

Matrix-Vector Equations

Geometric Algorithms
Lecture 5

Practice Problem

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ in $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$?

Answer

$$\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\} ?$$

Answer

Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix}$$

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 2R_1$$

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

Answer

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

no solution \equiv not in the span

Outline

- » Motivate the study of matrix–vector equations
- » Formally define matrix–vector multiplication
- » Revisit spans
- » Take stock of our perspectives on systems of linear equations

Keywords

matrix-vector multiplication

the matrix equation

inner-product

row-column rule

Recap

Recall: Vector "Interface"

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`equality` what does it mean for two vectors
to be equal?

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addition what does $\mathbf{u} + \mathbf{v}$ (adding two vectors
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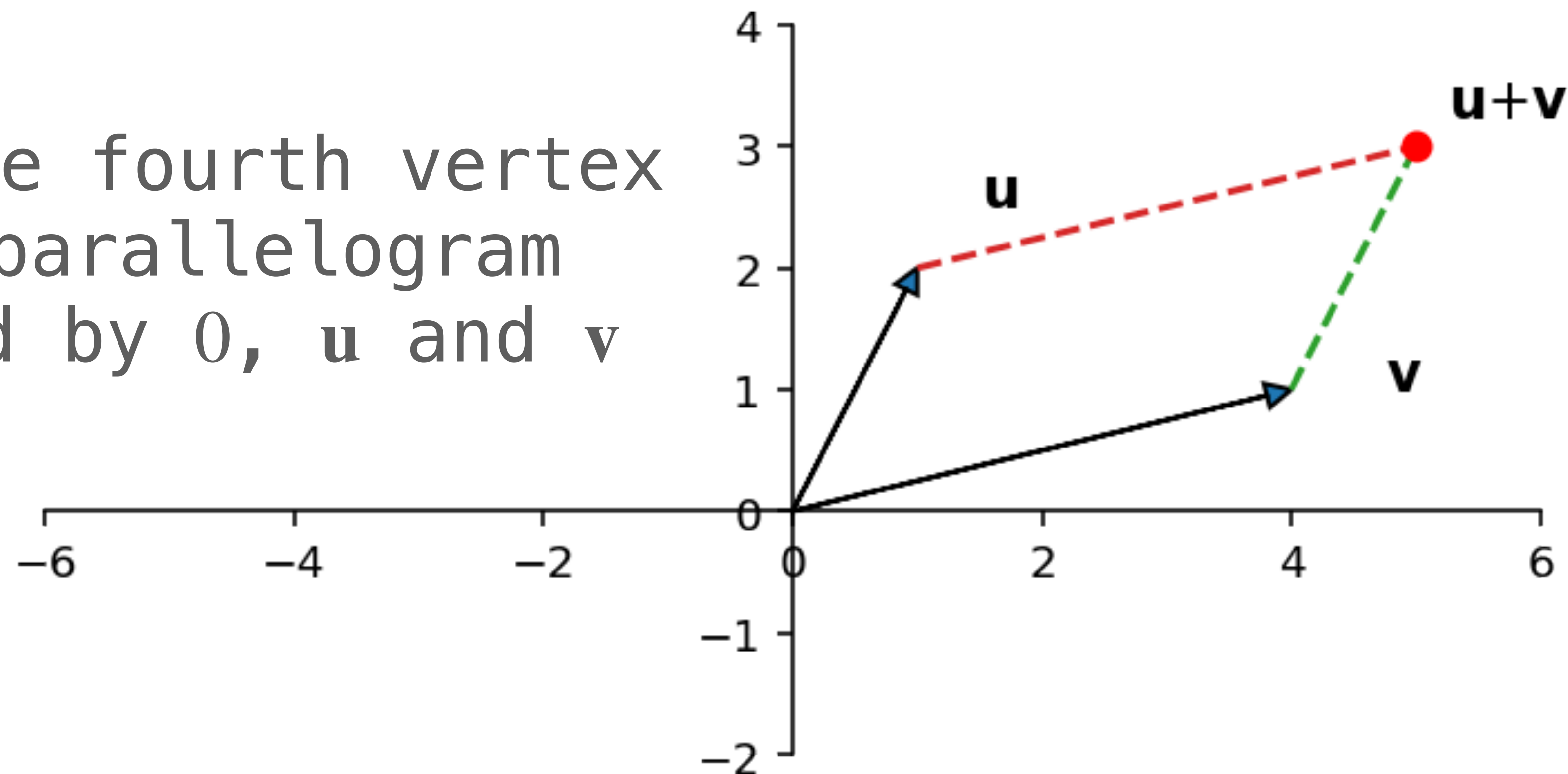
scaling what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Recall: Vector Addition (Geometrically)

in \mathbb{R}^2 it's called the *parallelogram rule*

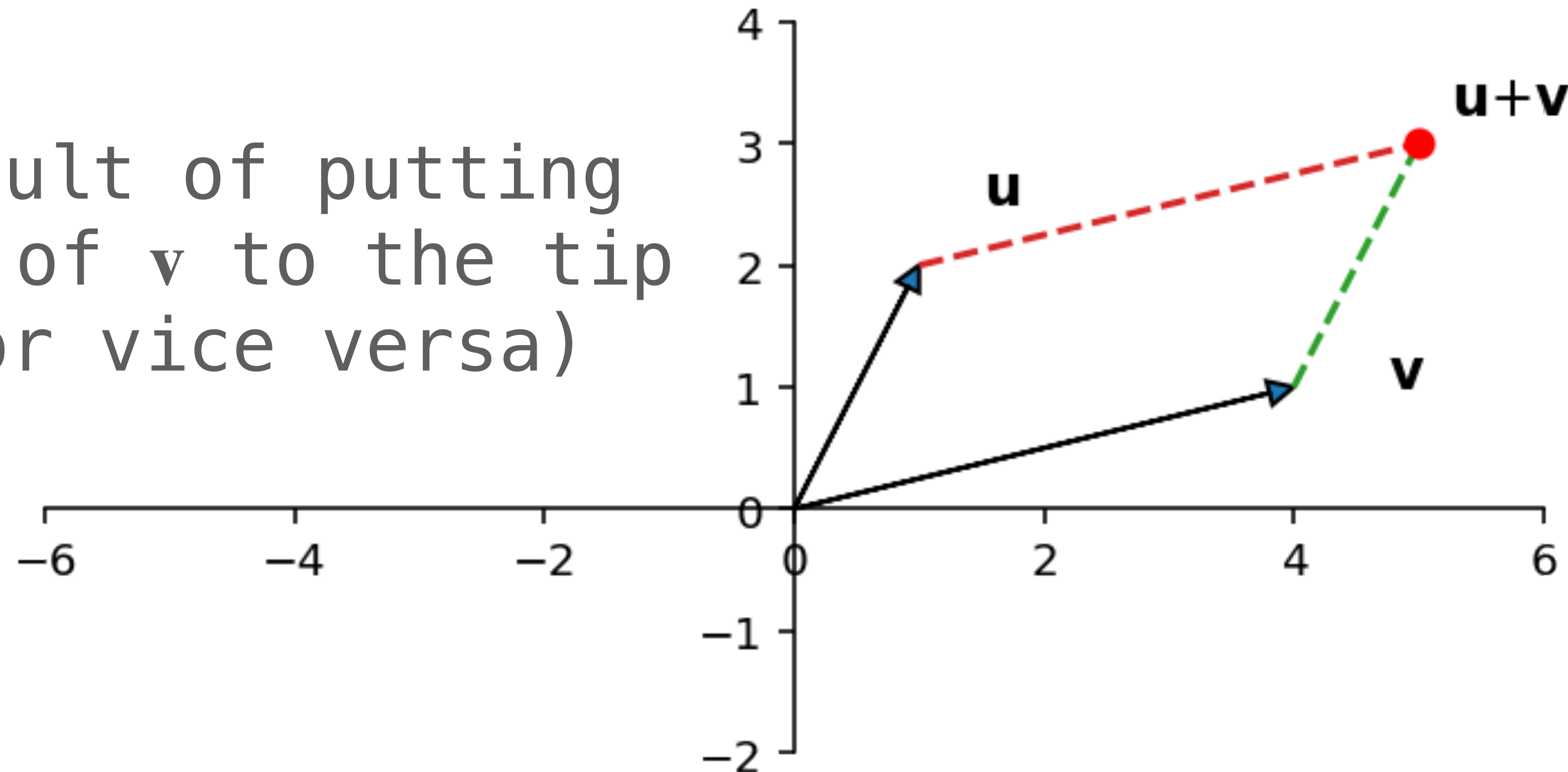
$\mathbf{u} + \mathbf{v}$ is the fourth vertex
of the parallelogram
generated by $\mathbf{0}$, \mathbf{u} and \mathbf{v}



Vector Addition (Geometrically)

or the *tip-to-tail rule*

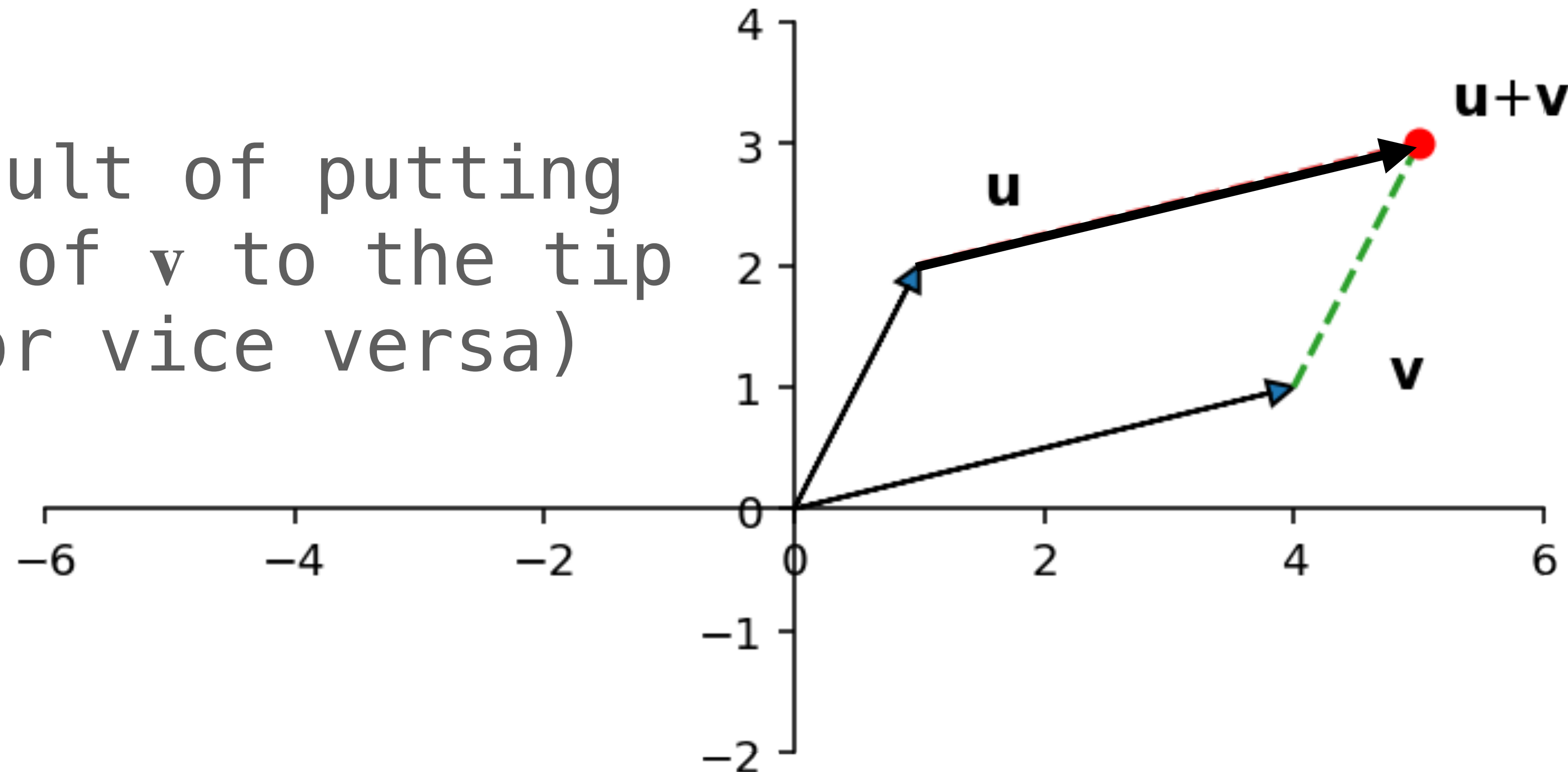
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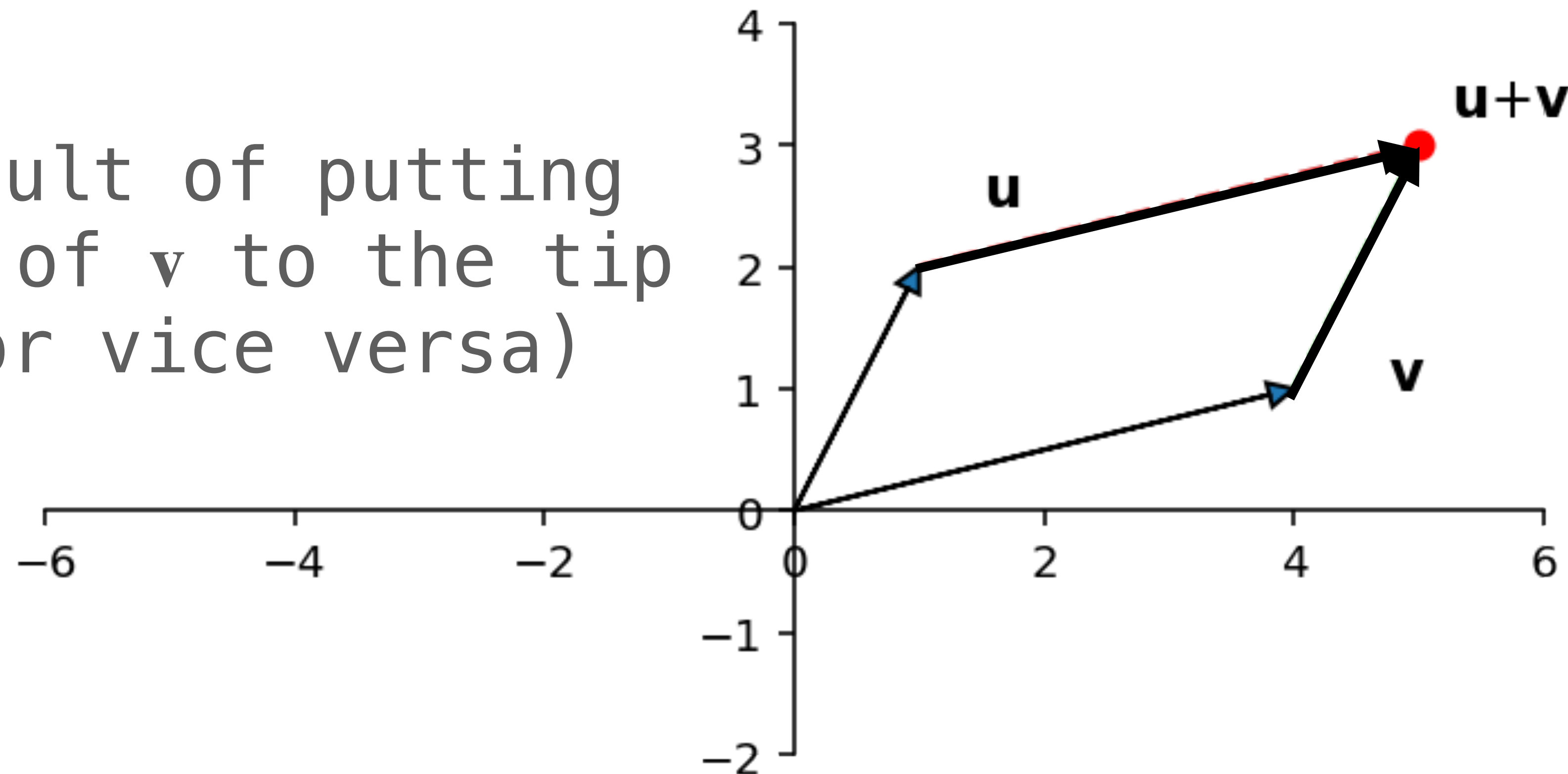
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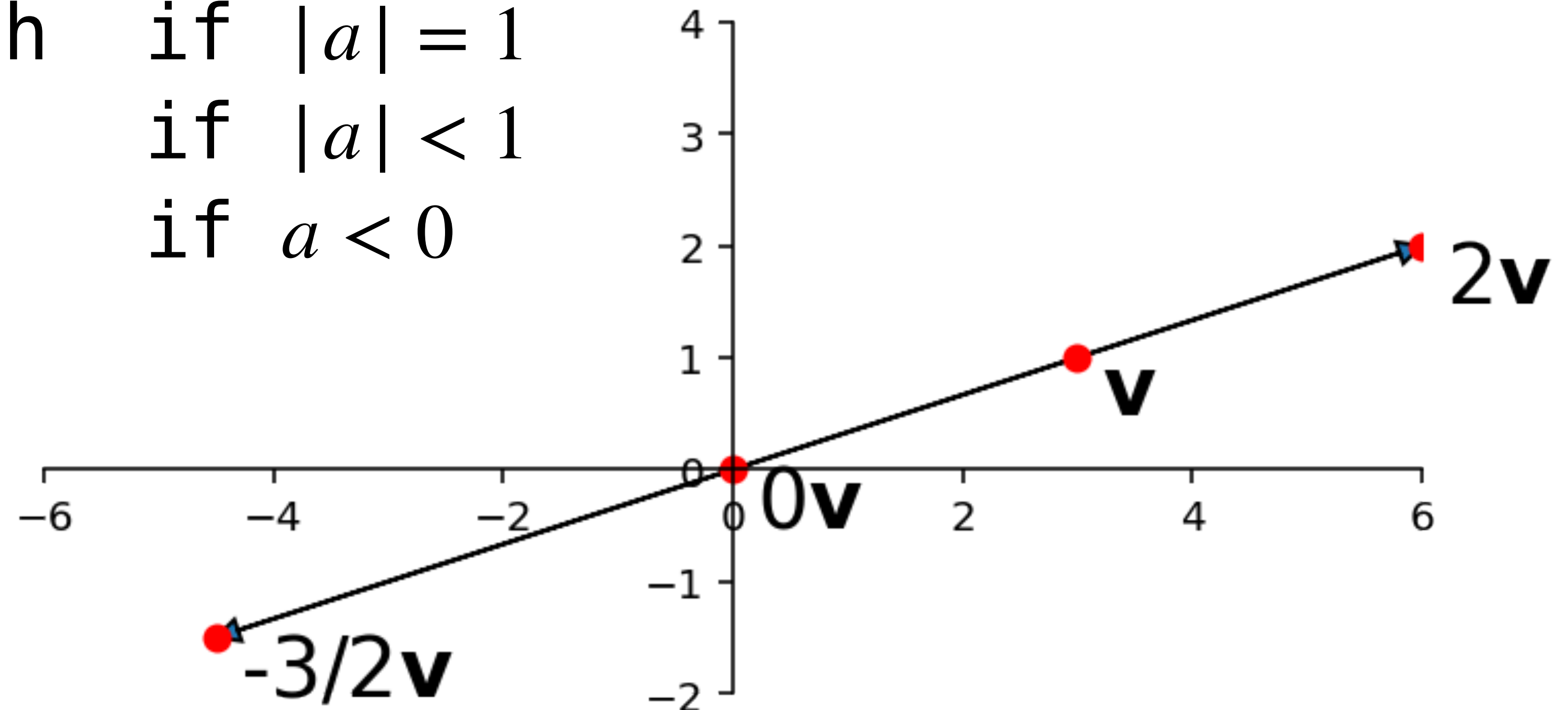
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Recall Vector Scaling (Geometrically)

| | |
|-----------------|--------------|
| longer | if $ a > 1$ |
| the same length | if $ a = 1$ |
| shorter | if $ a < 1$ |
| reversed | if $a < 0$ |



Recall Vector Scaling (Geometrically)

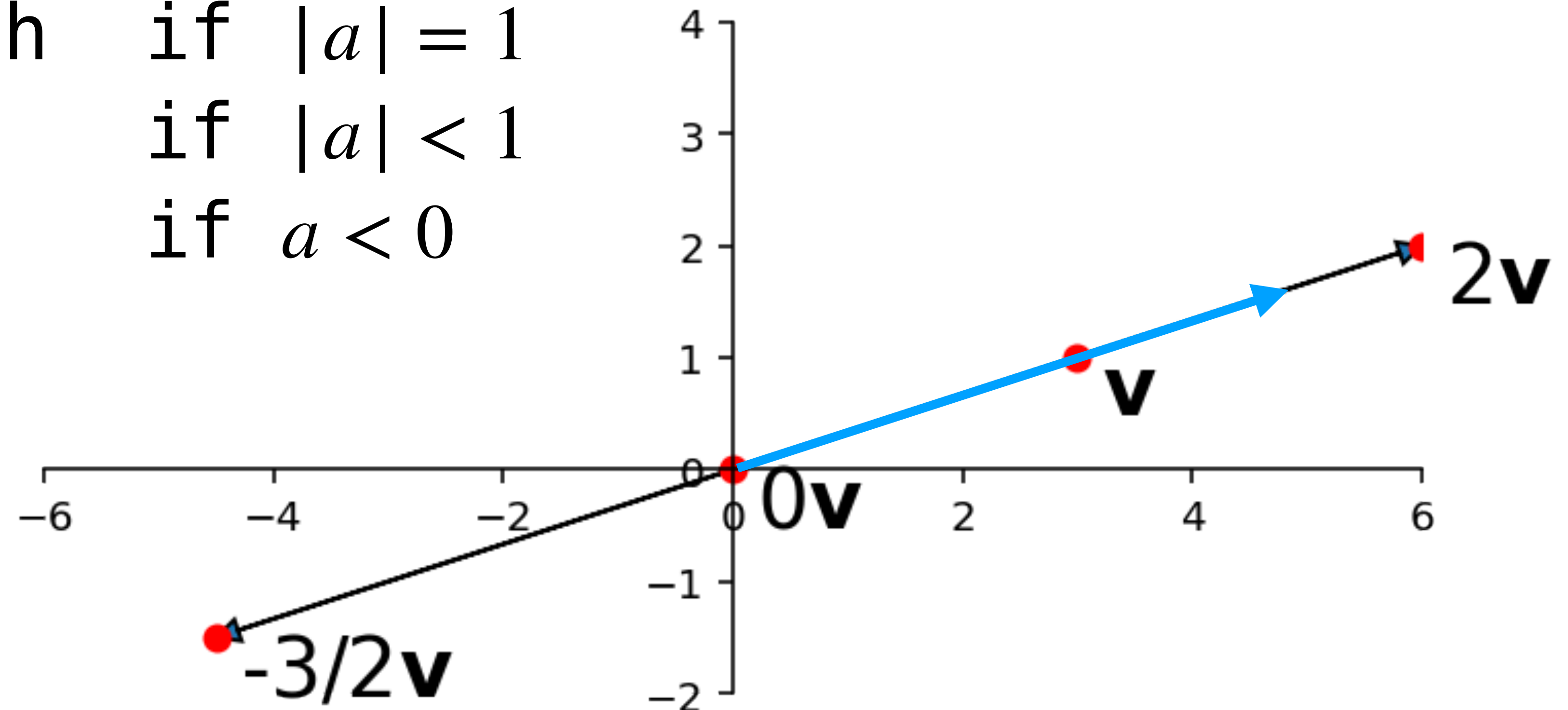
longer
the same length
shorter
reversed

if $|a| > 1$

if $|a| = 1$

if $|a| < 1$

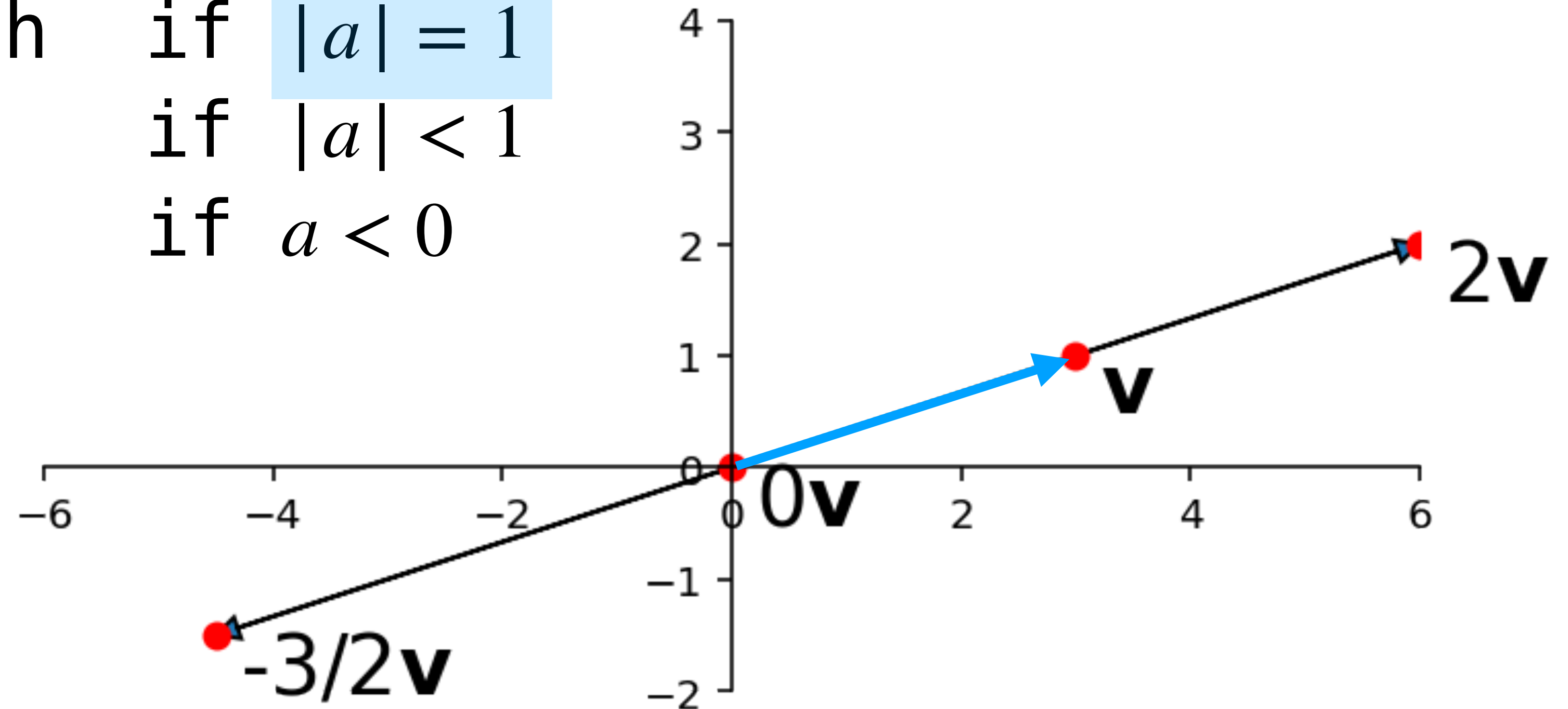
if $a < 0$



Recall Vector Scaling (Geometrically)

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Recall Vector Scaling (Geometrically)

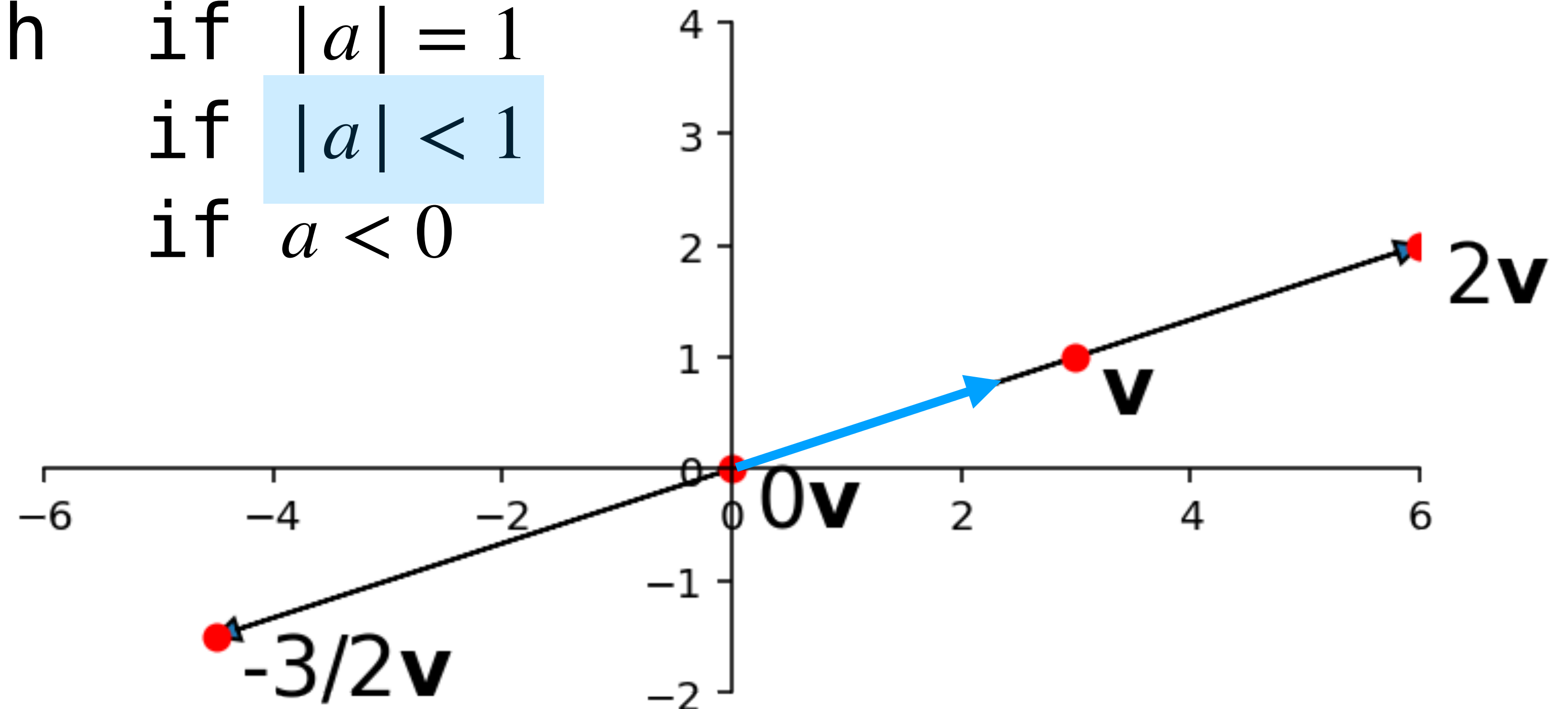
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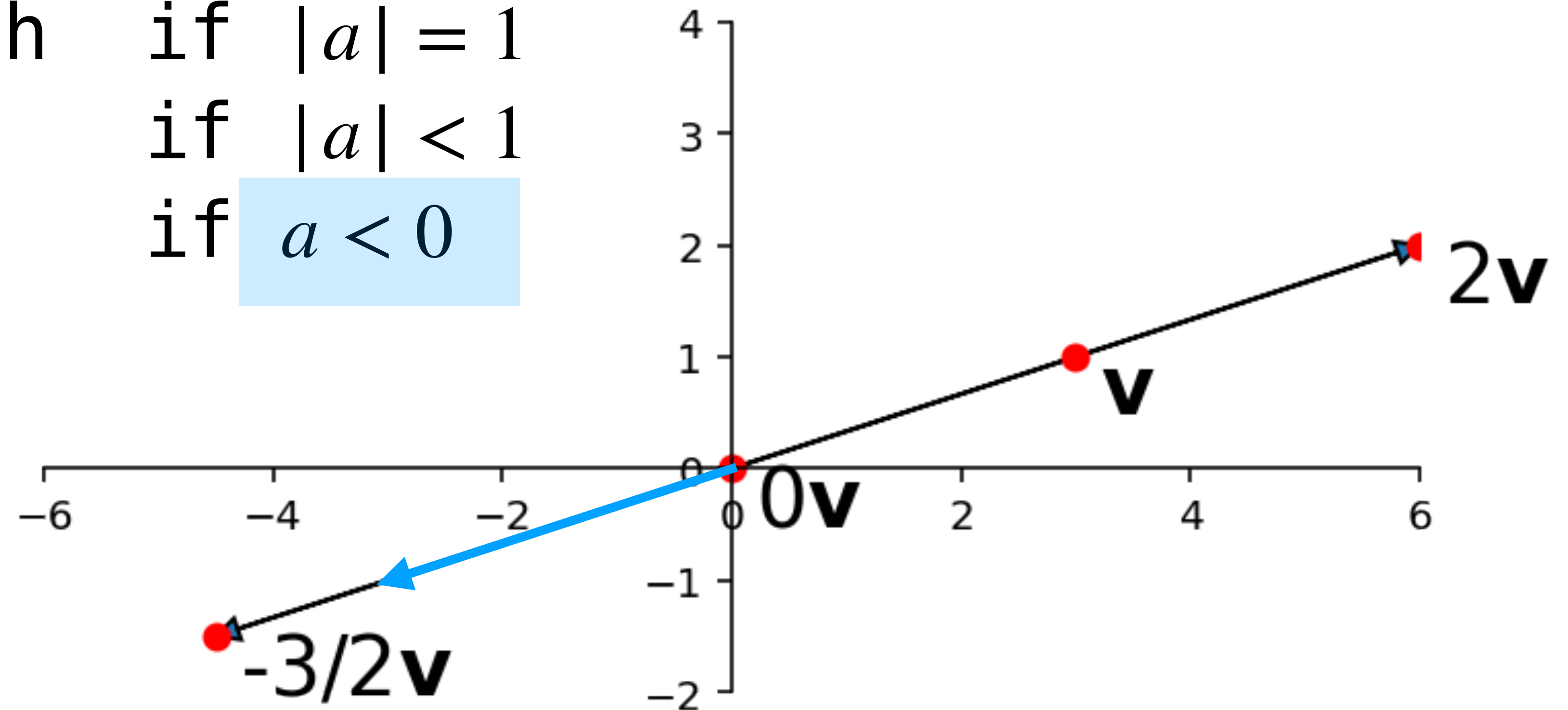
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Recall Vector Scaling (Geometrically)

longer if $|a| > 1$
the same length if $|a| = 1$
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reversed if $a < 0$



Recall: Linear Combinations

Definition. a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are in \mathbb{R}

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where $\alpha_1, \alpha_2, \dots, \alpha_n$ are in \mathbb{R}
weights

Recall: Linear Combinations (Example)

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

Recall: The Fundamental Concern

Can \mathbf{u} be written as a linear combination of

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$?

That is, are there weights $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{u}?$$

Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

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this is our big
shift in
perspective

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vector equation

Motivation

Recall: The Fundamental Connection

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augmented matrix

Why not view
these as a
vector too?

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

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vector equation

Solutions as Vectors

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Can we view a linear system as a single equation
with matrices and vectors?

How do matrices and vectors "interface"?

Matrix-Vector Multiplication

Matrix-Vector "Interface"

multiplication

what does $A\mathbf{v}$ mean when A is a matrix and \mathbf{v} is a vector?

Matrix-Vector Multiplication (Pictorially)

AS

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

The diagram illustrates matrix-vector multiplication. On the left, a matrix is represented by a large left square bracket followed by four vertical columns. The first column contains a_{11} , a_{21} , a vertical ellipsis, and a_{m1} . The second column contains a_{12} , a_{22} , a vertical ellipsis, and a_{m2} . The third column contains an ellipsis, a diagonal ellipsis, and an ellipsis. The fourth column contains a_{1n} , a_{2n} , a vertical ellipsis, and a_{mn} . A large right square bracket follows the fourth column. To the right of the matrix is a vector, represented by a large left square bracket followed by four light blue rectangular boxes. The first box contains s_1 , the second contains s_2 , the third contains a vertical ellipsis, and the fourth contains s_n . A large right square bracket follows the fourth box.

$$\left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array} \begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array} \begin{array}{c} \dots \\ \dots \\ \ddots \\ \dots \end{array} \begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array} \right] \left[\begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_n \end{array} \right]$$

Matrix-Vector Multiplication (Pictorially)

$$s_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + s_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + s_n \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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a linear combination of the columns where
s defines the weights

Why keeping track of matrix size is important

this only works if the number of *columns* of the matrix matches the number of *rows* of the vector

$$\begin{bmatrix} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$(m \times n) \qquad (n \times 1) \qquad (m \times 1)$

Why keeping track of matrix size is important

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$$\begin{array}{ccc} \begin{array}{c} \textcolor{blue}{m} \end{array} \begin{array}{c} \textcolor{red}{n} \\ \left[\begin{array}{ccc} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{array} \right] \end{array} & \begin{array}{c} \textcolor{red}{n} \begin{array}{c} \textcolor{purple}{1} \\ \left[\begin{array}{c} * \\ \vdots \\ * \end{array} \right] \end{array} \end{array} = \begin{array}{c} \textcolor{blue}{m} \begin{array}{c} \textcolor{purple}{1} \\ \left[\begin{array}{c} * \\ * \\ \vdots \\ * \\ * \end{array} \right] \end{array} \end{array} \\ \begin{array}{c} (\textcolor{blue}{m} \times \textcolor{red}{n}) \end{array} & \begin{array}{c} (\textcolor{red}{n} \times \textcolor{purple}{1}) \end{array} & \begin{array}{c} (\textcolor{blue}{m} \times \textcolor{purple}{1}) \end{array} \end{array}$$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3 ???$$

Non-Example

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THESE DON'T MATCH

(2×2) (3×1)

Example

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THESE MATCH

(2×2) (2×1)

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and a vector \mathbf{v} in \mathbb{R}^n , we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by \mathbf{v}

Algebraic Properties

The algebraic properties of matrix–vector multiplication are **very important**.

$$1. \quad A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$

$$2. \quad A(c\mathbf{v}) = c(A\mathbf{v})$$

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$$2. \quad A(c\mathbf{v}) = c(A\mathbf{v})$$

There are only two, please memorize them...

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$$

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{pmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \end{pmatrix}$$

by vector addition

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$(u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

by matrix vector multiplication

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$u_1 \mathbf{a}_1 + v_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + v_2 \mathbf{a}_2 + u_3 \mathbf{a}_3 + v_3 \mathbf{a}_3$$

by vector scaling (distribution)

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$(u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + u_3 \mathbf{a}_3) + (v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + v_3 \mathbf{a}_3)$$

by rearranging

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

by matrix vector multiplication

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$$

equals

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

fin

A Common Error

$$Av \neq vA$$

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It's **important** that we write our matrix–vectors multiplications with the matrix on the left

A Common Error

$$Av \neq vA$$

It's **important** that we write our matrix–vectors multiplications with the matrix on the left

This may feel artificial now, since the RHS is meaningless to us now, but it won't be for long

Looking forward a bit

$$\begin{bmatrix} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

Remember. column vectors are matrices with 1 column

Eventually we'll be able to view all of these as
matrix operations

Question

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Compute the above matrix-vector multiplication

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$5 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 1 \\ 0 \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$5(2) + 5(-3) + 4(4) = 11$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ ? \end{bmatrix}$$

$$5(-1) + 5(1) + 4(0) = 0$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_1 = a_{11}s_1 + a_{12}s_2 + \dots + a_{1n}s_n = \sum_{i=1}^n a_{1i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_2 = a_{21}s_1 + a_{22}s_2 + \dots + a_{2n}s_n = \sum_{i=1}^n a_{2i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ ? \end{bmatrix}$$

$$v_m = a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n = \sum_{i=1}^n a_{mi}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}s_i \\ \sum_{i=1}^n a_{2i}s_i \\ \vdots \\ \sum_{i=1}^n a_{mi}s_i \end{bmatrix}$$

Inner product: $[a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \sum_{i=1}^n a_i s_i$

Inner Product

Definition. The inner product of vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n is defined the

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}s_i \\ \sum_{i=1}^n a_{2i}s_i \\ \vdots \\ \sum_{i=1}^n a_{mi}s_i \end{bmatrix}$$

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The i th entry of the As is the inner product of the i th row of A and s

The Matrix Equation

Recall: Vector Equations

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

Recall: Vector Equations

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

Question. Can \mathbf{b} be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$?

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The Idea. think of the weights as *unknowns*

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we can use the same idea for matrix–vector
multiplication

The Matrix Equation

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \mathbf{x} = \mathbf{b}$$

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Can \mathbf{b} be written as a linear combination of **the columns of A** ?

The Matrix Equation

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Can \mathbf{b} be written as a linear combination of **the columns of A** ?

The Idea. write the "vector part" of our matrix-vector multiplication as an *unknown*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

How To: The Matrix Equation

Question. Does $A\mathbf{x} = \mathbf{b}$ have a solution?

Question. Is $A\mathbf{x} = \mathbf{b}$ consistent?

Question. Write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

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(augmented matrix)

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \quad \mathbf{b}]$$

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(augmented matrix)

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ \mathbf{b}]$$

!!they all have the same solution set!!

HOW TO: The Matrix Equation

Question. Write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution.

Use Gaussian elimination (or other means) to convert $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ \mathbf{b}]$ to reduced echelon form

Then read off a solution from the reduced echelon form

Full Span

Recall: Span

Recall: Span

Definition. the *span* of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots \alpha_n \mathbf{v}_n : \alpha_1, \alpha_2, \dots, \alpha_n \text{ are in } \mathbb{R}\}$$

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$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ exactly when \mathbf{u} can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. the *span* of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is:

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the span of the columns of a matrix A
is the set of vectors resulting
from multiplying A by any vector

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

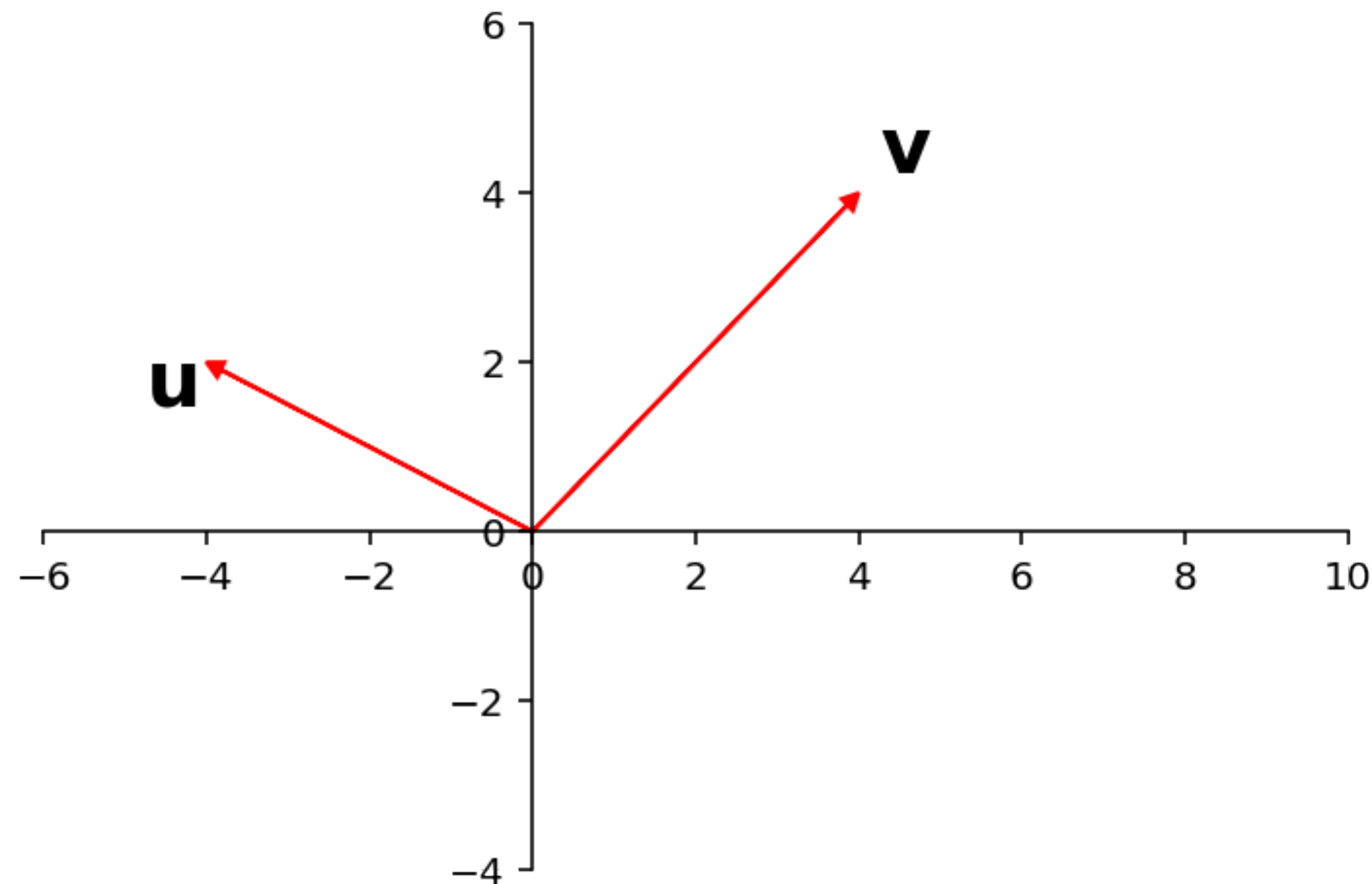
Spanning all of \mathbb{R}^2

Spanning all of \mathbb{R}^2

if two (or more) vectors in \mathbb{R}^2 span a plane,
they must span all of \mathbb{R}^2 . They "fill up" \mathbb{R}^2

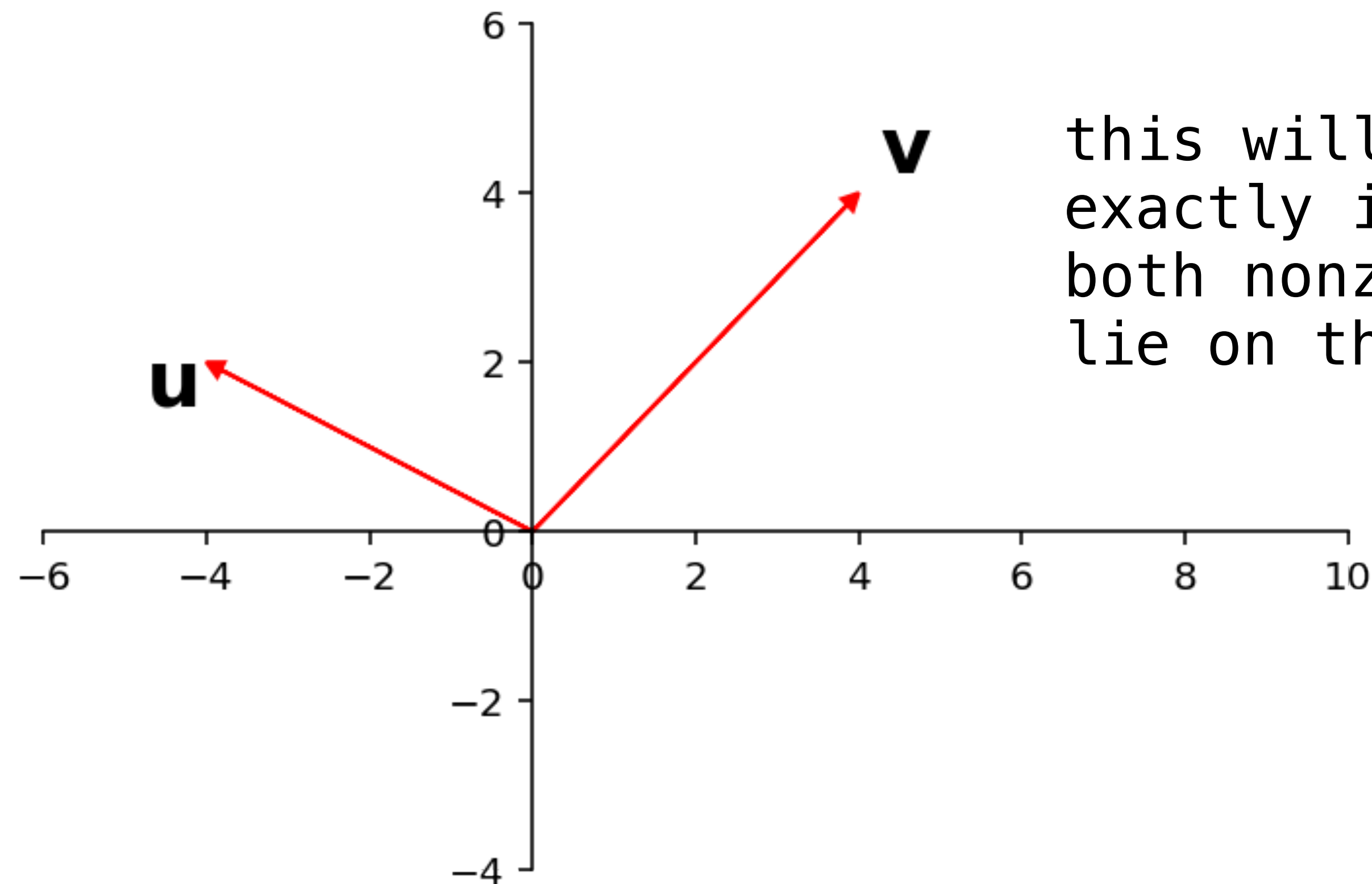
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Spanning all of \mathbb{R}^2

if two (or more) vectors in \mathbb{R}^2 span a plane,
they must span all of \mathbb{R}^2 . They "fill up" \mathbb{R}^2



this will happen
exactly if they are
both nonzero and don't
lie on the same line

What about \mathbb{R}^n ?

When do a set of vectors span all of \mathbb{R}^n ?

When do a set of vectors "fill up" \mathbb{R}^n ?

A Few Questions

Can two vectors in \mathbb{R}^3 span all of \mathbb{R}^3 ?

Is it required that five vectors \mathbb{R}^3 span all of \mathbb{R}^3 ?

A Thought Experiment

suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{array} \right]$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{bmatrix}$$

A Thought Experiment

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$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{array} \right]$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \begin{array}{c} \blacksquare \\ \blacksquare \end{array}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

Yes. It doesn't have an inconsistent row

A Thought Experiment

what about this system?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix}$$

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A Thought Experiment

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it depends...

Pivots and Spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

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If it doesn't matter what the last column is,
then **every choice must be possible**

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If it doesn't matter what the last column is,
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**Every vector in \mathbb{R}^2 can be written as a linear
combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$**

Spanning \mathbb{R}^m

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every \mathbf{b} in \mathbb{R}^m , $A\mathbf{x} = \mathbf{b}$ has a solution
2. The columns of A span \mathbb{R}^m
3. A has a pivot position in every row

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1. For every \mathbf{b} in \mathbb{R}^m , $A\mathbf{x} = \mathbf{b}$ has a solution
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HOW TO: Spanning \mathbb{R}^m

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ from \mathbb{R}^m span all of \mathbb{R}^m ?

Solution. Reduce $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

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Solution. Reduce $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

!! We only need the echelon form !!

Question

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ span all of \mathbb{R}^3 ?

Answer: No

the matrix

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 3 & 2023 \end{bmatrix}$$

cannot have more than 2 pivot positions

Not spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

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In this case the choice matters

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We can't make the last column $[0 \ 0 \ 0 \ \blacksquare]$ for nonzero \blacksquare

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But we can make the last column parameters to find equations that must hold

Not spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

Not spanning \mathbb{R}^m

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As long as $(-2)b_1 + b_2 = 0$, the system is consistent

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**This gives use a linear equation which describes
the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$**

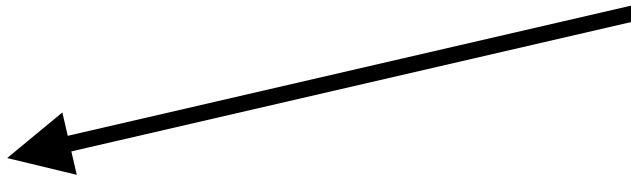
Question (Understanding Check)

True or False, the echelon form of any matrix has at most one row of the form $[0 \ 0 \ \dots \ 0 \ \blacksquare]$ where \blacksquare is nonzero.

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading
entry not
to the
right



this is not in echelon form

Question (More Challenging)

Give a linear equation for the span of the
vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Summary

Matrix and vectors can be multiplied together to get new vectors

The matrix equation is another representation of systems of linear equations

Looking forward: Matrices *transform* vectors