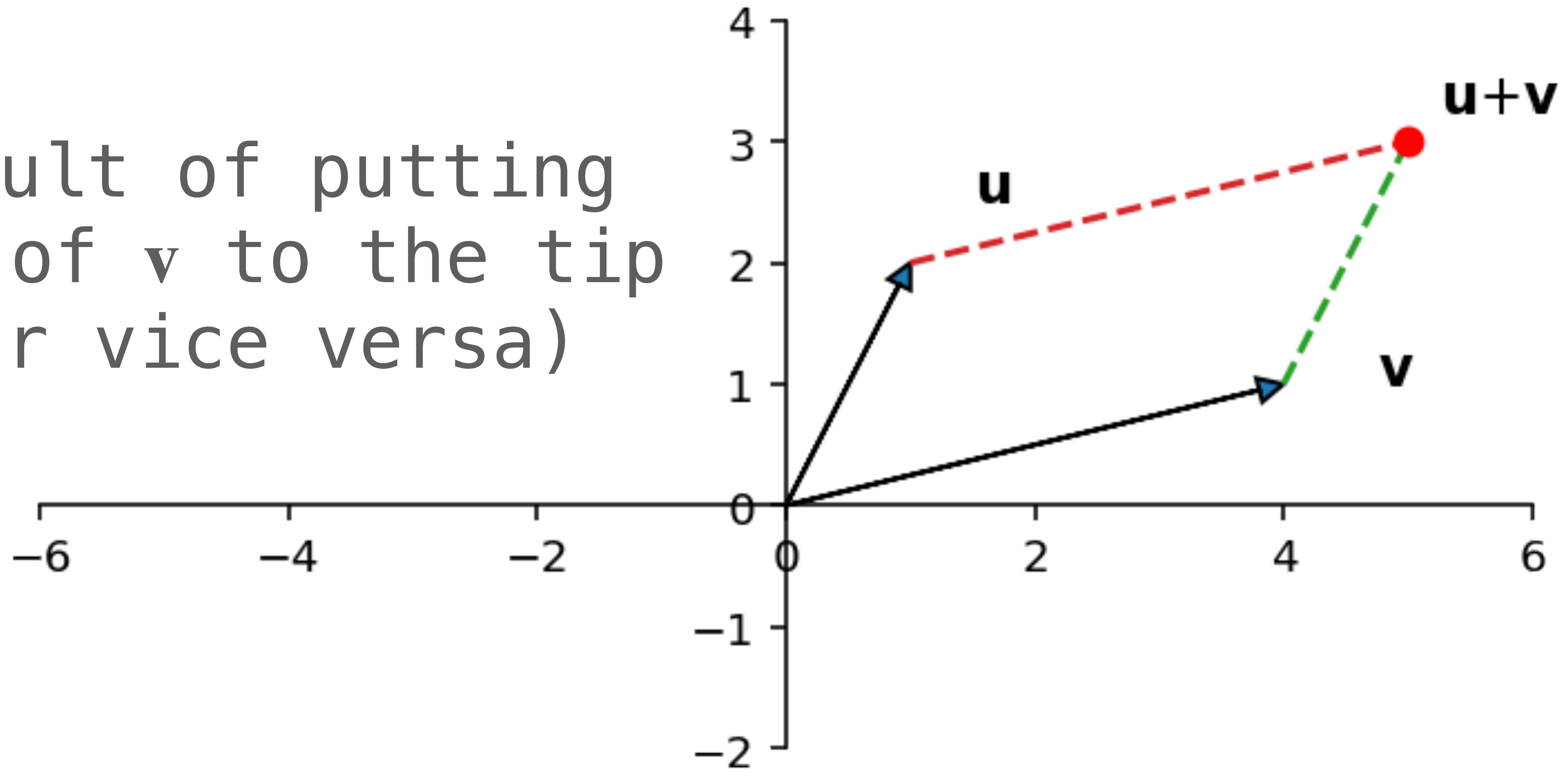
or the *tip–to–tail rule*



Vector Addition (Geometrically)

or the *tip–to–tail rule*

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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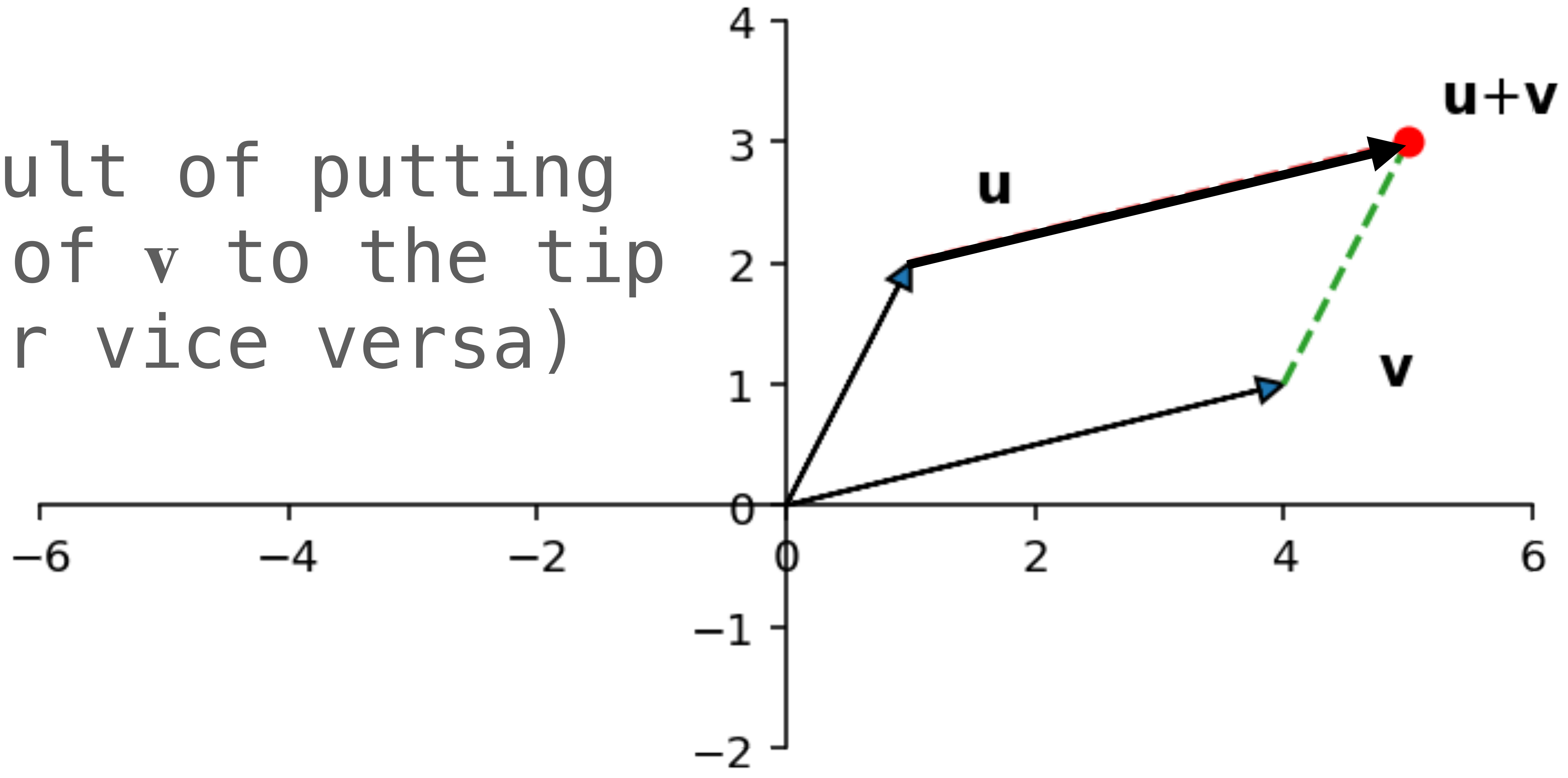

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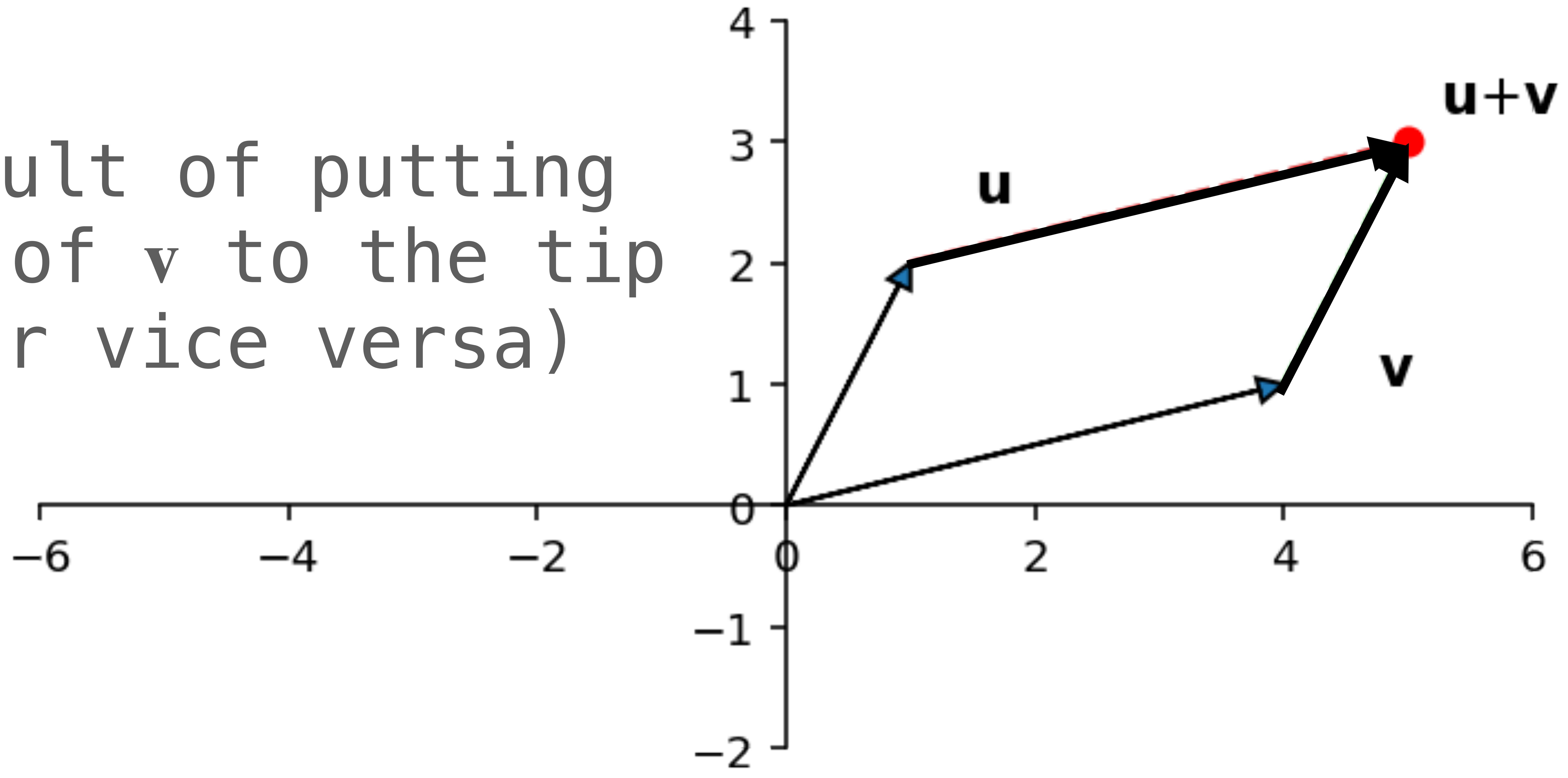

demo

(from ILA)

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What properties do they need to satisfy?

Vector Scaling/Multiplication

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

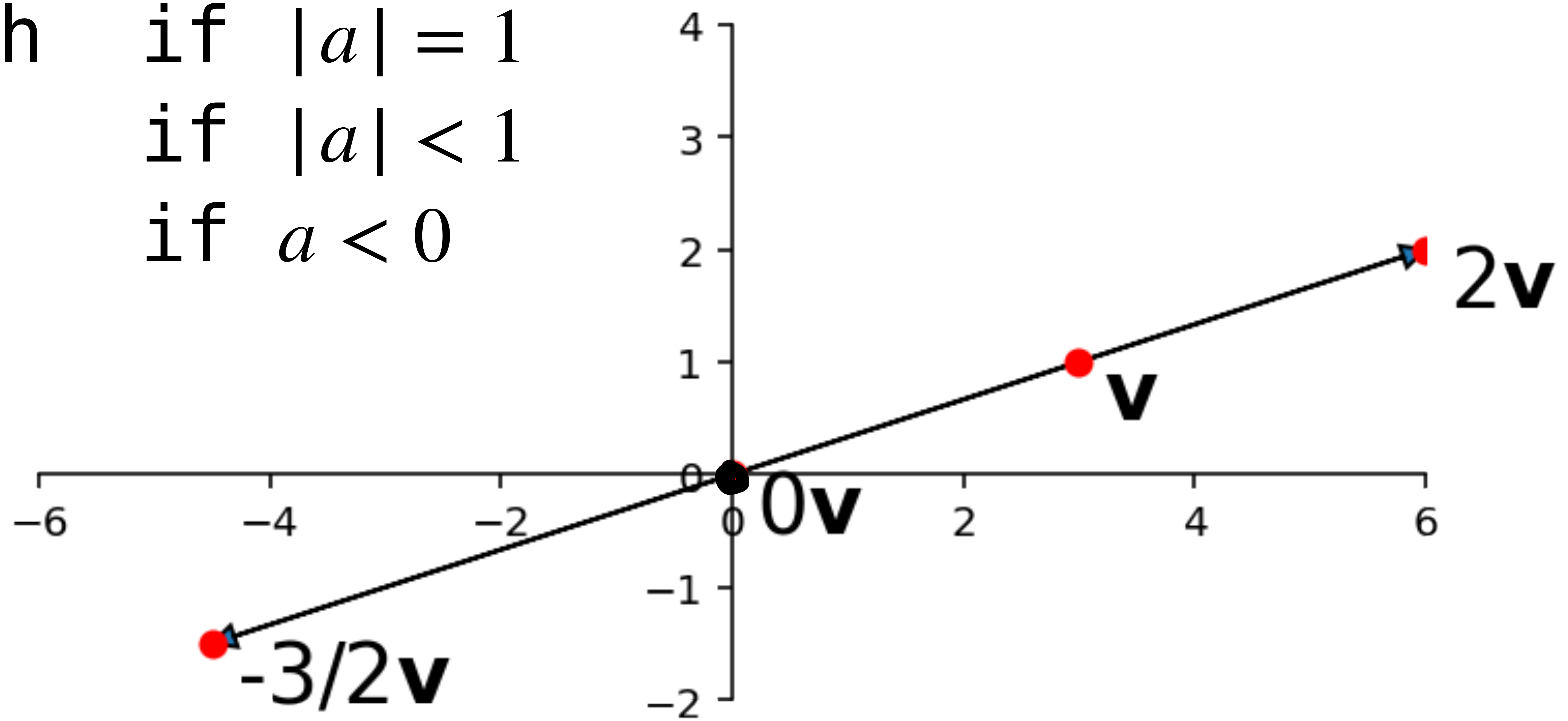
Scaling/multiplying a vector by a number means multiplying each of it's elements

$$3\begin{bmatrix}2\\1\\3.5\\4\end{bmatrix}=\begin{bmatrix}3\cdot 2\\3\cdot 1\\3\cdot 3.5\\3\cdot 4\end{bmatrix}=\begin{bmatrix}6\\3\\10.5\\12\end{bmatrix}$$

Vector Scaling (Geometrically)

$$0\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\cdot 3\\0\cdot 1\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

longer	if $\lvert a\rvert > 1$
the same length	if $\lvert a\rvert = 1$
shorter	if $\lvert a\rvert < 1$
reversed	if $a < 0$



demo

(from ILA)

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

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these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Example "Proof"

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1 + u_1 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_1 \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Question (Practice)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Compute the value of the above vector

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

In Sum

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```
we can add vectors
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this gives us a way of generating new vectors from old ones

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What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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weights

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Looks suspiciously like a linear equation...

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

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where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

demo

(from ILA)

The Fundamental Concern

$$2\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$$

Can $\mathbf{u}$ be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n = \mathbf{u}$?

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

Why is this fundamental?

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For now, how do we solve this problem?

Vector Equations and Linear Systems

The Fundamental Connection

The Fundamental Connection

We don't know the weights, that's want we want to find

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What if we write them as *unknowns*?

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What if we write them as *unknowns*?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing…

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$$\begin{bmatrix} x_1 \\ (-2)x_1 \\ (-5)x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

$$x_1 + 2x_2 = 7$$

$$(-2)x_1 + 5x_2 = 4$$

$$-5x_1 + 6x_2 = -3$$

Some Symbol Pushing...

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

General Vector Equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

General Vector Equations

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

General Vector Equations

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

General Vector Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

by vector equality

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

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augmented matrix

this is our big shift in perspective

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How To: Linear Combination Problems

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this is notation for building a matrix out of column vectors

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Question

Exercise

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

Answer

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix}+2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

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read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

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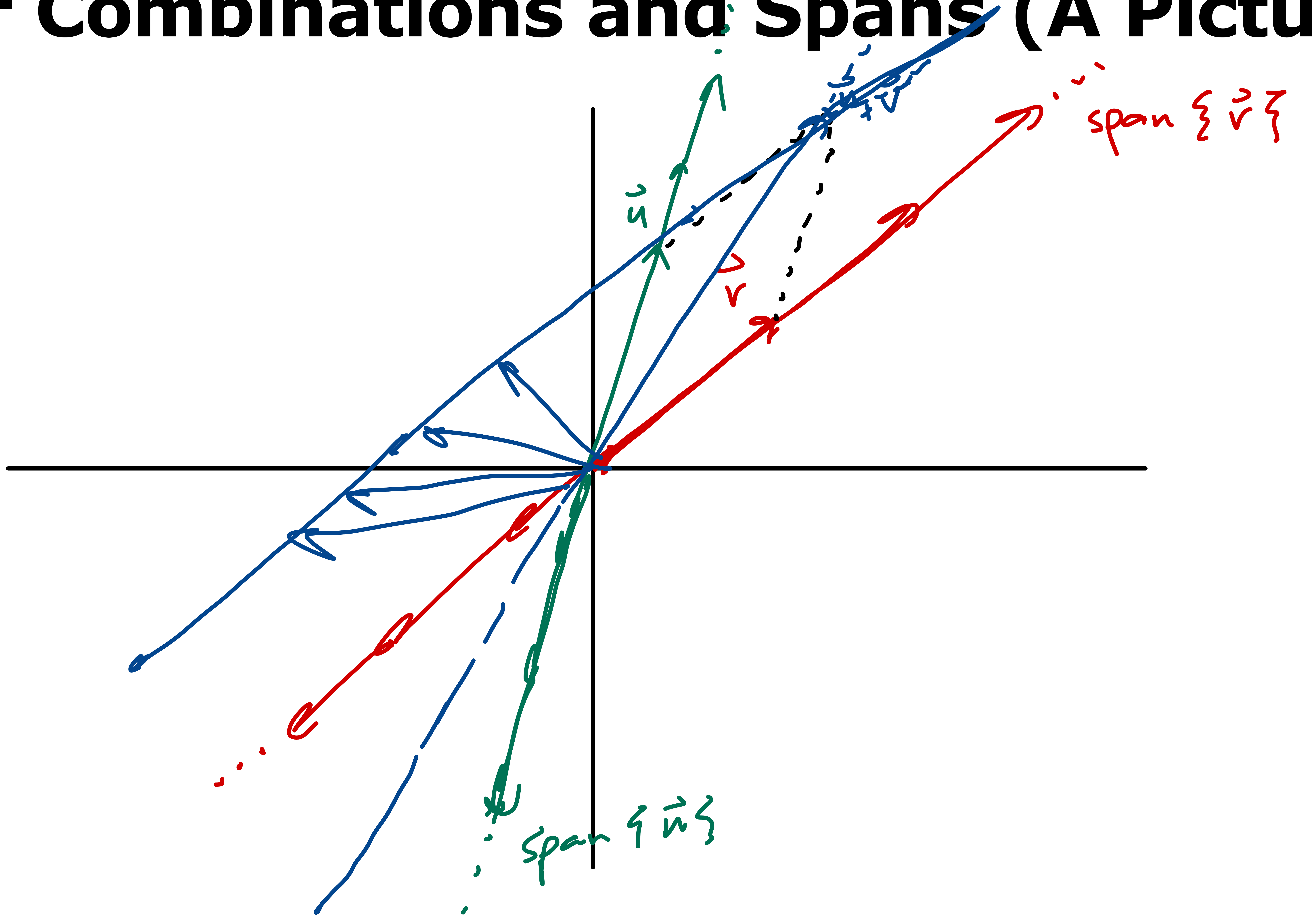
Linear Combinations and Spans (A Picture)



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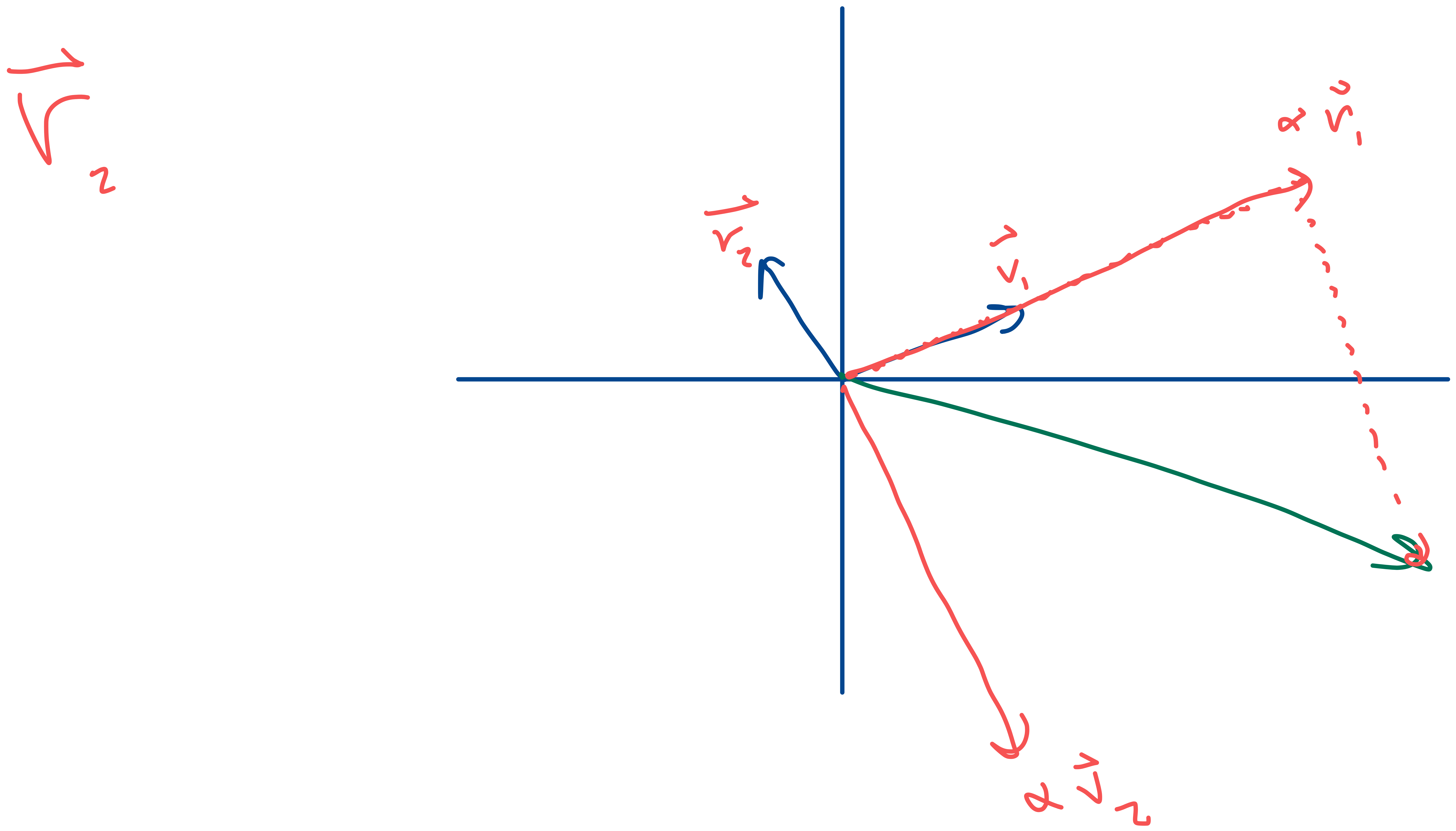

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the span of one vector is a **line**

Spans (Geometrically)

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the span of **two** vectors can be a **plane**

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Spans (Geometrically)

$\vec{0} \in \text{span } V$

the span of **two** vectors can be a **plane**

the span of **three** vectors can be a **hyperplane**

!!IMPORTANT!!
In all cases they pass through the origin

demo

(from ILA)

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you know how to do this now

Example

$$\textit{Is} \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \textit{in} \ \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\} ?$$

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo

(from ILA)

How To: Inconsistency and Spans

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There is **no way** to write $\mathbf{b}$ as a linear combination

Example

Find a vector ***not*** *in* $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$

Summary

Vectors are fundamental objects

We can think of them as the **columns** of a linear system

We can **scale** them and **add** them together

They can **span** spaces which represent **hyperplanes**