Vector Equations

Geometric Algorithms
Lecture 4

CAS CS 132

Practice Problem

Suppose that A *is a* 322×245 *augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that* A *can have?*

Answer

Outline

» Formally define vectors

» Discuss vector operations and vector algebra

» Draw the connection between vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Motivation (An Aside)

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all n

Changing Perspective

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$$100...000 - 000...001 = 011...111$$

`show that this holds for all` n

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100\ldots000 - 000\ldots001 = 011\ldots111$$

show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

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maybe I'm reaching...

Big Data (More Practical Motivation)

A piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

Maybe if they are **close together** in a geometric sense

A Note on Algebra

$$\mathbf{v} = \mathbf{w}$$

$$\mathbf{v} + \mathbf{w}$$

$$a\mathbf{v}$$

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In programming an *interface* is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

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We're defining an new thing called a *column vector*

We need to define what **equality** and **adding** and **multiplying by a number** means for column vectors

$$\mathbf{v} = \mathbf{w}$$

$$\mathbf{v} + \mathbf{w}$$

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Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

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it's common to conflate points and vectors

Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix} \end{array} \qquad \begin{array}{cc} & 1 \\ 4 & \begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix} \end{array}$$

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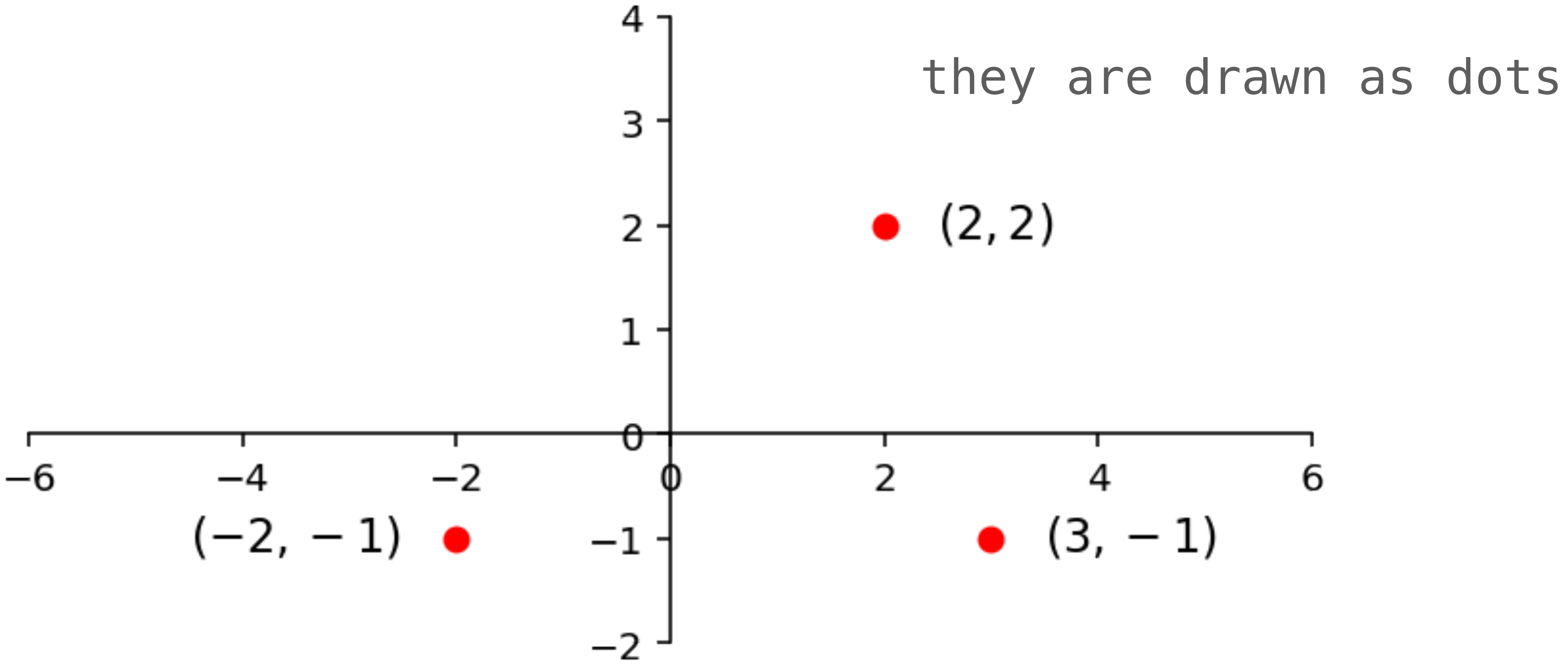
$$\begin{array}{cc} & 1 \\ 4 & \begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix} \end{array}$$

the number of rows of a vectors is called its **dimension**

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

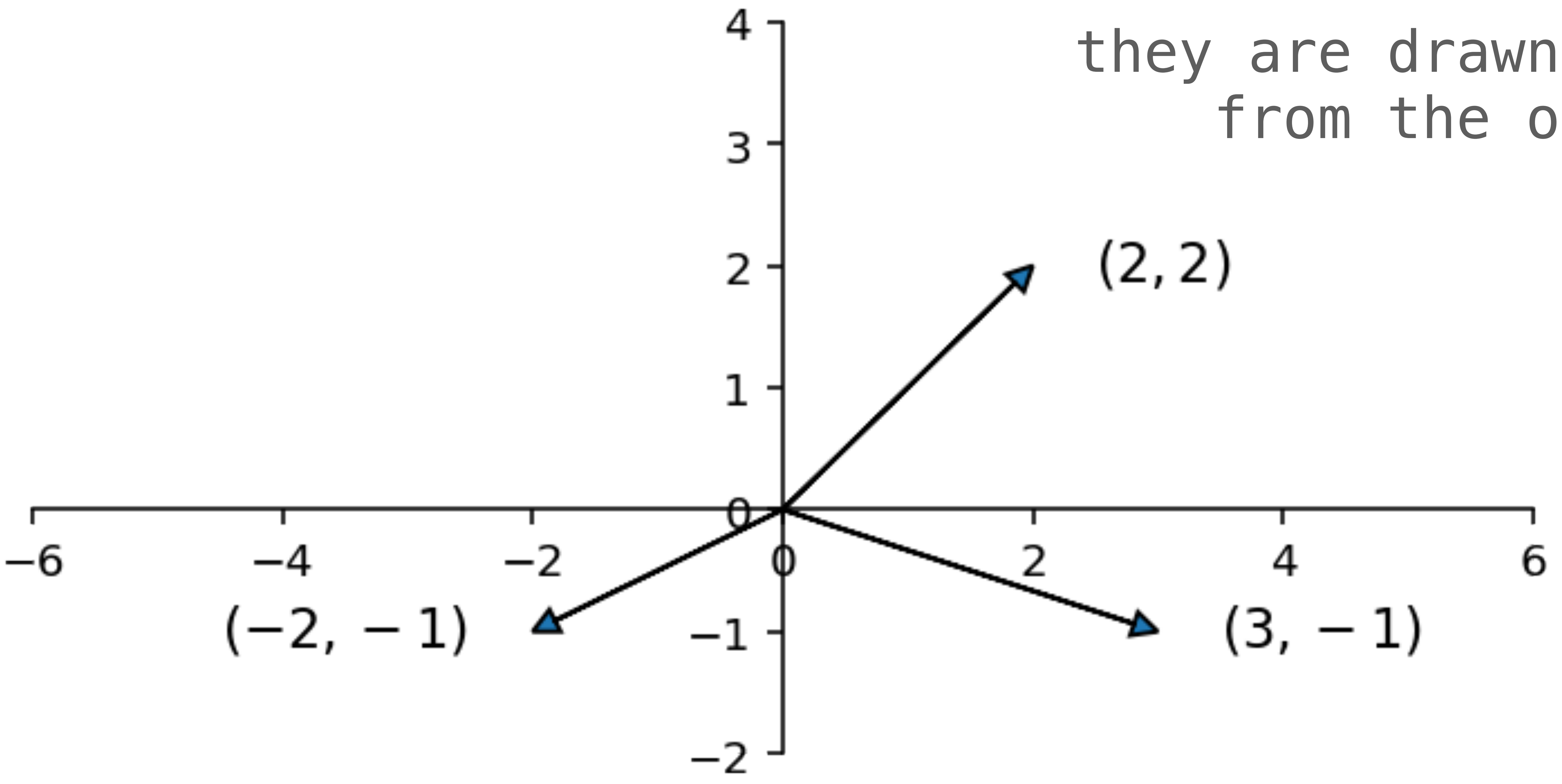
Examples

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



they are drawn as arrow from the origin

vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation (Looking ahead)

we will often write $[a_1\ a_2\ \dots\ a_n]^T$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

!!IMPORTANT!!

$(a_1, a_2, \dots, a_n)$ is not the same as $[a_1\ \ a_2\ \ \dots\ \ a_n]$

Vector Operations

Vector "Interface"

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`equality`	`what does it mean for two vectors to be equal?`

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equality what does it mean for two vectors to be equal?

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Vector Equality

two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!
ORDER MATTERS

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Vector Equality

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is the same as

$$\begin{gathered} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{gathered}$$

Examples

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Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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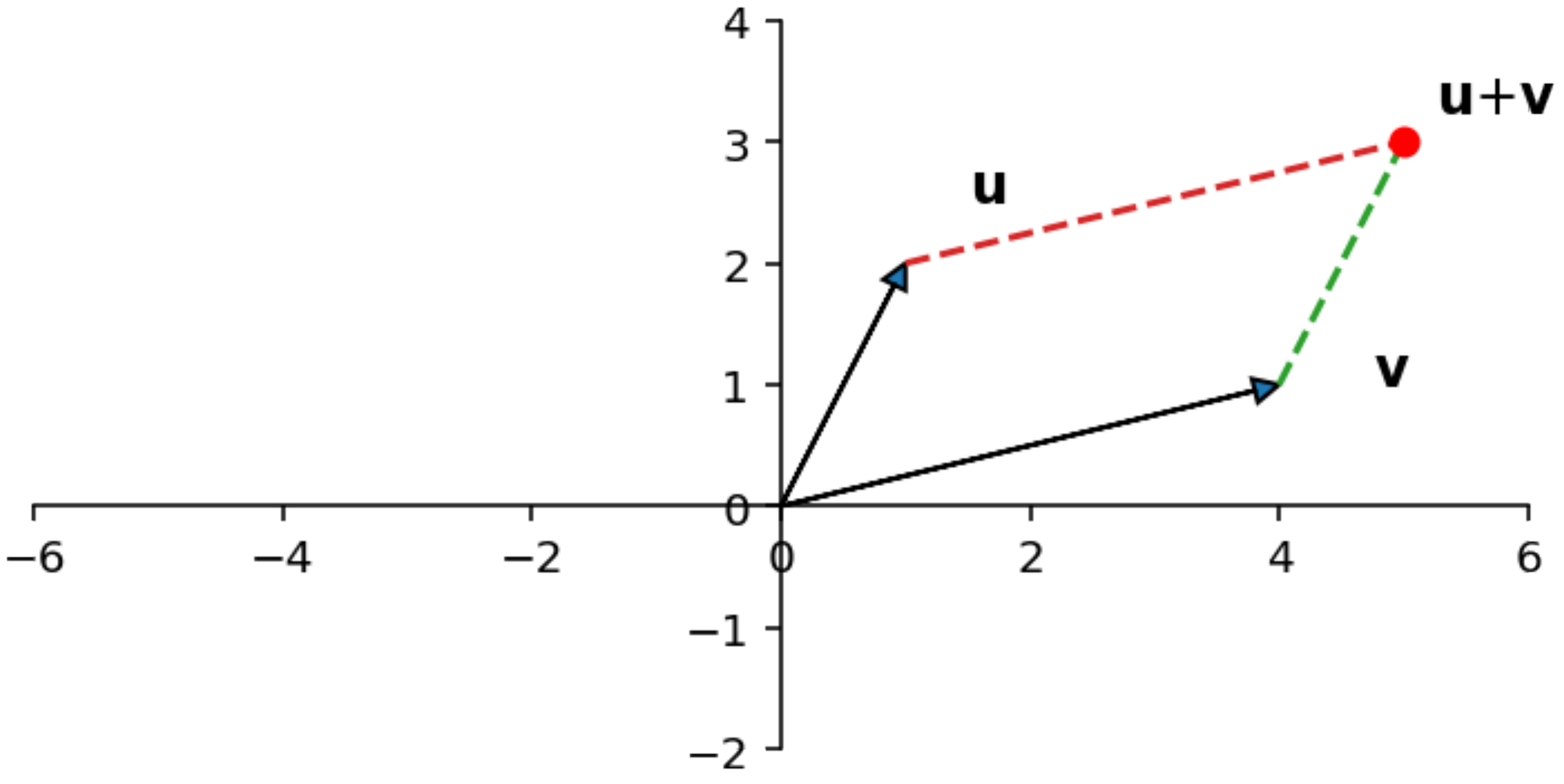
!!IMPORTANT!!
WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Examples

Vector Addition (Geometrically)

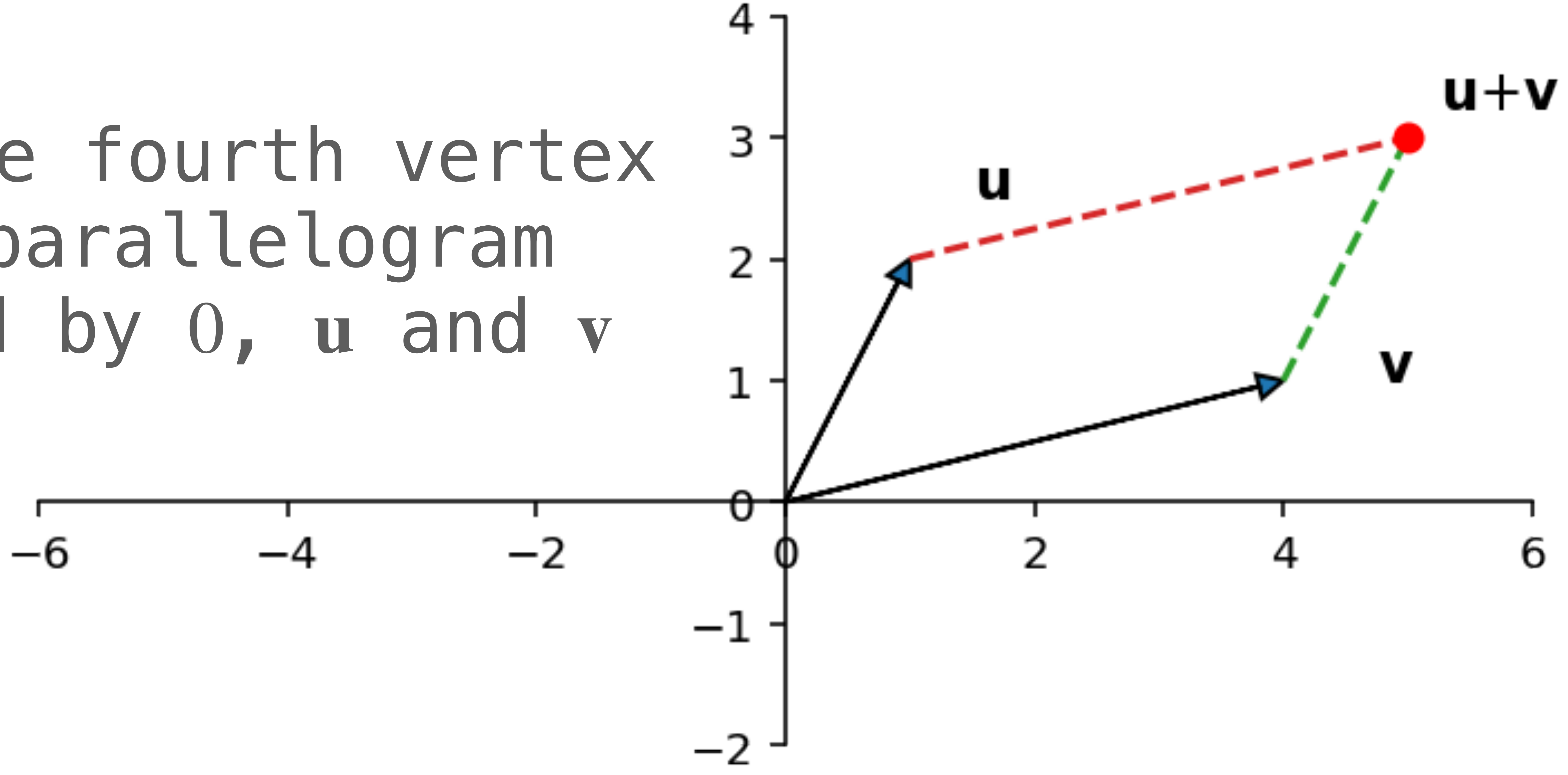
in $\mathbb{R}^2$ it's called the *parallelogram rule*



Vector Addition (Geometrically)

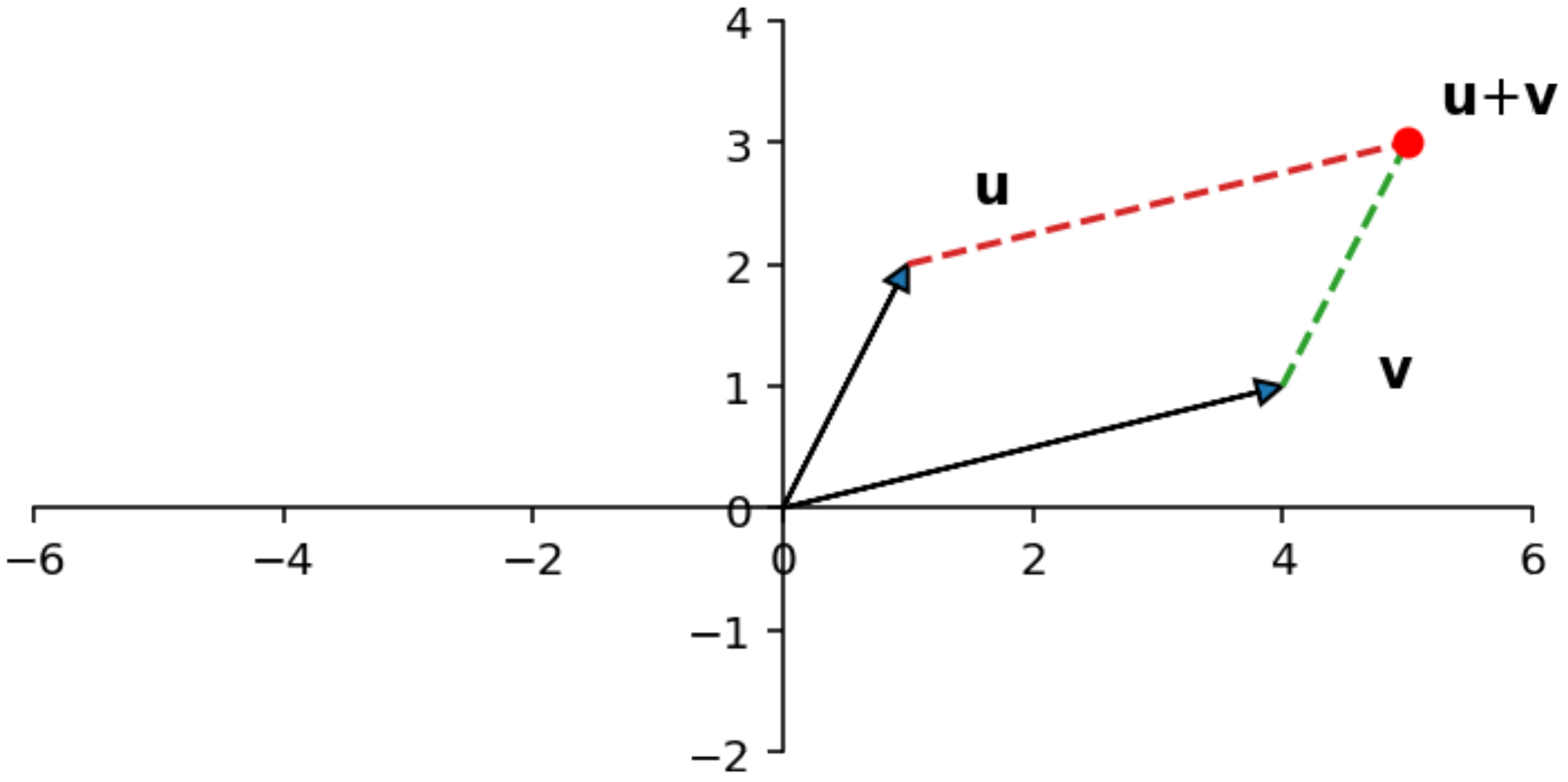
in $\mathbb{R}^2$ it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip-to-tail rule*



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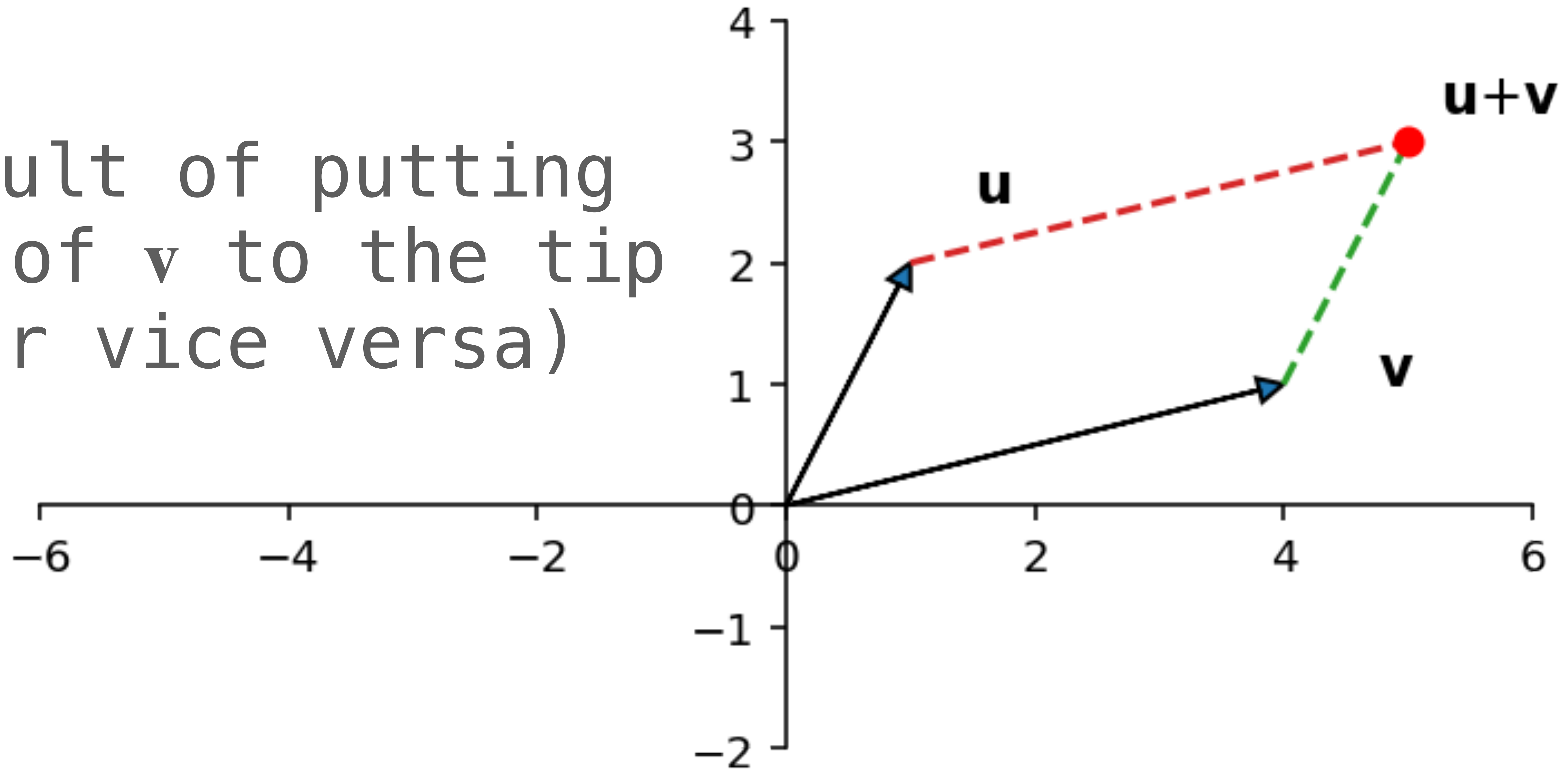
$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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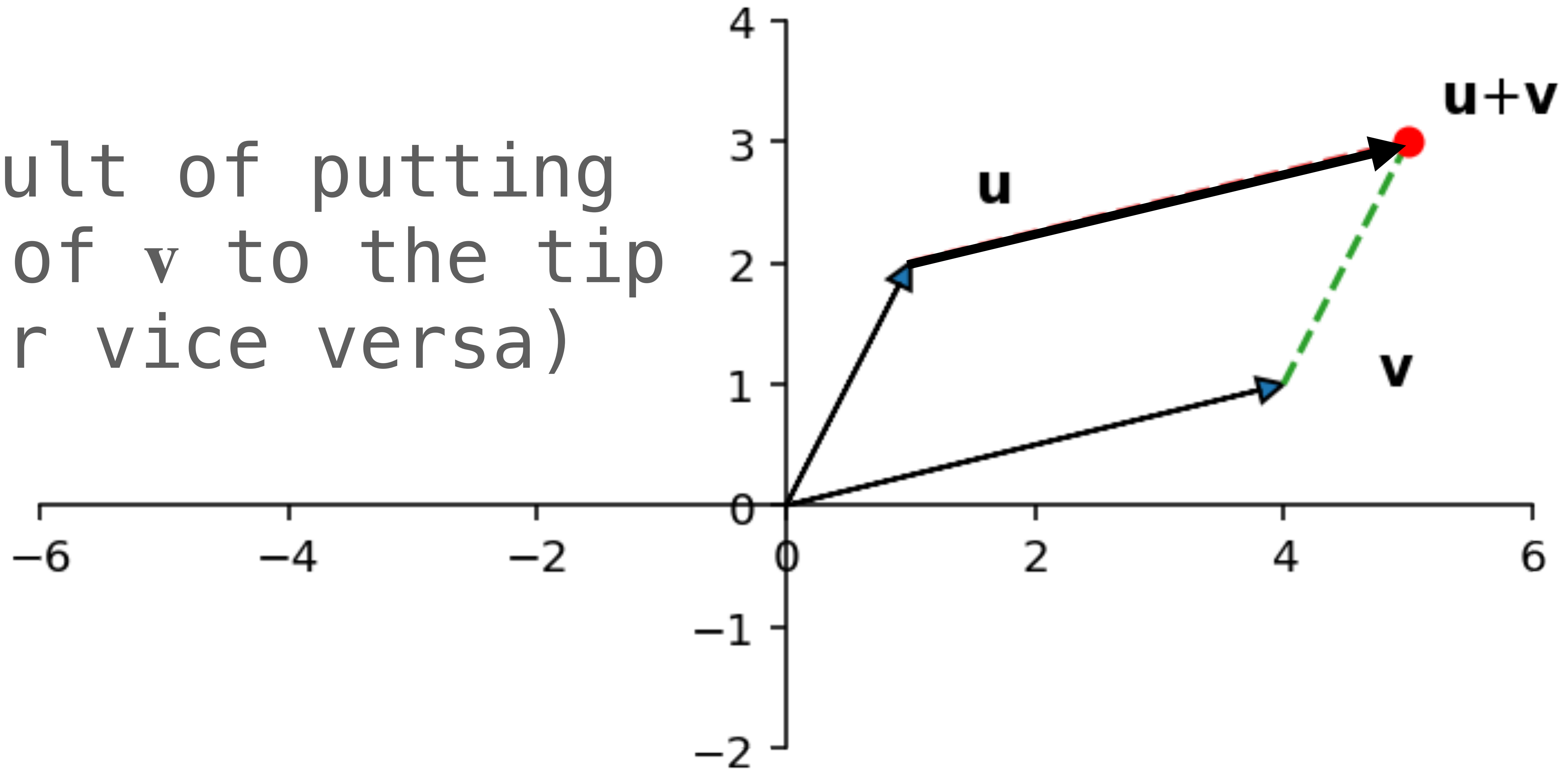

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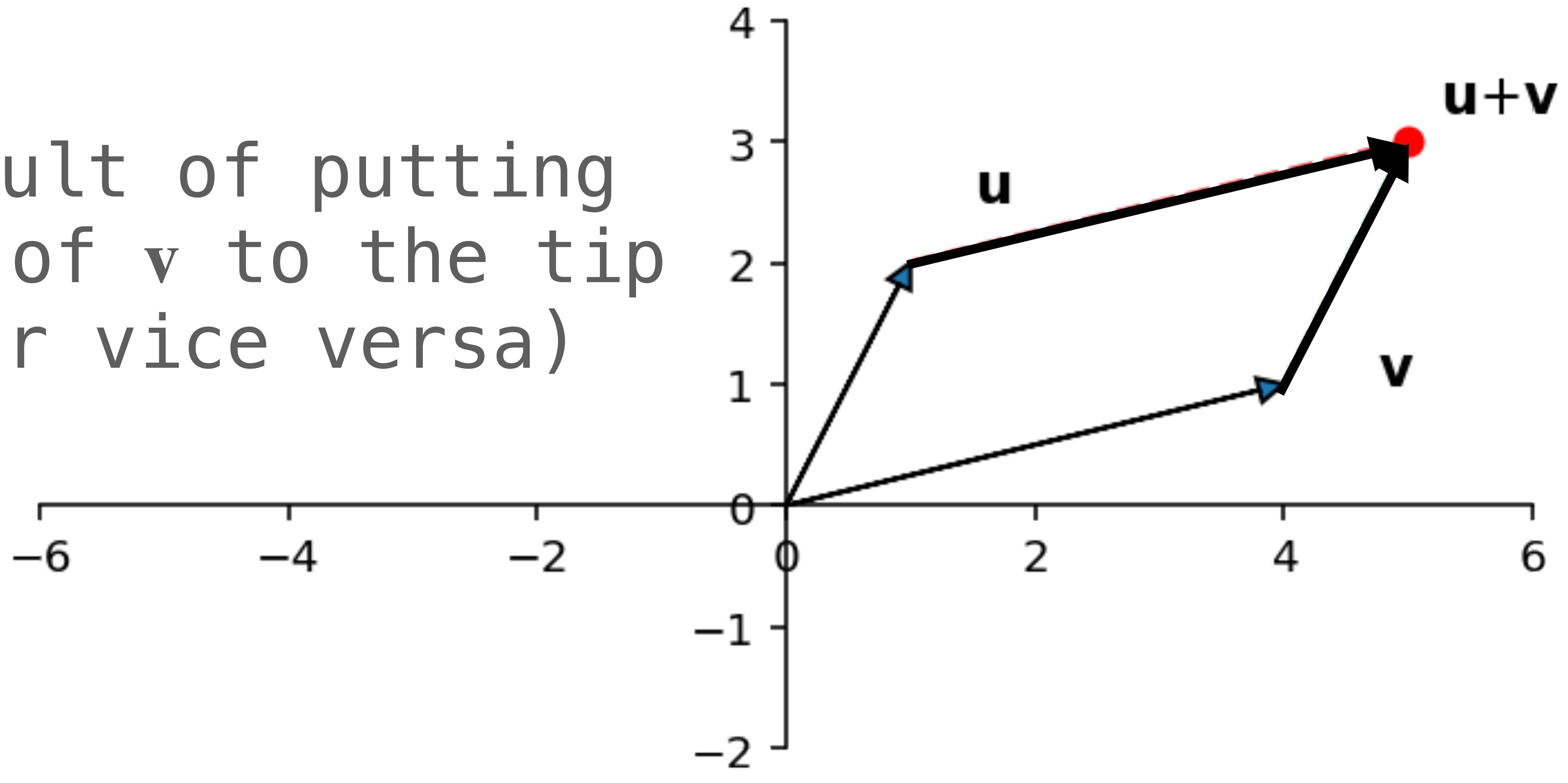

demo

(from ILA)

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What properties do they need to satisfy?

Vector Scaling/Multiplication

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

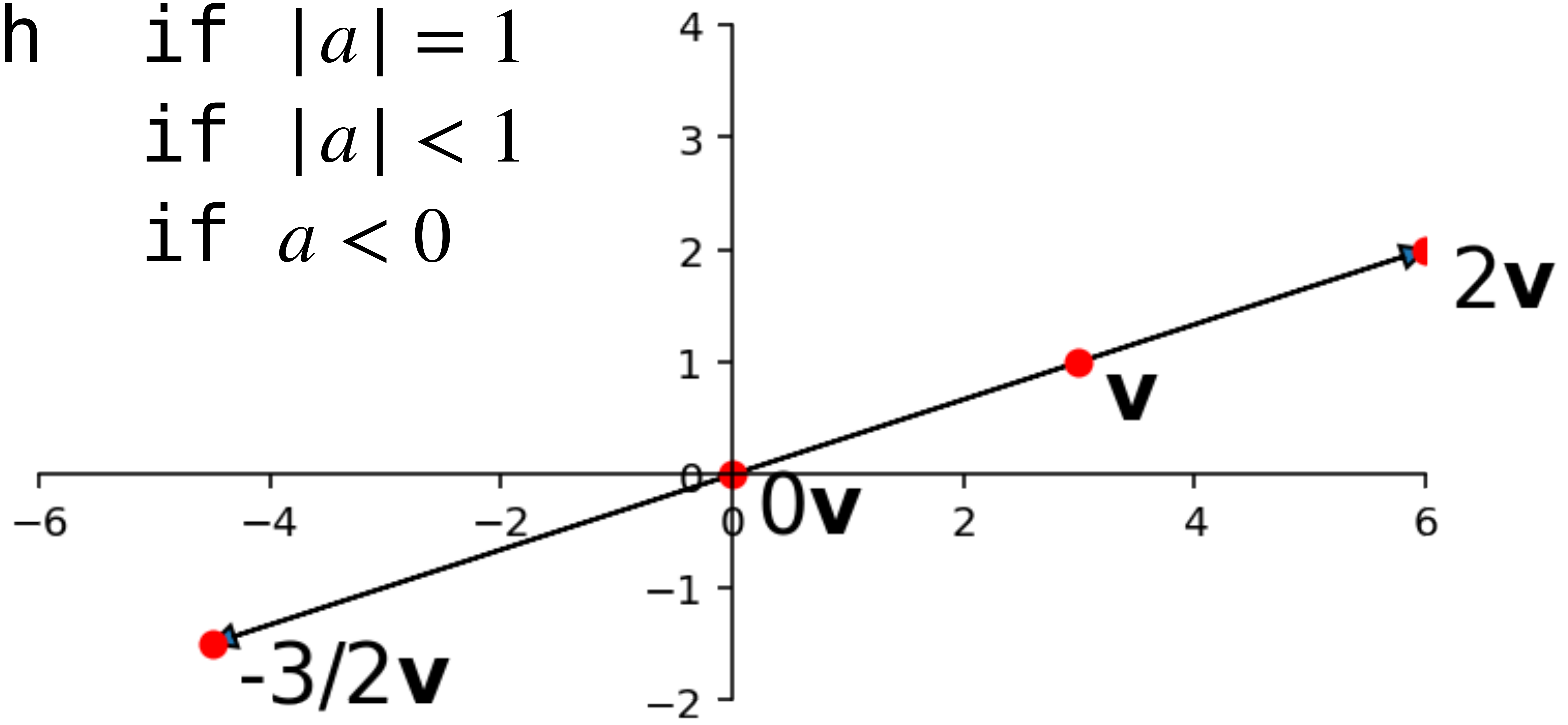
Vector Scaling/Multiplication (Example)

Scaling/multiplying a vector by a number means multiplying each of it's elements

$$3\begin{bmatrix}2\\1\\3.5\\4\end{bmatrix}=\begin{bmatrix}3\cdot 2\\3\cdot 1\\3\cdot 3.5\\3\cdot 4\end{bmatrix}=\begin{bmatrix}6\\3\\10.5\\12\end{bmatrix}$$

Vector Scaling (Geometrically)

longer	if $\lvert a \rvert > 1$
the same length	if $\lvert a \rvert = 1$
shorter	if $\lvert a \rvert < 1$
reversed	if $a < 0$



demo

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Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

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these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Example "Proof"

Question (Practice)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Compute the value of the above vector

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

In Sum

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`we can add vectors`

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this gives us a way of generating new vectors from old ones

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What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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Looks suspiciously like a linear equation...

where $\underbrace{\alpha_1, \alpha_2, \ldots, \alpha_n}_{\text{weights}}$ are in $\mathbb{R}$

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

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where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

demo

(from ILA)

The Fundamental Concern

$$2\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$$

Can $\mathbf{u}$ be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n = \mathbf{u}$?

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

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For now, how do we solve this problem?

Vector Equations and Linear Systems

The Fundamental Connection

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We don't know the weights, that's want we want to find

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What if we write them as *unknowns*?

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

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$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

$$
\begin{aligned}
x_1 + 2x_2 &= 7 \\
(-2)x_1 + 5x_2 &= 4 \\
-5x_1 + 6x_2 &= -3
\end{aligned}
$$

Some Symbol Pushing...

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

General Vector Equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

General Vector Equations

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

General Vector Equations

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

General Vector Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

by vector equality

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

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vector equation

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augmented matrix

this is our big shift in perspective

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

How To: Linear Combination Problems

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this is notation for building a matrix out of column vectors

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Question

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

Answer

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix}+2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

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read: $\mathbf{u}$ is an element of $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

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Linear Combinations and Spans (A Picture)

Spans (Geometrically)

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`for one vector`

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the span of one vector is a **line**

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the span of **two** vectors can be a **plane**

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!!IMPORTANT!!

In all cases they pass through the origin demo

(from ILA)

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you know how to do this now

Example

$$\textit{Is} \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \textit{ in } \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}?$$

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo

(from ILA)

How To: Inconsistency and Spans

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There is **no way** to write $\mathbf{b}$ as a linear combination

Example

Find a vector ***not*** *in* $\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$

Summary

Vectors are fundamental objects

We can think of them as the **columns** of a linear system

We can **scale** them and **add** them together

They can **span** spaces which represent **hyperplanes**