Matrices of Linear Transformations

Geometric Algorithms
Lecture 8

CAS CS 132

Practice Problem

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 9 \qquad\qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2$$

Suppose that T is a linear transformation with the above input–output behavior.

What is the domain of T? What is the codomain of T?

What is the value of $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right)$?

Answer

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 9 \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2$$

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^1 = \mathbb{R}$

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$$

$$T(c\vec{x}) = cT(\vec{x})$$

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$$

$x_1 = 2$, $x_2 = -3$

$$T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) =$$

$$T\left(2\begin{bmatrix}1\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left((-3)\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - 3T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$2(9) - 3(2) = \boxed{12}$$

Objectives

- » Look at more examples of linear transformations
- » Show that matrix transformations and linear transformations are really the same thing
- » See more the geometry of linear transformations
- » Relate the properties of matrix equations to properties of linear transformations

Keywords

matrix of a linear transformation

standard basis vectors (standard coordinate vectors)

2D linear transformations

the unit square

one–to–one

onto

Recap

Recap: Matrices as Transformations

Matrices allow us to *transform* vectors

The transformed vector lies in the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector [illegible] to the vector $A\mathbf{v}$

$\vec{v}$

Recap: Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

$$T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Recap: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recap: Linear Transformations

Definition. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)

2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Recap: Linear Transformations

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Matrix transformations are linear transformations

Verification

any matrix transformation:

rotation about the origin:

translation *(non-example)*:

Recap: Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

We can generalize linearity to any linear combination

Our Next Motivating Question

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We know that matrix transformations are linear transformations

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We know that matrix transformations are linear transformations

Are there any other kinds of linear transformations?

Our Next Motivating Question

We know that matrix transformations are linear transformations

Are there any other kinds of linear transformations?

NO

Matrix of a Linear Transformation

Theorem. A transformation T is linear if and only if there is a matrix whose corresponding transformation is T (which "implements" T)

Matrix of a Linear Transformation

Theorem. A transformation T is linear if and only if there is a matrix whose corresponding transformation is T (which "implements" T)

Linear transformations are **exactly** matrix transformations

A Fundamental Concern

Given a linear transformation T, how do we find the matrix A such that $T(\mathbf{v}) = A\mathbf{v}$?

A Thought Experiment

Suppose I tell you T is a linear transformation and

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}5\\6\end{bmatrix}$$

Do we know what $T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)$ is?

Answer: Yes

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}$$

Because of additivity:

$$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) =$$

$$\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}8\\10\end{bmatrix}$$

A Thought Experiment

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}5\\6\end{bmatrix}$$

What about: $2\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$

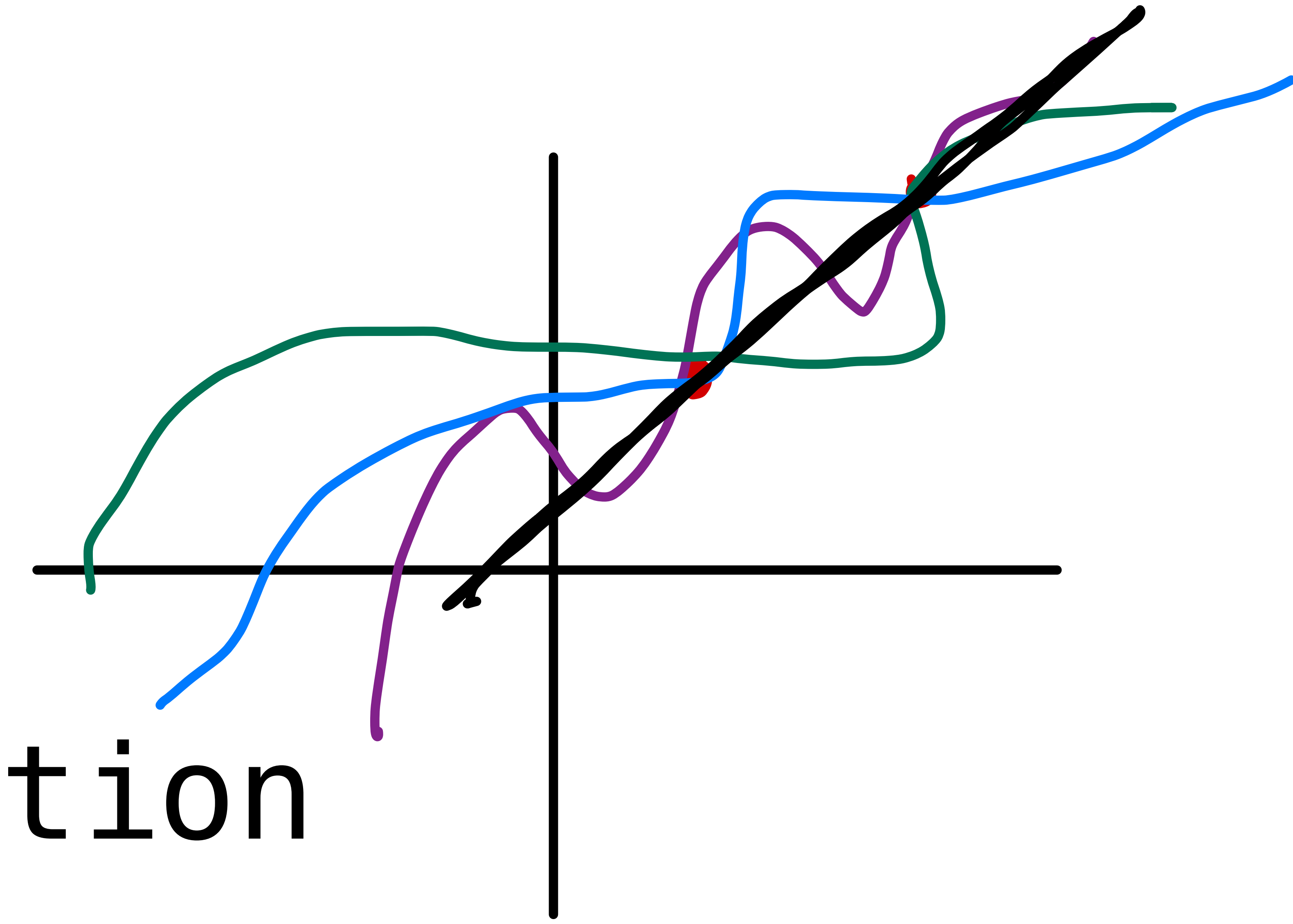
$$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=T\left(\frac{1}{2}\begin{bmatrix}4\\6\end{bmatrix}\right)=\frac{1}{2}T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}8\\10\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

$$\begin{bmatrix}1&3&1\\2&4&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2\\0&1&1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}3\\4\end{bmatrix}-2\begin{bmatrix}1\\2\end{bmatrix}\right)=T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)-2T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$$

$$=\begin{bmatrix}5\\6\end{bmatrix}-2\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

The Takeaway



Linearity is a **very** strong restriction

If we know the values of $T:\boxed{\mathbb{R}^n}\to\mathbb{R}^m$ on **any** set of vectors which spans all of $\boxed{\mathbb{R}^n}$, then we know T.

$\operatorname{span}\{\vec{u}_1,\ldots,\vec{u}_n\}=\mathbb{R}^n \qquad v\in\mathbb{R}^n$

why?

$\vec{v}=\sum_{i=1}^{n}\alpha_i\vec{u}_i \qquad T(\vec{v})=T\left(\sum_{i=1}^{n}\alpha_i\vec{u}_i\right)=\sum_{i=1}^{n}\alpha_i T(u_i)$

Another Thought Experiment (Game)

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Suppose I am holding a matrix A

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Your objective is to figure out what A is

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But you're only allowed to ask the question:

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What is $A\mathbf{v}$?

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What is $A\mathbf{v}$?

(you pick the $\mathbf{v}$'s, and I have to tell the truth)

Another Thought Experiment (Game)

Suppose I am holding a matrix A

Your objective is to figure out what A is

But you're only allowed to ask the question:

What is $A\mathbf{v}$?

(you pick the $\mathbf{v}$'s, and I have to tell the truth)

This is basically linear algebraic battleship

Recall: Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$b_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = \sum_{i=1}^{n} a_{1i}v_i$$

Recall: Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by $\mathbf{v}$

Isolating a_{11}

$$b_1 = \boxed{a_{11}}\overset{1}{v_1} + a_{12}\overset{0}{v_2} + \ldots + a_{1n}\overset{0}{v_n} = \sum_{i=1}^{n} a_{1i}v_i$$

$$b_2 = a_{21}\overset{1}{v_1} + a_{22}\overset{0}{v_2} \quad a_{2n}\overset{0}{v_n}$$

$$A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ \vdots \end{bmatrix}$$

a_{21}

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(1)+3(0) \\ 2(1)+4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & , & \ldots & \vec{a}_n \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1\vec{a}_1 + 0\vec{a}_2 + \ldots 0\vec{a}_n = \vec{a}_1$$

Isolating a_{11}

$$b_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = \sum_{i=1}^{n} a_{1i}v_i$$

We actually get the whole column $\mathbf{a}_1$

So its like battleship, but you get to choose one column at a time.

The Takeaway

We can learn the first column of the matrix implementing T by looking at $T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$

Matrix of a Linear Transformation

Standard Basis

Definition. The n***–dimensional standard basis vectors*** (or standard coordinate vectors) are the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ where

$\mathbb{R}_3 \ni \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\mathbf{e}_i = \begin{bmatrix}0\\0\\\vdots\\0\\1\\0\\\vdots\\0\\0\end{bmatrix}\begin{array}{c}1\\2\\\vdots\\i-1\\i\\i+1\\\vdots\\n-1\\n\end{array}$$

Standard Basis

Definition (Alternative). The n–dimensional standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix

$$I = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]$$

$$\begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \in \mathbb{R}^{3 \times 3}$$

Standard Basis and the Matrix Equation

The key points: $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \cdots \;\; \mathbf{a}_n]\,\mathbf{e}_i = \mathbf{a}_i$

The standard basis vectors gives us a way to "look into" a matrix

Standard Basis and Vector Coordinates

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Column vectors can be viewed as describing how to write a vector as a linear combination of the standard basis

Standard Basis and Linear Transformations

Theorem. For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$$

is the <u>unique</u> matrix such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v}$ in $\mathbb{R}^n$

More Formally

$$v = \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$T(\mathbf{v}) =$$

$$T\left(\alpha_1 \vec{e}_1 + \cdots + \alpha_n \vec{e}_n\right) =$$

$$\alpha_1 T(\vec{e}_1) + \cdots + \alpha_n T(\vec{e}_n) =$$

$$\begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix} \mathbf{v}$$

How To: Matrices of Linear Transformations

Question. Find the matrix which implements the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

Solution. Determine the images of standard basis under T. Then write down

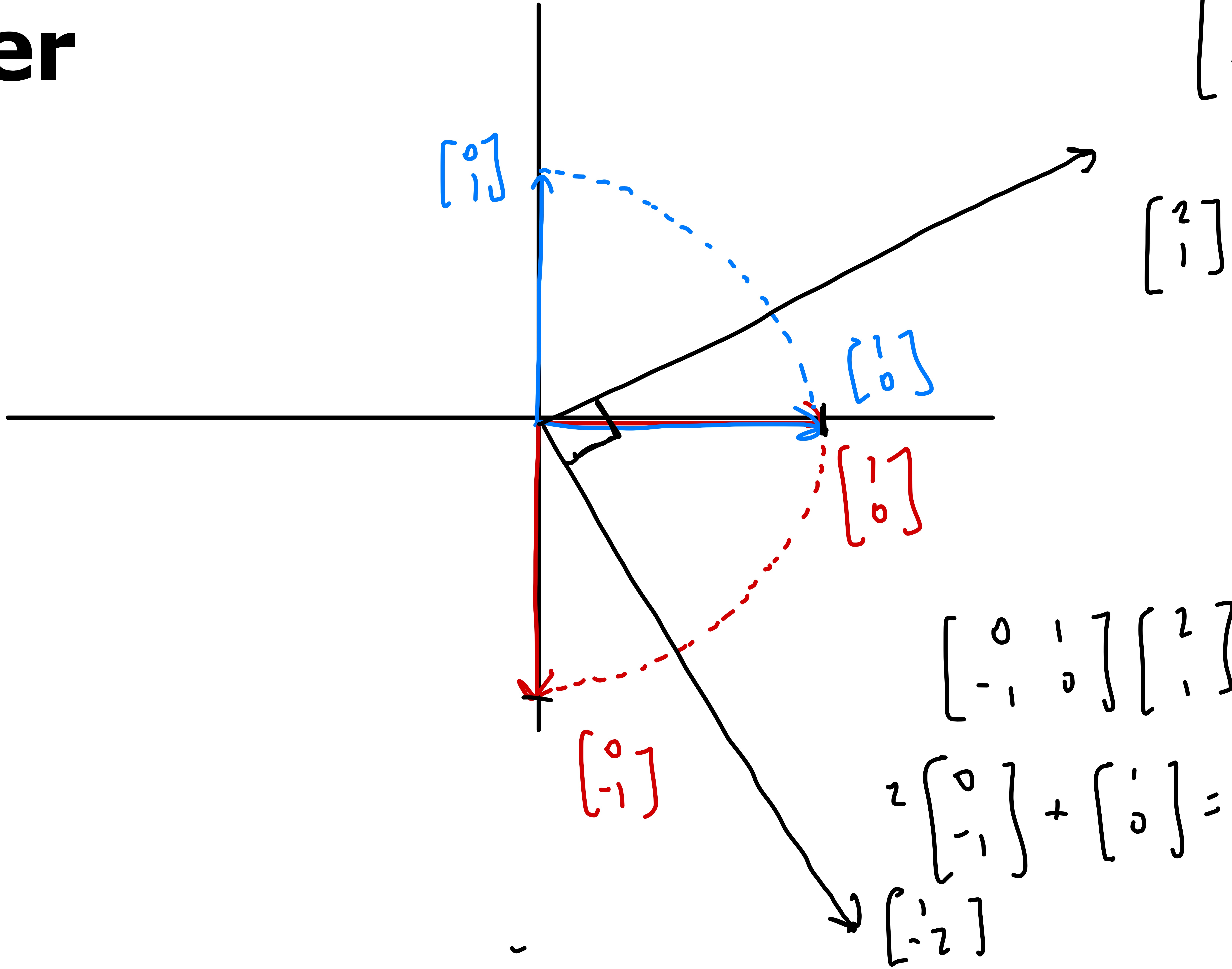
$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

Question

*Write done the matrix which implements the linear transformation T which **rotates** vectors by 90 degrees clockwise*

Answer

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

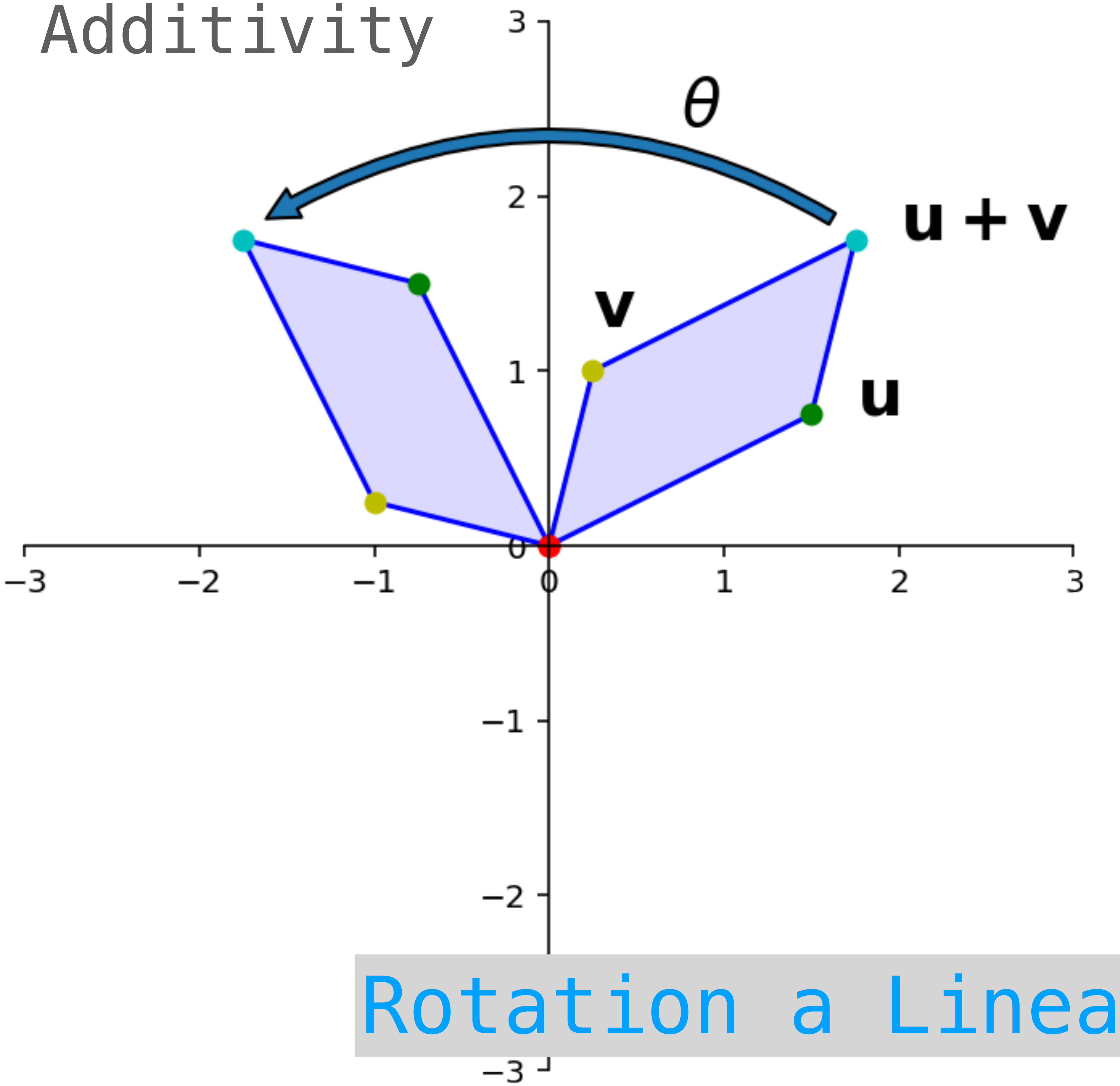


$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=$$

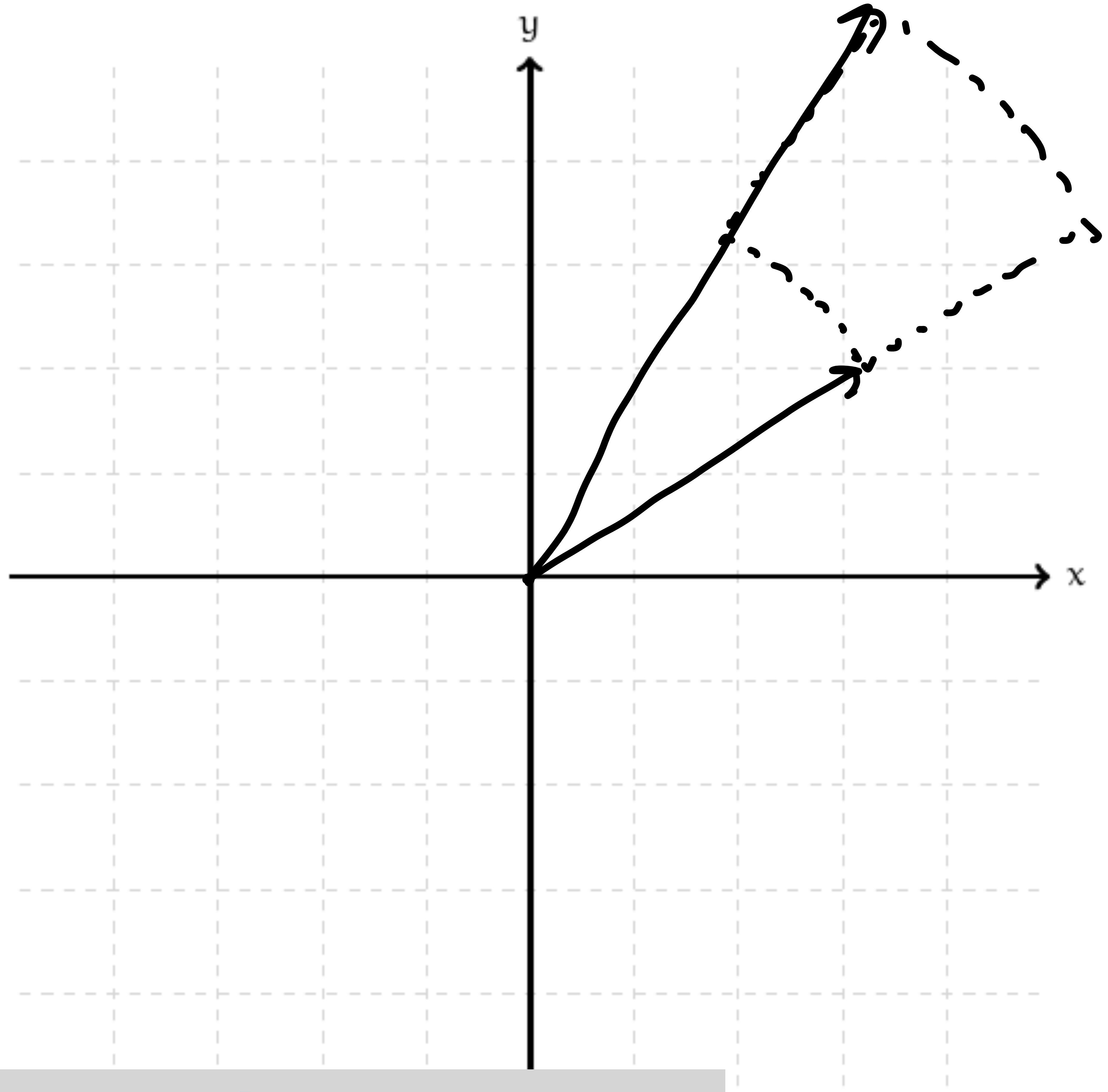
$$2\begin{bmatrix}0\\-1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$$

General Rotation

Additivity



Homogeneity



Rotation a Linear Transformation

Geometry of Matrix Transformations

Motivating Questions

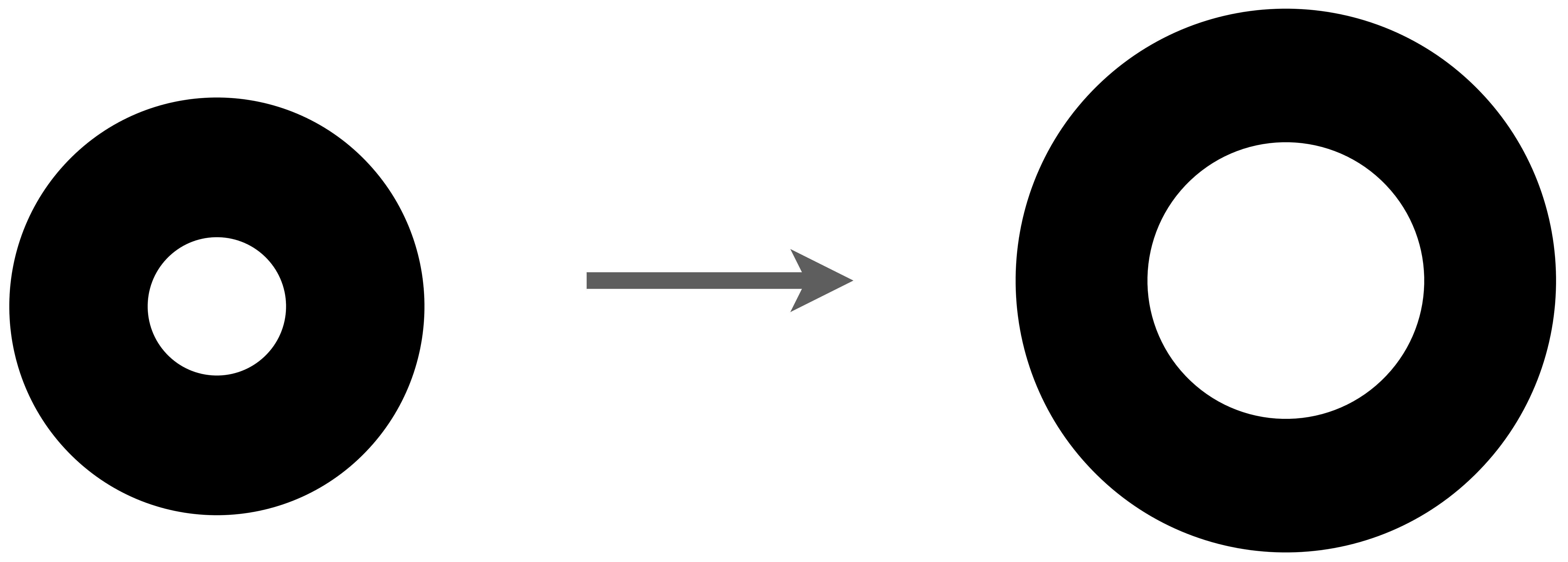
What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

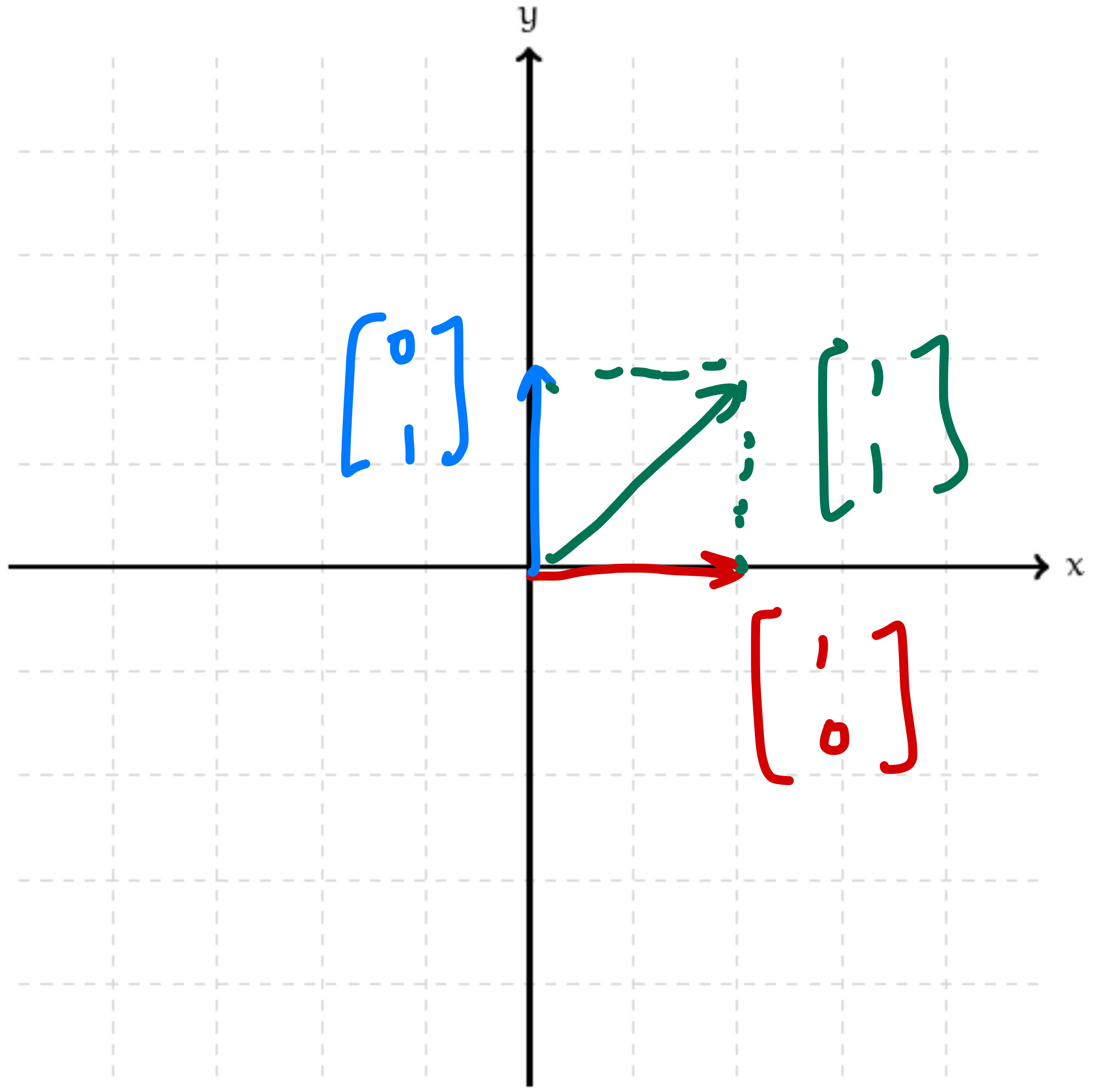
Matrix transformations change the "shape" of a set of set of vectors (points).

Example: Dilation

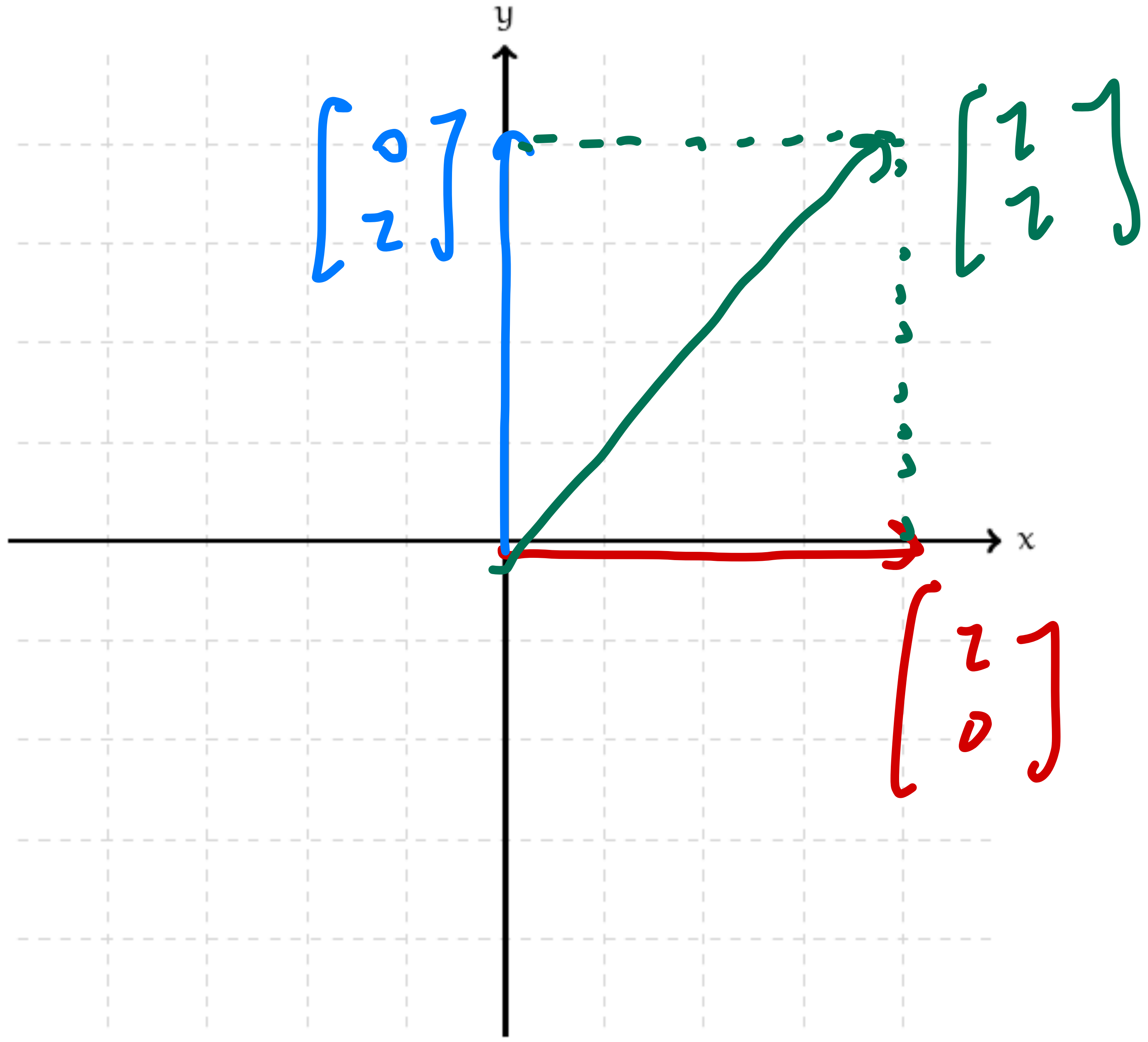


Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$

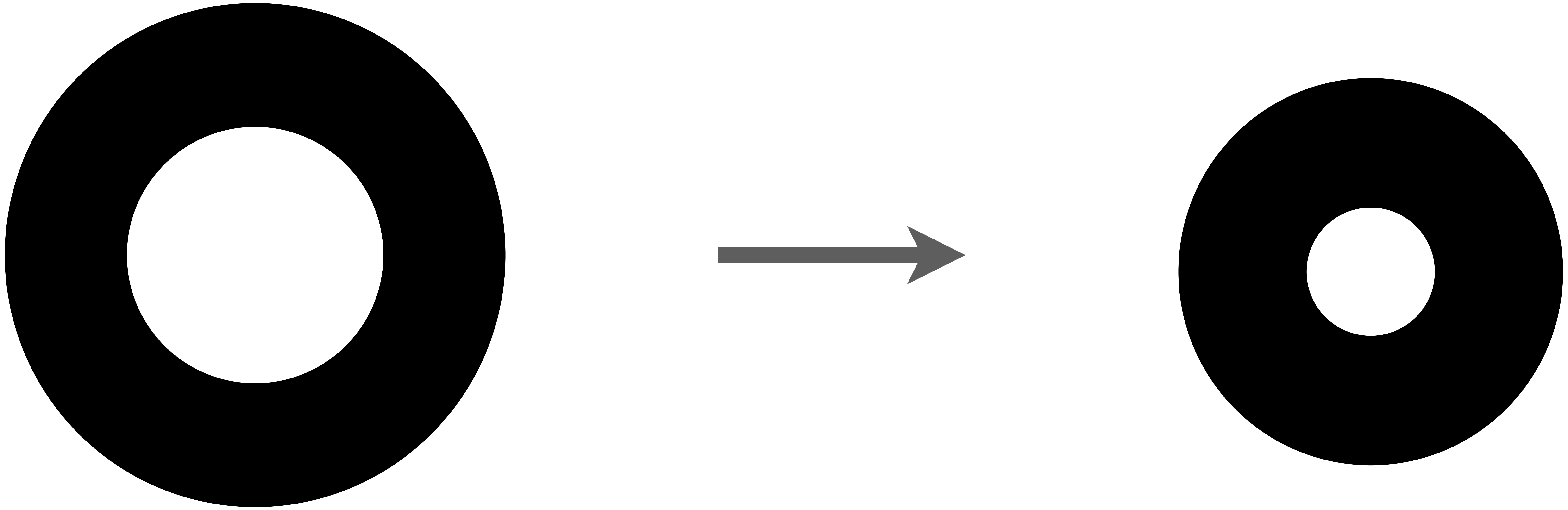


$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$



if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



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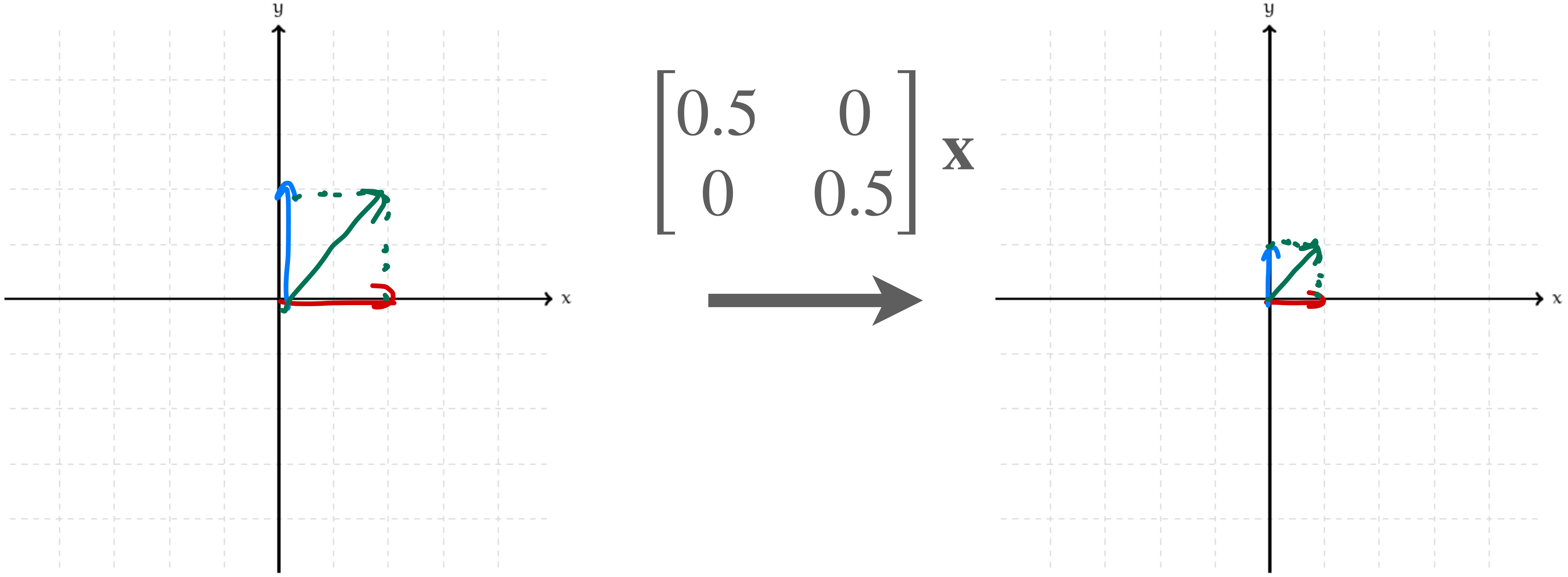


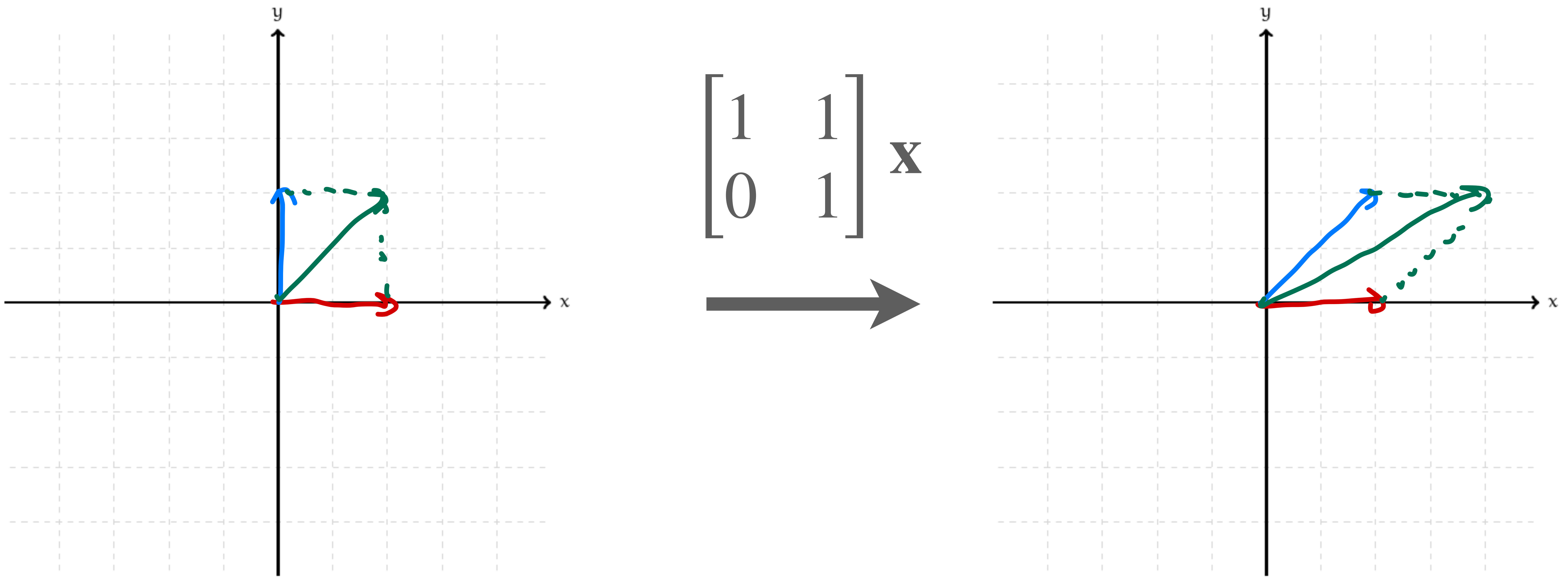
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}$$

if $0 \le r \le 1$, then the transformation pulls points towards the origin.

Example: Shearing

Example: Shearing

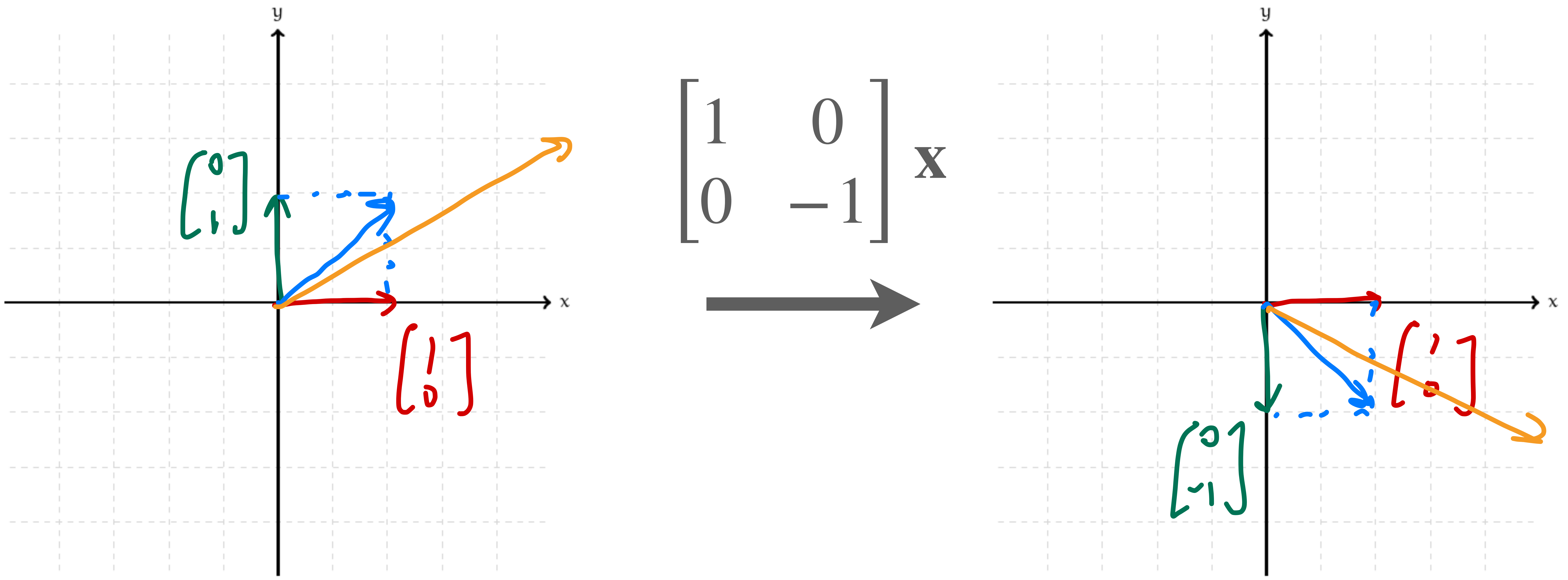
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



Imagine shearing like with rocks or metal.

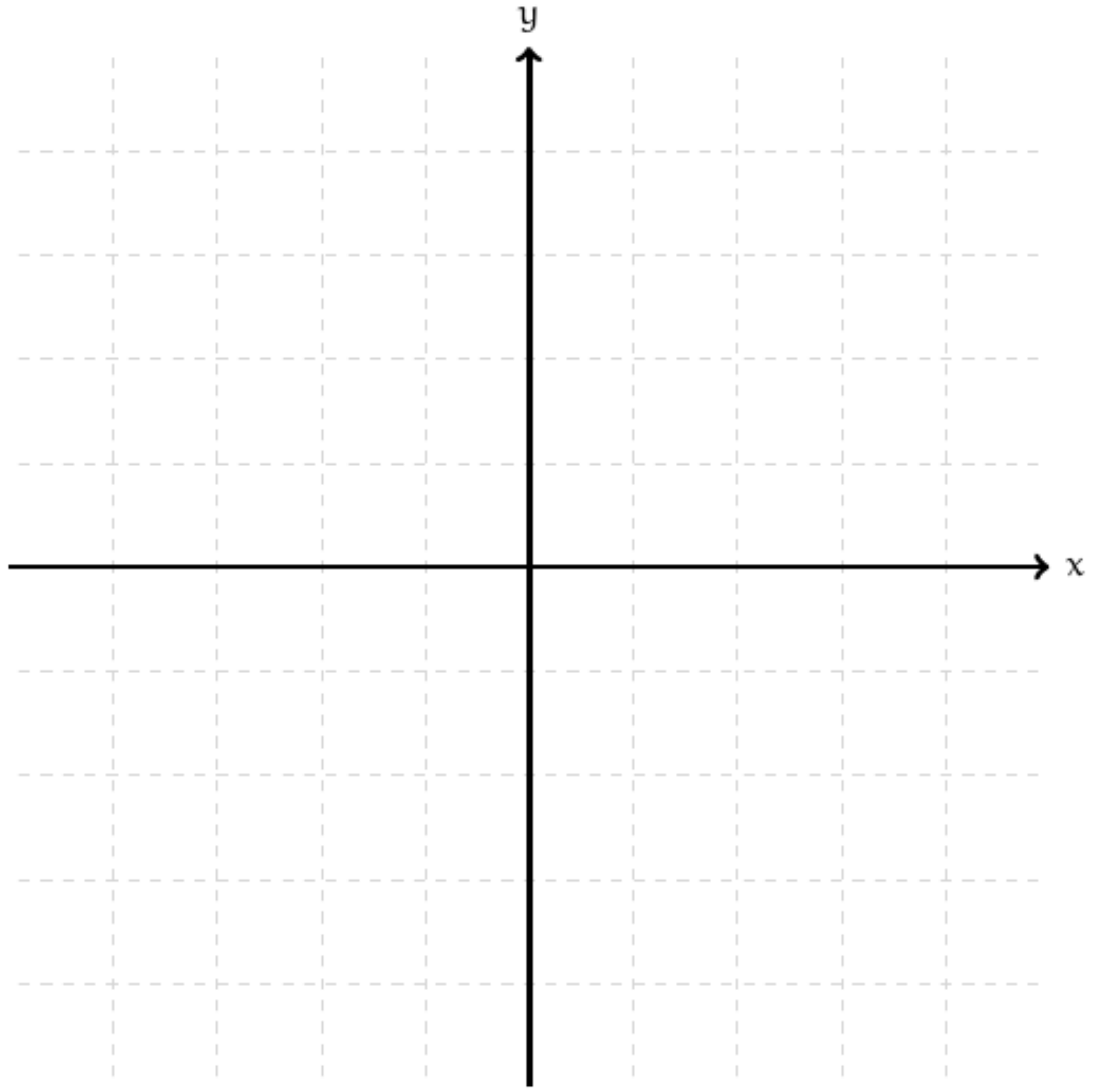
Question

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ -y\end{bmatrix}$$

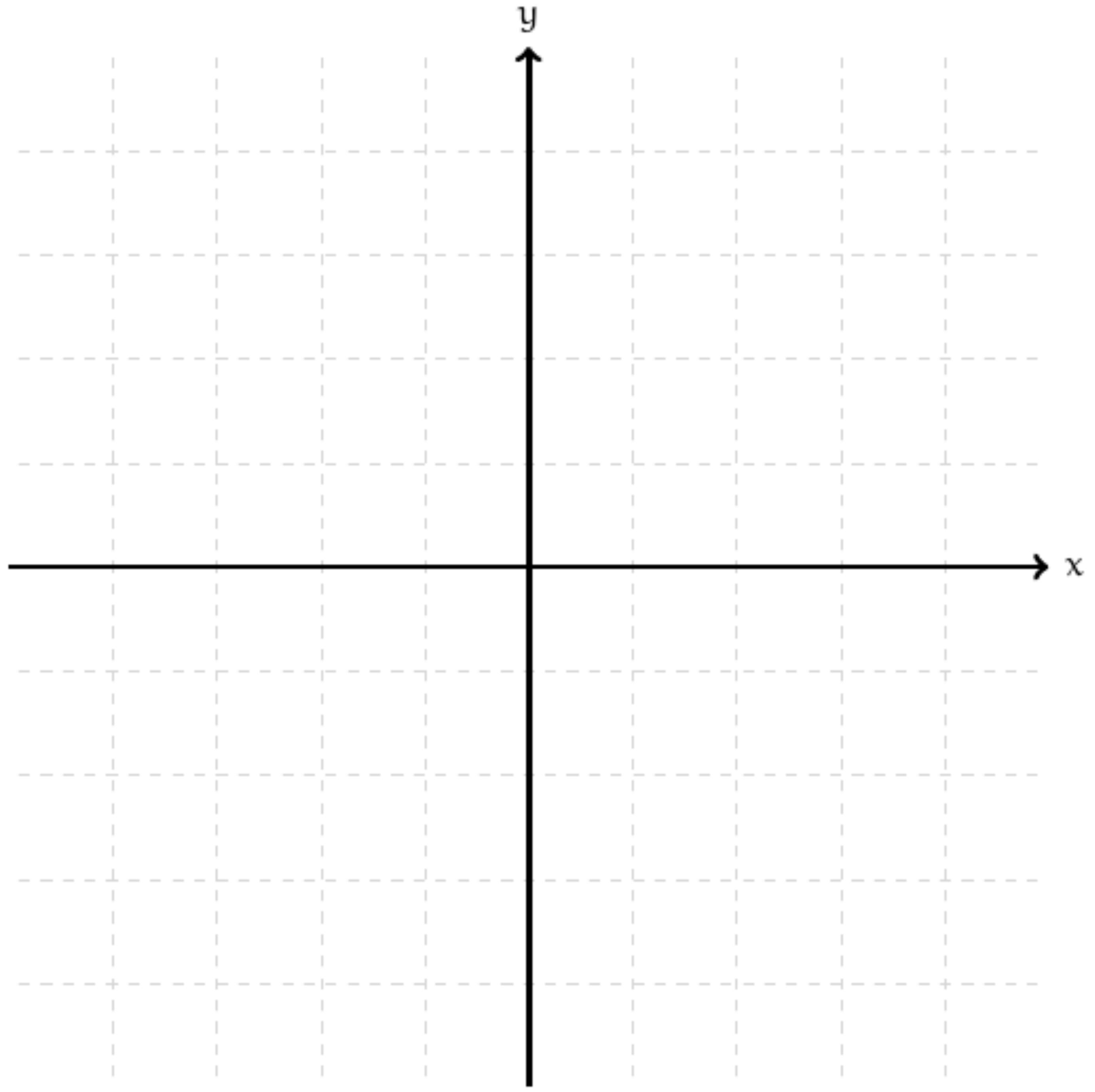


Draw how this matrix transforms points. What kind of transformation does it represent?

Answer: Reflection



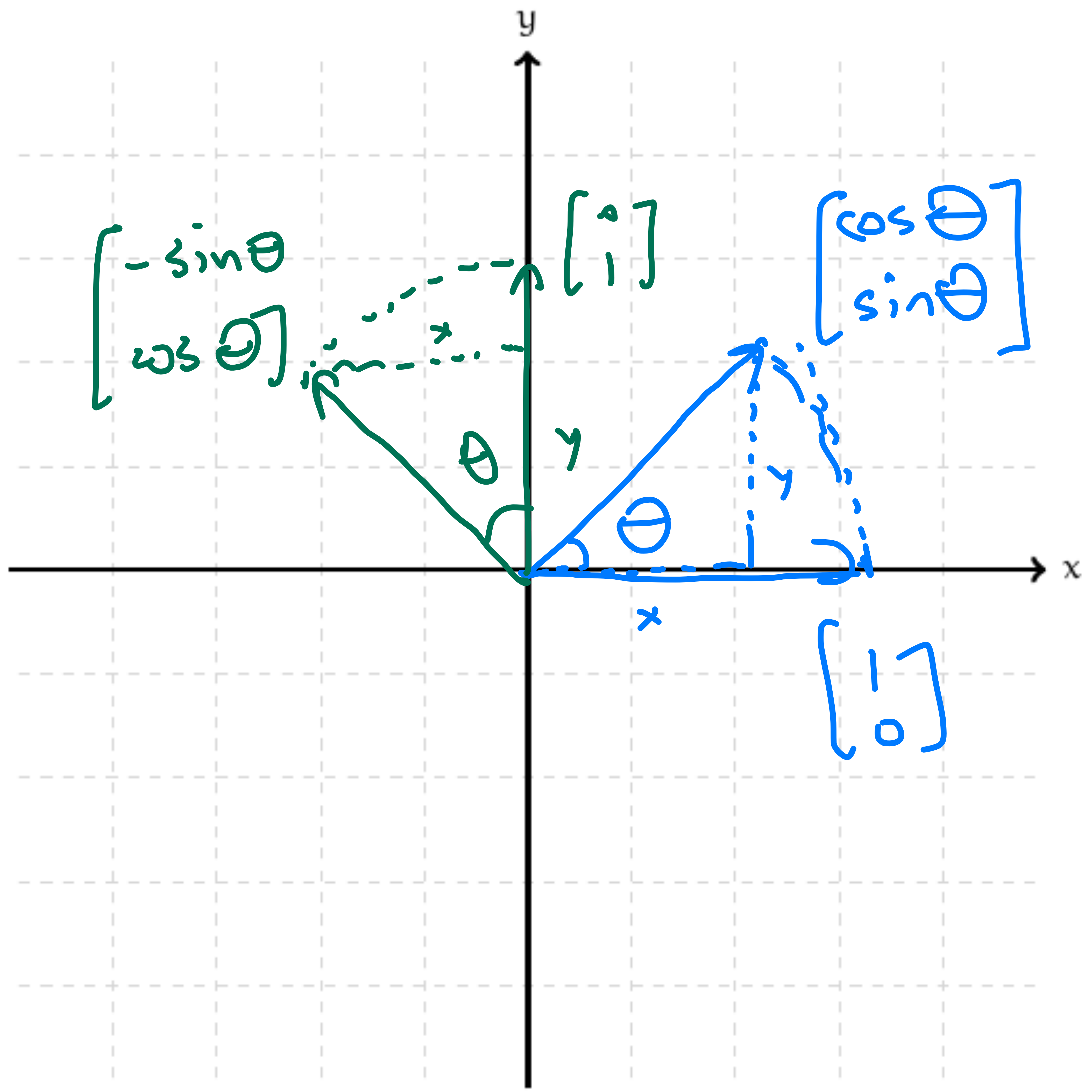
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



General Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

How does rotation affect the standard basis?



Rotation Matrix

Rotation Matrix

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Note: This is rotation about the origin

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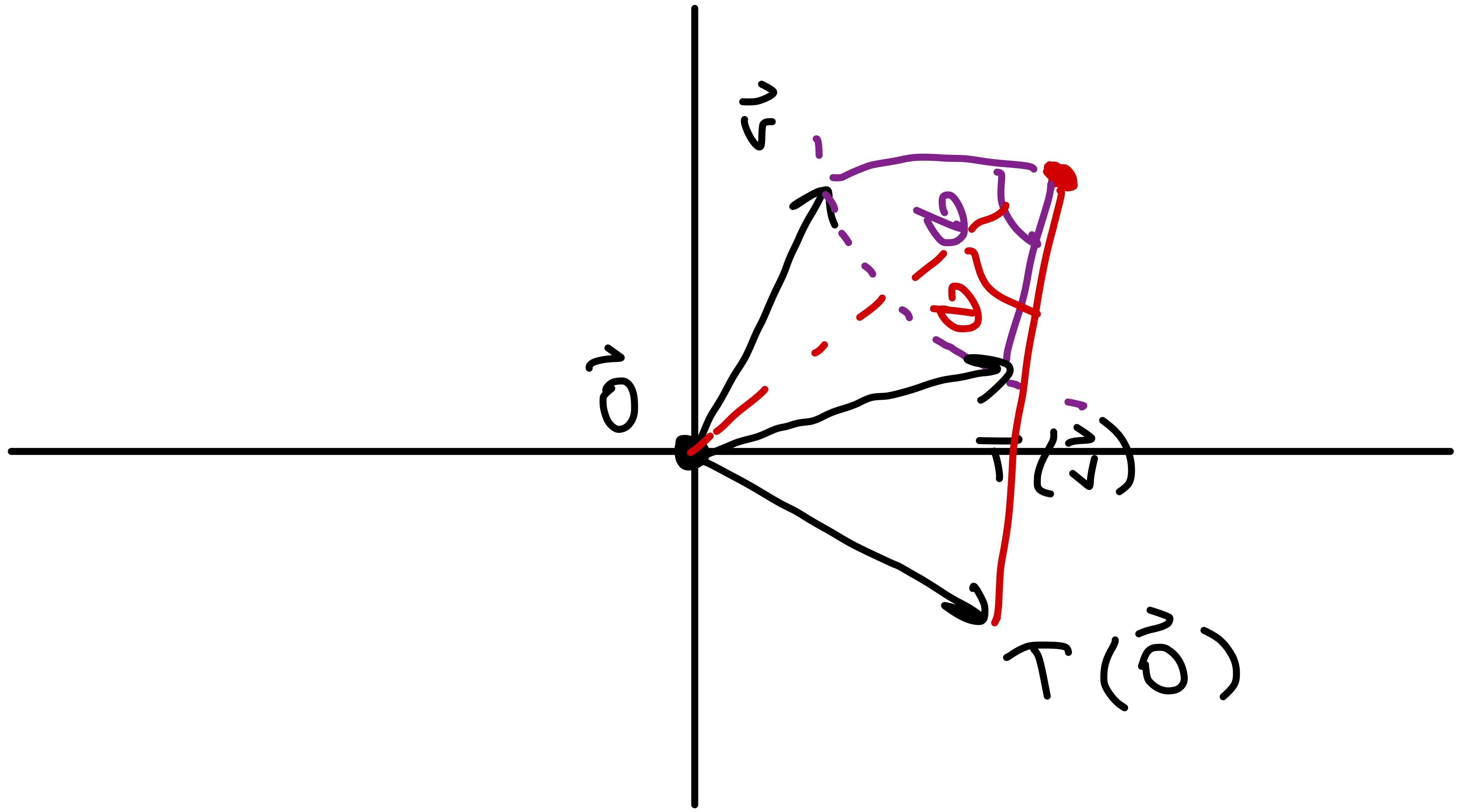
Note: This is rotation about the origin

The Takeaway: We can figure out the matrices which implement complex linear transformations by understanding what they do to the standard basis

Question (Conceptual)

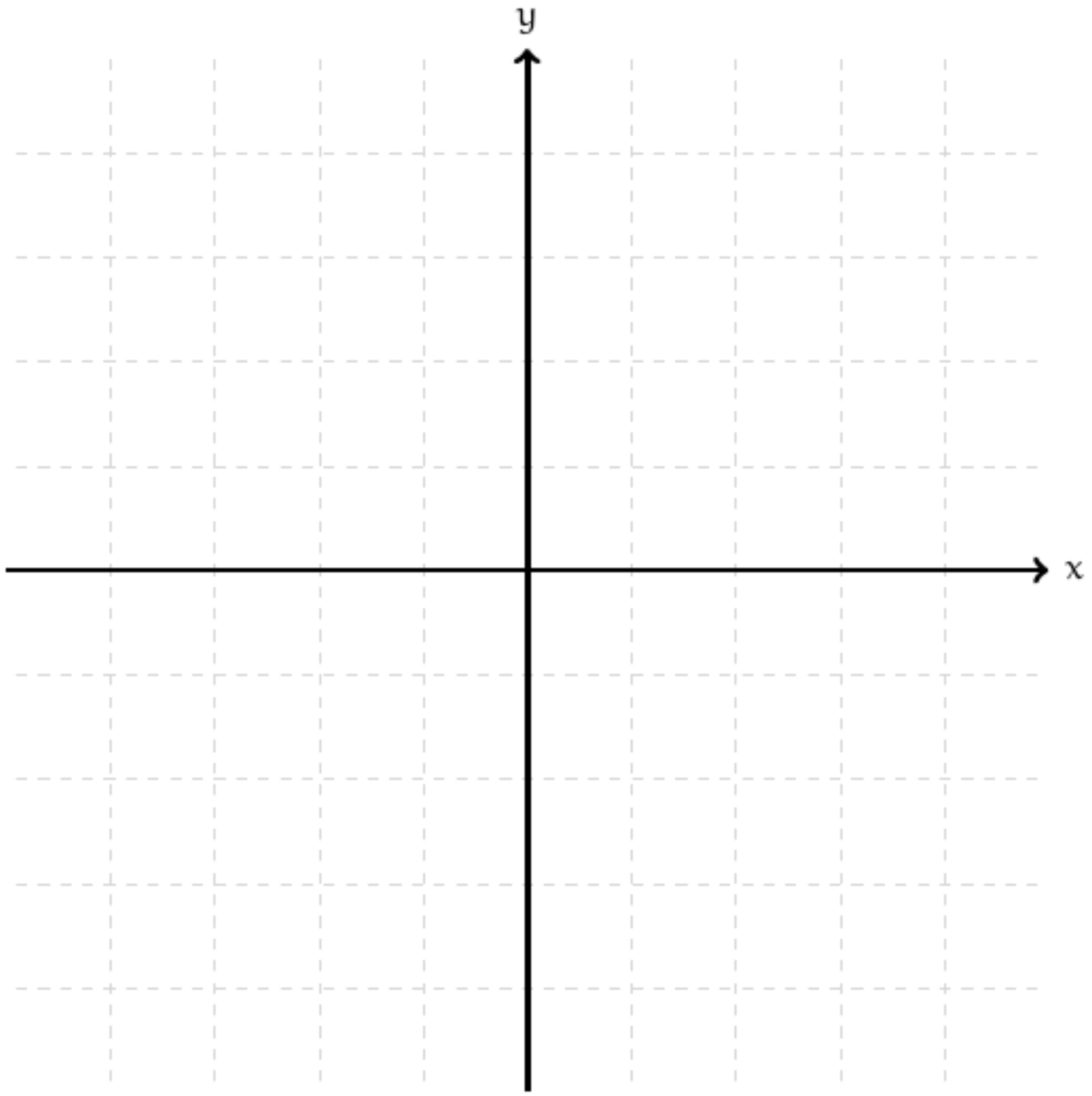
Is rotation about a point other than the origin a linear transformation?

$T(\vec{0}) = \vec{0}$



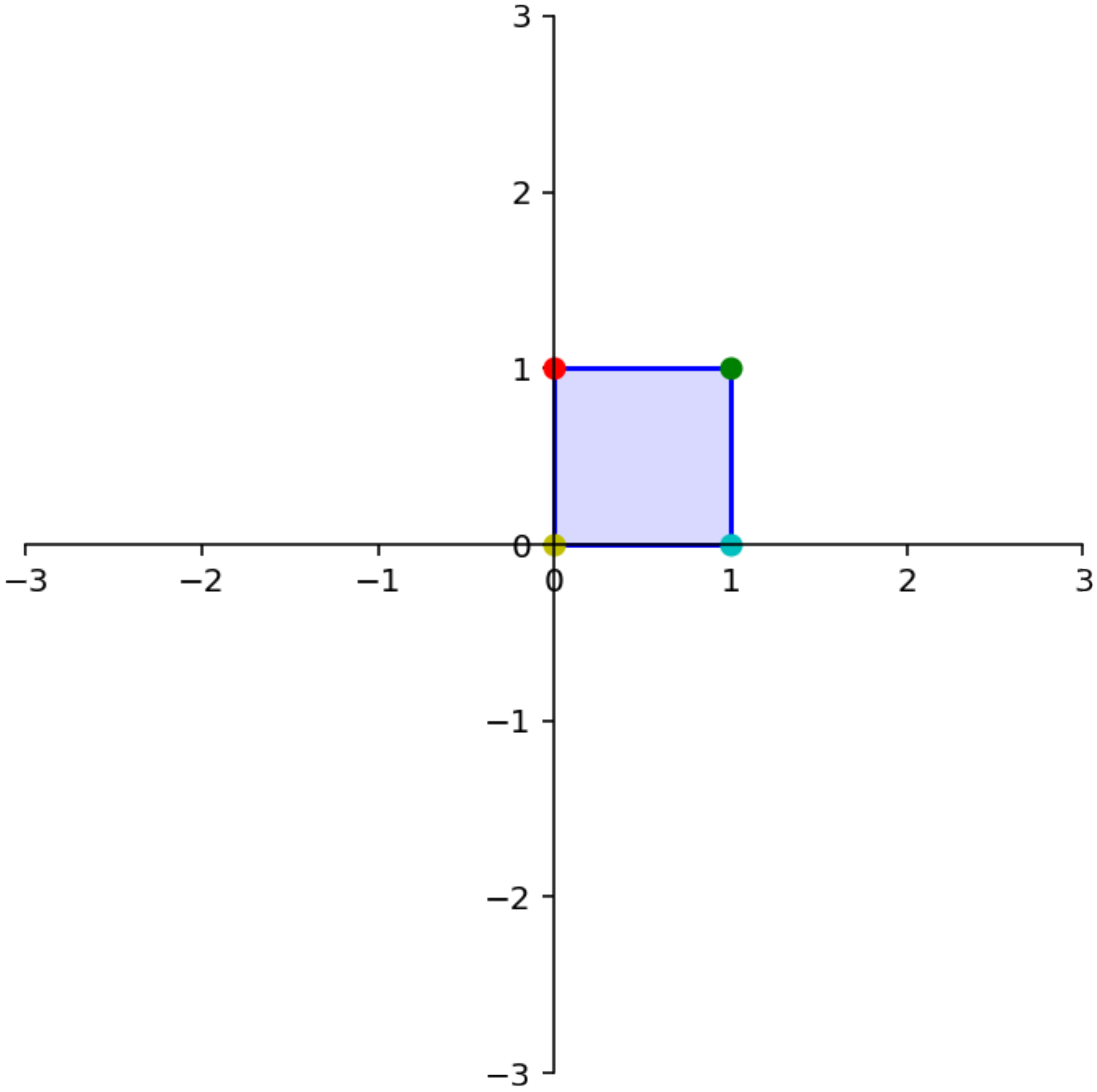
Answer: No

The origin is not fixed by this transformation



The Unit Square

The ***unit square*** is the set of points in $\mathbb{R}^2$ enclosed by the points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.



How To: The Unit Square and Matrices

How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

How To: The Unit Square and Matrices

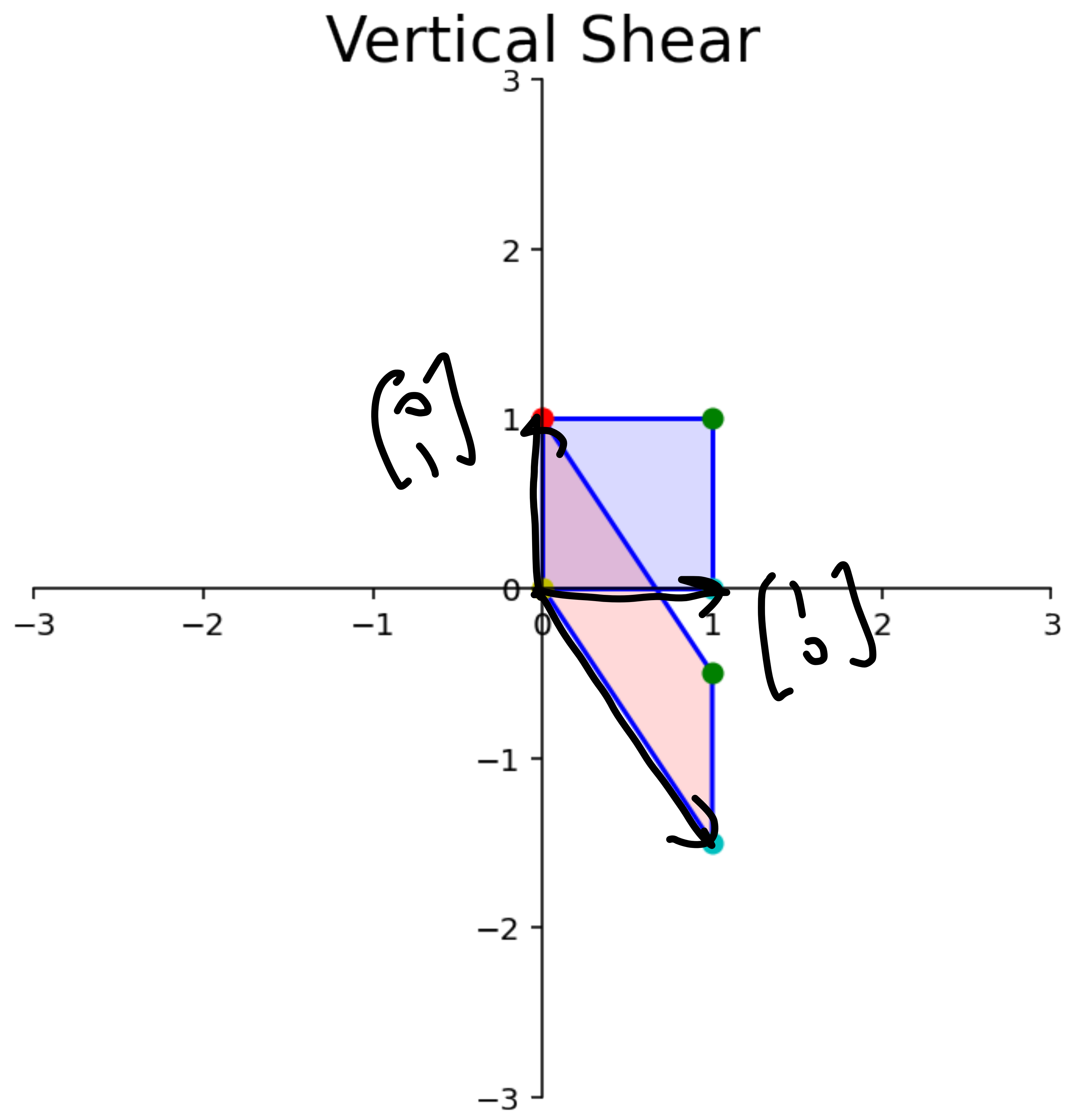
Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

Solution. Find where the standard basis vectors go

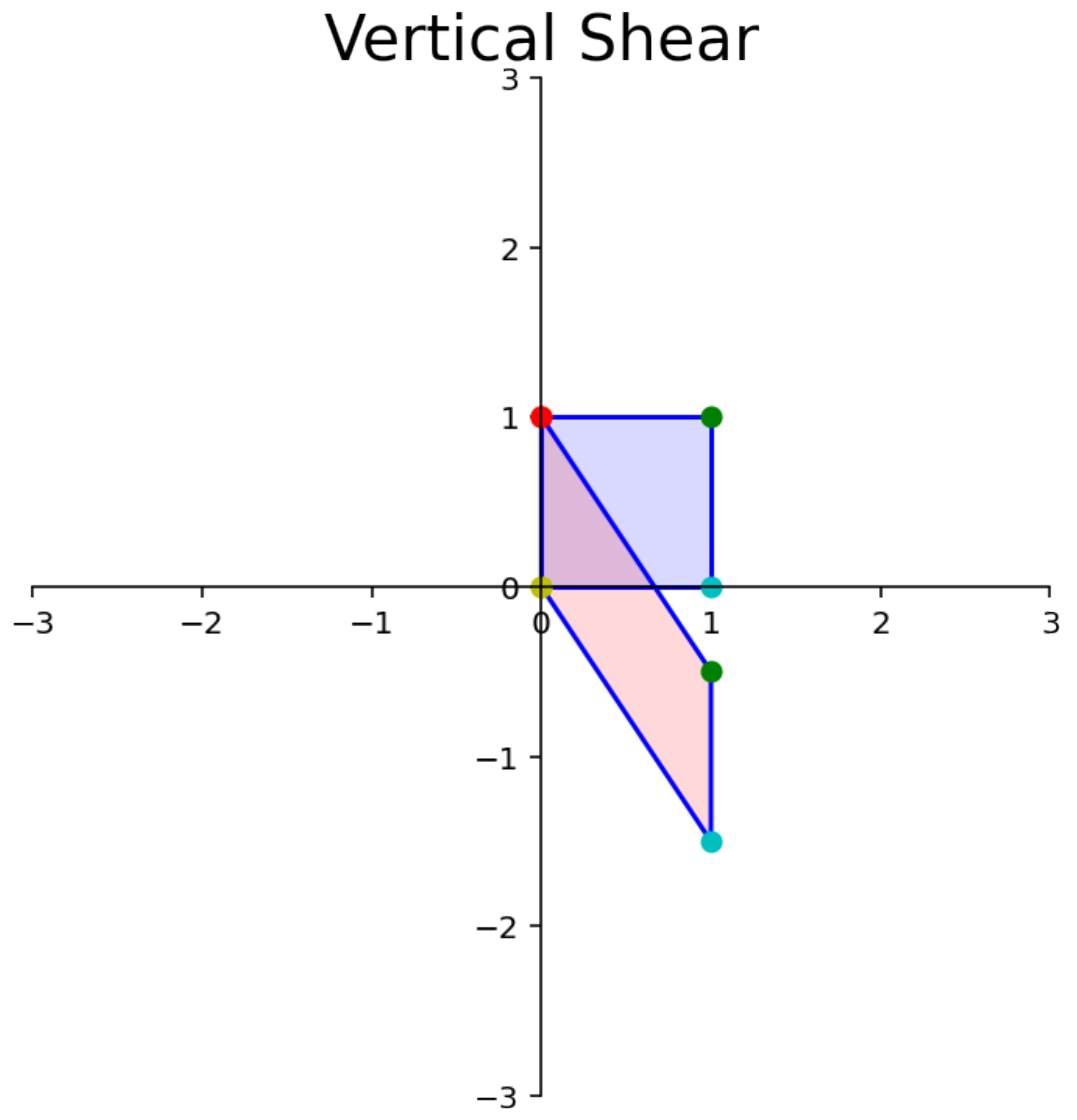
Question

Write down the matrix for the following shearing operation using this method

$$\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}$$



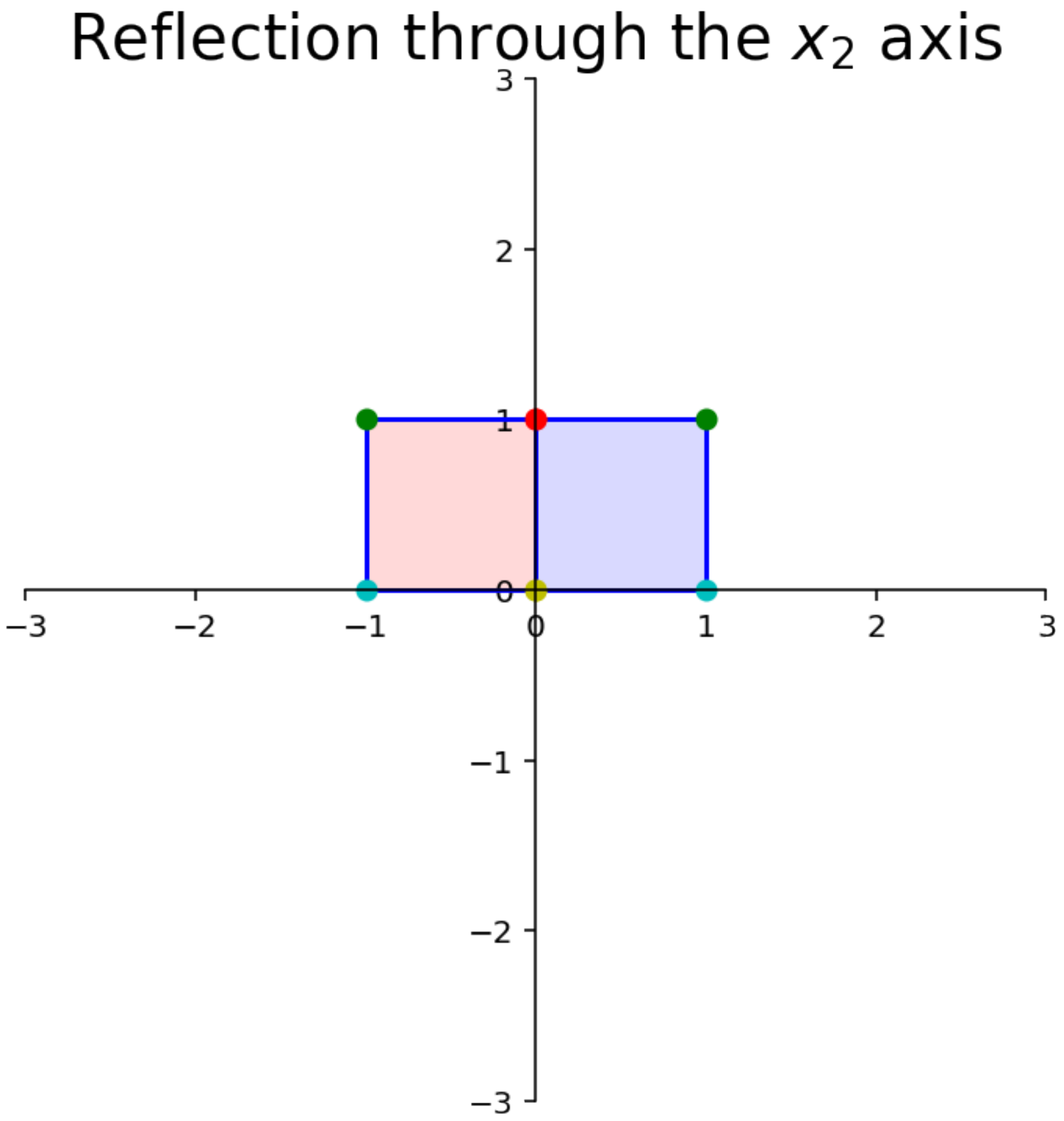
Answer



You need to know these matrices, but you don't need to memorize them

Remember: What does this matrix do to the unit square? Then build the matrix from there

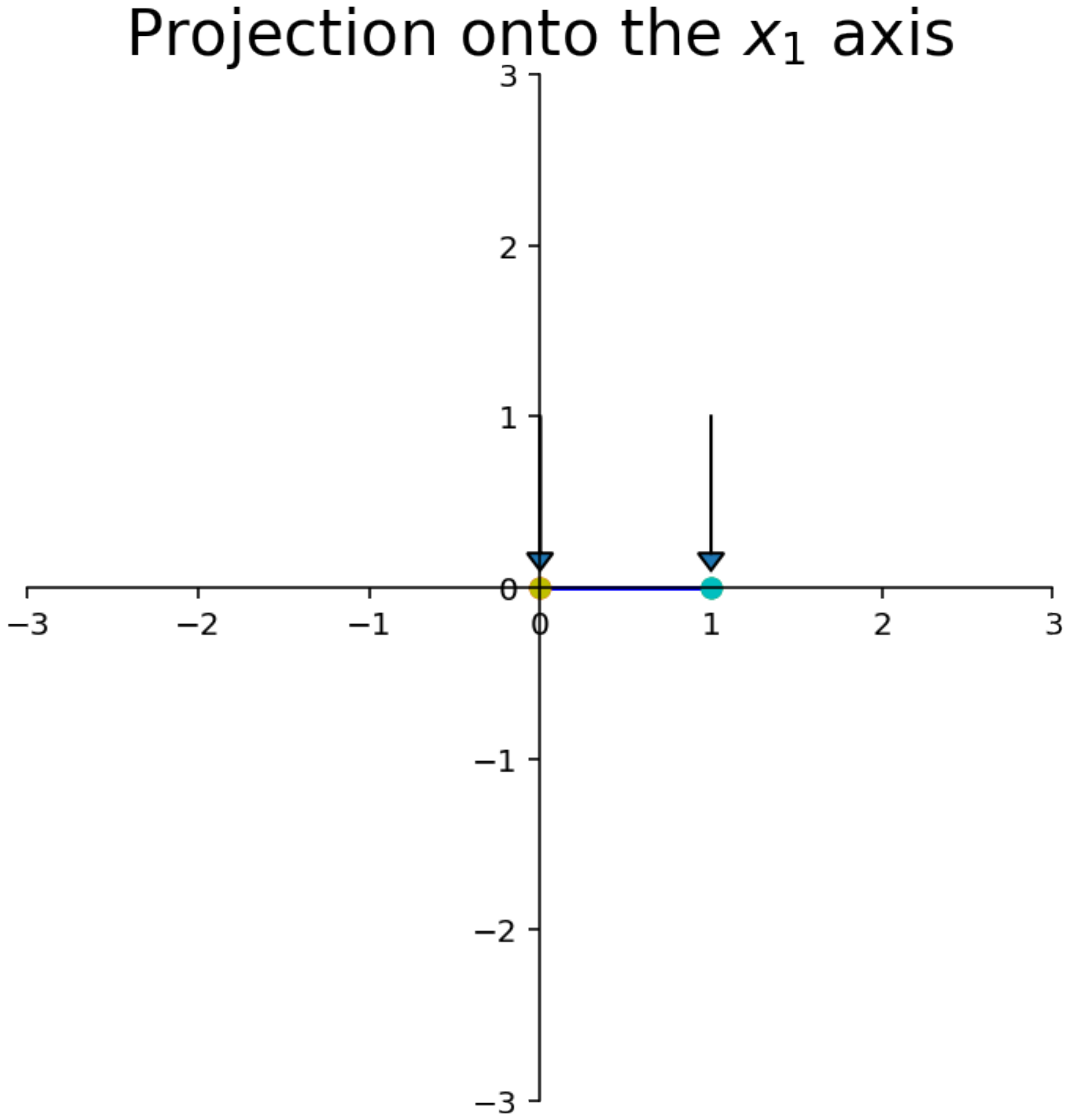
Reflection through the x_2-axis



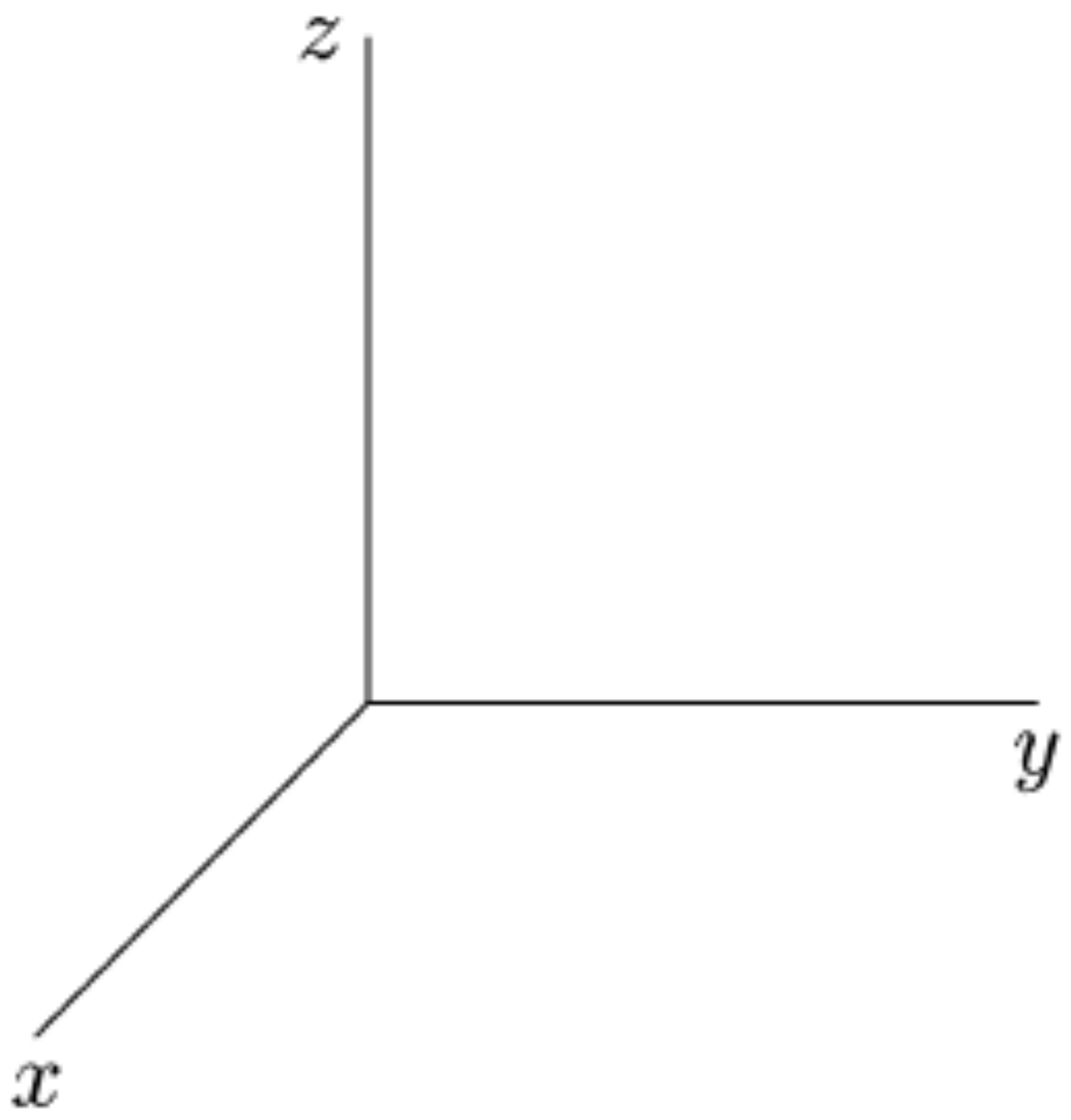
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Projections

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$



A 3D Example: Rotation about the x_3-Axis (z-Axis)



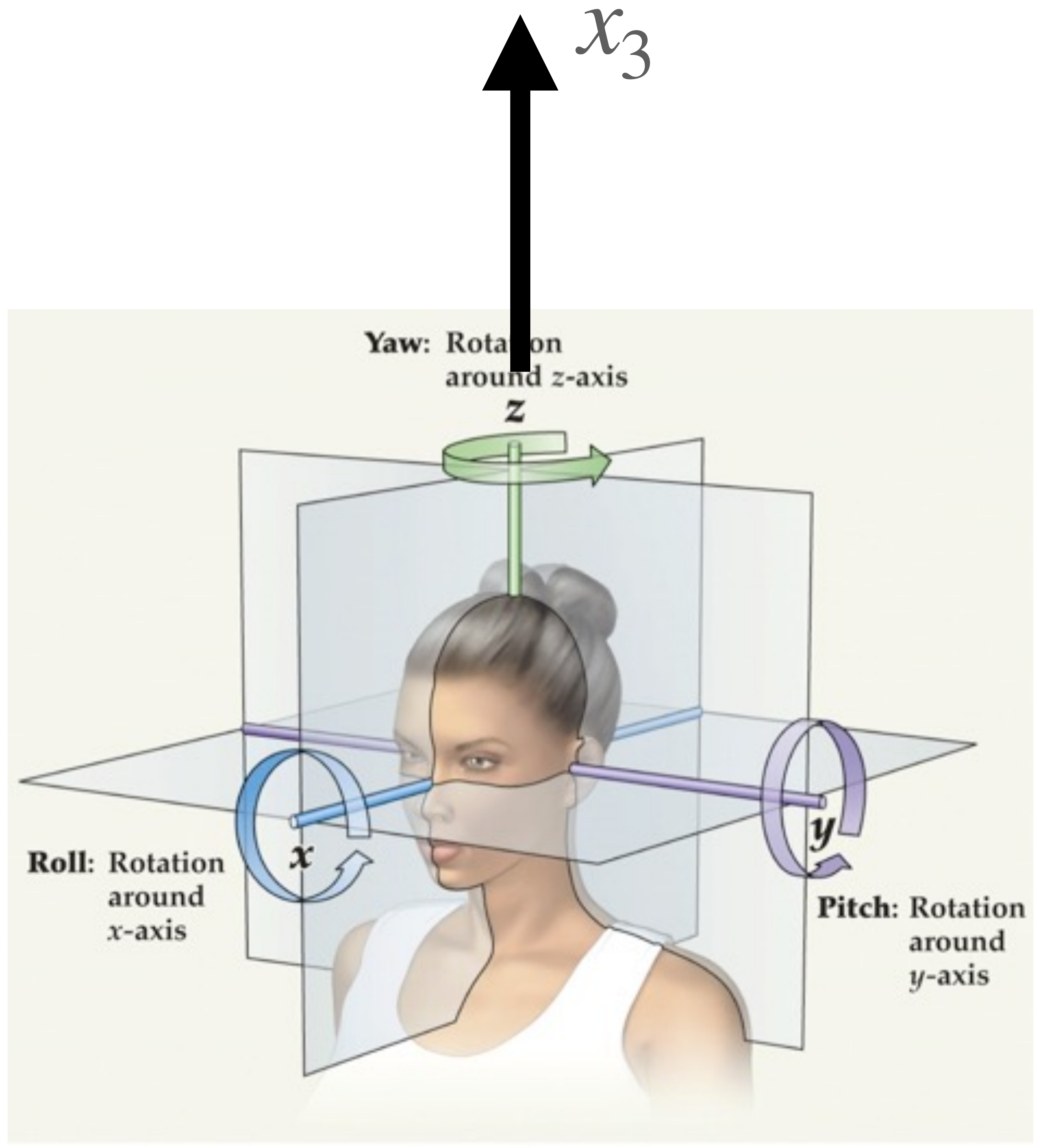


image source

List of Important 2D Linear Transformations

» dilation, contraction
» reflections
» projections
» horizontal/vertical contractions
» horizontal/vertical shearing

Look through the notes for a comprehensive collection of pictures or...

demo

One-to-One and Onto

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? $\equiv$ is there a vector which A transforms into $\mathbf{b}$?

Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

Recall: A New Interpretation of the Matrix Equation

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Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

What about other questions?

Other Questions Like…

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of $\mathbf{b}$?

Wait

$A\mathbf{x} = \mathbf{0}$ has a unique solution $\equiv$ $A\mathbf{x} = \mathbf{b}$ has at most one solution

why?:

Onto and One-to-One

Onto and One-to-One

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$)

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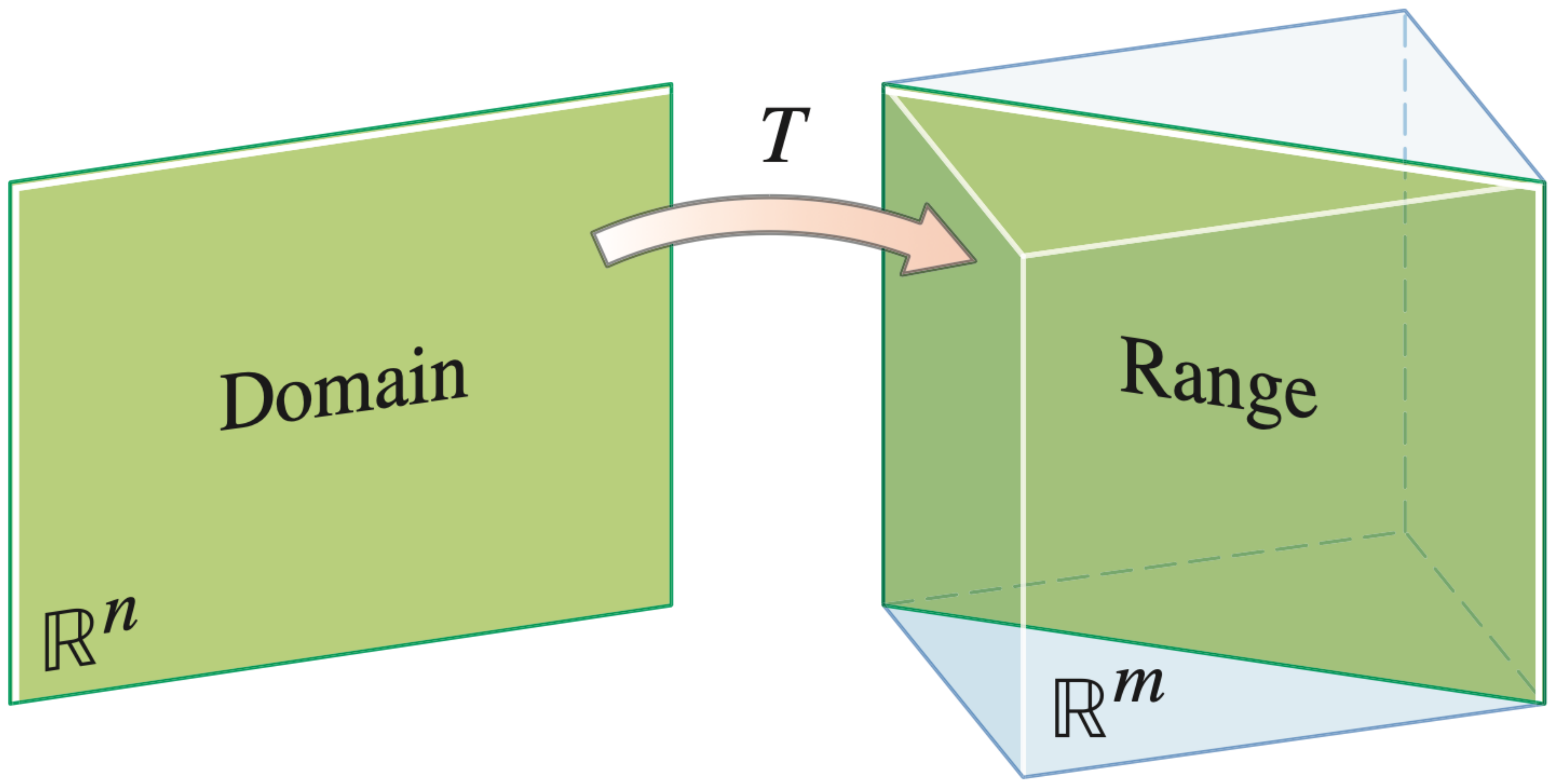
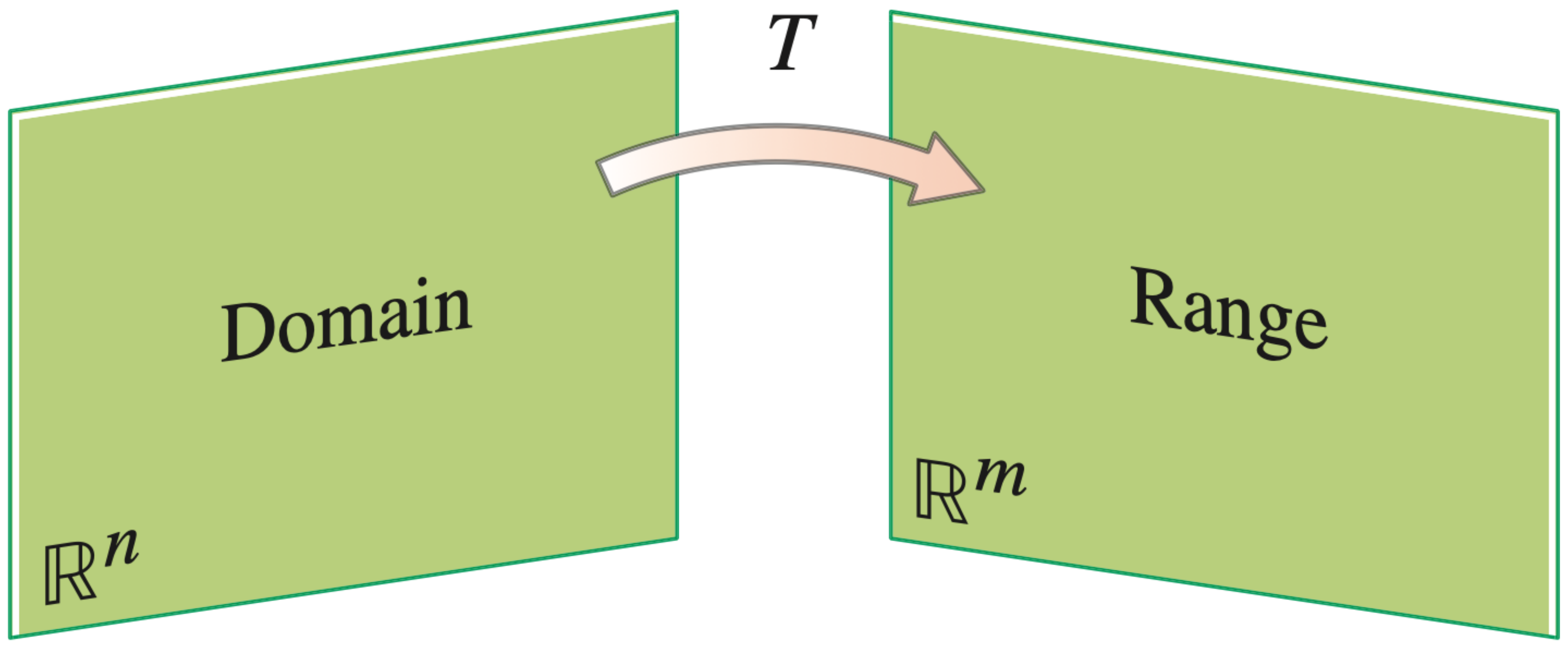
Onto (Pictorially)



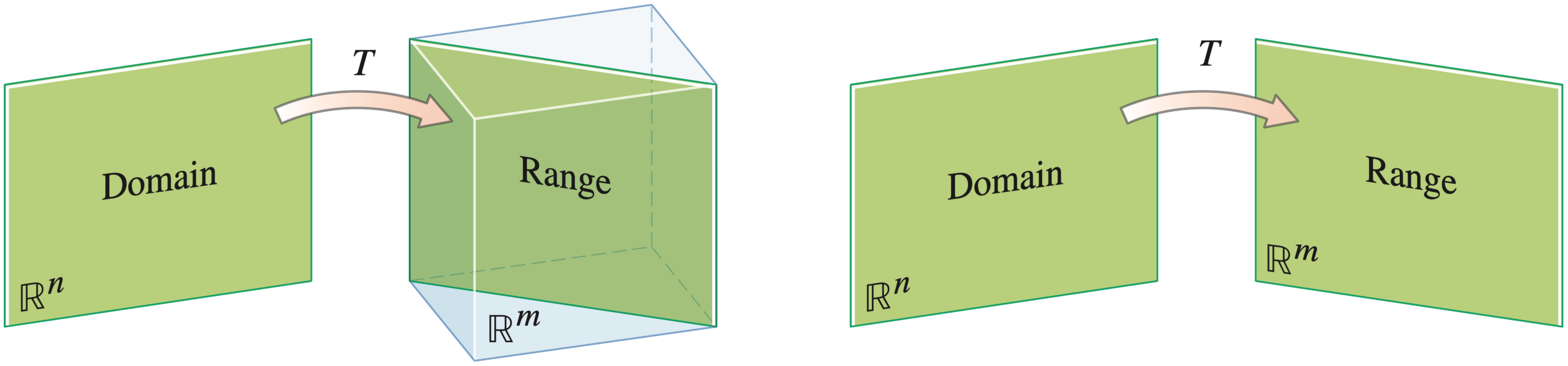
T is *not* onto $\mathbb{R}^m$



T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

Onto (Pictorially)



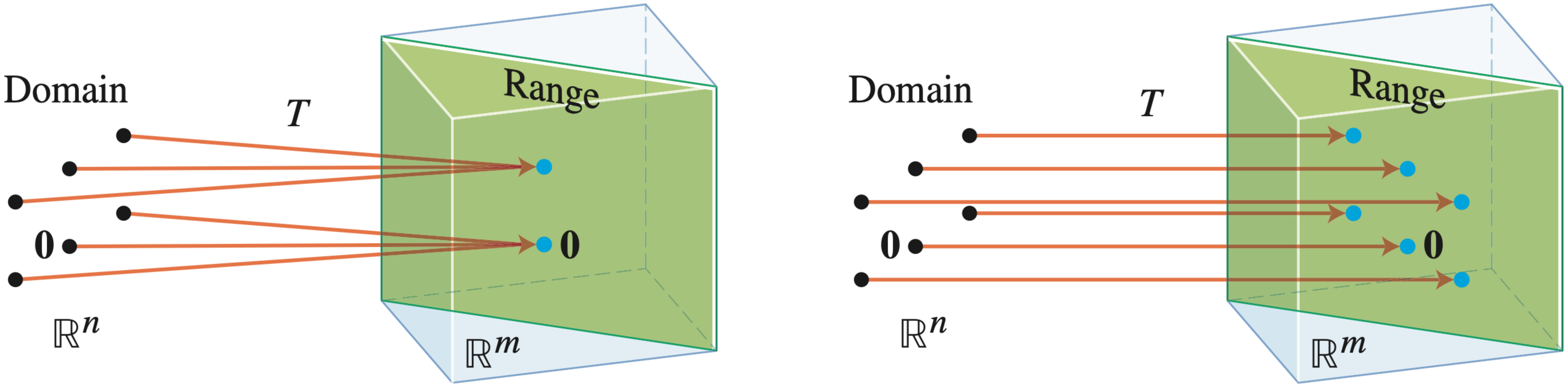
T is *not* onto $\mathbb{R}^m$

T is onto $\mathbb{R}^m$

T is onto if its range = its codomain

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

One-to-One (Pictorially)



T is *not* one-to-one

T is one-to-one

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

Taking Stock: Onto

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- » The columns of A are linearly independent
- » A has a pivot position in every <u>column</u>

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

why?:

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

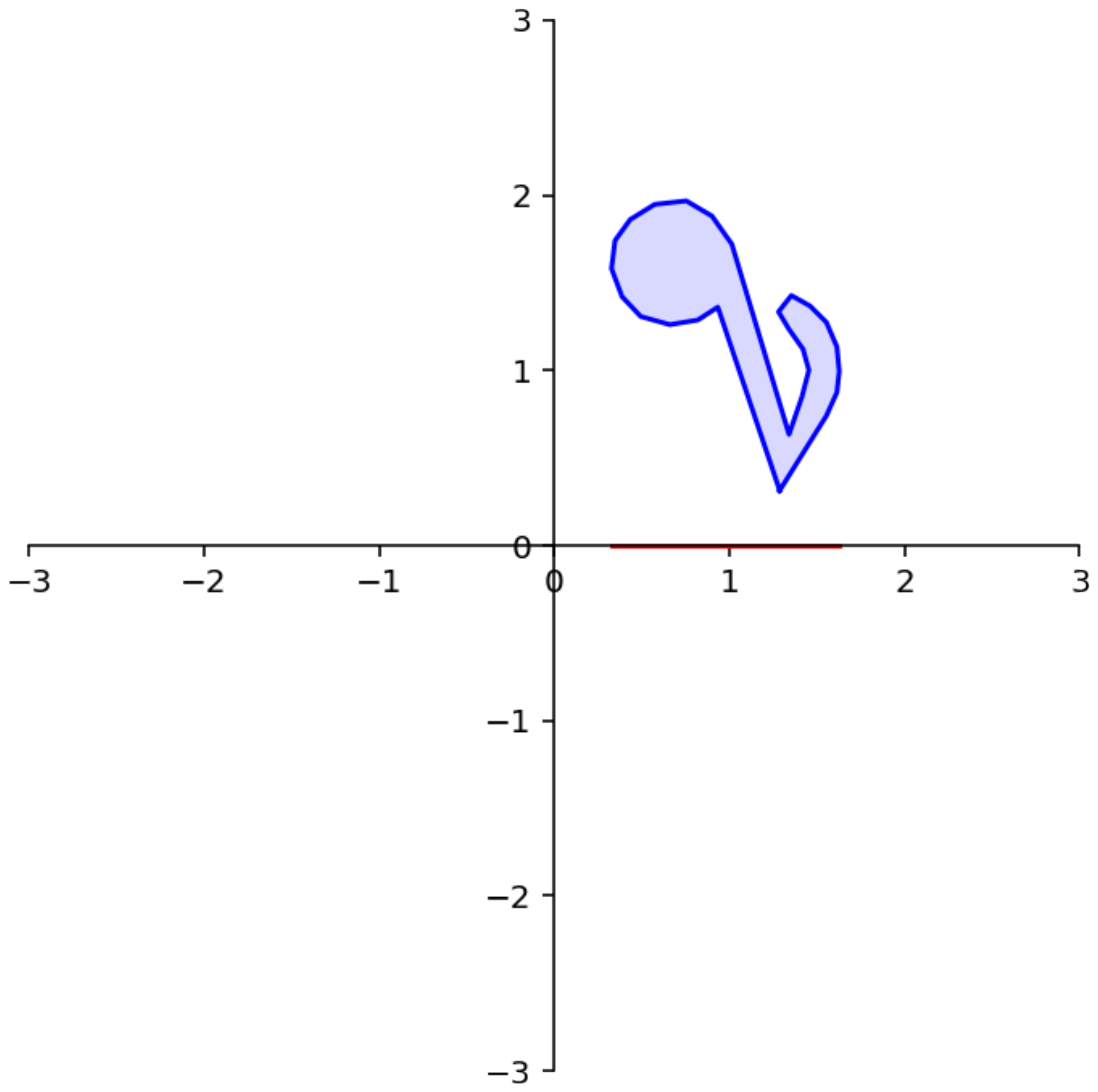
why?:

Example: not 1-1, not onto

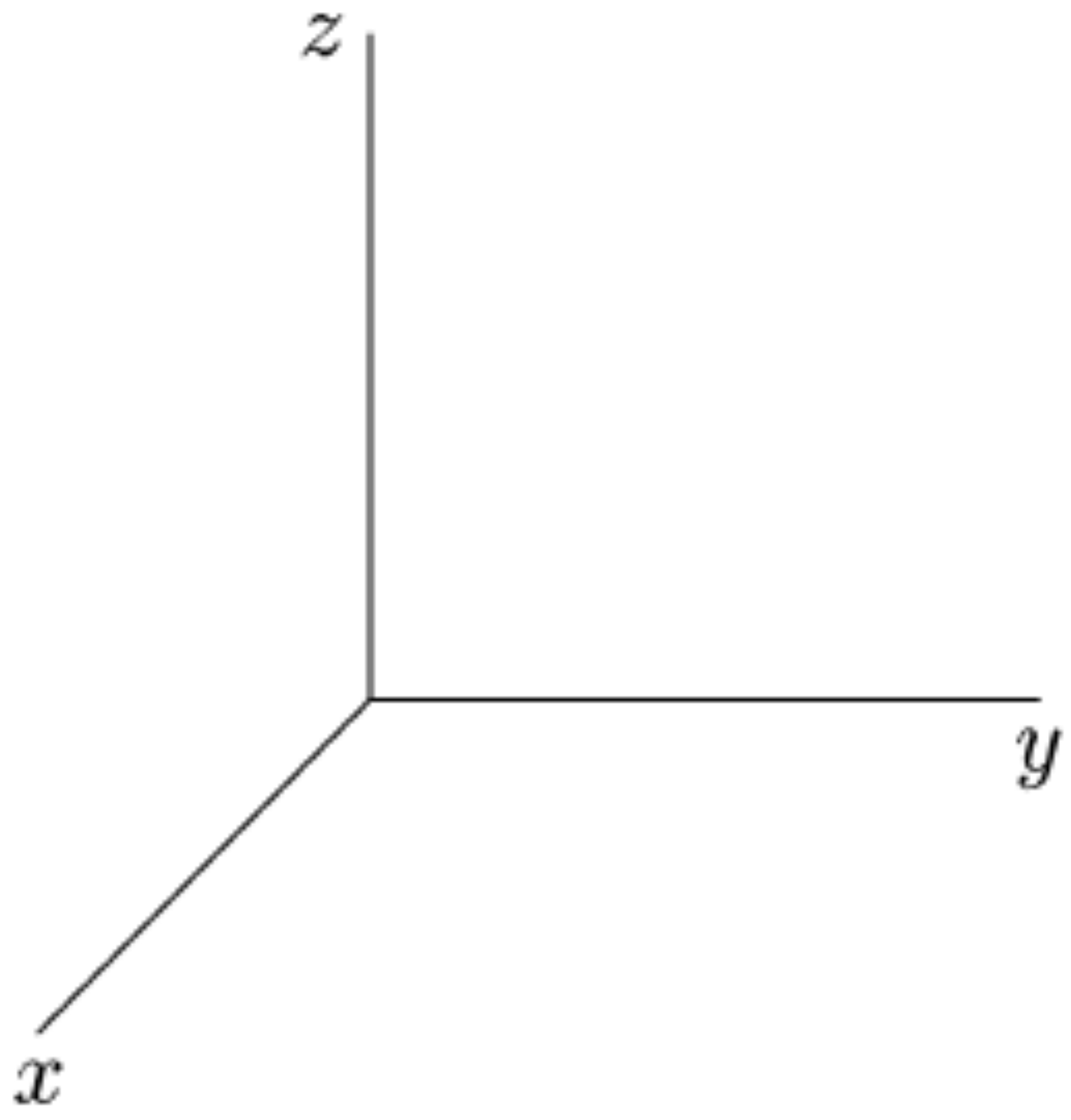
Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why?:



Example: onto, not 1-1



Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why?:

Summary

Matrix transformations and linear transformations are **the same thing**

We can find these matrices by looking at how the transformation behaves on the **standard basis**

We can reason about matrix equations by directly reasoning about the linear transformations