Matrix-Vector Equations

Geometric Algorithms
Lecture 5

CAS CS 132

Practice Problem

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$?

Answer

$$\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}?$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 \; {\scriptstyle +2} & -1 \; {\scriptstyle +2} & -4 \; {\scriptstyle +6} & -14 \; {\scriptstyle -18} \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 \; {\scriptstyle -0} & 1 \; {\scriptstyle -1} & 2 \; {\scriptstyle -2} & 4 \; {\scriptstyle -3} \end{bmatrix}$$

NO

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer

Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix}$$

Answer

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$$R_3 \leftarrow R_3 + 2R_1$$

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$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

Answer

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

`no solution  ≡  not in the span`

Outline

- Motivate the study of matrix–vector equations
- Formally define matrix–vector multiplication
- Revisit spans
- Take stock of our perspectives on systems of linear equations

Keywords

matrix–vector multiplication

the matrix equation

inner–product

row–column rule

Recap

Recall: Vector "Interface"

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equality	what does it mean for two vectors to be equal?

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Recall: Vector "Interface"

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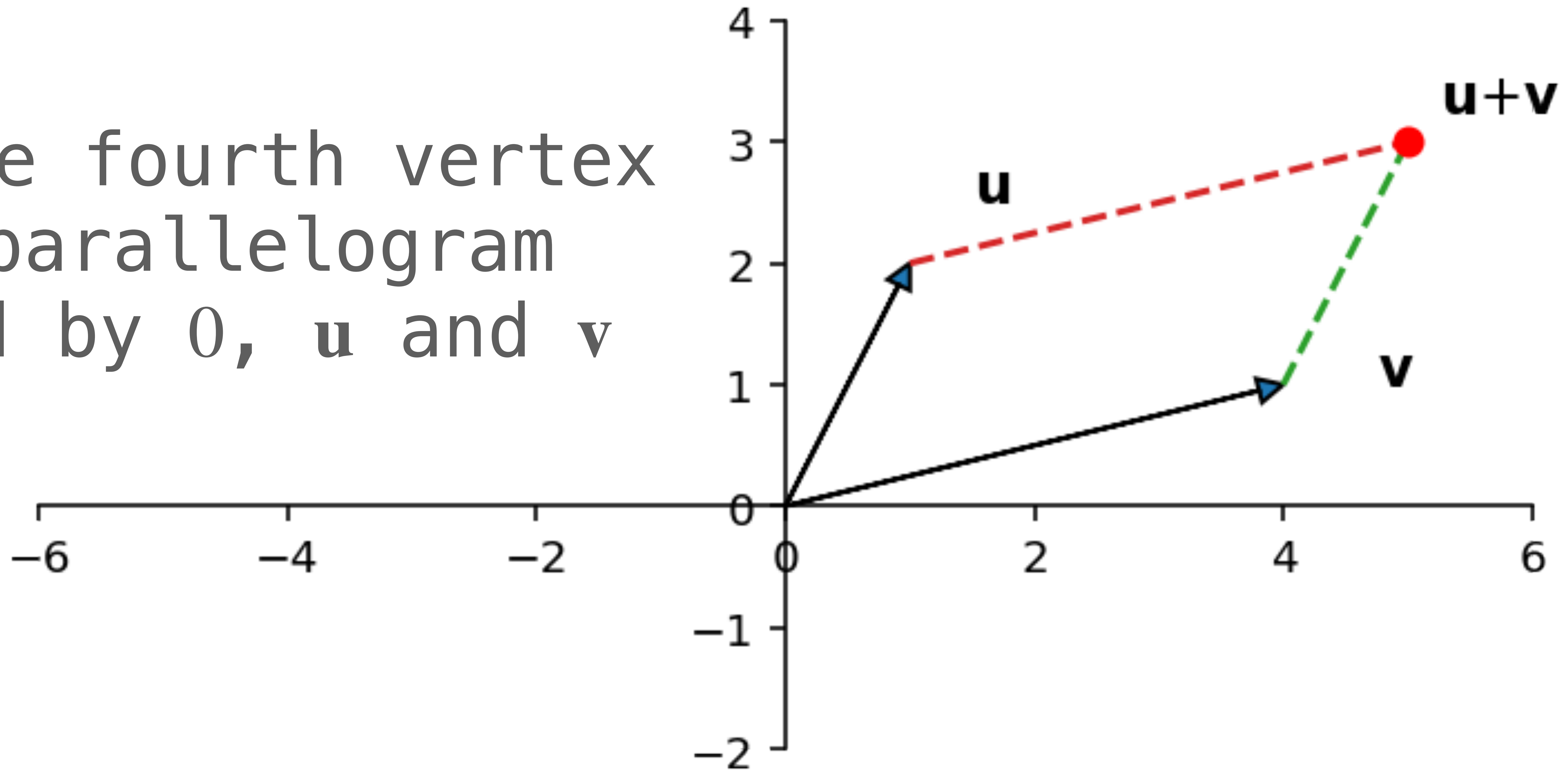
scaling — what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Recall: Vector Addition (Geometrically)

in $\mathbb{R}^2$ it's called the *parallelogram rule*

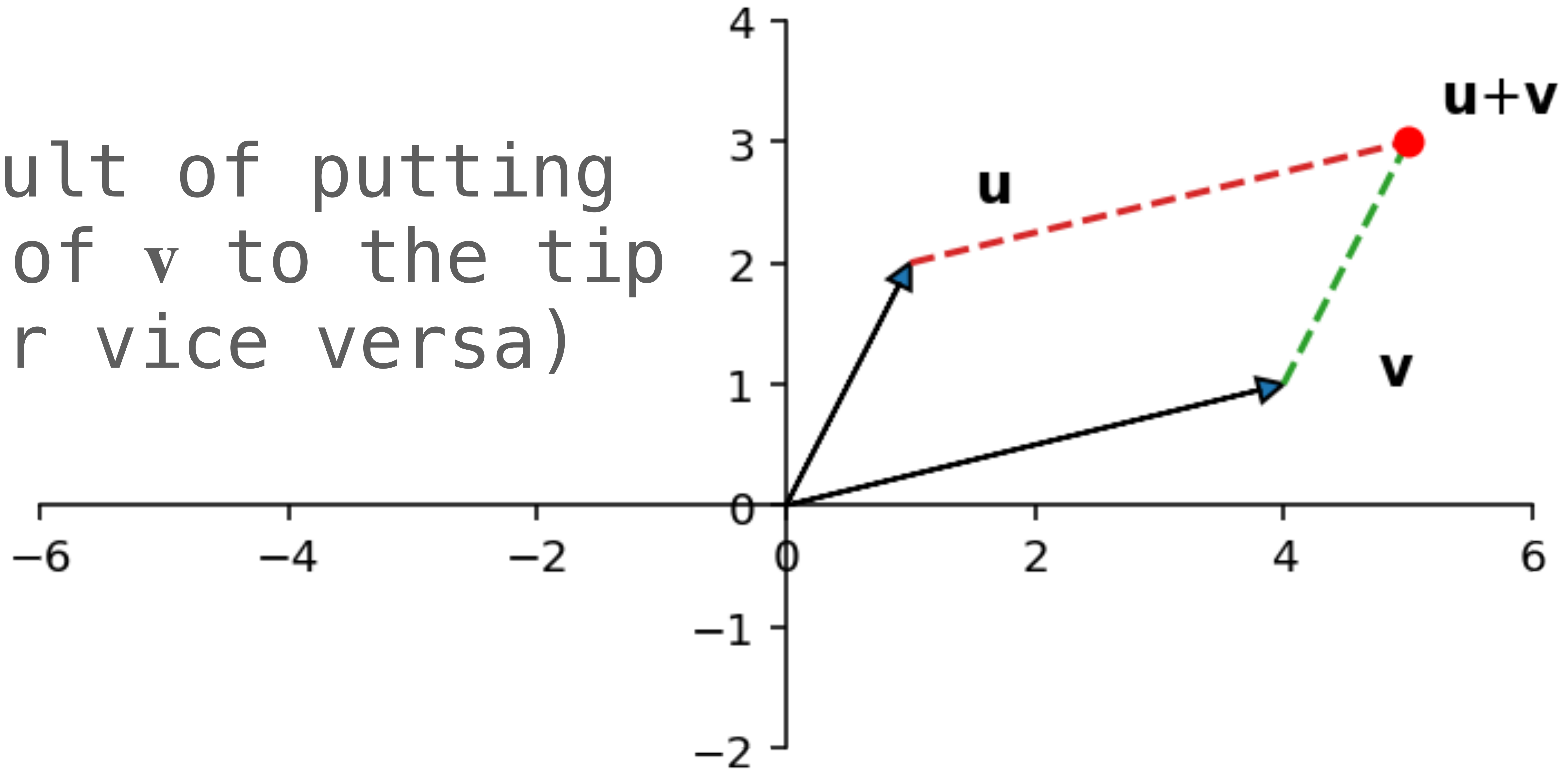
$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip–to–tail rule*

$\mathbf{u}+\mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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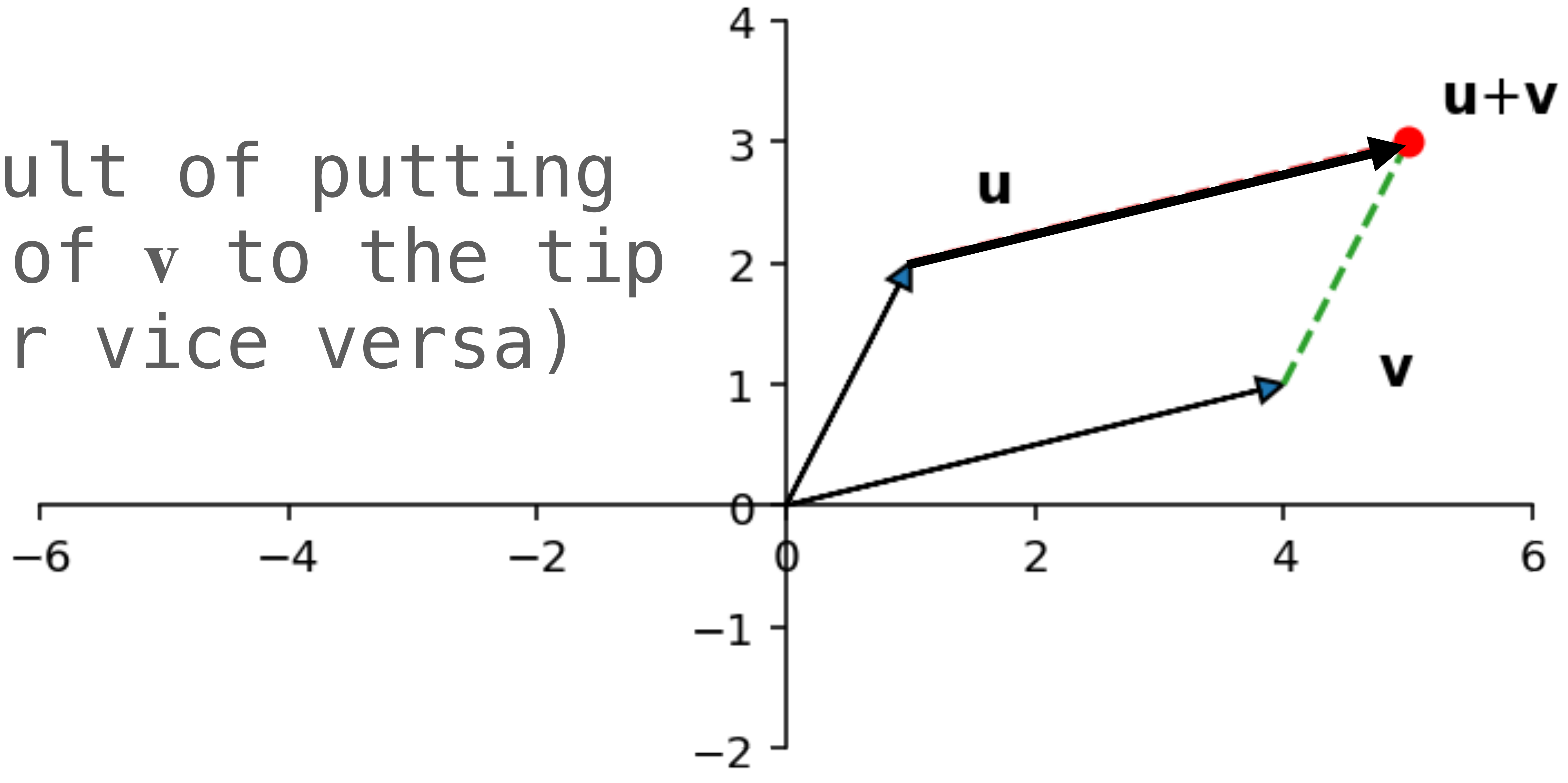

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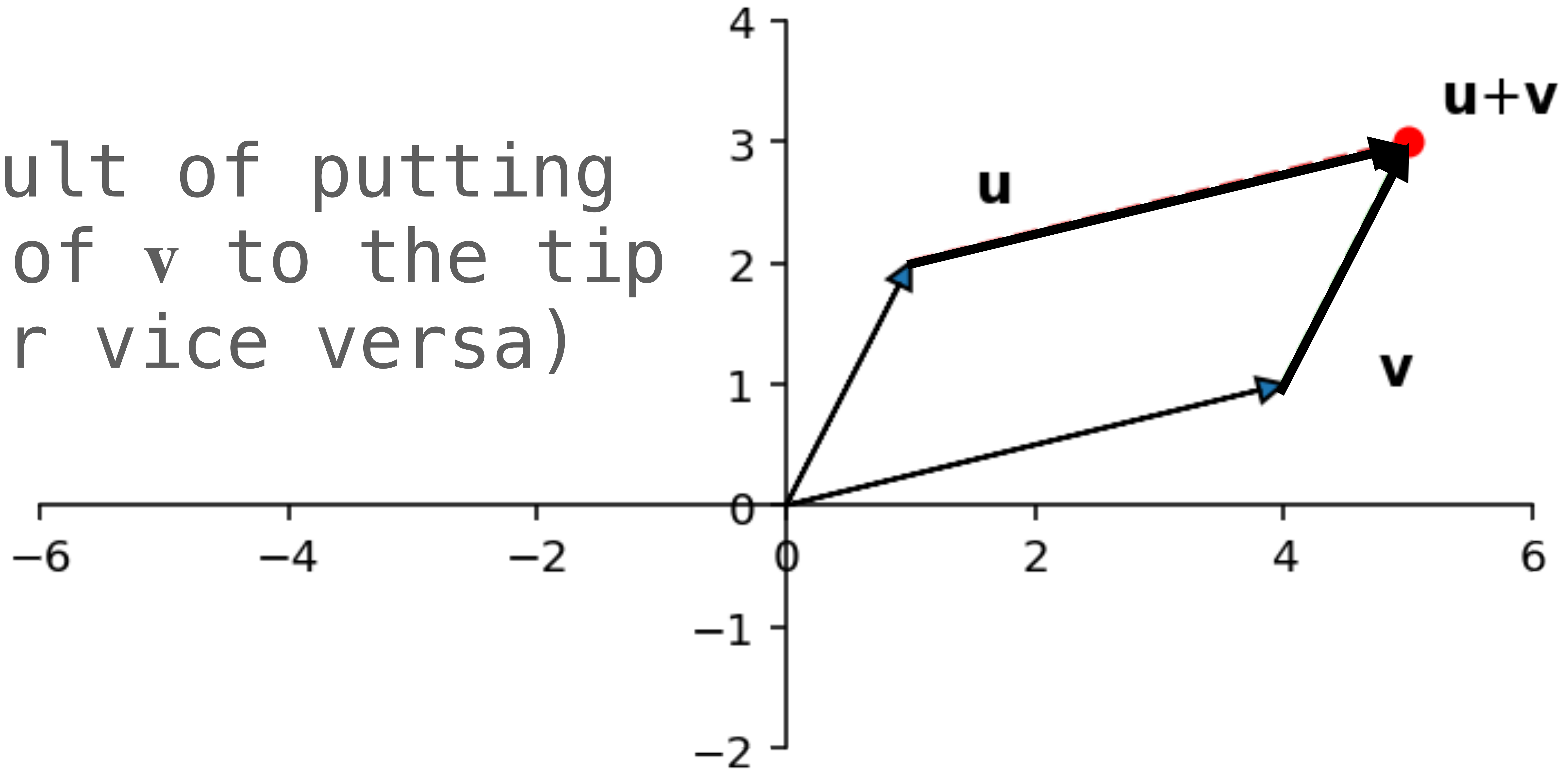
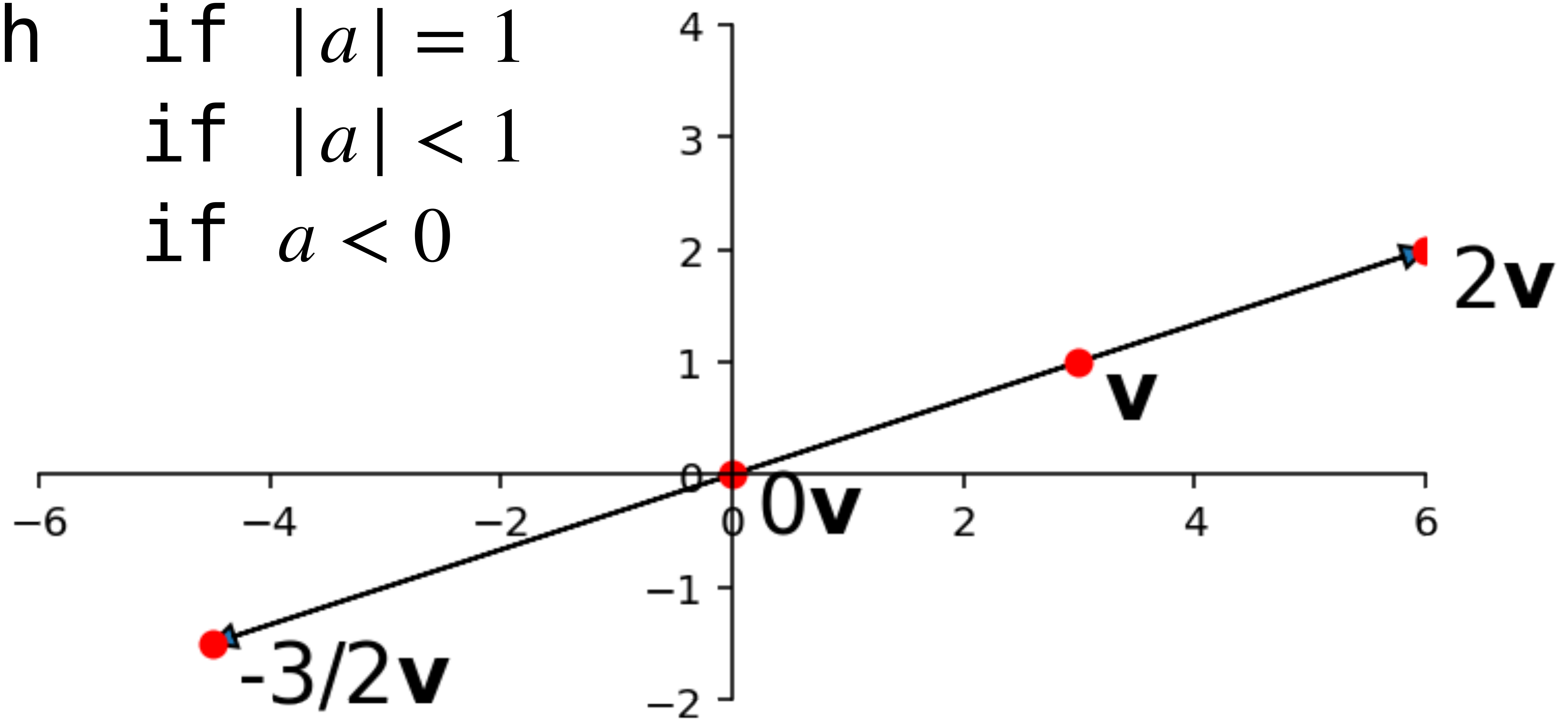


Recall Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



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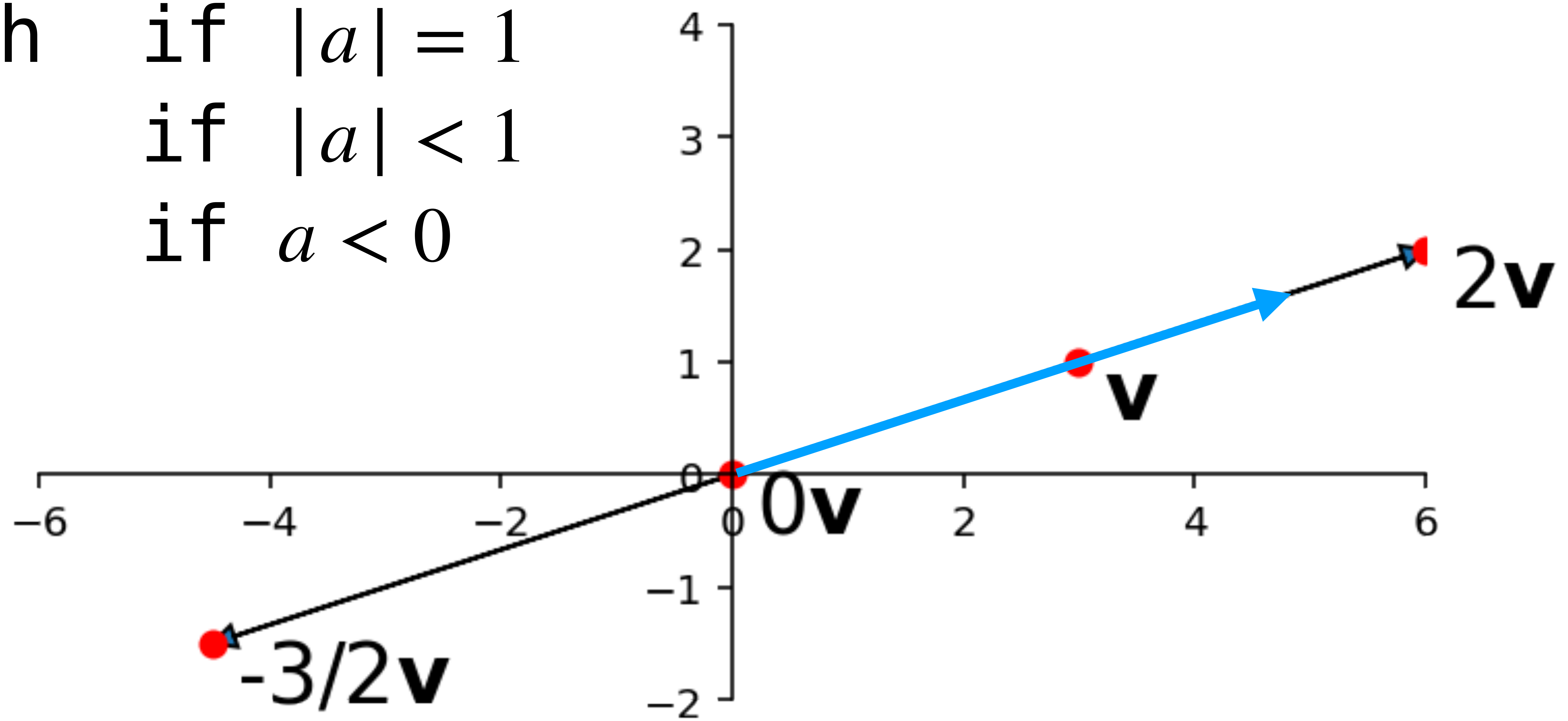

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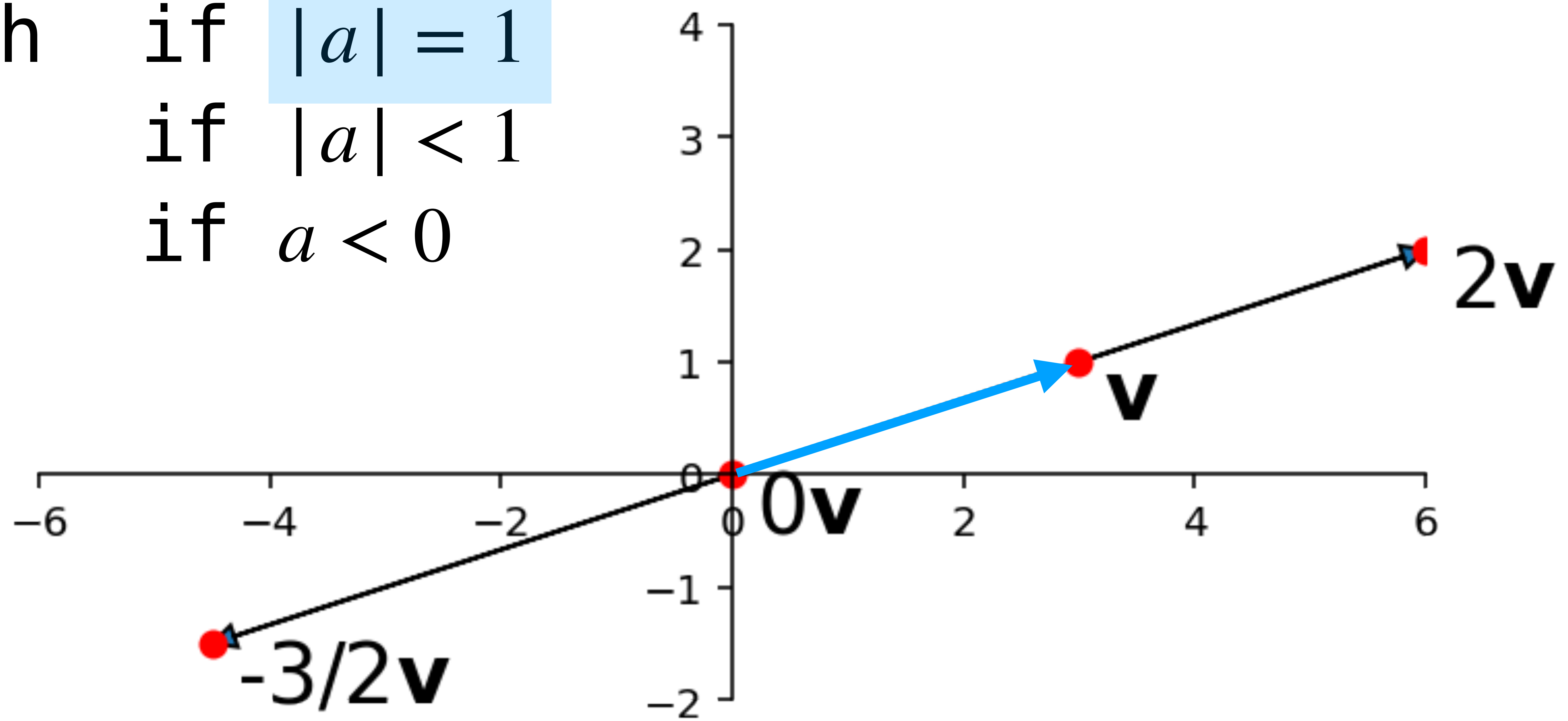

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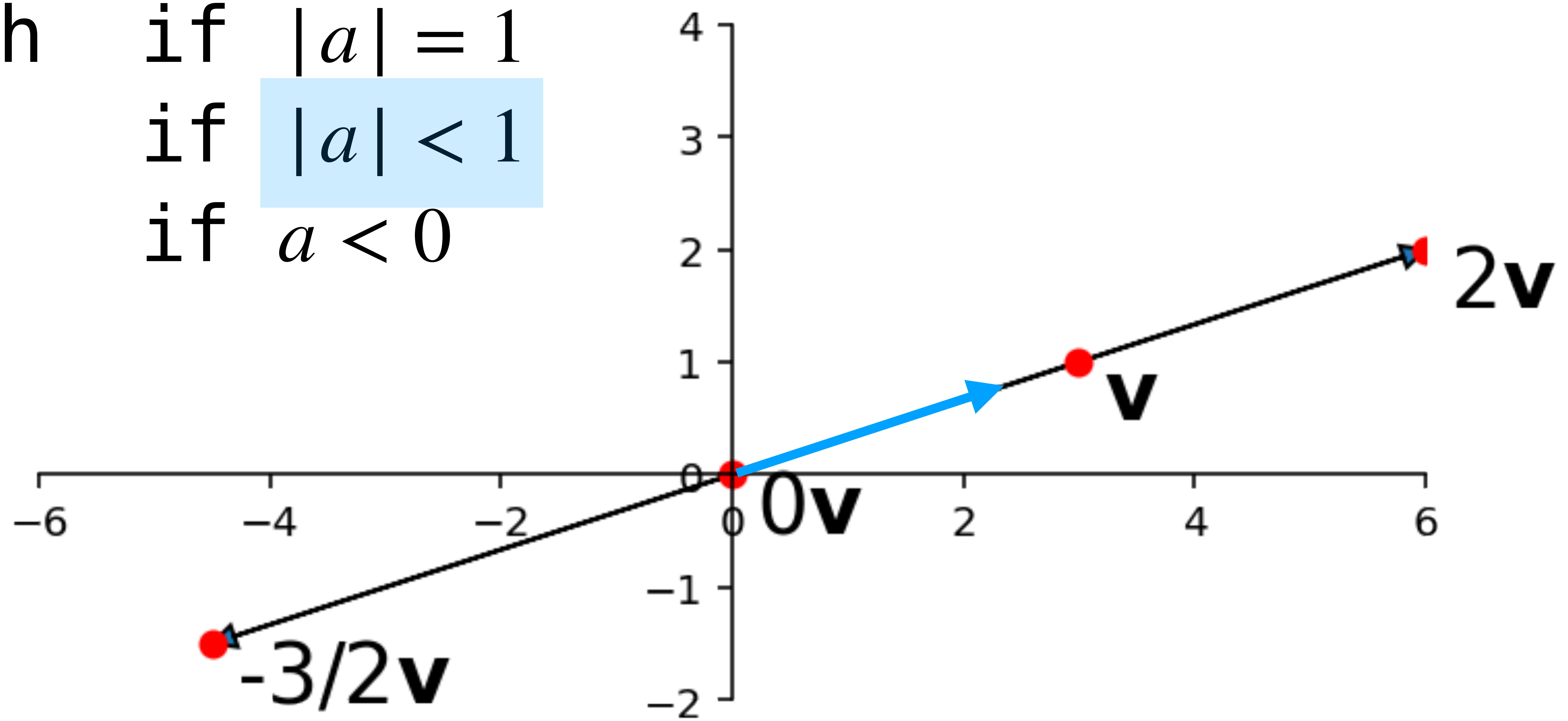

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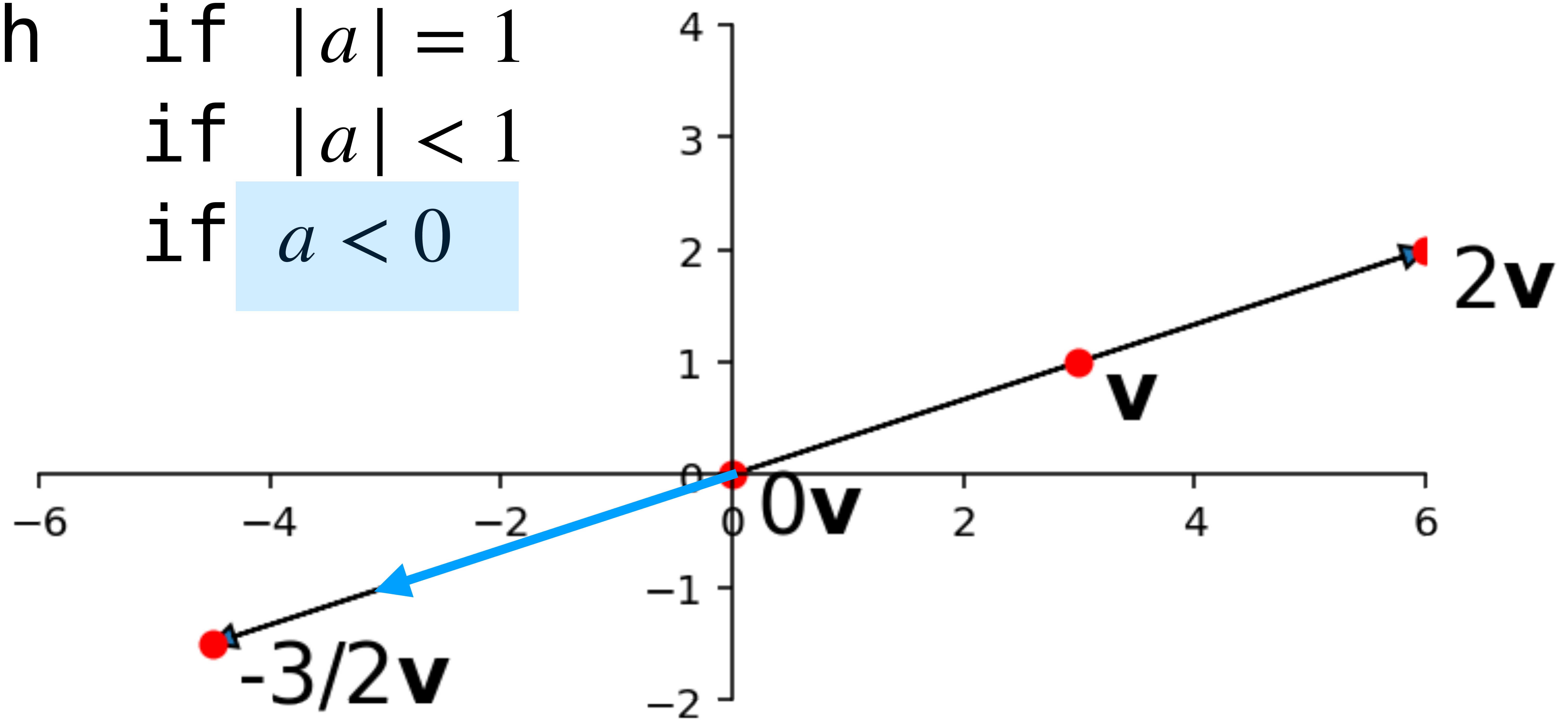

Recall: Linear Combinations

Definition. a ***linear combination*** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Recall: Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Recall: The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n = \mathbf{u}$?

Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

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this is our big shift in perspective

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vector equation

Motivation

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augmented matrix

Why not view these as a vector too?

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

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vector equation

Solutions as Vectors

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so it can be represented as a vector

Can we view a linear system as a single equation with matrices and vectors?

How do matrices and vectors "interface"?

Matrix-Vector Multiplication

Matrix-Vector "Interface"

multiplication   what does $A\mathbf{v}$ mean when A is a matrix and $\mathbf{v}$ is a vector?

Matrix-Vector Multiplication (Pictorially)

$$As$$

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

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a linear combination of the columns where $\mathbf{s}$ defines the weights

Why keeping track of matrix size is important

this only works if the number of *columns* of the matrix matches the number of *rows* of the vector

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$$(m \times n) \qquad (n \times 1) \qquad (m \times 1)$$

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$$
m\overbrace{\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}}^{n}
\; n\overbrace{\begin{bmatrix} * \\ \vdots \\ * \end{bmatrix}}^{1}
= m\overbrace{\begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}}^{1}
$$

$$(m \times n) \qquad (n \times 1) \qquad (m \times 1)$$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3???$$

Non-Example

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$(2 \times 2) \quad (3 \times 1)$

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THESE DON'T MATCH

(2×2) (3×1)

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Example

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THESE MATCH

$$(2 \times 2) \quad (2 \times 1)$$

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$$A\mathbf{v} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n}_{\text{lin. comb.}}$$

Matrix-Vector Multiplication

Euclidean Space

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ = set of $n \times 1$ matrices = set of n-element vectors

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by $\mathbf{v}$

Algebraic Properties

The algebraic properties of matrix–vector multiplication are **very important.**

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{v}) = c(A\mathbf{v})$

Algebraic Properties

The algebraic properties of matrix-vector multiplication are **very important.**

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

There are only two, please memorize them...

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$A(\vec{u} + \vec{v})$$

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$$

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$

by vector addition

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

by matrix vector multiplication

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(5+2)\begin{bmatrix}1\\1\\1\end{bmatrix} = 5\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$u_1\mathbf{a}_1 + v_1\mathbf{a}_1 + u_2\mathbf{a}_2 + v_2\mathbf{a}_2 + u_3\mathbf{a}_3 + v_3\mathbf{a}_3$$

`by vector scaling (distribution)`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

by rearranging

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$A\vec{u} + A\vec{v}$

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

`by matrix vector multiplication`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)$$

equals

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

fin

A Common Error

$$A\mathbf{v} \neq \mathbf{v}A$$

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It's **important** that we write our matrix-vectors multiplications with the matrix on the left

A Common Error

$$A\mathbf{v} \neq \mathbf{v}A$$

It's **important** that we write our matrix–vectors multiplications with the matrix on the <u>left</u>

This may feel artificial now, since the RHS is meaningless to us now, but it won't be for long

Looking forward a bit

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

Remember. column vectors are matrices with 1 column

Eventually we'll be able to view all of these as *matrix operations*

Question

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Compute the above matrix-vector multiplication

$$5\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

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Answer

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Answer

$$\begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$5(2) + 5(-3) + 4(4) = 11$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ ? \end{bmatrix}$$

$$5(-1) + 5(1) + 4(0) = 0$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_1 = a_{11}s_1 + a_{12}s_2 + \ldots + a_{1n}s_n = \sum_{i=1}^{n} a_{1i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_2 = a_{21}s_1 + a_{22}s_2 + \ldots + a_{2n}s_n = \sum_{i=1}^{n} a_{2i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ ? \end{bmatrix}$$

$$v_m = a_{m1}s_1 + a_{m2}s_2 + \ldots + a_{mn}s_n = \sum_{i=1}^{n} a_{mi}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}s_i \\ \sum_{i=1}^{n} a_{2i}s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}s_i \end{bmatrix}$$

Inner product: $[a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \sum_{i=1}^{n} a_i s_i \in \mathbb{R}$

$1 \times n \qquad n \times 1 \qquad 1 \times 1$

Inner Product

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Definition. The **inner product** of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined the

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

$$\left\langle \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix} \right\rangle = 1(4) + 2(5) + 3(6) = 32$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} s_i \\ \sum_{i=1}^{n} a_{2i} s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} s_i \end{bmatrix}$$

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Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} s_i \\ \sum_{i=1}^{n} a_{2i} s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} s_i \end{bmatrix}$$

The ith entry of the $A\mathbf{s}$ is the inner product of the ith row of A and $\mathbf{s}$

The Matrix Equation

Recall: Vector Equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Recall: Vector Equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$?

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The Idea. think of the weights as *unknowns*

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The Idea. think of the weights as *unknowns*

we can use the same idea for matrix-vector multiplication

The Matrix Equation

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{b}$$

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Can $\mathbf{b}$ be written as a linear combination of **the columns of A**?

The Matrix Equation

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Can $\mathbf{b}$ be written as a linear combination of **the columns of A**?

The Idea. write the "vector part" of our matrix-vector multiplication as an *unknown*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

How To: The Matrix Equation

Question. Does $A\mathbf{x} = \mathbf{b}$ have a solution?

Question. Is $A\mathbf{x} = \mathbf{b}$ consistent?

Question. Write down a solution to the equation$A\mathbf{x} = \mathbf{b}$

How To: The Matrix Equation

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(augmented matrix) $\left[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}\right]$

How To: The Matrix Equation

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(vector equation) $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b}$

(augmented matrix) $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$

!!they all have the same solution set!!

HOW TO: The Matrix Equation

Question. Write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution.

Use Gaussian elimination (or other means) to convert $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ to reduced echelon form

Then read off a solution from the reduced echelon form

Full Span

Recall: Span

Recall: Span

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

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$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. the ***span*** of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is:

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

Spanning all of $\mathbb{R}^2$

Spanning all of $\mathbb{R}^2$

if two (or more) vectors in $\mathbb{R}^2$ span a plane, they must span all of $\mathbb{R}^2$. They "fill up" $\mathbb{R}^2$

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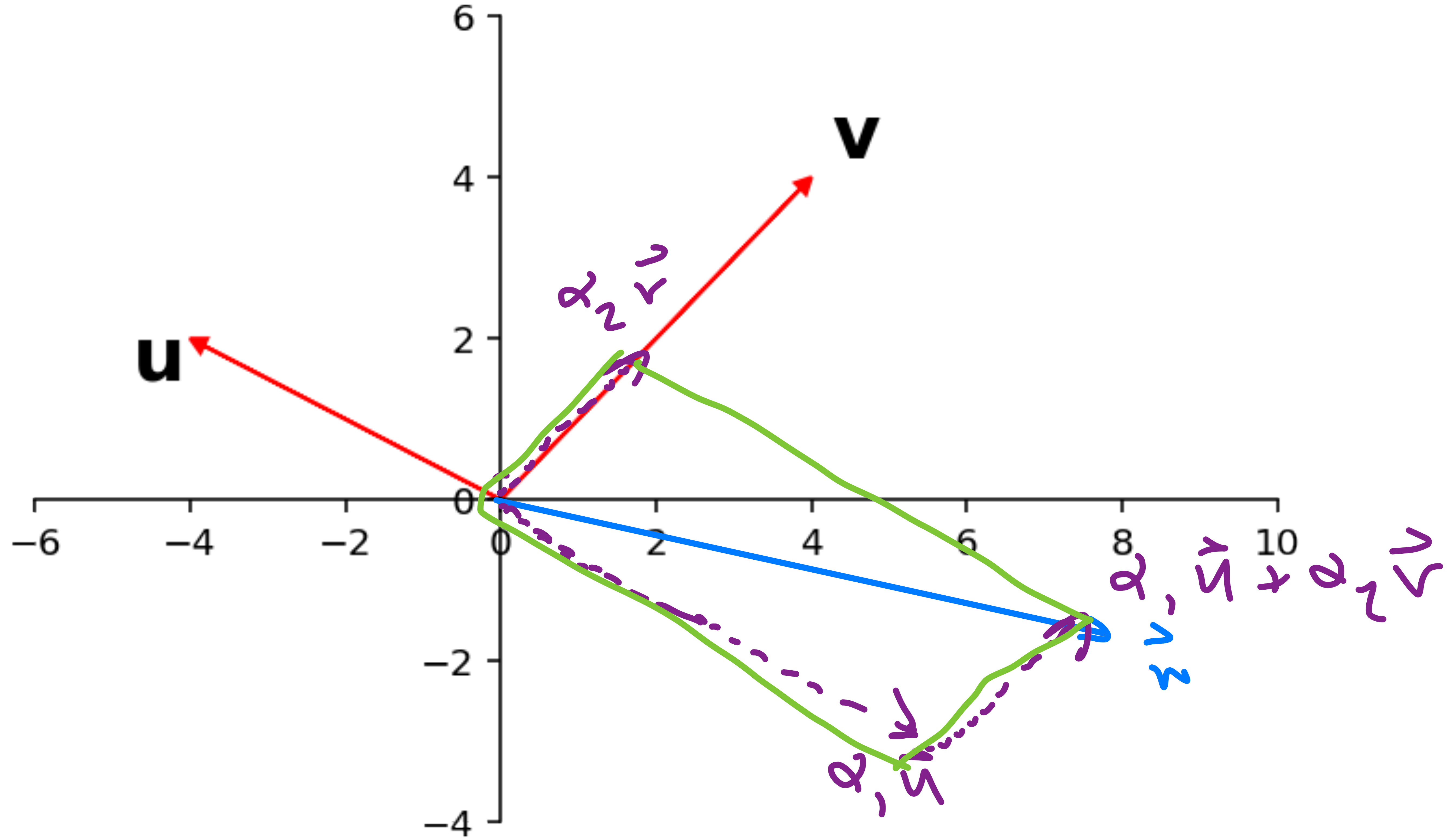

Spanning all of $\mathbb{R}^2$

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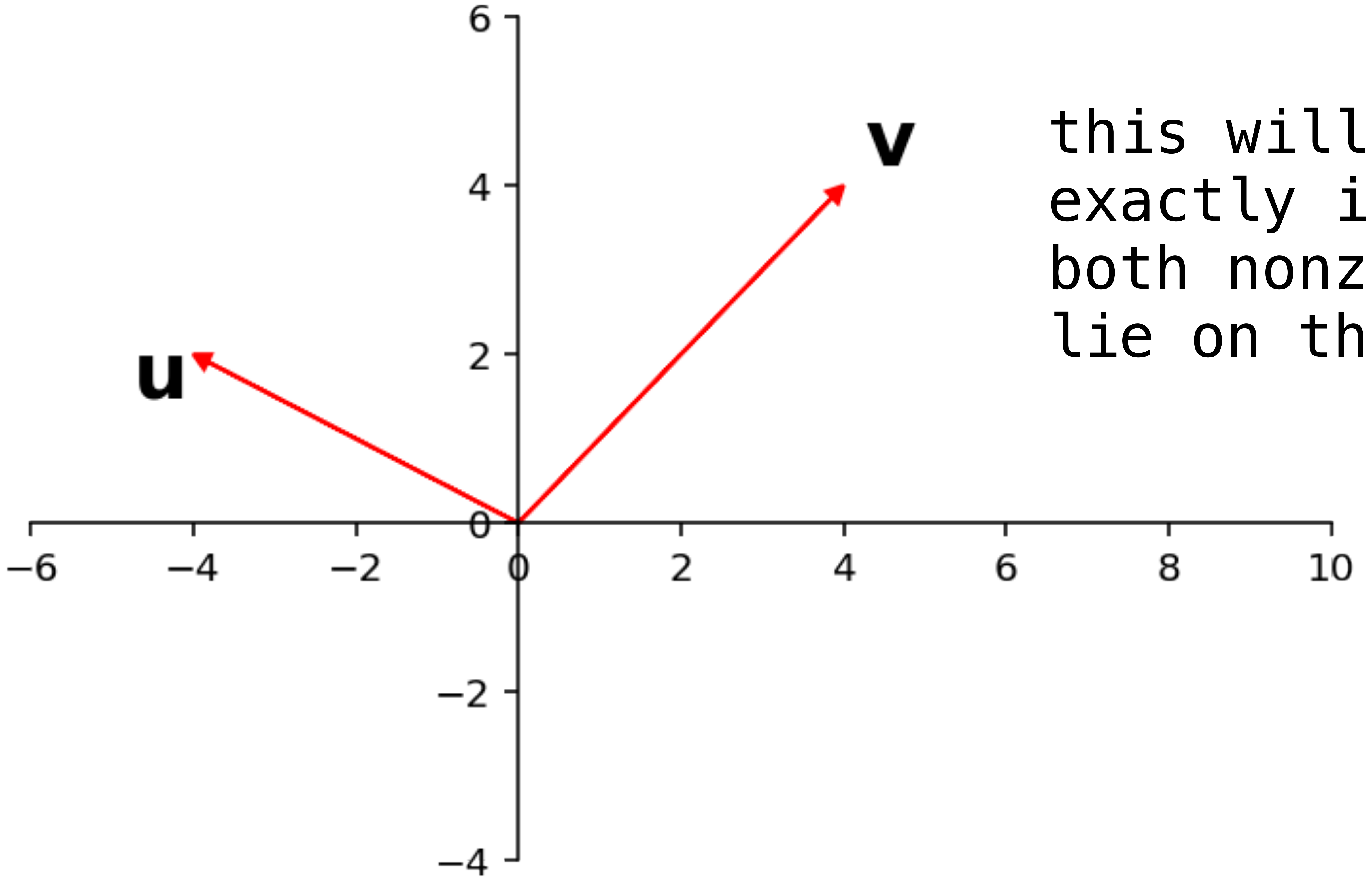
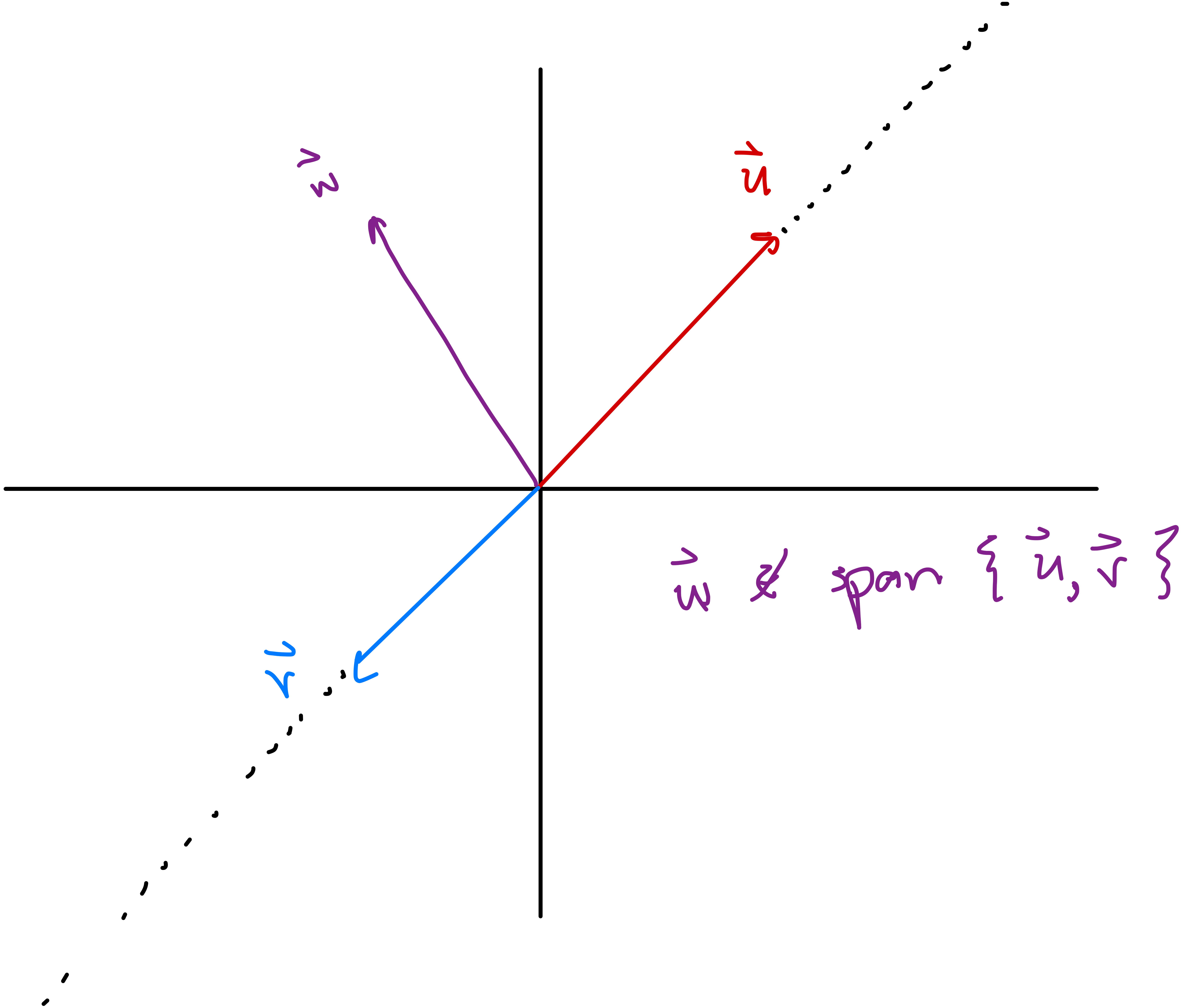


this will happen exactly if they are both nonzero and don't lie on the same line


$\vec{w}$
$\vec{u}$
$\vec{v}$
$\vec{w} \notin \text{span}\{\vec{u}, \vec{v}\}$

What about $\mathbb{R}^n$?

When do a set of vectors span all of $\mathbb{R}^n$?
When do a set of vectors "fill up" $\mathbb{R}^n$?

A Few Questions

Can two vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$?

Is it required that five vectors $\mathbb{R}^3$ span all of $\mathbb{R}^3$?

A Thought Experiment

suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Yes. It doesn't have an inconsistent row

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \text{[redacted]} \end{bmatrix}$$

A Thought Experiment

what about this system?

$$\left[\begin{matrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{matrix} \quad \text{[redacted]} \right]$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

A Thought Experiment

what about this system?

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

it depends...

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Pivots and Spanning $\mathbb{R}^m$

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If it doesn't matter what the last column is, then **every choice must be possible**

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If it doesn't matter what the last column is, then **every choice must be possible**

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Spanning $\mathbb{R}^m$

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution

2. The columns of A span $\mathbb{R}^m$

3. A has a pivot position in every row

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1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution

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HOW TO: Spanning $\mathbb{R}^m$

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

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!! We only need the echelon form !!

Question

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ span all of $\mathbb{R}^3$?

Answer: No

max pivots = min(m , n)

the matrix

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 3 & 2023 \end{bmatrix}$$

cannot have more than 2 pivot positions

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

In this case the choice matters

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

In this case the choice matters

We can't make the last column $[0 \ \ 0 \ \ 0 \ \ \blacksquare]$ for nonzero $\blacksquare$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

In this case the choice matters

We can't make the last column $\begin{bmatrix} 0 & 0 & 0 & \blacksquare \end{bmatrix}$ for nonzero $\blacksquare$

But we can make the last column <u>parameters</u> to find equations that must hold

Not spanning $\mathbb{R}^m$

echelon form

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

Not spanning $\mathbb{R}^m$

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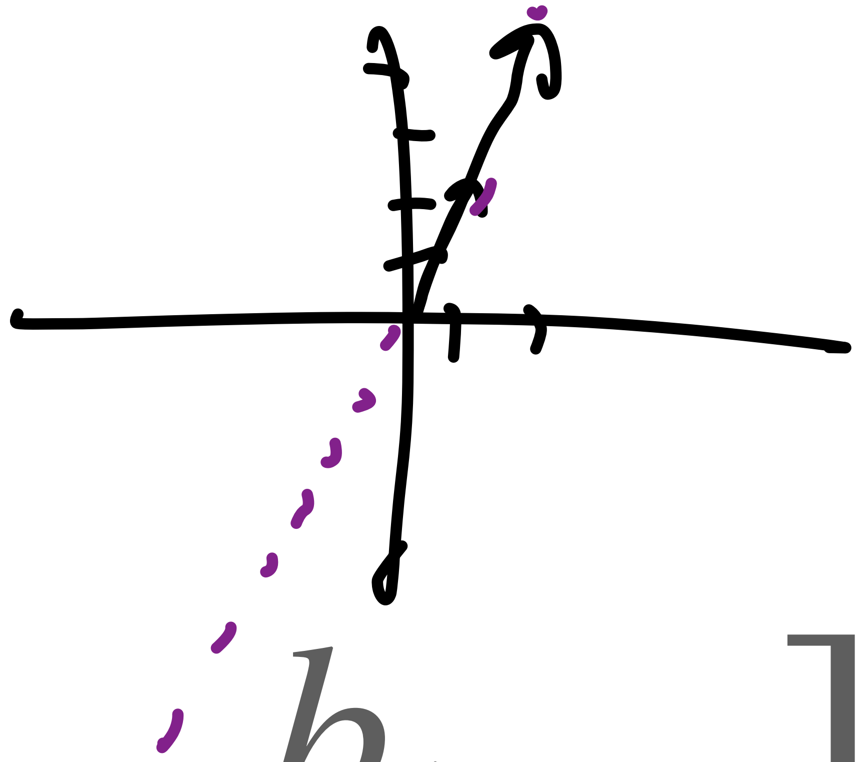
As long as $(-2)b_1 + b_2 = 0$, the system is consistent

Not spanning $\mathbb{R}^m$



$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

As long as $(-2)b_1 + b_2 = 0$, the system is consistent

This gives use a <u>linear equation</u> which describes the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **and** $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$-2x_1 + x_2 = 0$$

$$x_1 = 7$$

$$x_2 = 14$$

$$\begin{bmatrix} 7 \\ 14 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : -2x_1 + x_2 = 0 \right\}$$

Question (Understanding Check)

True or **False**, the echelon form of any matrix has at most one row of the form $[0 \;\; 0 \;\; \ldots \;\; 0 \;\; \blacksquare]$ where ■ is nonzero.

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entry not to the right

this is not in echelon form

Question (More Challenging)

Give a linear equation for the span of the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$R_3 \leftarrow R_3 - (1/2)R_2$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Summary

Matrix and vectors can be multiplied together to get new vectors

The matrix equation is another representation of systems of linear equations

Looking forward: Matrices *transform* vectors