Linear Transformations

Geometric Algorithms
Lecture 7

CAS CS 132

Practice Problem

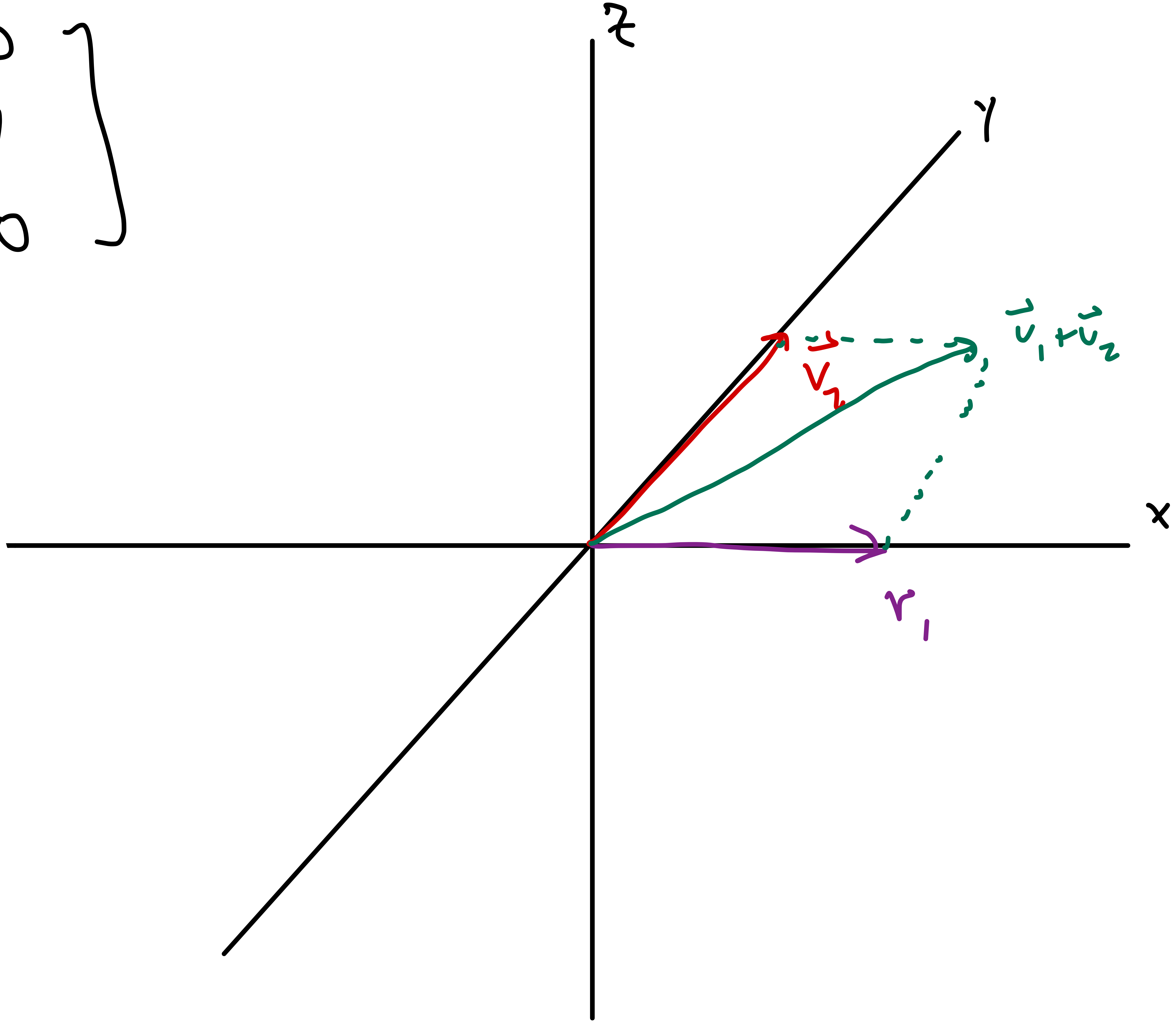
Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ such that

» every pair of vectors (i.e., $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$) are linearly independent not colinear

» $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent coplanar

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$



Objectives

- Introduce Matrix Transformations
- Define Linear Transformations
- Start looking at the Geometry of Linear Transformations

Keywords

Transformations

Domain, Codomain

Image, Range

Matrix Transformations

Linear Transformations

Additivity, Homogeneity

Dilation, Contraction, Shearing, Rotation

Recap

Recap: Homogenous Linear Systems

Definition. A system of linear equations is called ***homogeneous*** if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

Recap: Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution).

$$\vec{x} = \vec{0}$$

Recap: Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution).

The columns of A are linearly independent if $A\mathrm{x} = 0$ has exactly one solution.

Recap: Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has a nontrivial solution.

Recap: Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

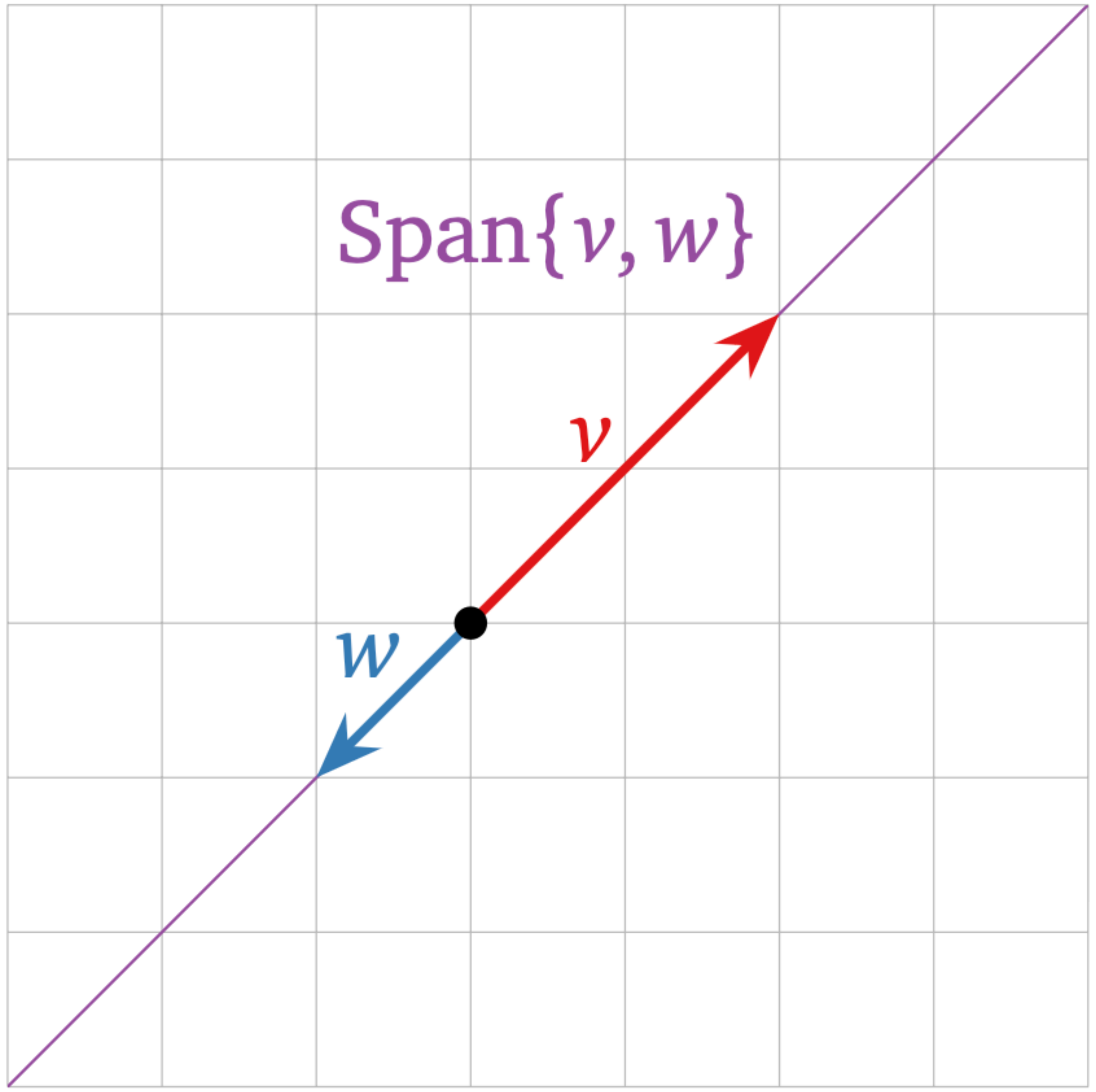
has a nontrivial solution.

A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals $\mathbf{0}$.

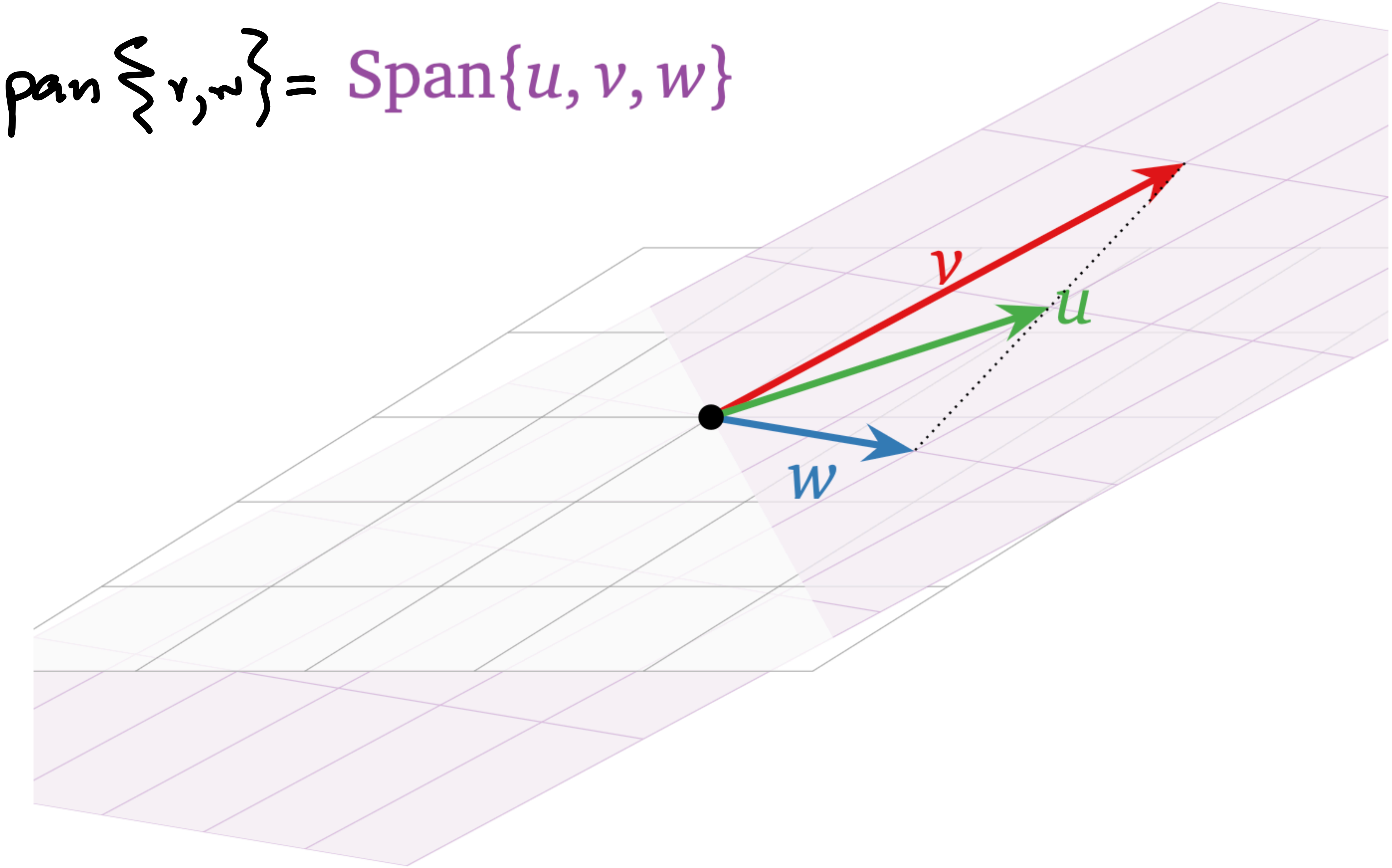
Recap: Linear Dependence

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself).

Linear Dependence (Pictorally)



$\mathrm{Span}\{v, w\} = \mathrm{Span}\{u, v, w\}$



Recall: Linear Dependence Relation

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

A linear dependence relation *witnesses* the linear dependence.

Example

Write down the linear dependence relation for the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Example

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 2 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 4^{-4} & 6^{+6} & 8^{+4} \\ -4^{+4} & -3^{-6} & -5^{-4} \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 0 & 1\,\cancel{12} & 1\,\cancel{12} \\ 0 & 1\,\cancel{-9} & 1\,\cancel{-9} \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & \overset{+3}{-3} & \overset{+3}{-2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -0.5x_3$

$x_2 = -x_3$

x_3 is free

$x_1 = -1$

$x_2 = -2$

$x_3 = 2$

$$\boxed{-\vec{v}_1 - 2\vec{v}_2 + 2\vec{v}_3 = \vec{0}}$$

Recap: Increasing Span

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only there is an $i \leq n$,

$$\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

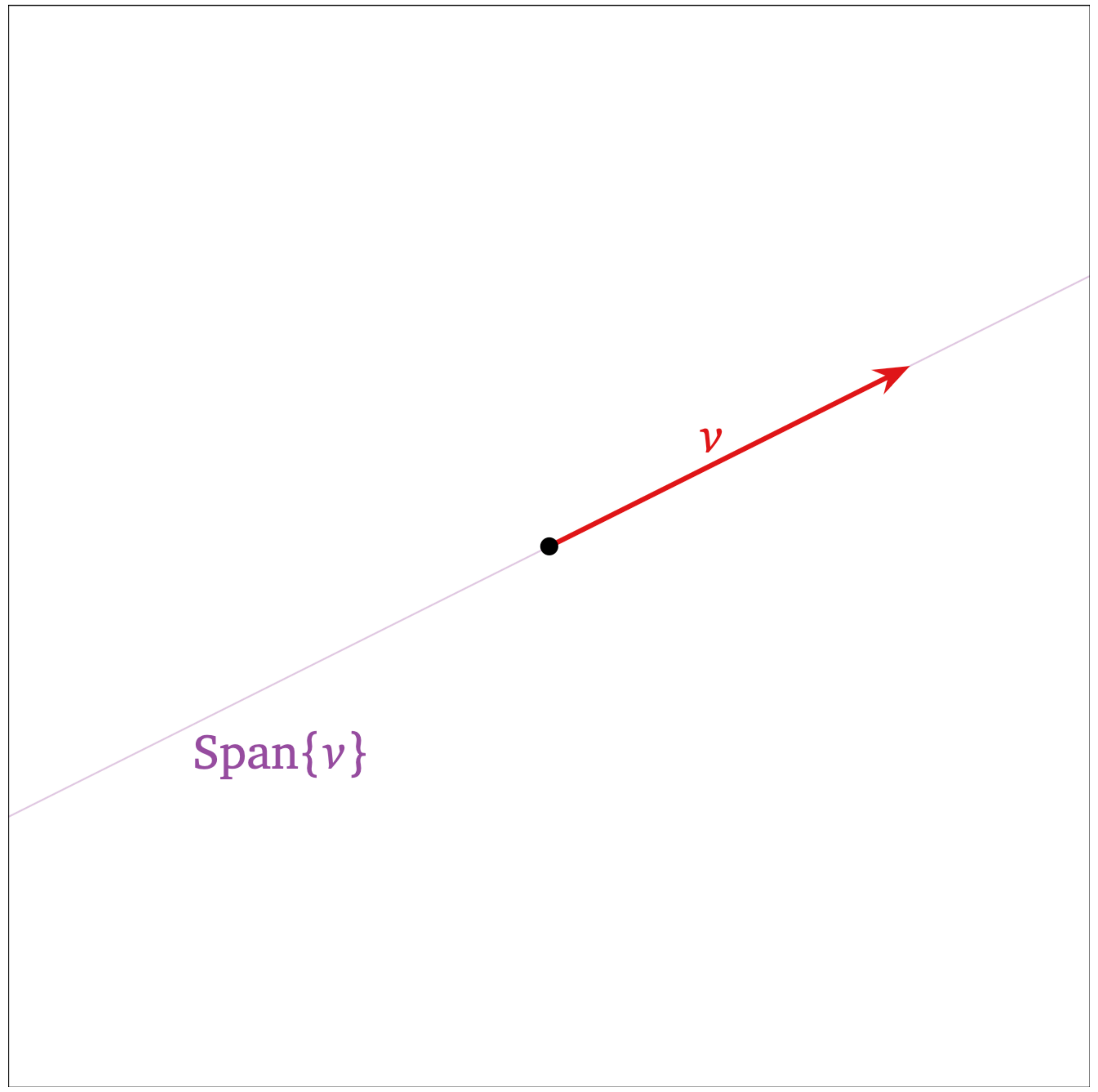
Recap: Increasing Span

Theorem. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only if there is an $i \leq n$,

$$\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

As we add vectors, we'll eventually find one in the span of the preceding ones.

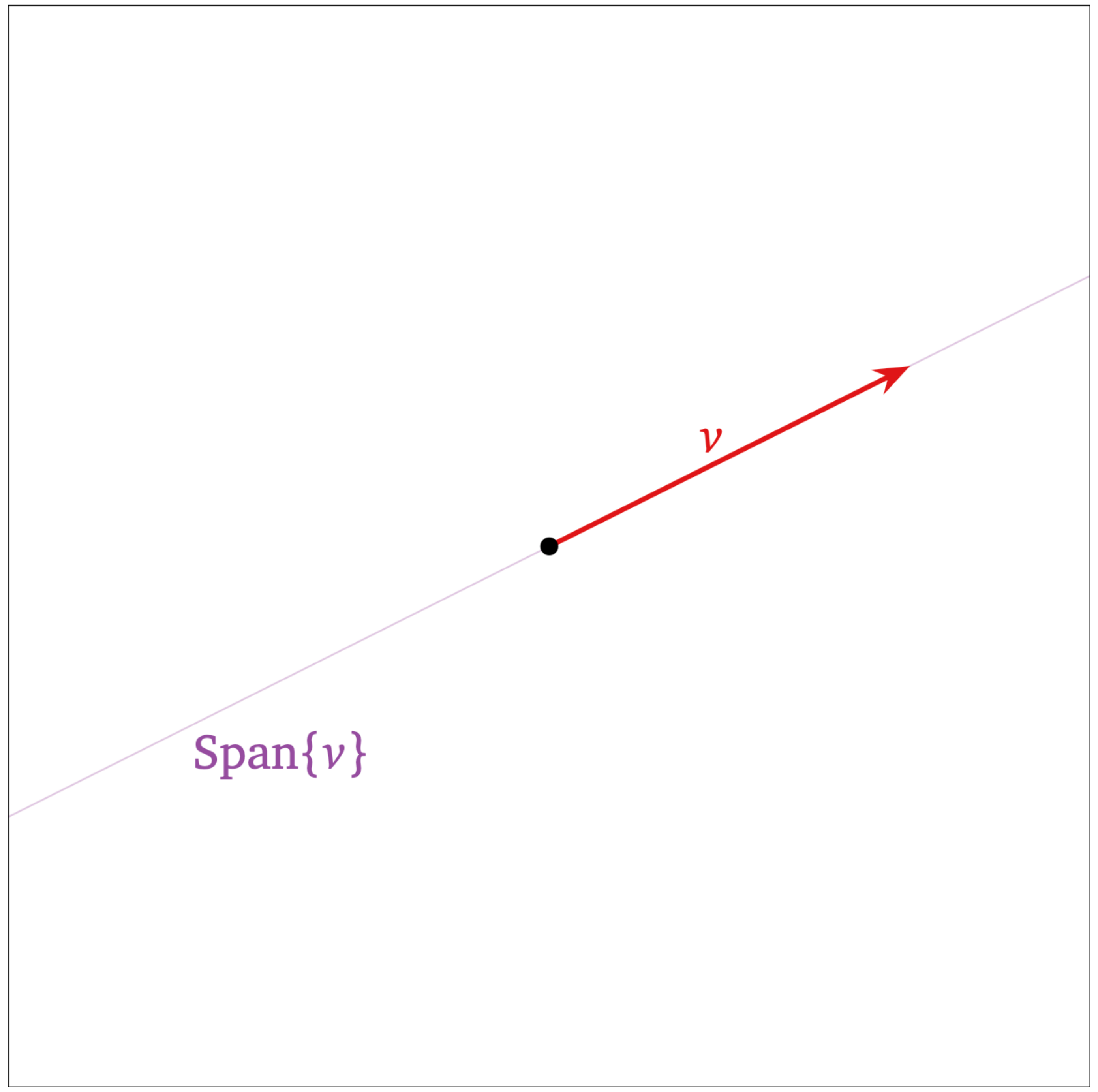
Recap: Increasing Span



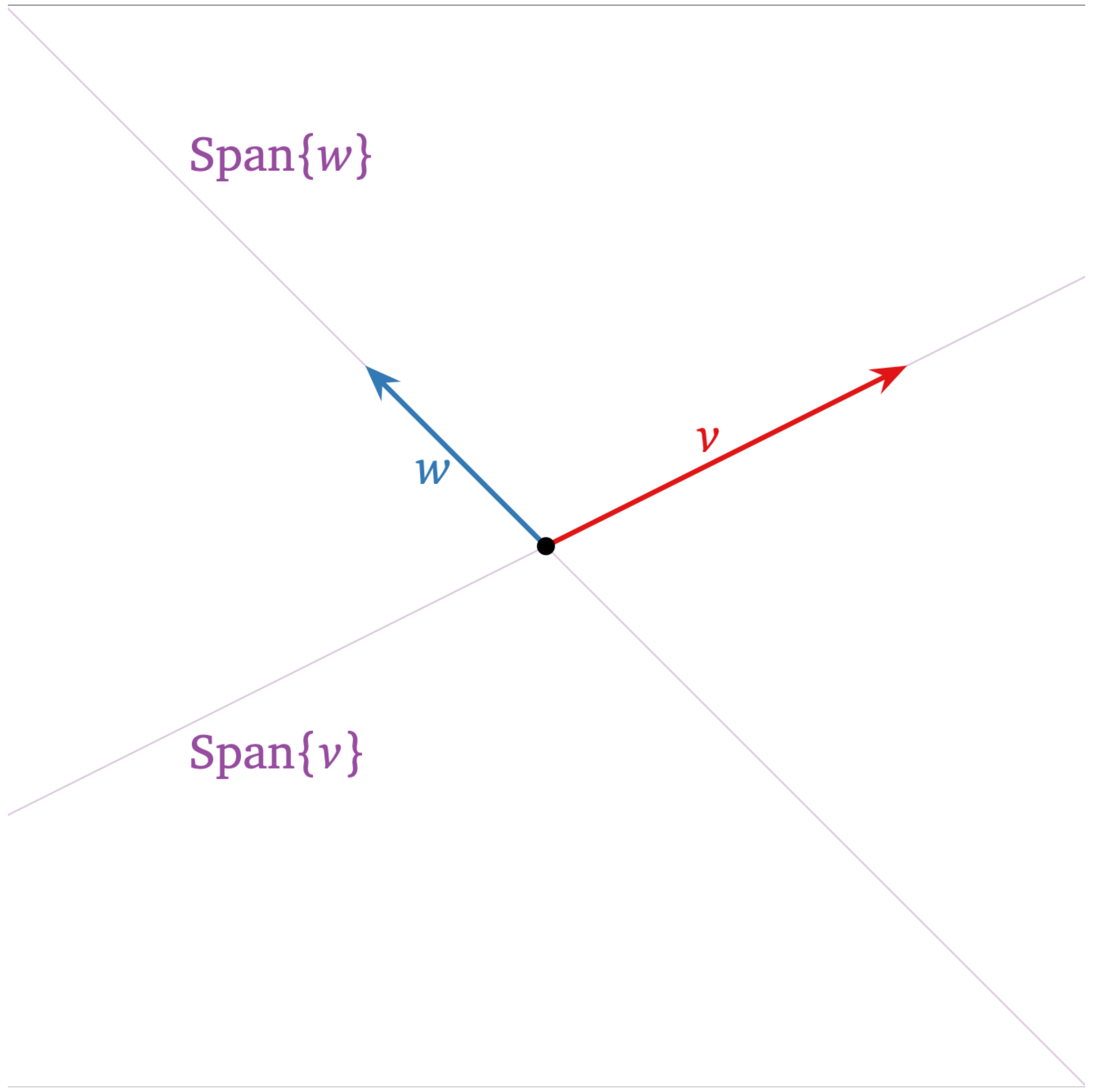
span of 1 vector
a line

image source

Recap: Increasing Span



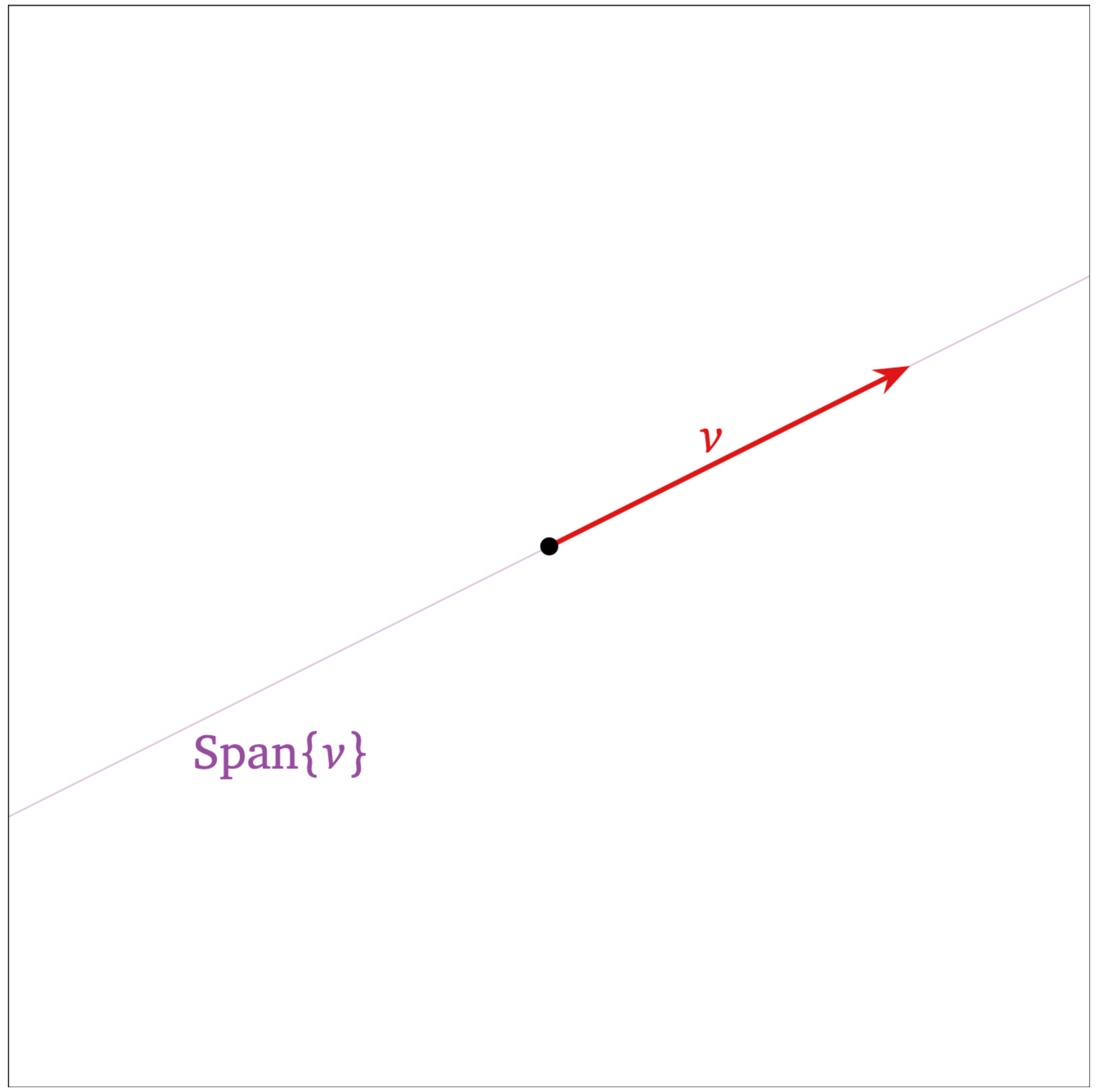
span of 1 vector
a line



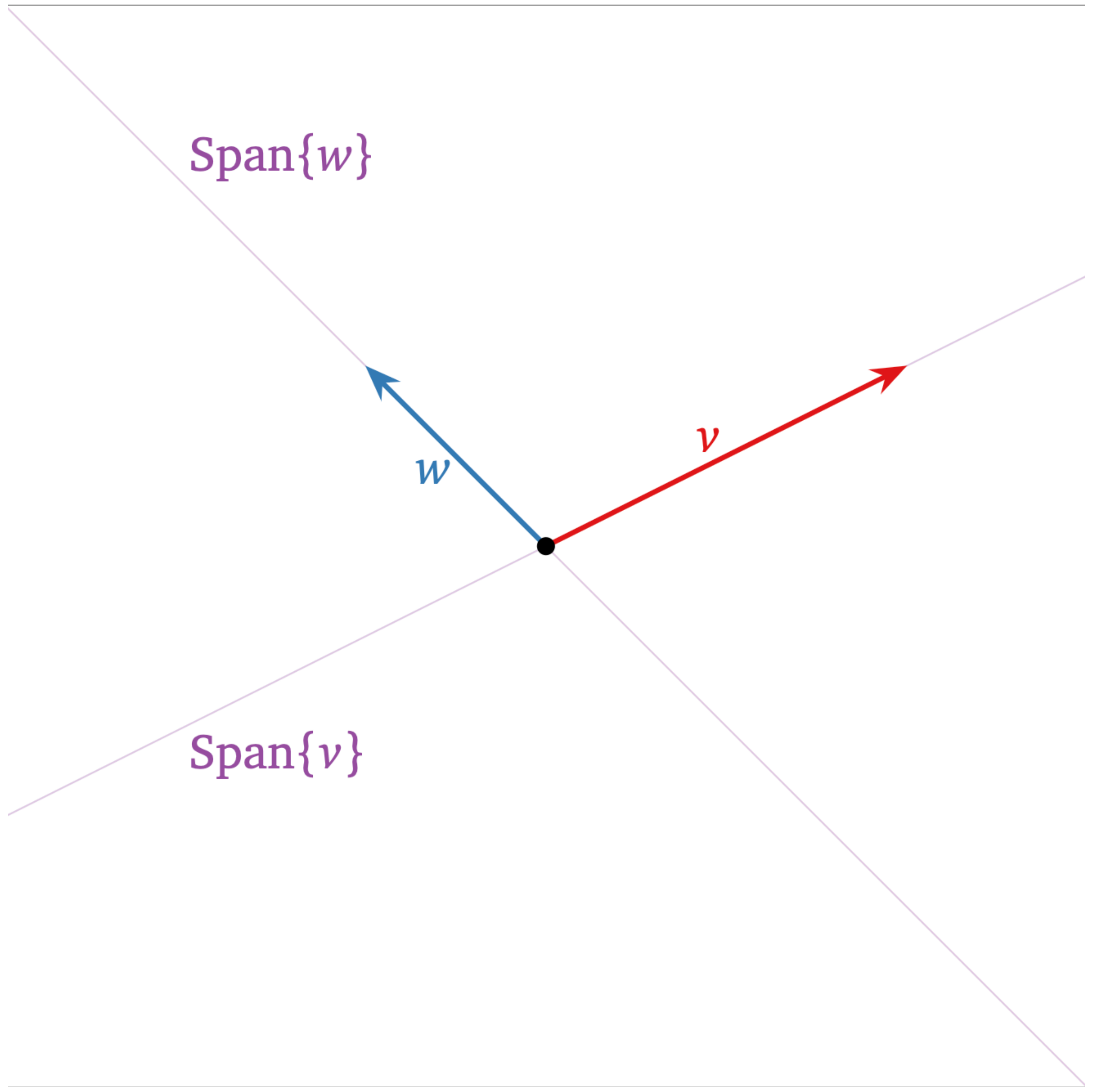
span of 2 vector
a plane

image source

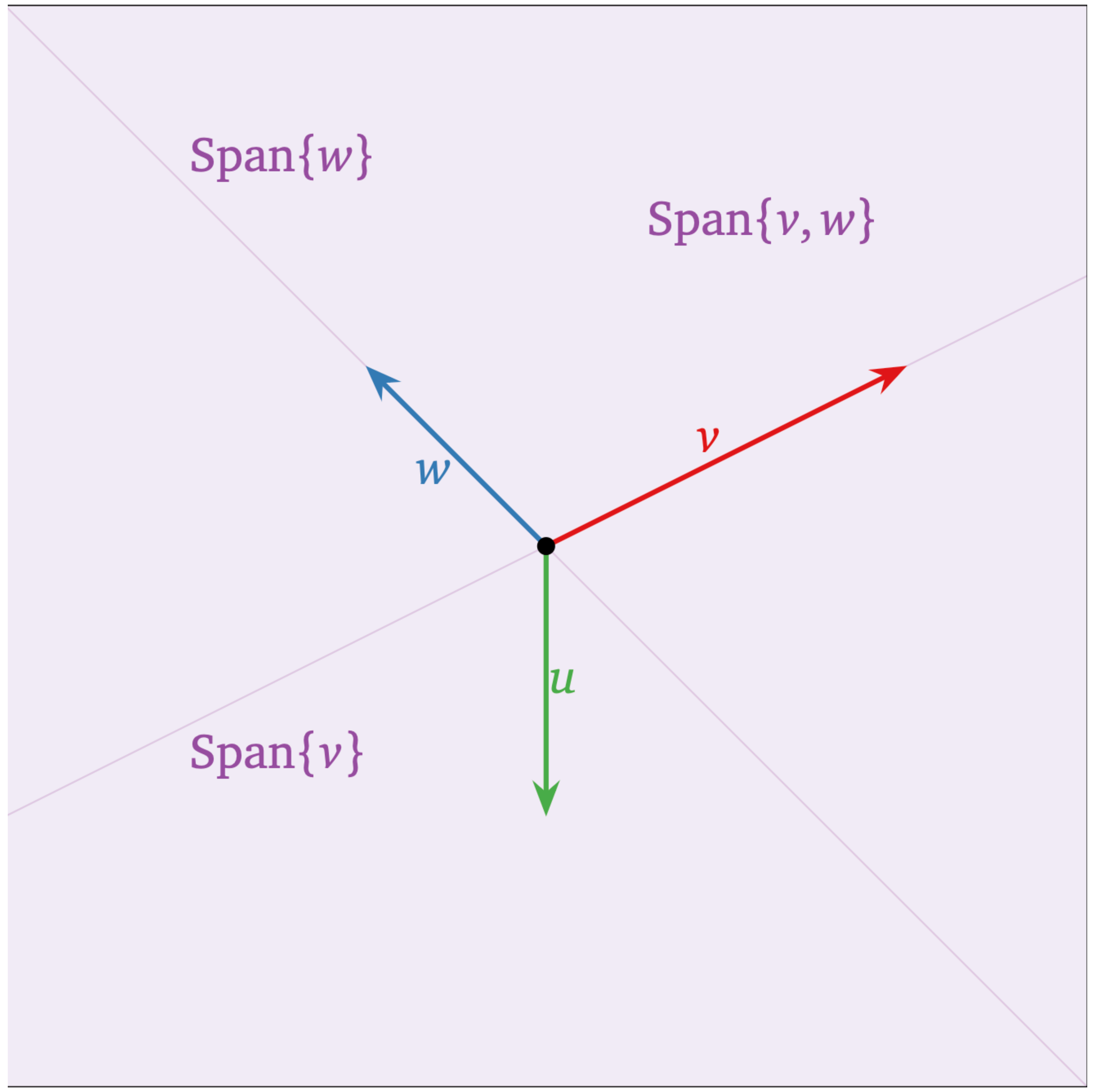
Recap: Increasing Span



span of 1 vector
a line



span of 2 vector
a plane



span of 3 vector
still a plane

image source

Recap: Linear Dependence Relations

When finding a linear dependence relation, we came across a system which a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take x_3 to be free

Recap: Pivots and Linear Dependence

Recap: Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every <u>column</u>.

Recap: Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every <u>column</u>.

Remember that we choose our free variables to be the ones whose columns don't have pivots.

Recap: Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column.

Remember that we choose our free variables to be the ones whose columns don't have pivots.

Free variables allow for infinitely many (nontrivial) solution.

Recap: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent?

Solution. Reduce $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \cdots \;\; \mathbf{a}_n]$ to echelon form and check if **has a pivot position in every column.**

Recap: Example

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column without a pivot

Recap: Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row.

The columns of a matrix are linearly independent if there is a pivot in every column.

Don't confuse these!

Matrix Transformations

Recall: Spans (with Matrices)

Definition. The ***span*** of a set of vectors is the set of all possible linear combinations of them.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

Recall: Spans (with Matrices)

Definition. The ***span*** of a set of vectors is the set of all possible linear combinations of them.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

The span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector.

Matrices as Transformations

Matrices allow us to *transform* vectors

The transformed vector lies in the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

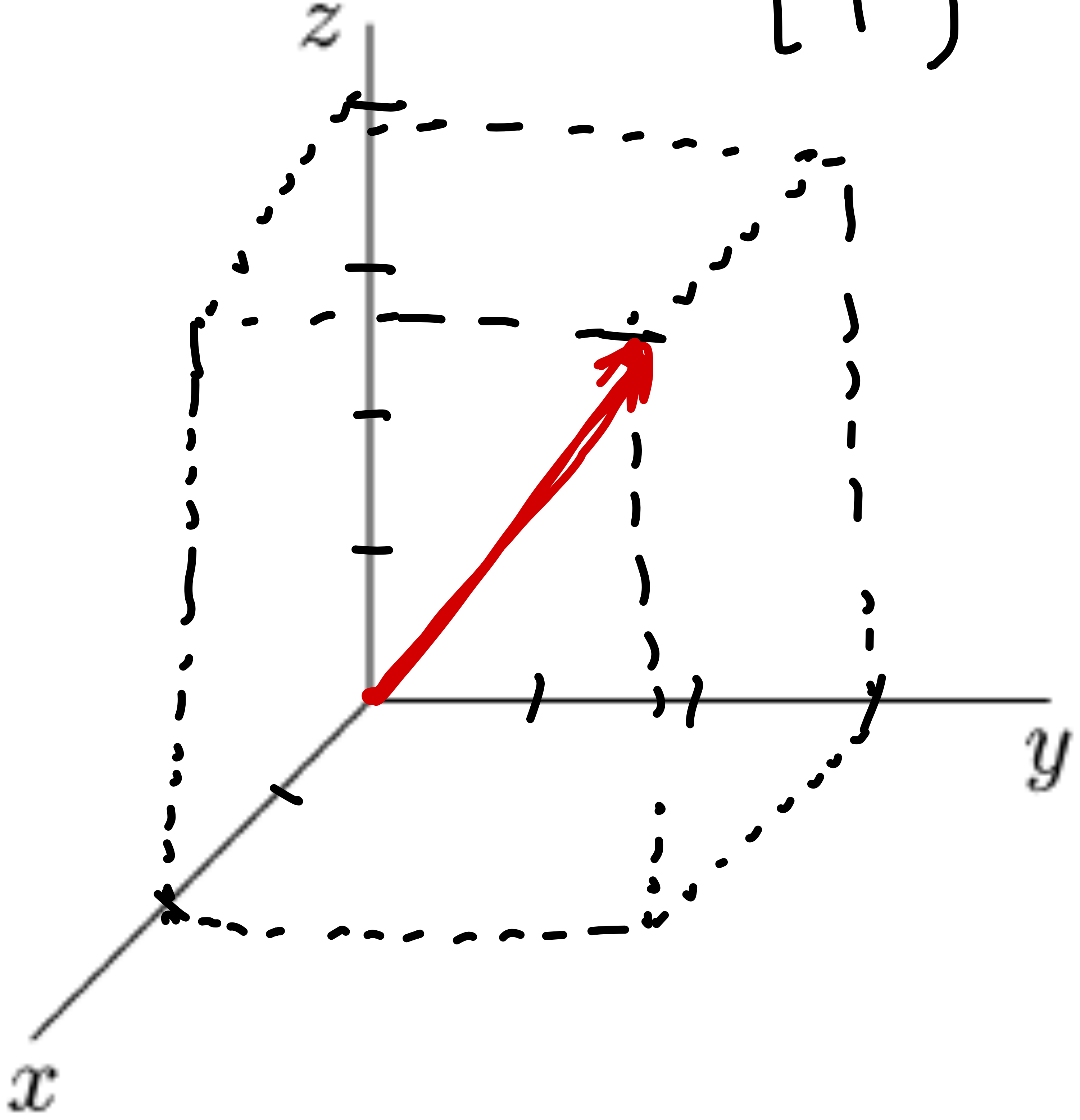
map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \overset{\mathbb{R}^3}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \overset{\mathbb{R}^2}{\begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}}$$

$$\vec{v} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

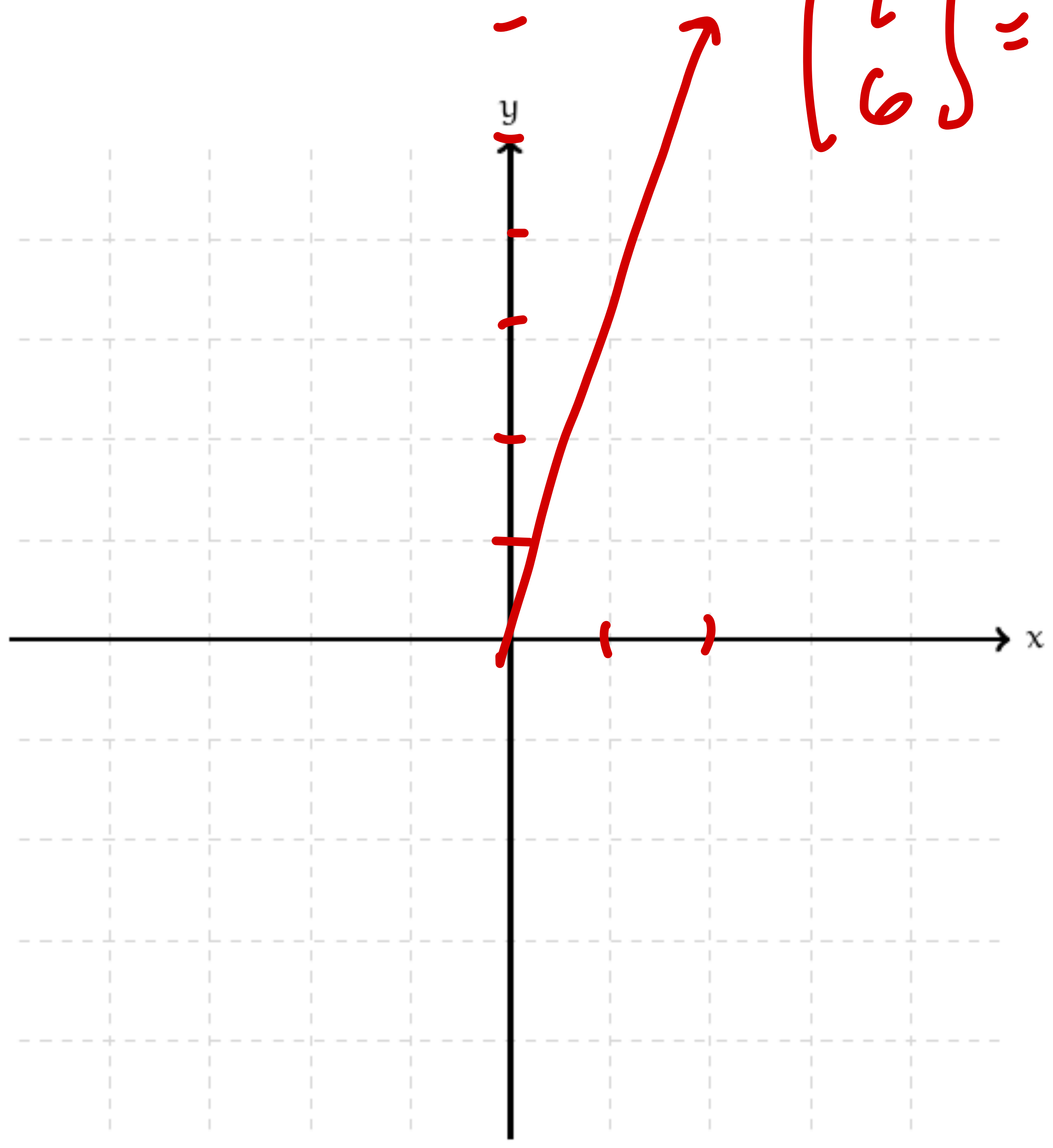


$$\overset{A}{\overset{\shortparallel}{\begin{bmatrix}1&0&0\\0&2&0\end{bmatrix}}}\overset{\vec{v}}{\overset{\shortparallel}{\begin{bmatrix}2\\3\\4\end{bmatrix}}} = \begin{bmatrix}2\\6\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&2&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\2x_2\end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix}1&0&0\\0&2&0\end{bmatrix}\mathbf{x}}$$

$$\xcancel{\begin{bmatrix}2\\6\end{bmatrix} \neq \begin{bmatrix}2\\6\\0\end{bmatrix}}$$



$$\begin{bmatrix}2\\6\end{bmatrix} = A\vec{v}$$

!!Important!!

The vector may be a different size after translation.

Recall: Matrix-Vector Multiplication and Dimension

matrix-vector multiplication only works if the number of *columns* of the matrix matches the dimension of the vector

$$m\overset{n}{\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}} \; n\begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = m\begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$$(m \times n) \qquad \mathbb{R}^n \qquad \mathbb{R}^m$$

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

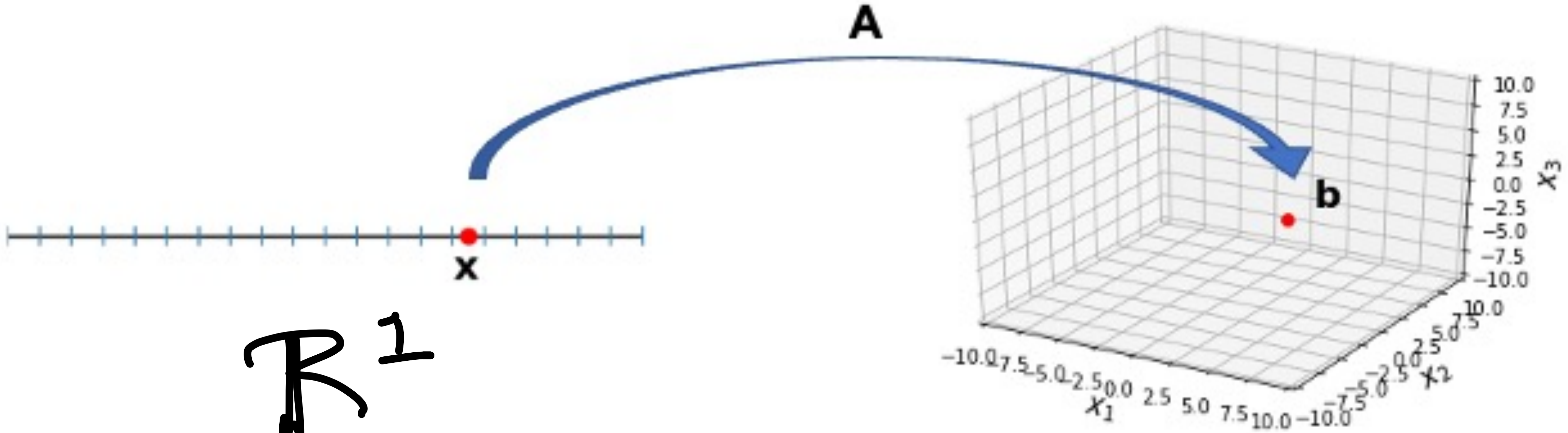
A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? $\equiv$ is there a vector which A transforms into $\mathbf{b}$?

Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

Question (Conceptual)

Suppose a matrix transforms a vector according to the following picture. What is the size of the matrix?



$\mathbb{R}^1$

$\mathbb{R}^3$

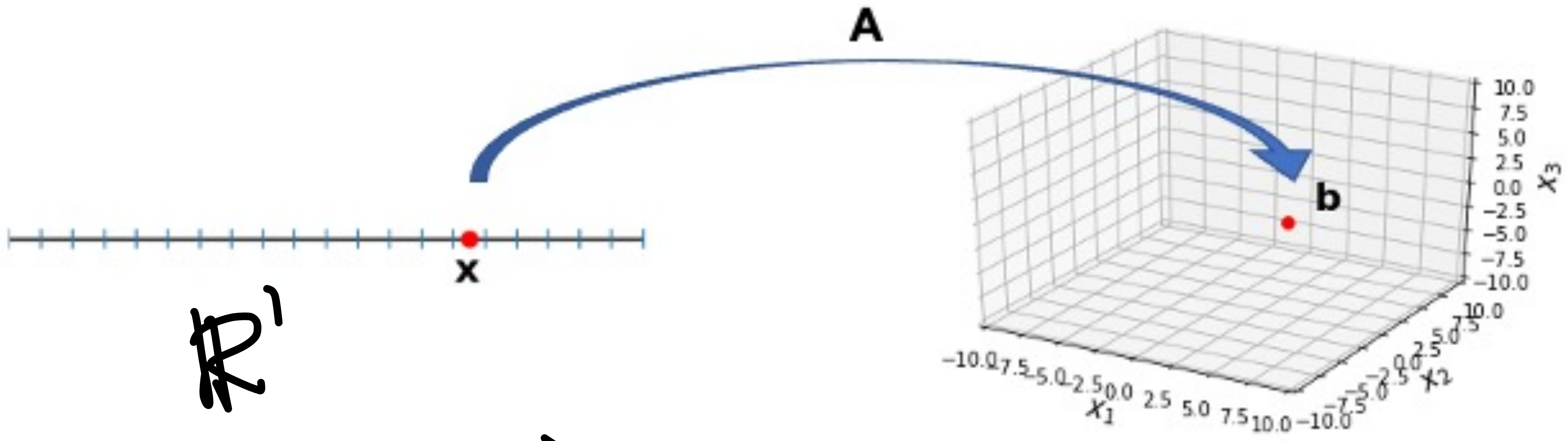
$\vec{b} = A\vec{x}$

$A \in \mathbb{R}^{m \times n}$

$m = 3$

$n = 1$

$$x \mapsto \begin{bmatrix}1\\1\\1\end{bmatrix} x$$



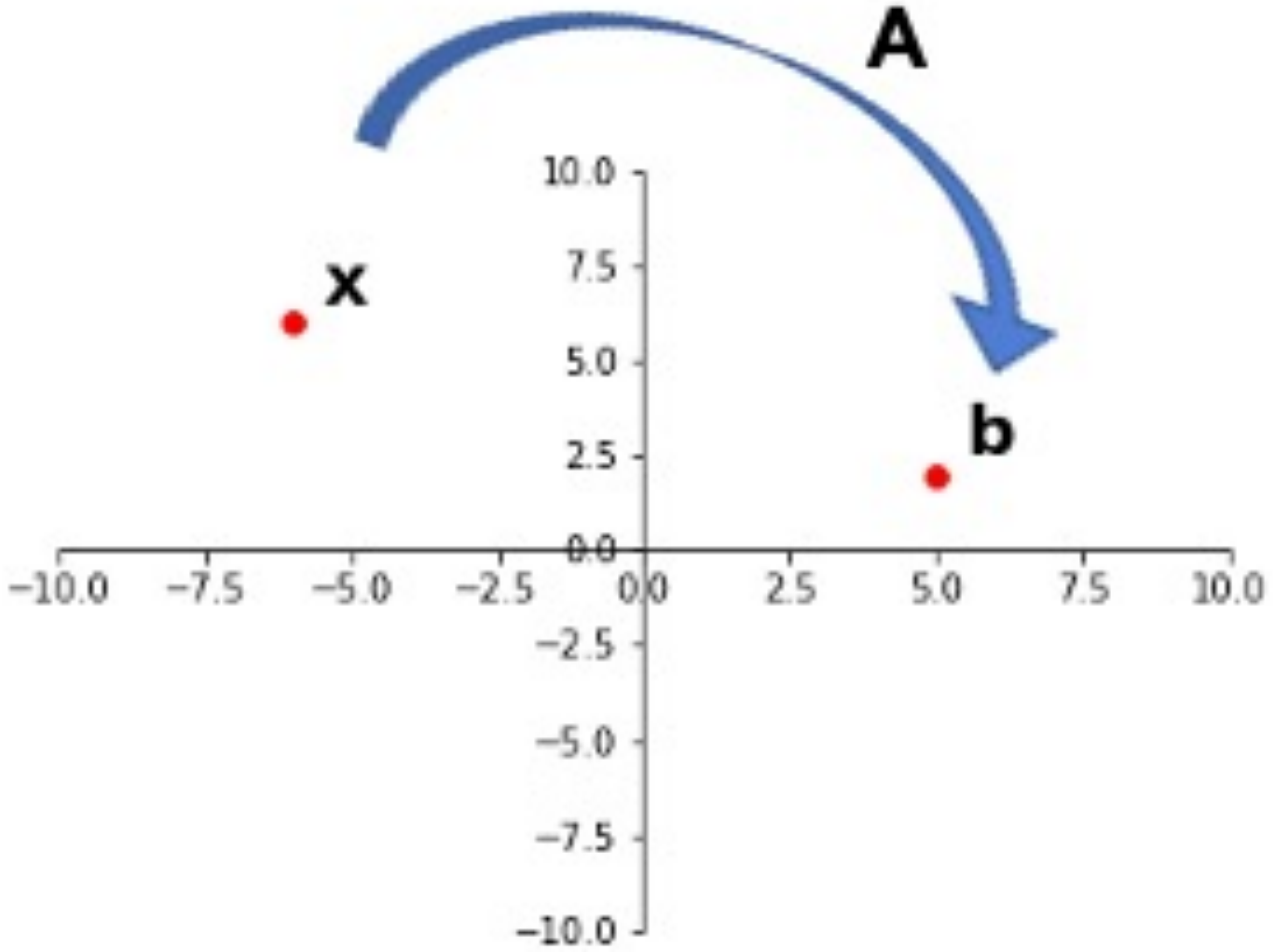
$\mathbb{R}^1$

$$3 \mapsto \begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$\vec{x} \mapsto \underset{\uparrow\ \mathbb{R}^{3\times 1}}{A}\vec{x} \qquad \mathbb{R}^3$$

$\mathbb{R}^n \to \mathbb{R}^n$

Mapping between the same space can be viewed as a way of moving around points.



Transformations

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

$$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

$\vec{v} \mapsto T(\vec{v})$

$$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

It's just a function, like in calculus.

Image and Range

Image and Range

Definition. For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation T is the vector $T(\mathbf{v})$

Image and Range

Definition. For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation T is the vector $T(\mathbf{v})$

Definition. The ***range*** of a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under T

Image and Range

Definition. For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation T is the vector $T(\mathbf{v})$

Definition. The ***range*** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under T

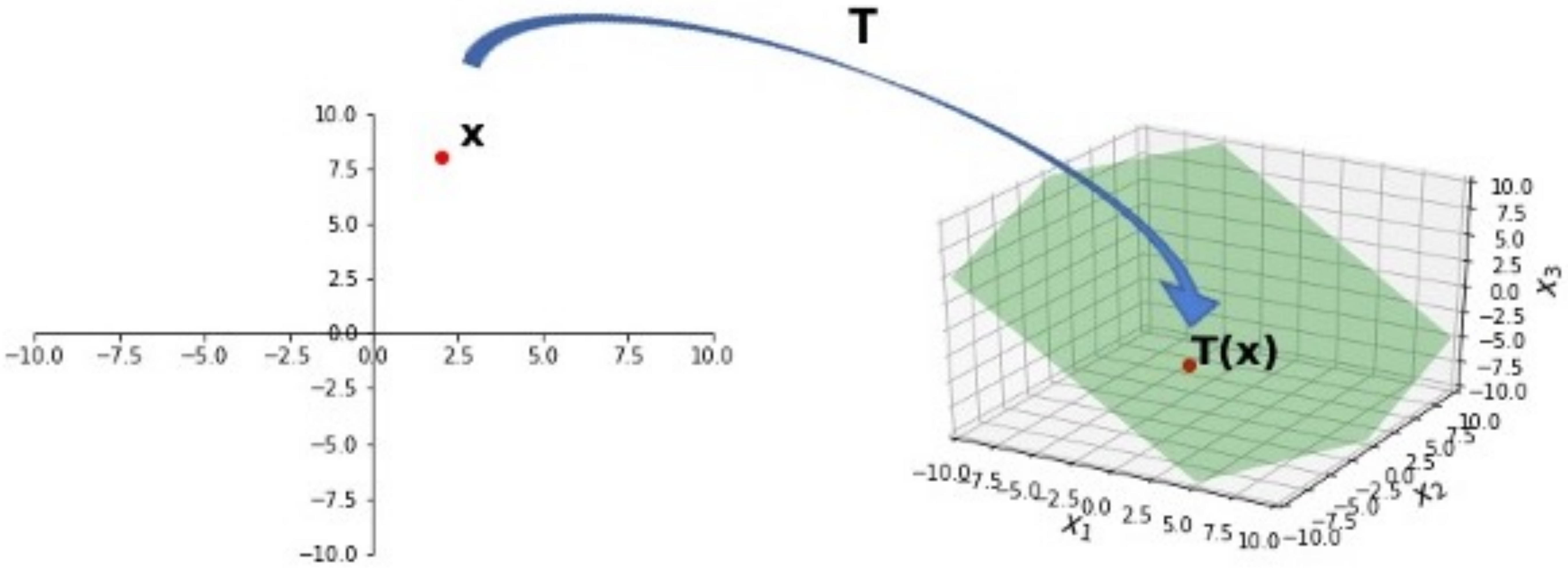
$$\text{ran}(T) = \{T(\mathbf{v}) : v \in \overset{\text{dom}(T)}{\overset{\shortparallel}{\mathbb{R}^n}}\}$$

image of $\mathbf{v}$ under T $\equiv$ output of T applied to $\mathbf{v}$

range of T $\equiv$ all possible output of T

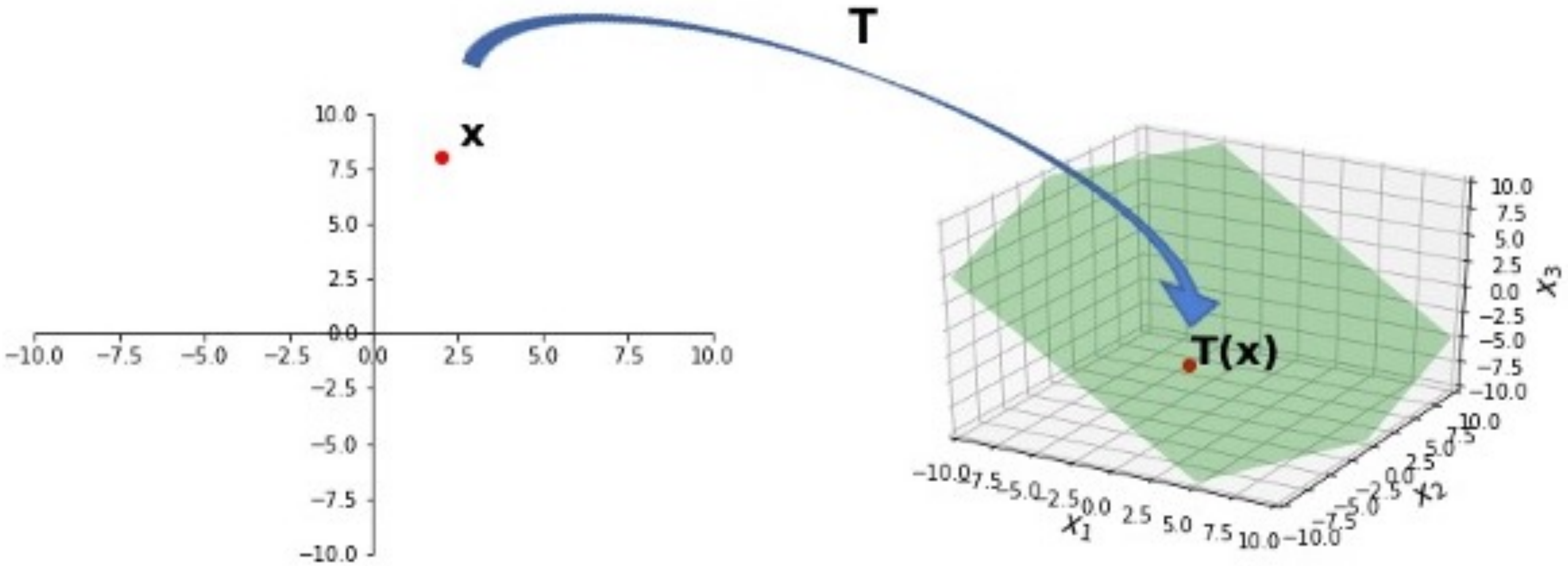
Codomain and Range

The codomain and range of a transformation may or may not be the same



Codomain and Range

The codomain and range of a transformation may or may not be the same



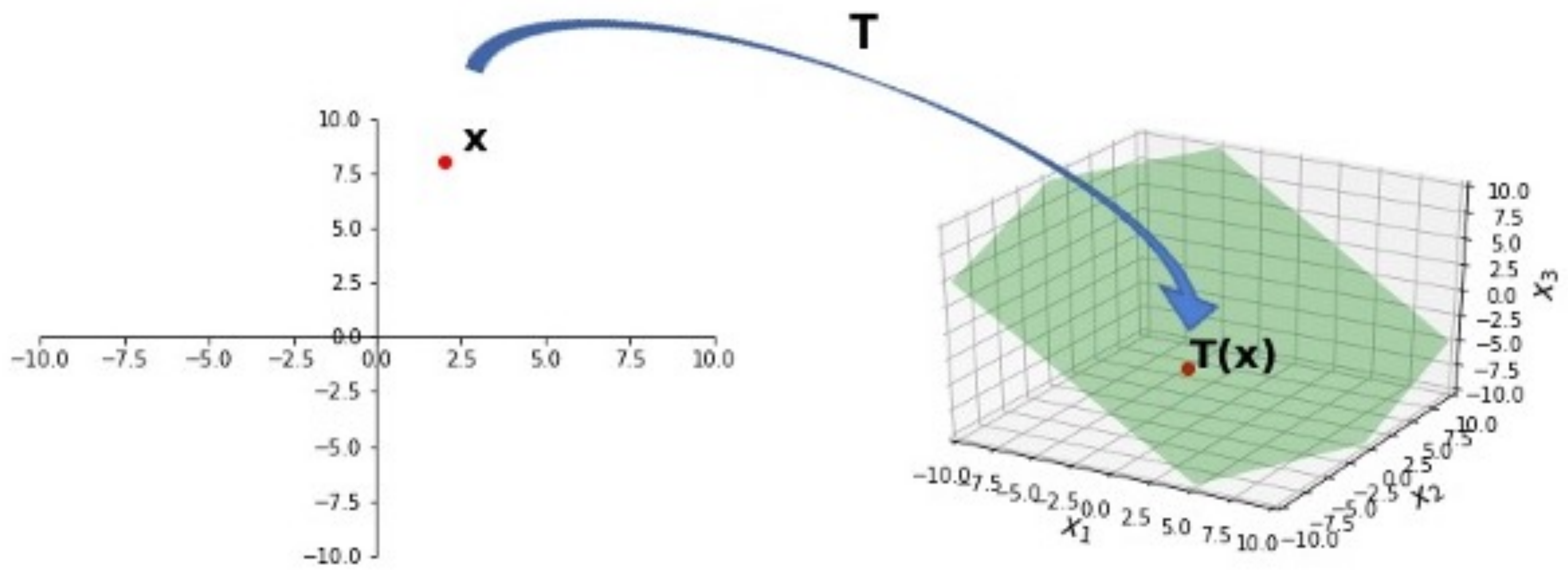
range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

Codomain and Range

The codomain and range of a transformation may or may not be the same



range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

The range is always contained in the codomain

Example

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix}\longmapsto\begin{bmatrix}x_1^2\\x_2\\0\end{bmatrix}$$

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

$$\begin{bmatrix}-1\\5\end{bmatrix}\longmapsto\begin{bmatrix}1\\5\\0\end{bmatrix}$$

$\vec{v}$

image of $\vec{v}$

$$\{T(\vec{v}) : \vec{v}\in\mathbb{R}^2\} = \operatorname{ran}(T) = \left\{\begin{bmatrix}z_1\\z_2\\0\end{bmatrix} : z_2\in\mathbb{R},\ z_1\in\mathbb{R},\ z_1\geq 0\right\}$$

$$\begin{bmatrix}5\\4\end{bmatrix}\longmapsto\begin{bmatrix}5\\4\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\notin \operatorname{range}(T)$$

$$\begin{bmatrix}-1\\0\\0\end{bmatrix}\notin \operatorname{range}(T)$$

Matrix Transformations

Transformation of a Matrix

Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\operatorname{ran}(T) = ?$

$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) =$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Range and Span

Range and Span

The span of the columns of a matrix A is the set of all possible *images* under A

Range and Span

The span of the columns of a matrix A is the set of all possible *images* under A

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix})$$

Range and Span

The span of the columns of a matrix A is the set of all possible *images* under A

$$\{[\vec{a}_1 \cdots \vec{a}_n]\vec{v} : v \in \mathbb{R}^n\} = \{[\vec{a}_1 \cdots \vec{a}_n]\vec{v} : \vec{v} \in \mathbb{R}^n\}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

The transformation of a vector $\mathbf{v}$ under the matrix A always lies in the span of its columns

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Linear Transformations

Recall: Algebraic Properties

Matrix–vector multiplication satisfies the following two properties:

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = 2 \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

Exercise

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Exercise

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) =$$

Linear Transformations

Definition. A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Linear Transformations

Definition. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Matrix transformations are linear transformations

Example: Identity

$$T(\mathbf{v}) = \mathbf{v}$$

$$T(\vec{u} + \vec{v}) = \vec{u} + \vec{v} = T(\vec{u}) + T(\vec{v})$$ ✓

$$T(c\vec{u}) = c\vec{u} = c\,T(\vec{u})$$ ✓

Example: Zero

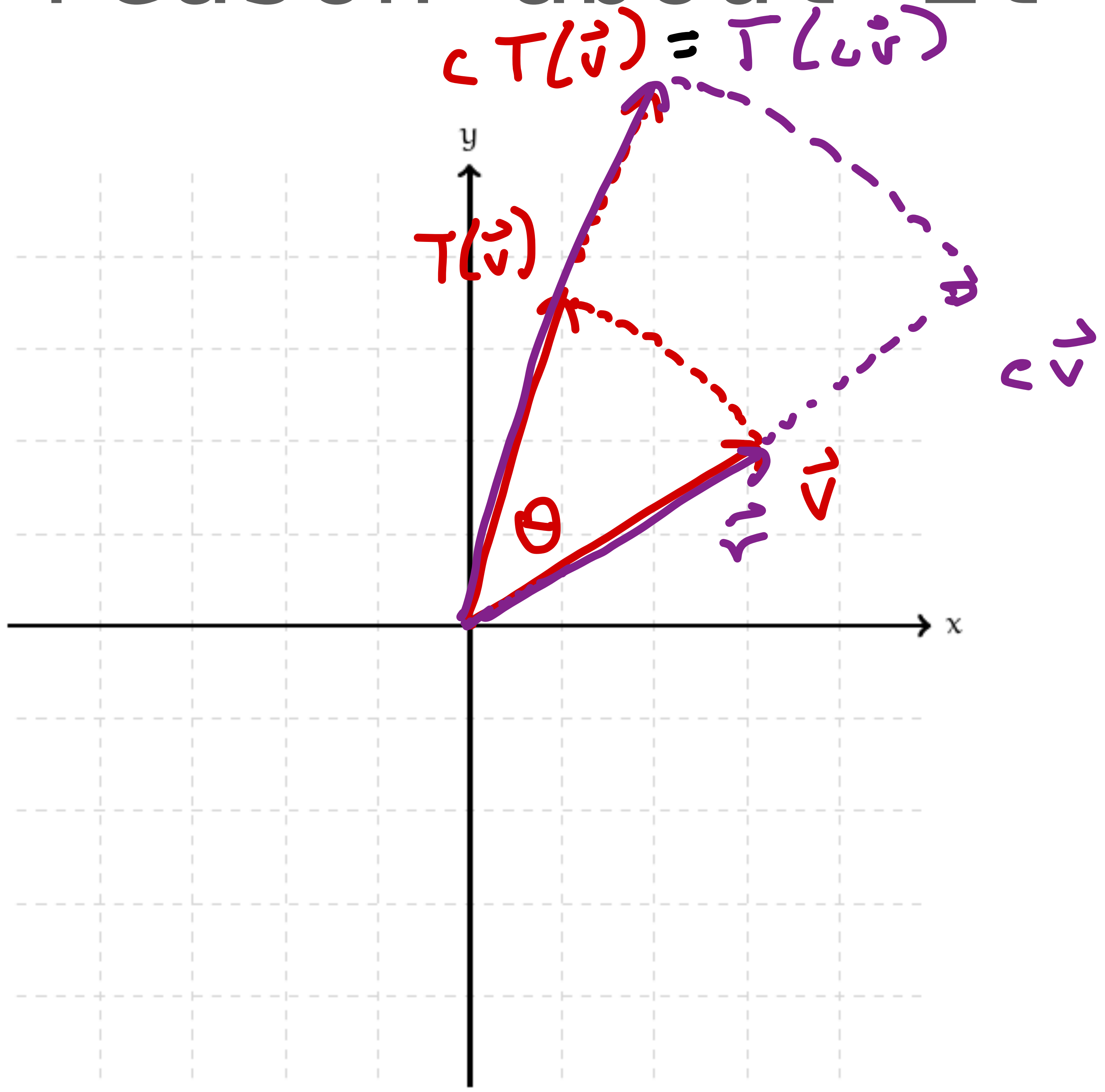
$$T(\mathbf{v}) = \mathbf{0}$$

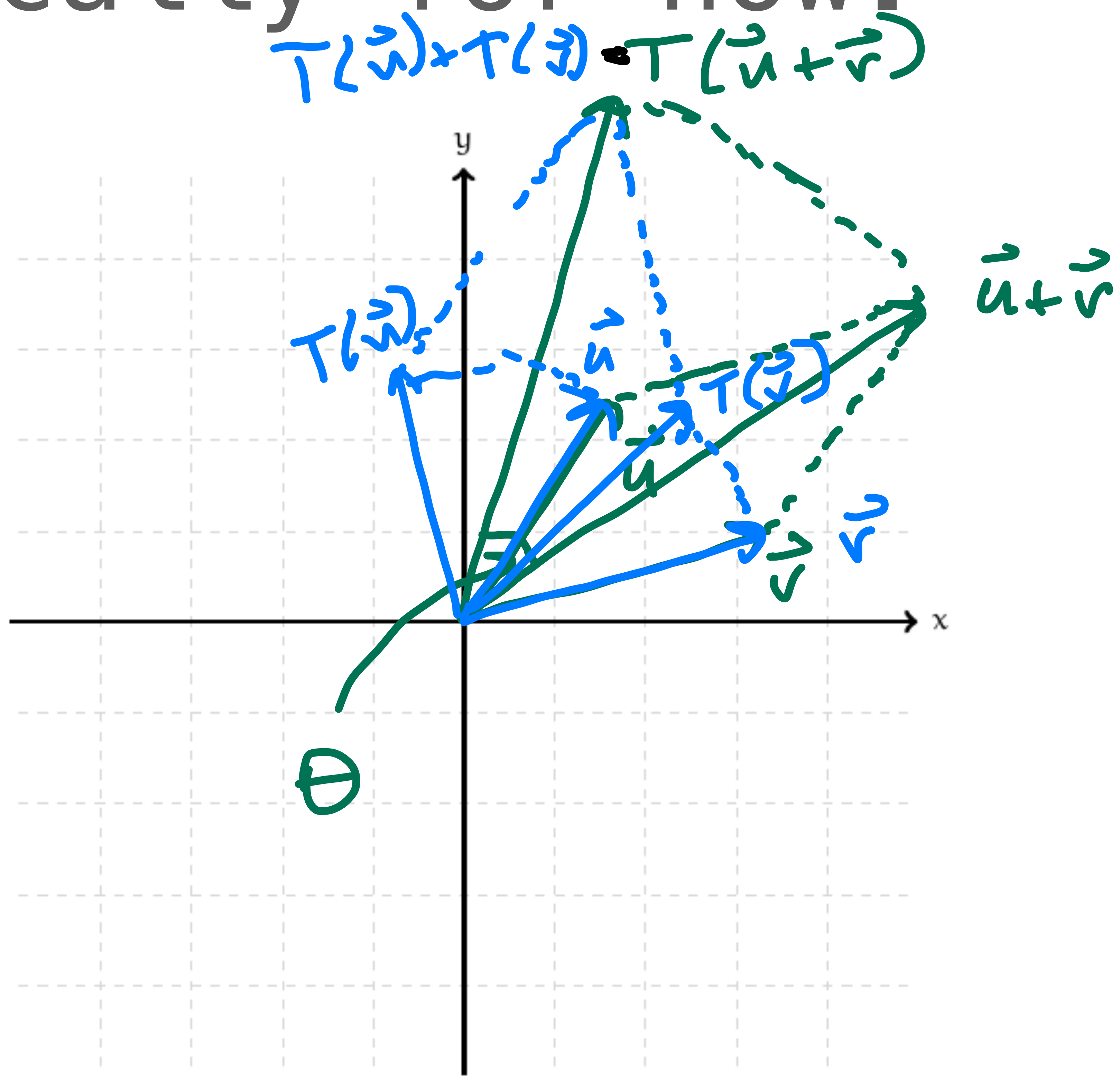
$$T(\vec{u}+\vec{v}) = \vec{0} = \vec{0} + \vec{0} = T(\vec{u}) + T(\vec{v}) \quad ✓$$

$$T(c\vec{u}) = \vec{0} = c\vec{0} = cT(\vec{u}) \quad ✓$$

Example: Rotation

We'll see this on Thursday, but we can reason about it geometrically for now.





Example: Indefinite Integrals

$$T(f) = \int f(x)dx$$

Disclaimer:
Advanced
Material

the same goes for derivatives
(how are functions vectors???)

Example: Expectation

$$T(X) = \mathbb{E}[X]$$

Disclaimer:
Advanced
Material

This is exactly linearity of expectation.

(how are random variables vectors???)

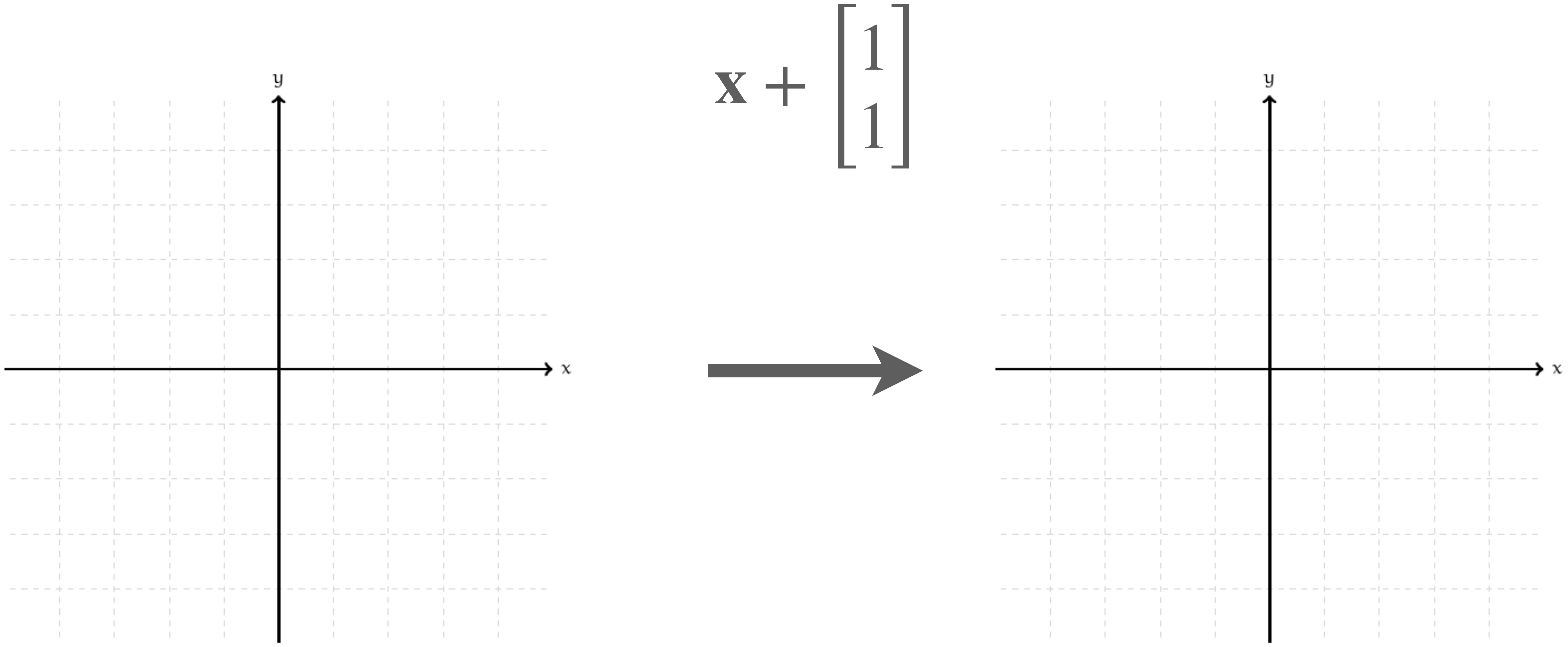
Non-Example: Squares

$$T(x) = x^2$$

Note that $T : \mathbb{R}^1 \to \mathbb{R}^1$

$$(5+1)^2 \neq 5^2 + 1^2$$

Non-Example: Translation



$$\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Properties of Linear Transformations

The Zero Vector

$$T(\mathbf{0}) = \ ???$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

$$T(\vec{0}) = T(0\vec{v}) = 0\,T(\vec{v}) = \vec{0}$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The zero vector is *fixed* by linear transformations.

The Zero Vector

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$

$$T\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

$$T(\mathbf{0}) = \mathbf{0}$$

Note: These may be different dimensions!

The zero vector is *fixed* by linear transformations.

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

$= T(a\mathbf{v}) + T(b\mathbf{u})$ (additivity)

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

$= T(a\mathbf{v}) + T(b\mathbf{u})$ (additivity)

$= aT(\mathbf{v}) + bT(\mathbf{u})$ (homogeneity for each term)

A Single Condition

Theorem. A transformation $T:\mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u}+b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

A Single Condition

Theorem. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

It's often easiest to show this single condition

Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_nT(\mathbf{v}_n)$$

We can generalize this condition to any linear combination

Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

We can generalize this condition to any linear combination

Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

We can generalize this condition to any linear combination

This is the most useful form

Geometry of Matrix Transformations

Motivating Questions

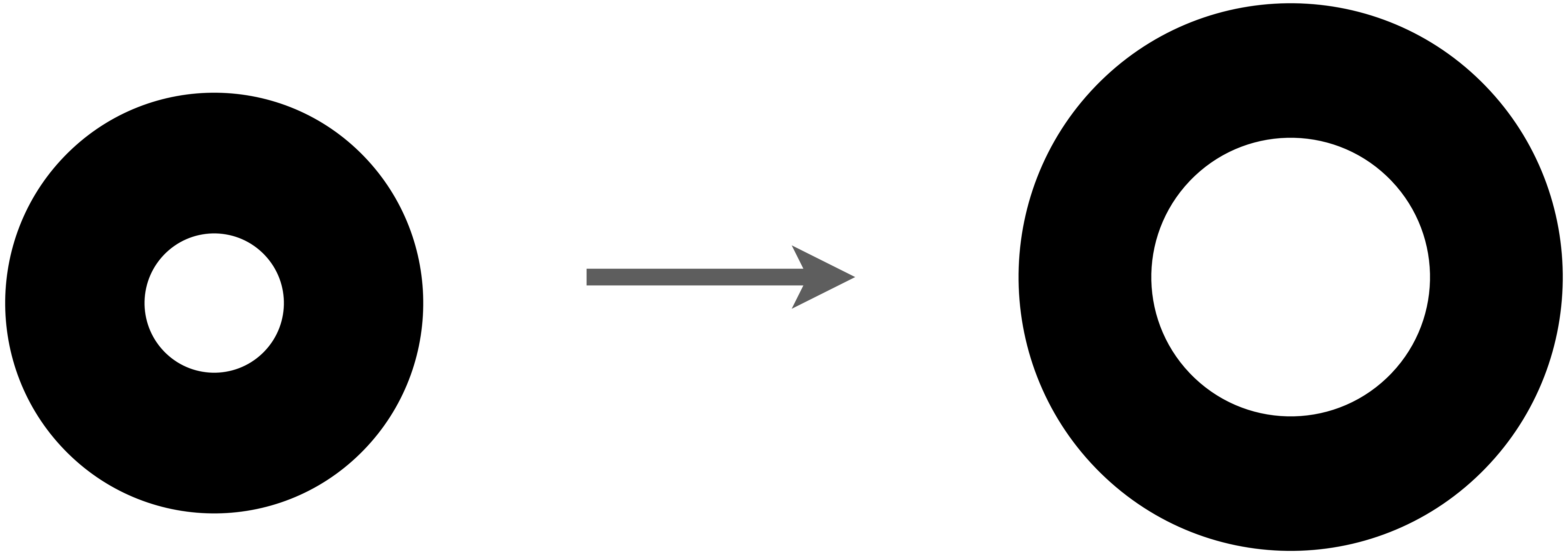
What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

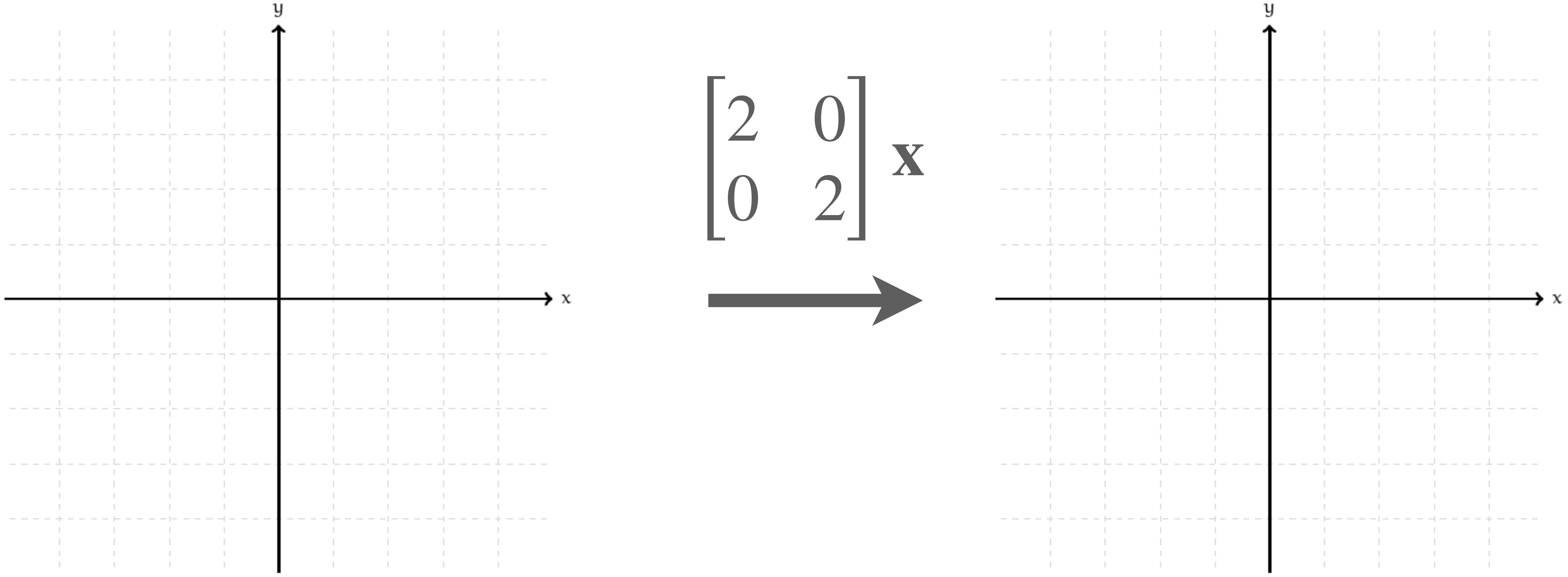
Matrix transformations change the "shape" of a set of set of vectors (points).

Example: Dilation



Example: Dilation

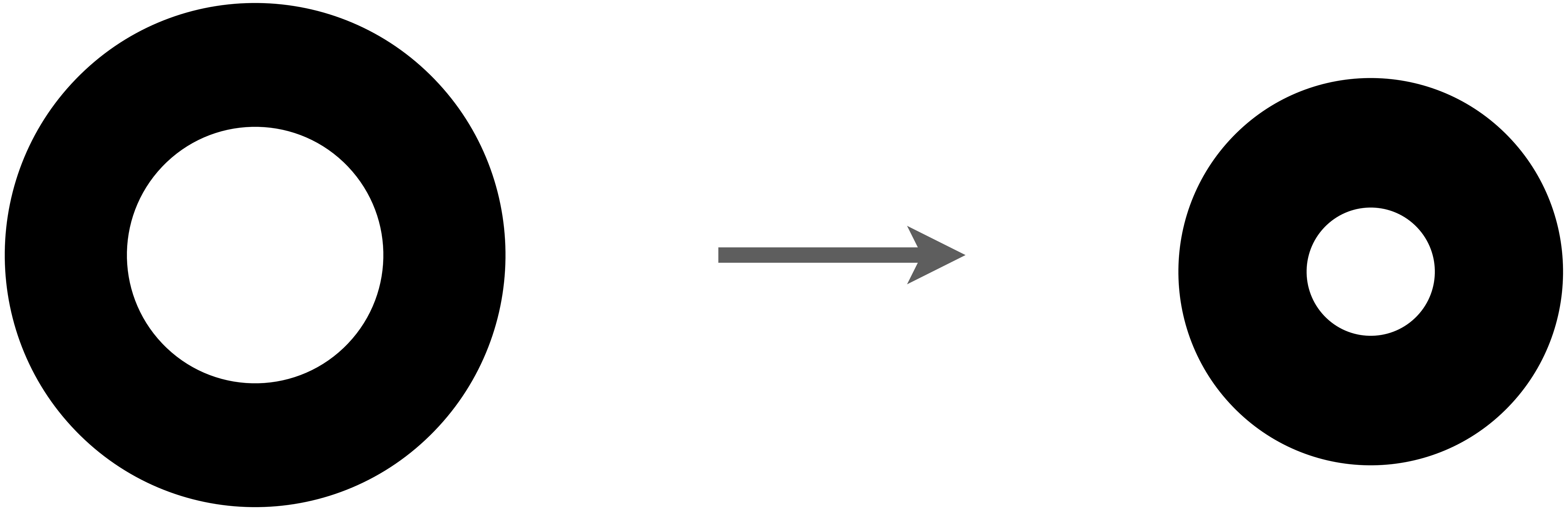
$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

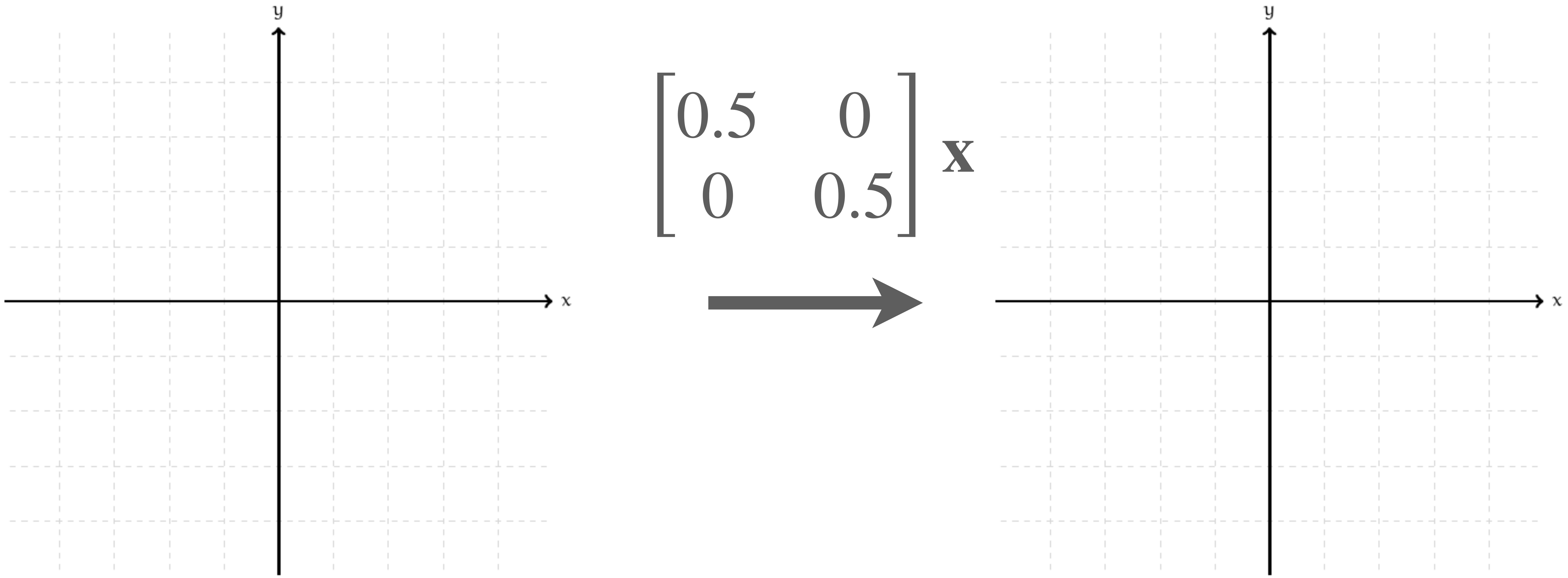
if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



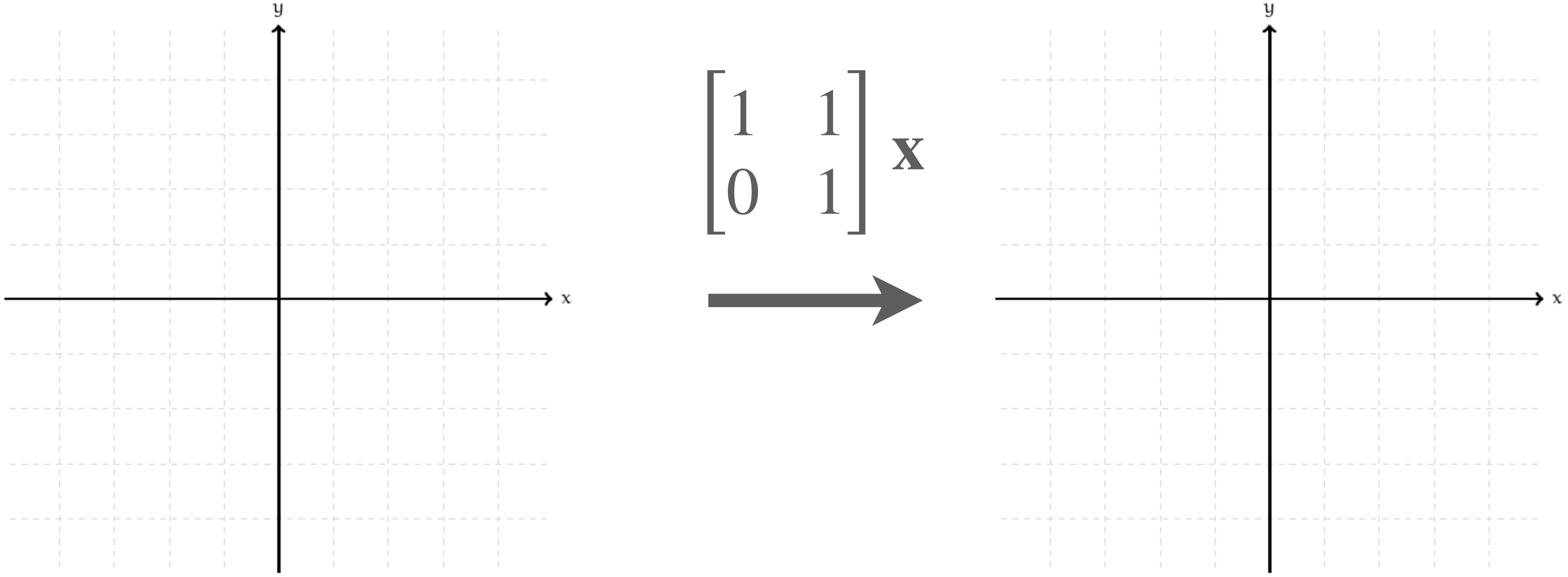
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}$$

if $0 \leq r \leq 1$, then the transformation pulls points towards the origin.

Example: Shearing

Example: Shearing

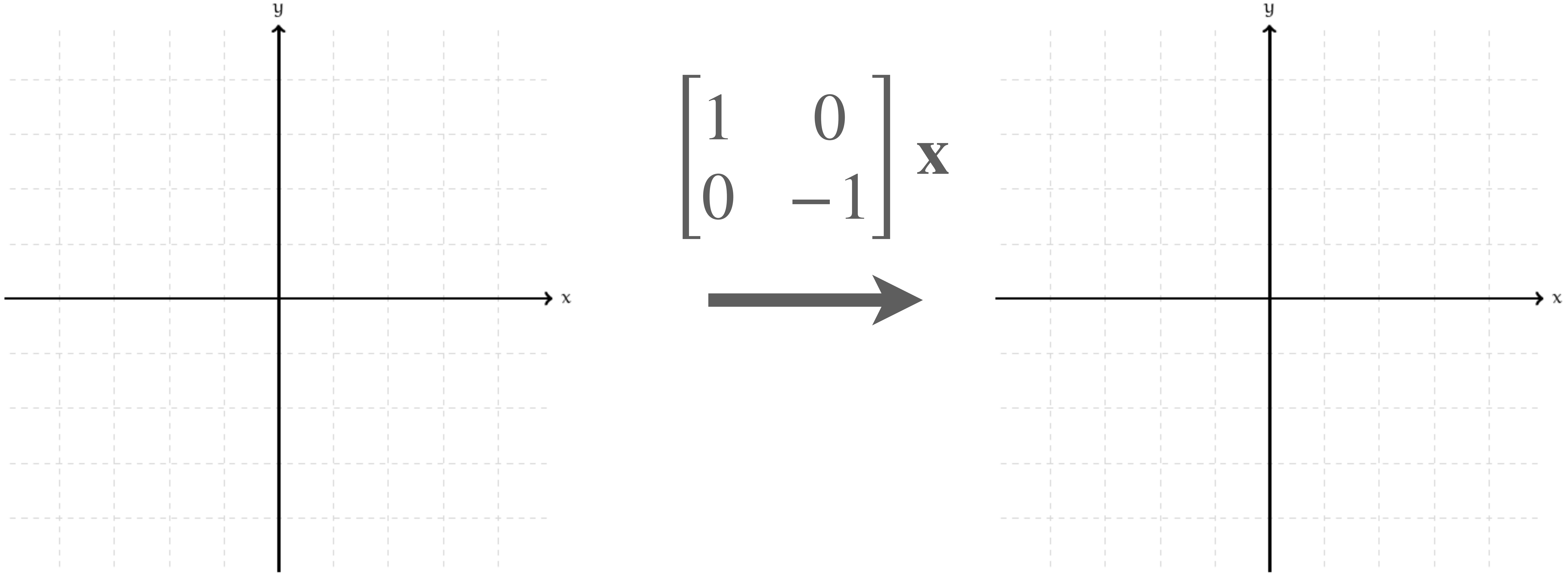
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \longrightarrow$$

Imagine shearing like with rocks or metal.

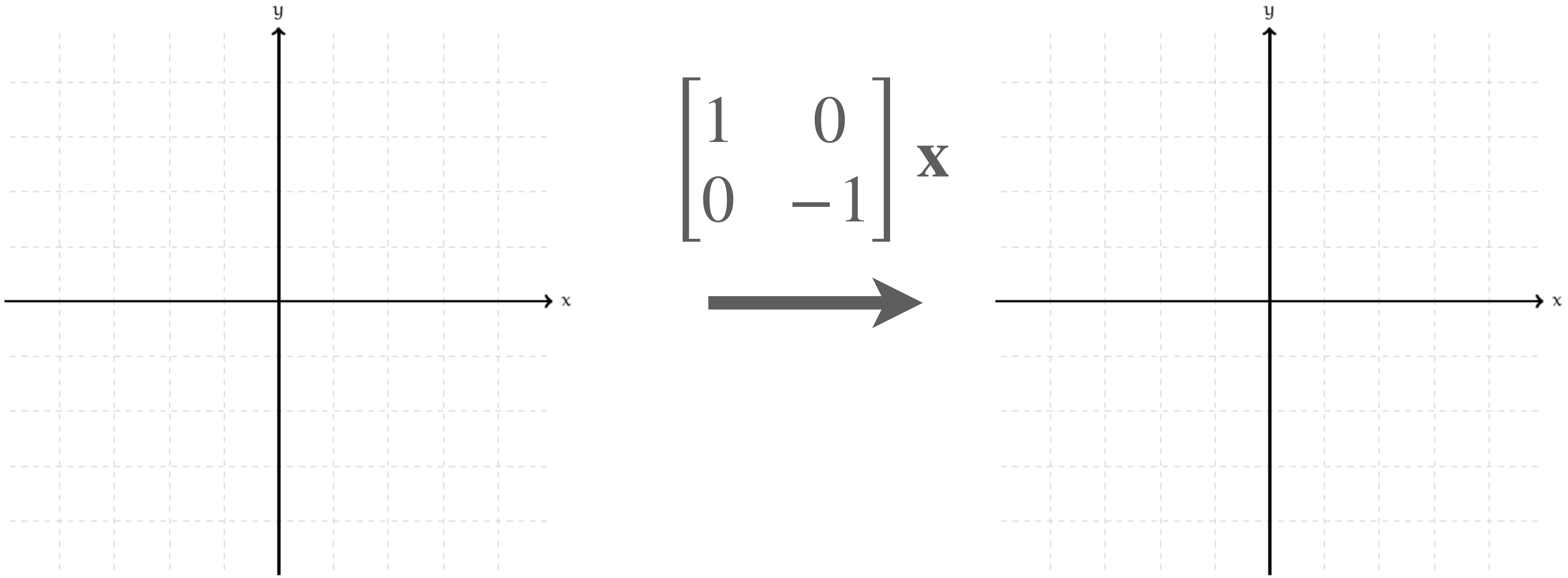
Question



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

Draw how this matrix transforms points. What kind of transformation does it represent?

Answer: Reflection



$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\mathbf{x}$$

demo

Summary

Matrices can be viewed as **linear transformations**

Matrix transformations change the **shape** of points sets

Linear transformations behave well with respect to **linear combinations**