

Matrices of Linear Transformations

Geometric Algorithms
Lecture 8

Practice Problem

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 9$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2$$

Suppose that T is a linear transformation with the above input-output behavior.

What is the domain of T ? What is the codomain of T ?

What is the value of $T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$?

Answer

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 9 \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2$$

Objectives

- » Look at more examples of linear transformations
- » Show that matrix transformations and linear transformations are really the same thing
- » See more the geometry of linear transformations
- » Relate the properties of matrix equations to properties of linear transformations

Keywords

matrix of a linear transformation

standard basis vectors (standard coordinate vectors)

2D linear transformations

the unit square

one-to-one

onto

Recap

Recap: Matrices as Transformations

Matrices allow us to *transform* vectors

The transformed vector lies in the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector \mathbf{x} to the vector $A\mathbf{x}$

Recap: Transformation of a Matrix

The *transformation of a* $(m \times n)$ *matrix* A is the function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given \mathbf{v} , return A multiplied by \mathbf{v}

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

Recap: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recap: Linear Transformations

Definition. A transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is *linear* if it satisfies the following two properties

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)

2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Recap: Linear Transformations

Definition. A transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is *linear* if it satisfies the following two properties

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Matrix transformations are linear transformations

Verification

any matrix transformation:

rotation about the origin:

translation (*non-example*):

Recap: Linear Combinations

$$T\left(\sum_{i=1}^n a_i \mathbf{v}_i\right) = \sum_{i=1}^n a_i T(\mathbf{v}_i)$$

We can generalize linearity to any linear combination

Our Next Motivating Question

Our Next Motivating Question

We know that matrix transformations are linear transformations

Our Next Motivating Question

We know that matrix transformations are linear transformations

Are there any other kinds of linear transformations?

Our Next Motivating Question

We know that matrix transformations are linear transformations

Are there any other kinds of linear transformations?

NO

Matrix of a Linear Transformation

Theorem. A transformation T is linear if and only if there is a matrix whose corresponding transformation is T (which "implements" T)

Matrix of a Linear Transformation

Theorem. A transformation T is linear if and only if there is a matrix whose corresponding transformation is T (which "implements" T)

Linear transformations are **exactly**
matrix transformations

A Fundamental Concern

Given a linear transformation T , how do we find the matrix A such that $T(\mathbf{v}) = A\mathbf{v}$?

A Thought Experiment

Suppose I tell you T is a linear transformation
and

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Do we know what $T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$ is?

Answer: Yes

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Because of additivity:

$$T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) =$$

A Thought Experiment

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

What about:

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) =$$

The Takeaway

Linearity is a **very** strong restriction

If we know the values of $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ on **any** set of vectors which spans all of \mathbb{R}^n , then we know T .

why?

Another Thought Experiment (Game)

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Suppose I am holding a matrix A

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Your objective is to figure out what A is

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But you're only allowed to ask the question:

Another Thought Experiment (Game)

Suppose I am holding a matrix A

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But you're only allowed to ask the question:

What is $A\mathbf{v}$?

Another Thought Experiment (Game)

Suppose I am holding a matrix A

Your objective is to figure out what A is

But you're only allowed to ask the question:

What is Av ?

(you pick the v 's, and I have to tell the truth)

Another Thought Experiment (Game)

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This is basically linear algebraic battleship

Recall: Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$b_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n = \sum_{i=1}^n a_{1i}v_i$$

Recall: Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and a vector \mathbf{v} in \mathbb{R}^n , we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots v_n\mathbf{a}_n$$

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$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + \dots + v_n \mathbf{a}_n$$

$A\mathbf{v}$ is a linear combination of the columns of A with weights given by \mathbf{v}

Isolating a_{11}

$$b_1 = a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n = \sum_{i=1}^n a_{1i}v_i$$

Isolating a_{11}

$$b_1 = a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n = \sum_{i=1}^n a_{1i}v_i$$

We actually get the whole column \mathbf{a}_1

So its like battleship, but you get to
choose one column at a time.

The Takeaway

We can learn the first column of the matrix

implementing T by looking at $T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Matrix of a Linear Transformation

Standard Basis

Definition. The *n -dimensional standard basis vectors* (or standard coordinate vectors) are the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ where

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ n-1 \\ n \end{matrix}$$

Standard Basis

Definition (Alternative). The n -dimensional standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix

$$I = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_n]$$

Standard Basis and the Matrix Equation

The key points: $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] \mathbf{e}_i = \mathbf{a}_i$

The standard basis vectors gives us a way to
"look into" a matrix

Standard Basis and Vector Coordinates

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n$$

Column vectors can be viewed as describing how to write a vector as a linear combination of the standard basis

Standard Basis and Linear Transformations

Theorem. For any linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, the matrix

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$$

is the unique matrix such that $T(\mathbf{v}) = A\mathbf{v}$ for all \mathbf{v} in \mathbb{R}^n

More Formally $T(\mathbf{v}) =$

$$= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix} \mathbf{v}$$

How To: Matrices of Linear Transformations

Question. Find the matrix which implements the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Solution. Determine the images of standard basis under T . Then write down

$$[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$$

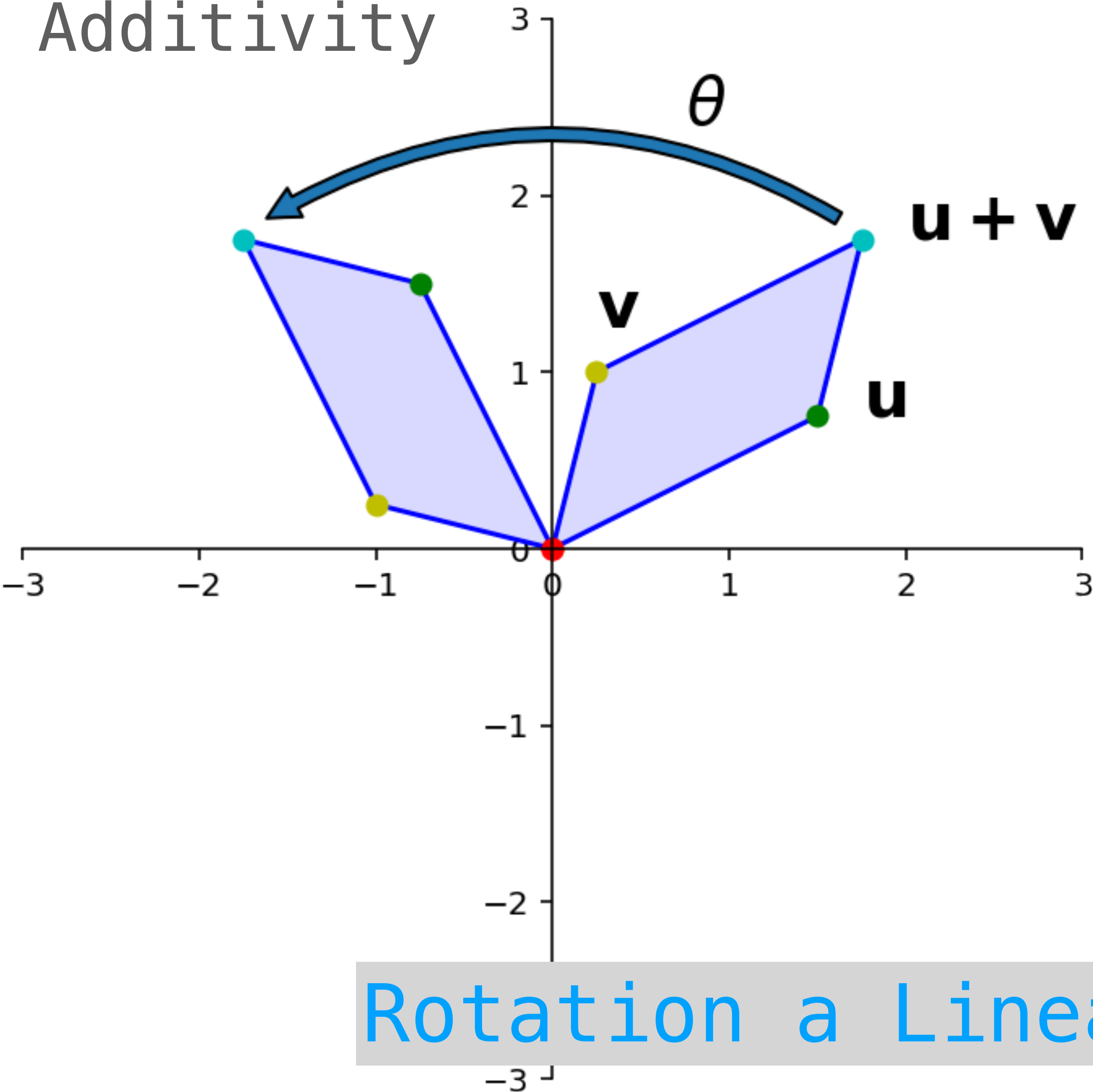
Question

*Write down the matrix which implements the linear transformation T which **rotates** vectors by 90 degrees clockwise*

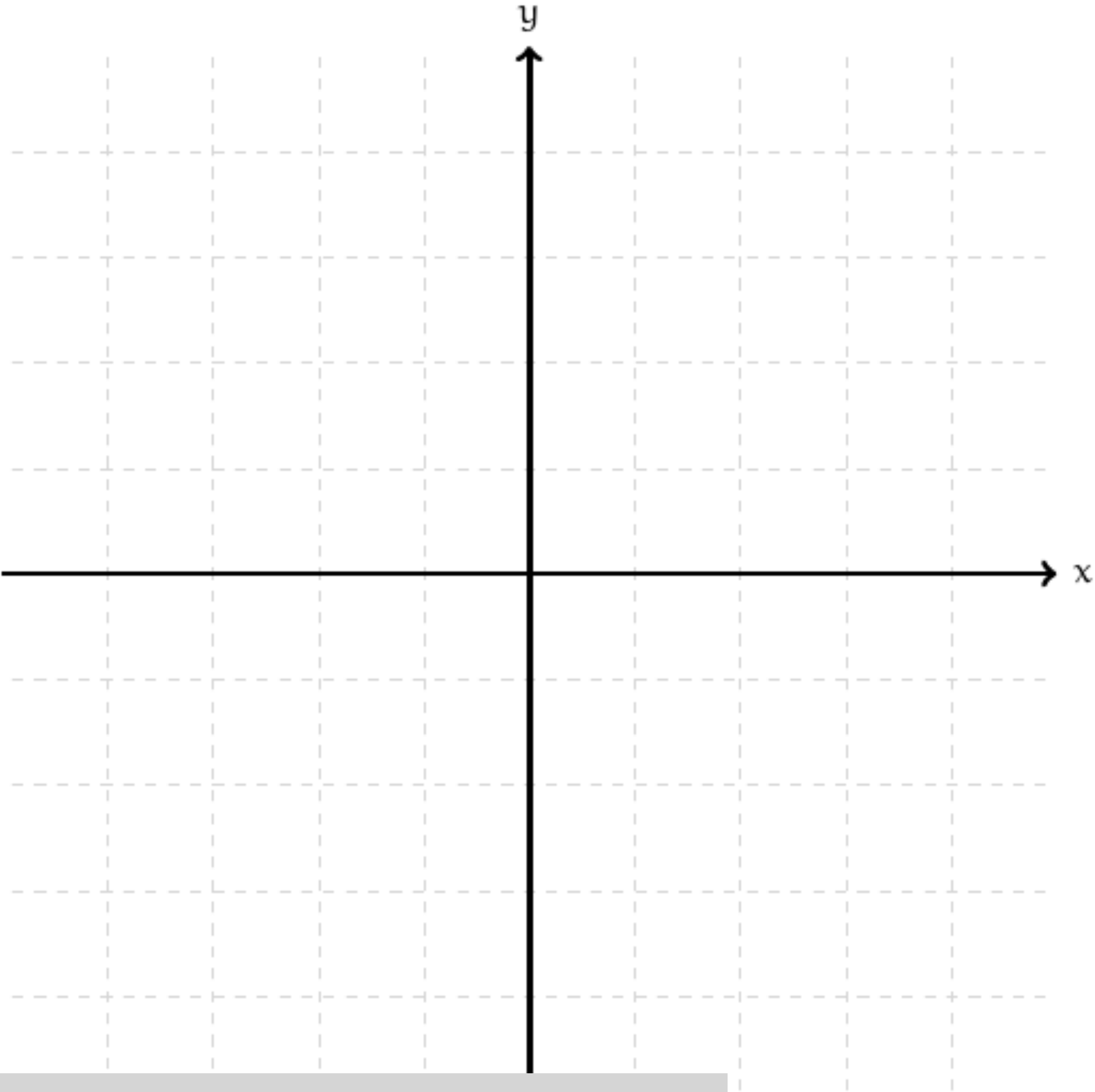
Answer

General Rotation

Additivity



Homogeneity



Rotation a Linear Transformation

Geometry of Matrix Transformations

Motivating Questions

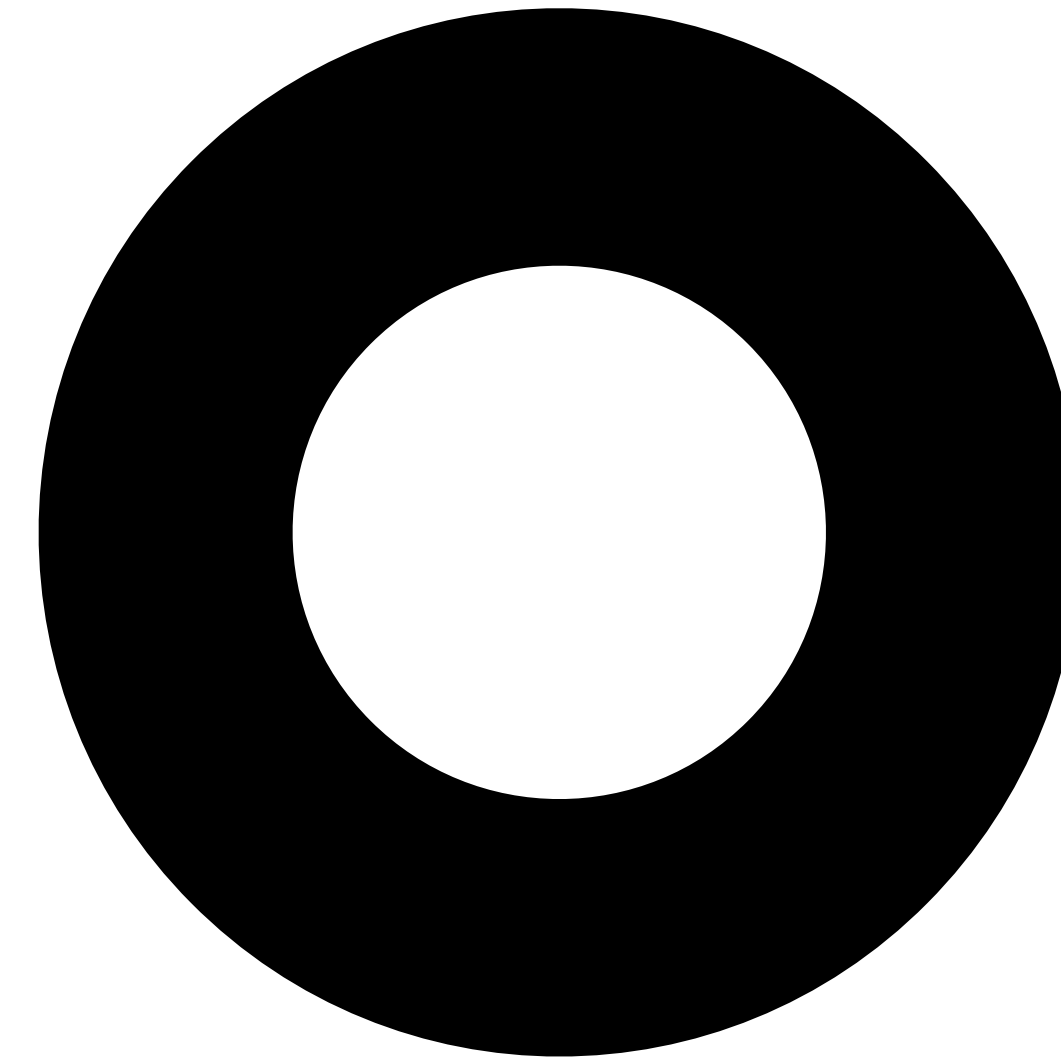
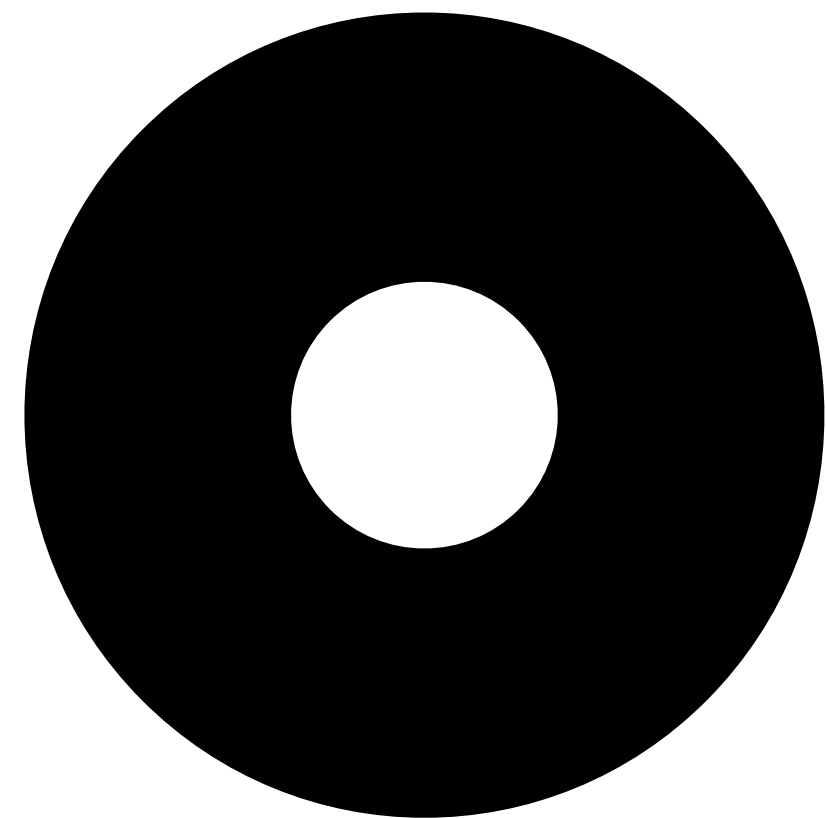
What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

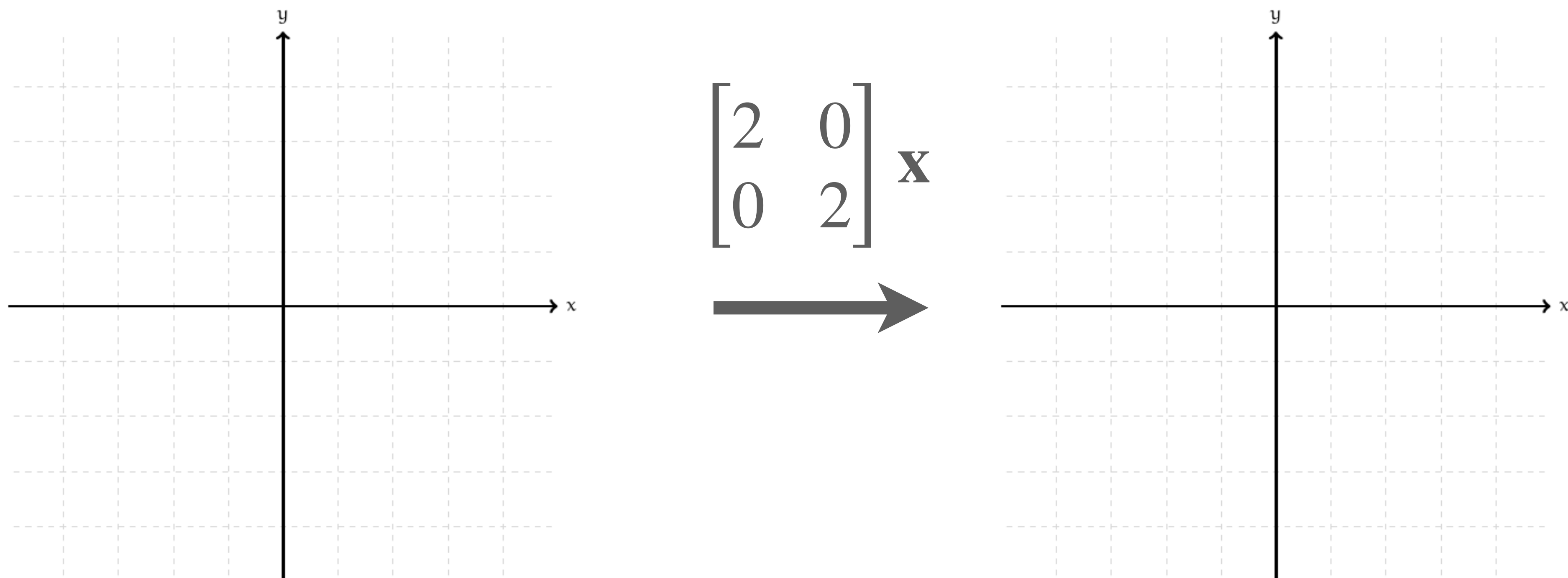
Matrix transformations change the
"shape" of a set of set of
vectors (points).

Example: Dilation



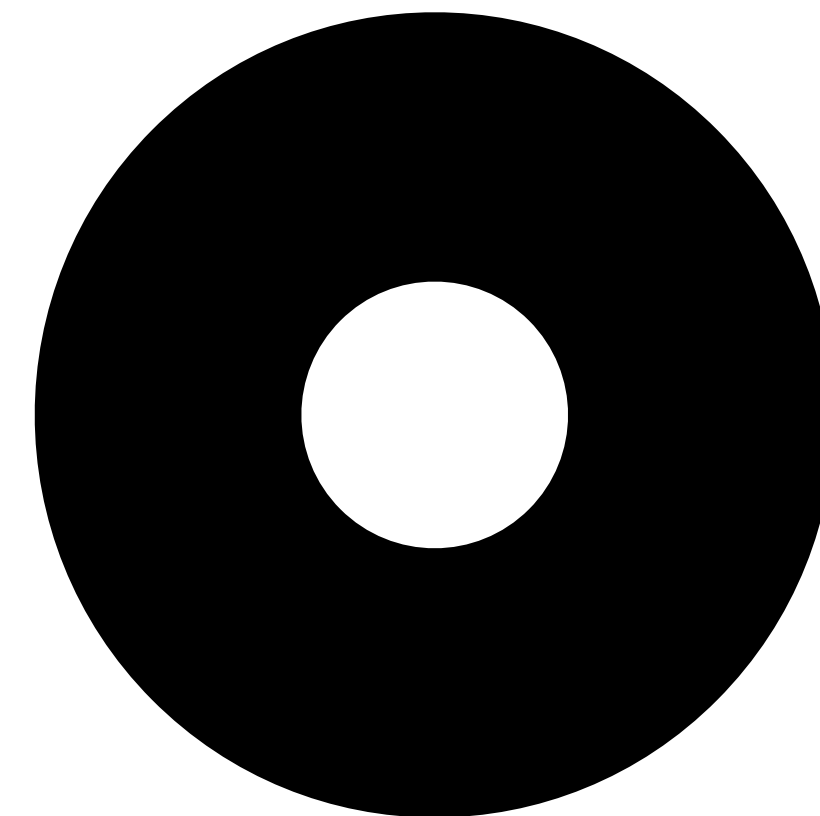
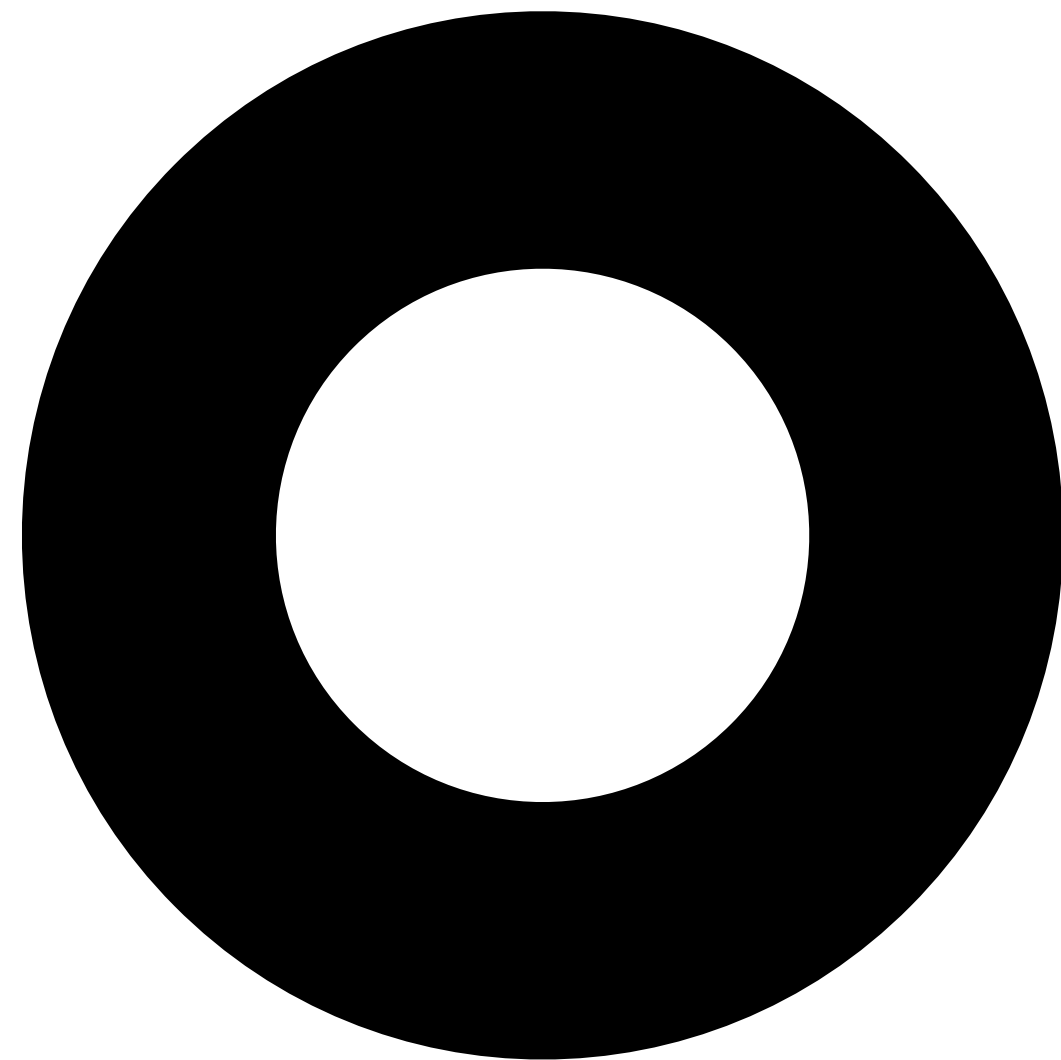
Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



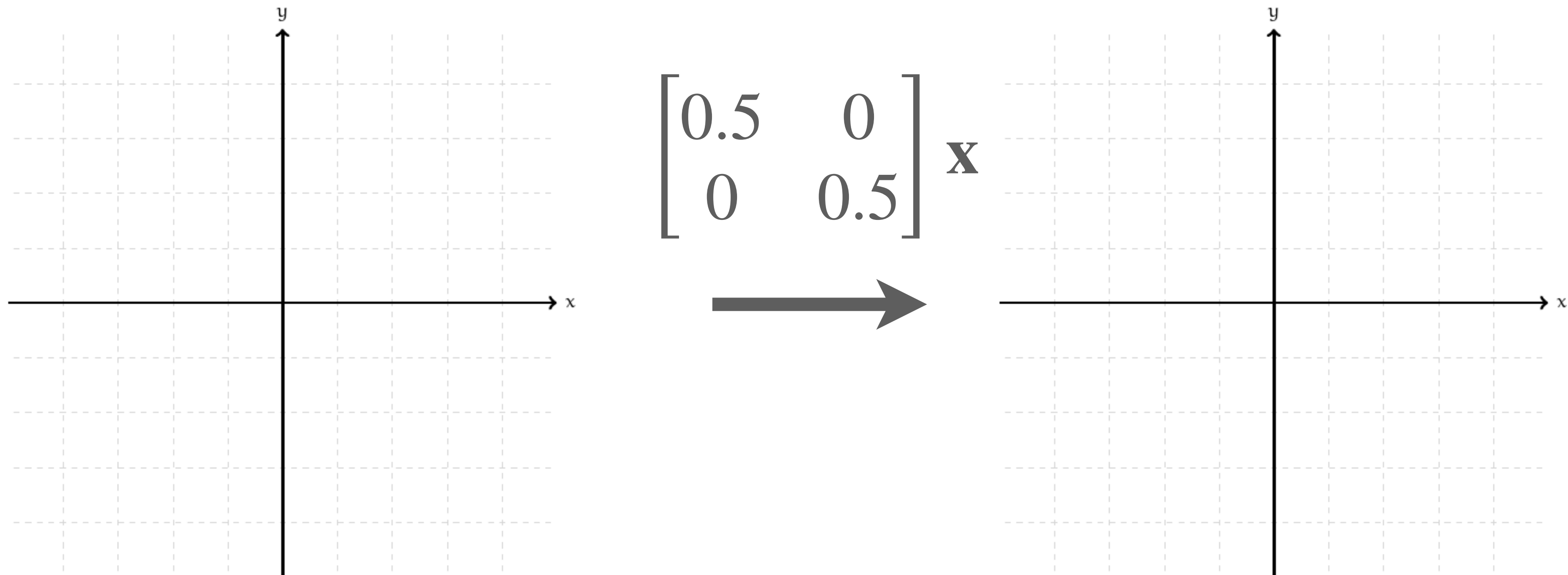
if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



Example: Contraction

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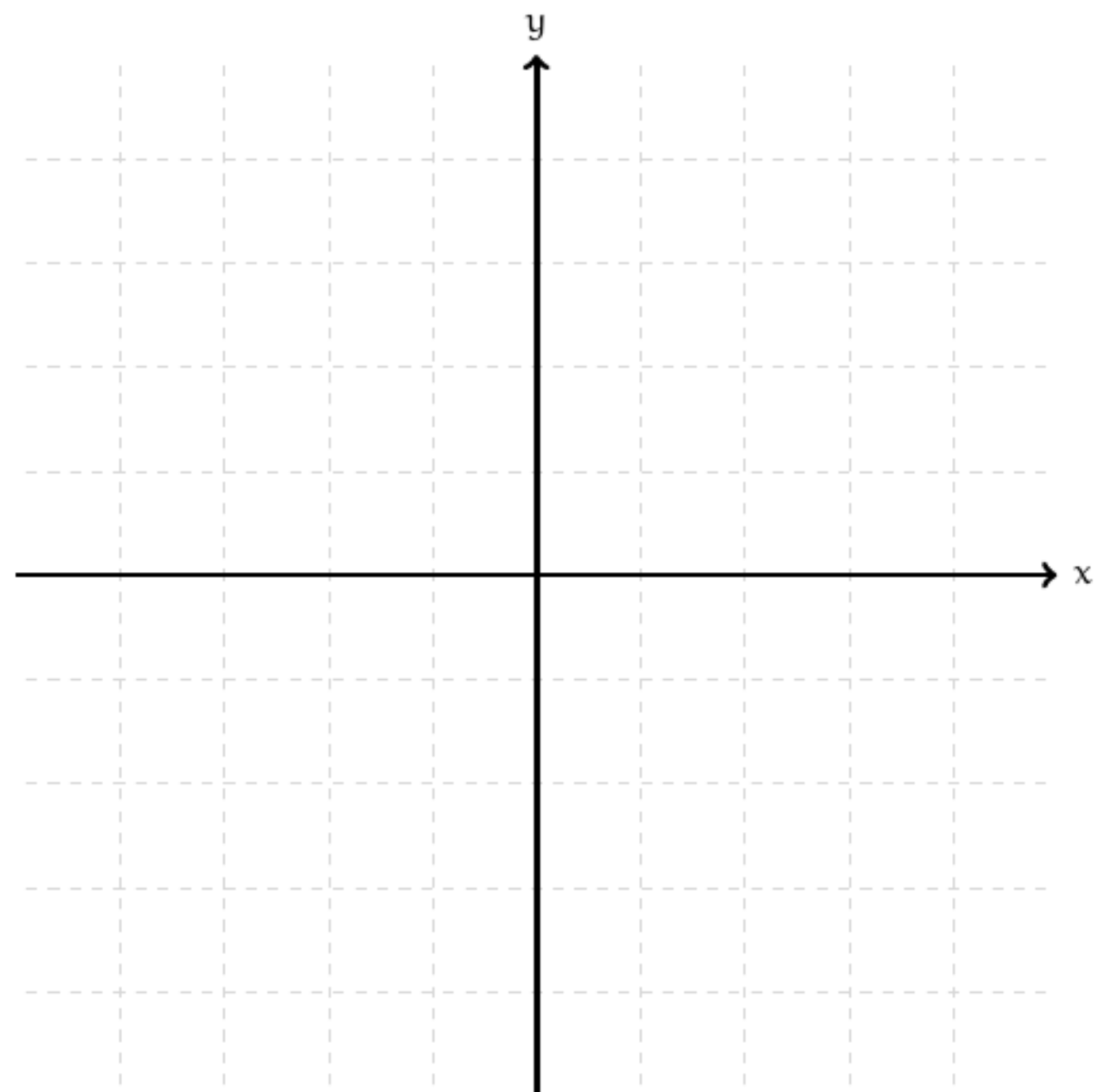
if $0 \leq r \leq 1$, then the transformation
pulls points towards the origin.

Example: Shearing



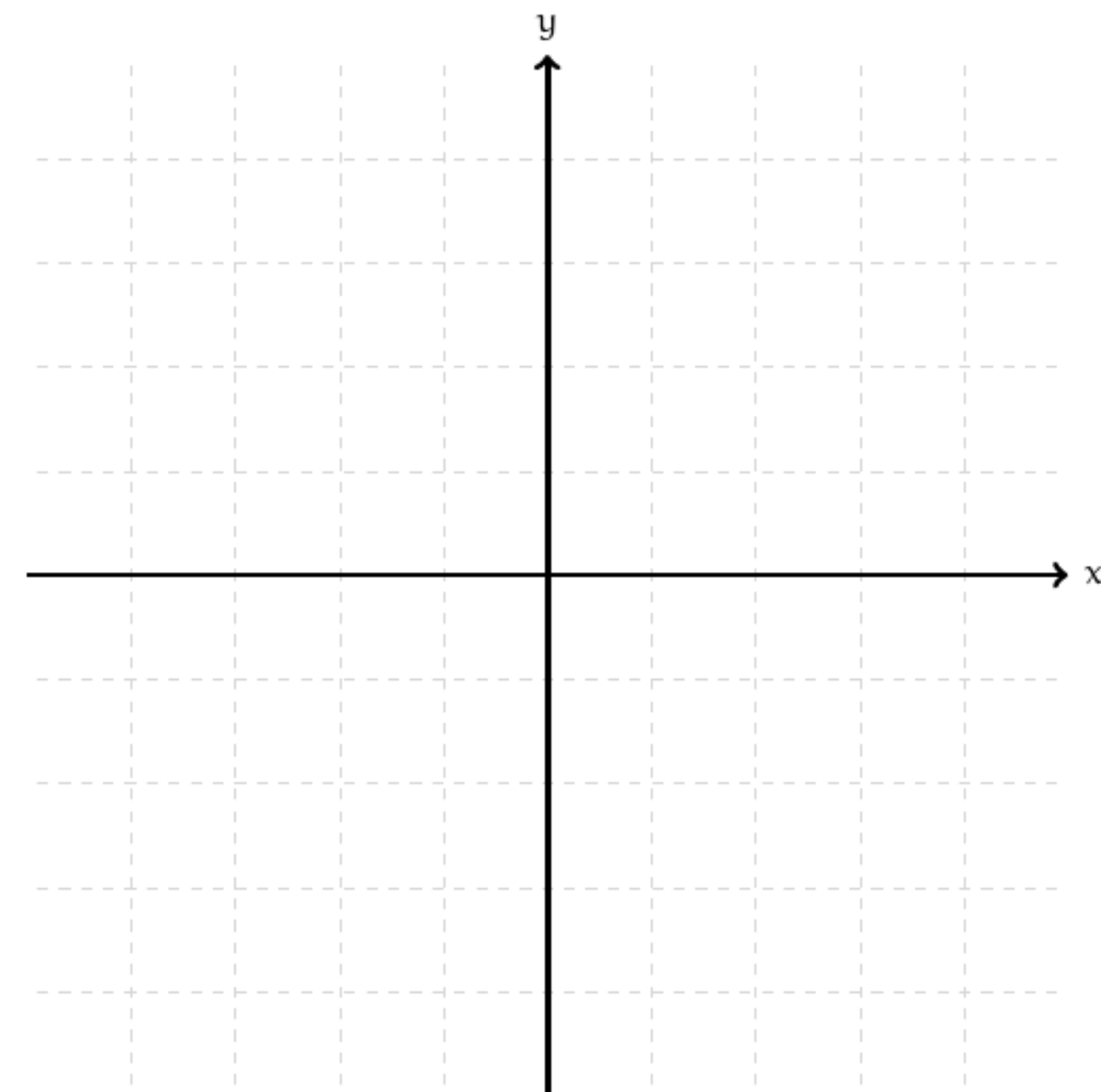
Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



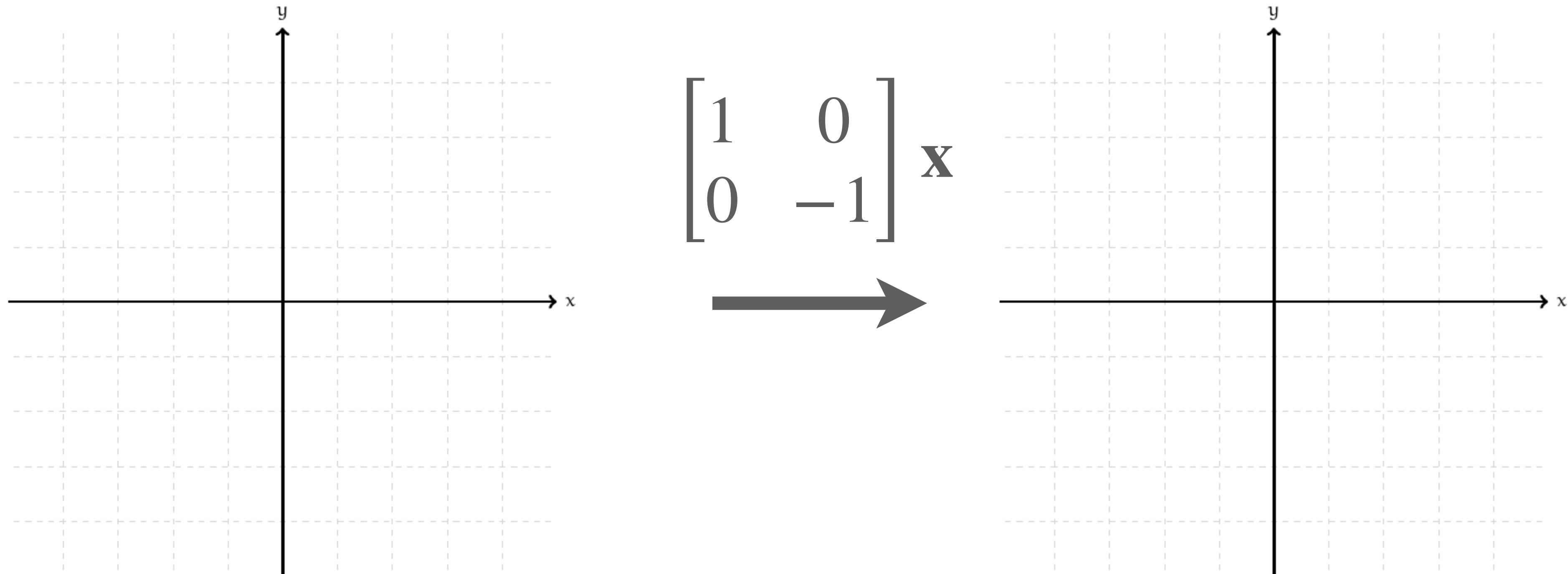
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

→



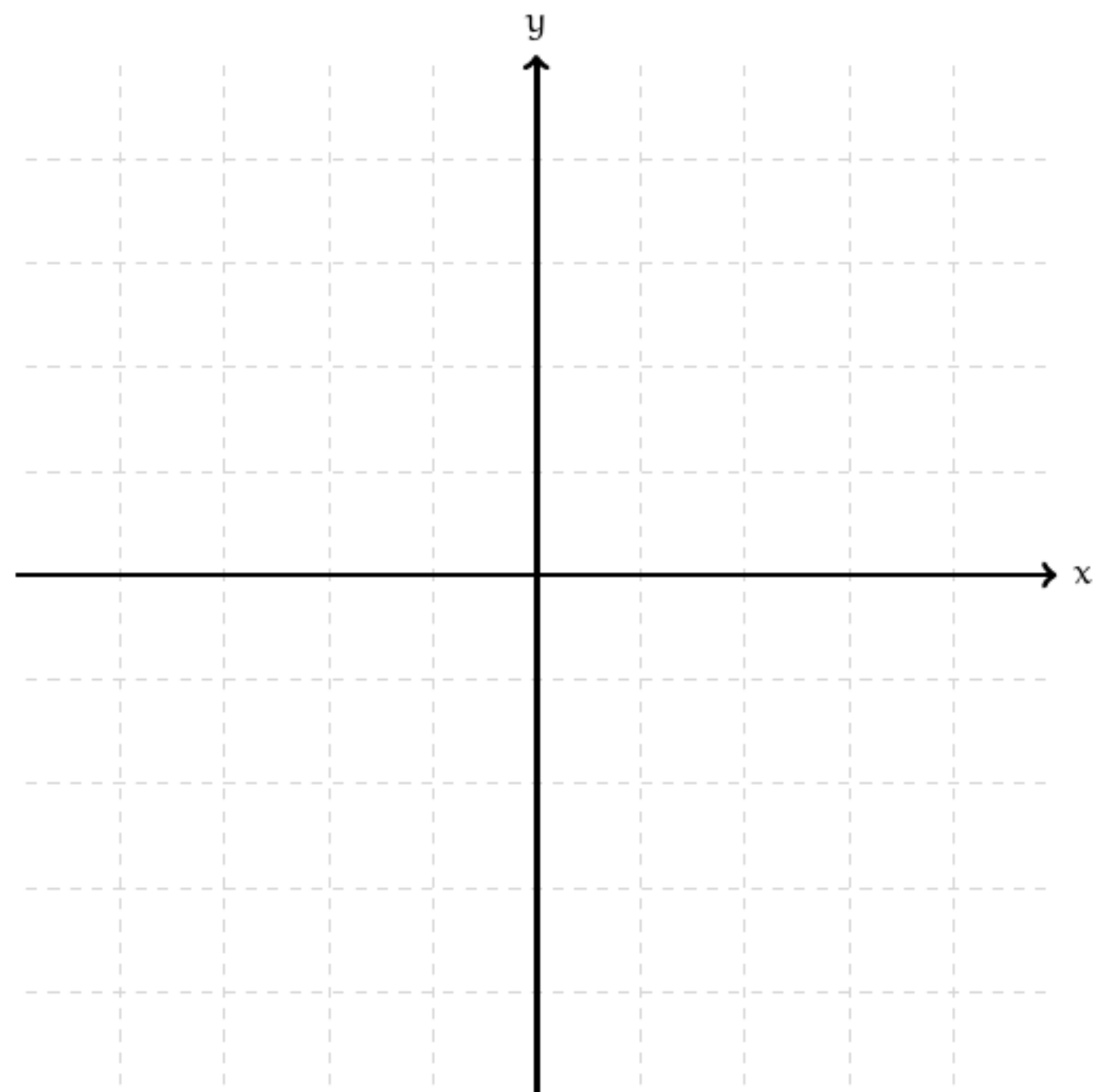
Imagine shearing like with rocks or metal.

Question

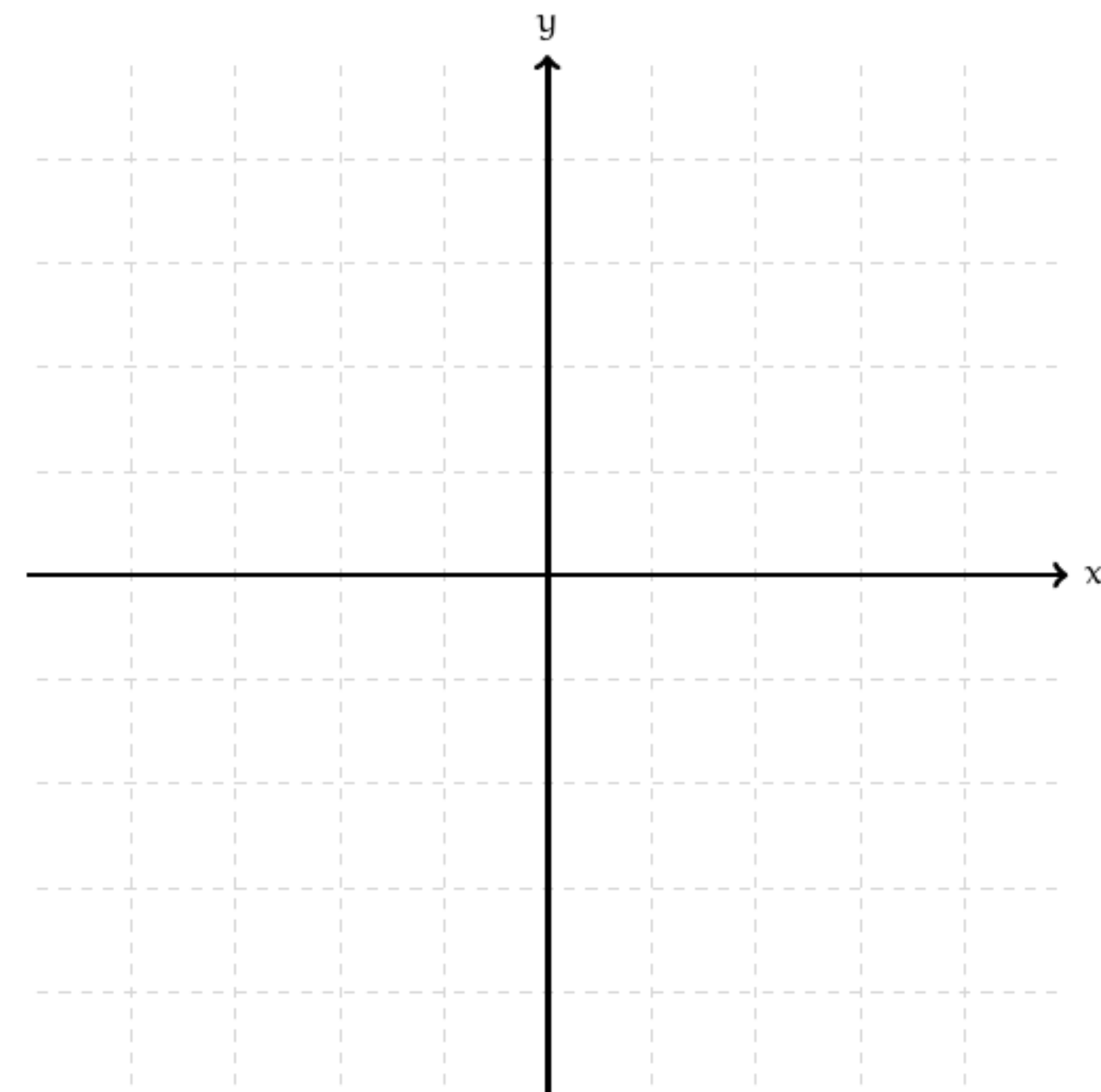


Draw how this matrix transforms points. What kind of transformation does it represent?

Answer: Reflection

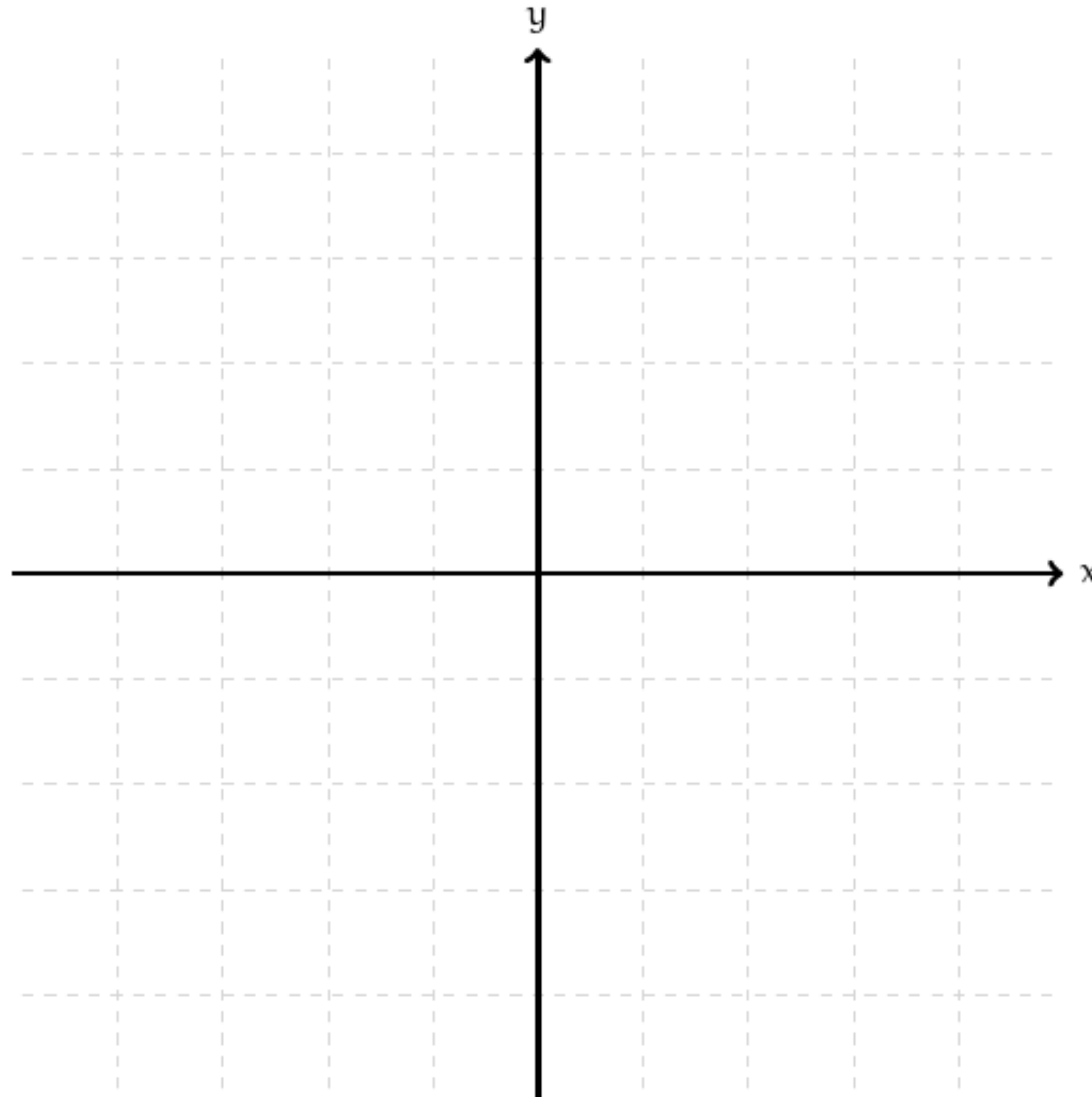


$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



General Rotation

How does
rotation
affect the
standard
basis?



Rotation Matrix

Rotation Matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Rotation Matrix

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Note: This is rotation about the origin

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Note: This is rotation about the origin

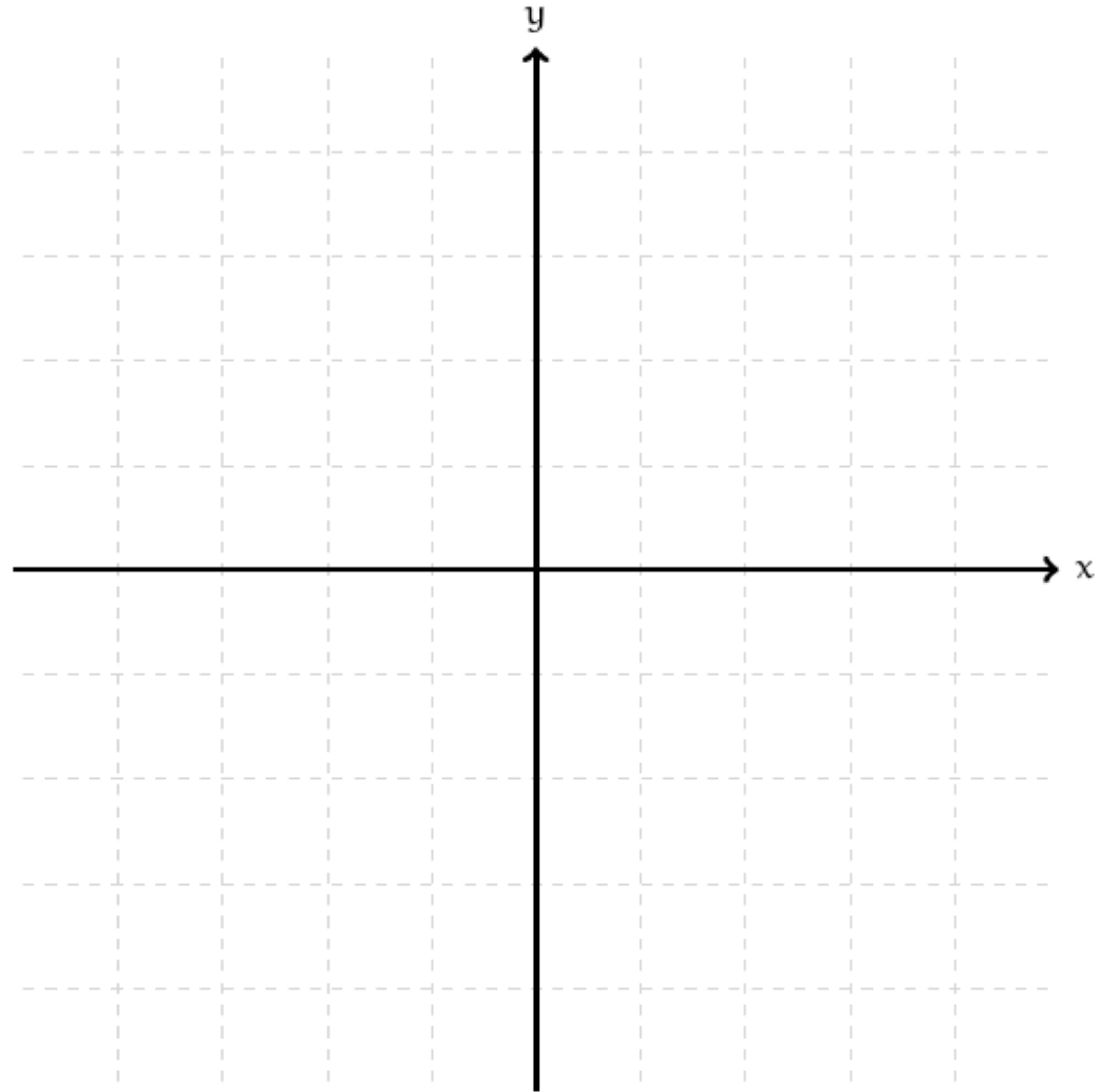
The Takeaway: We can figure out the matrices which implement complex linear transformations by understanding what they do to the standard basis

Question (Conceptual)

Is rotation about a point other than the origin
a linear transformation?

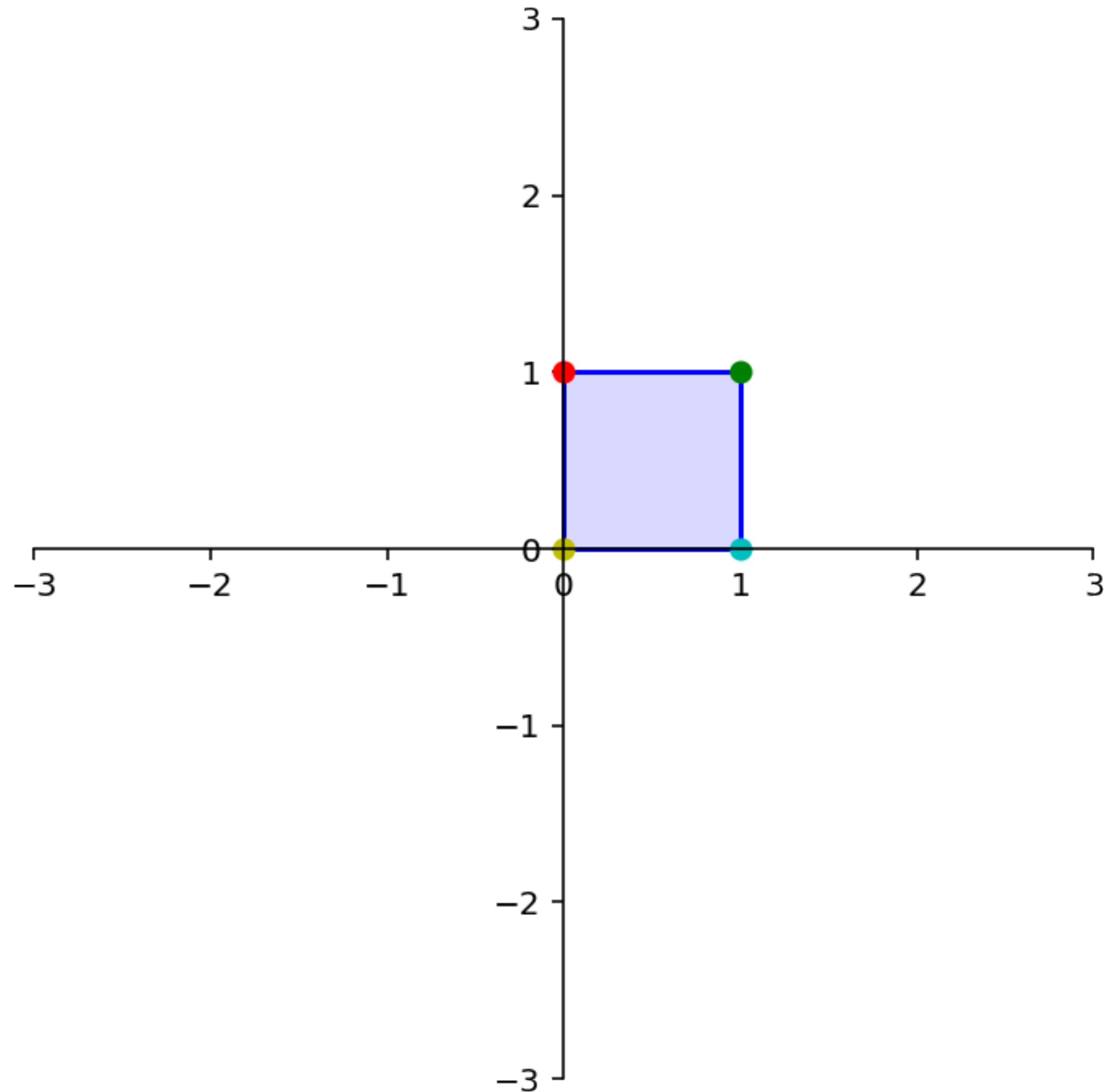
Answer: No

The origin is not
fixed by this
transformation



The Unit Square

The *unit square* is the set of points in \mathbb{R}^2 enclosed by the points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.



How To: The Unit Square and Matrices

How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

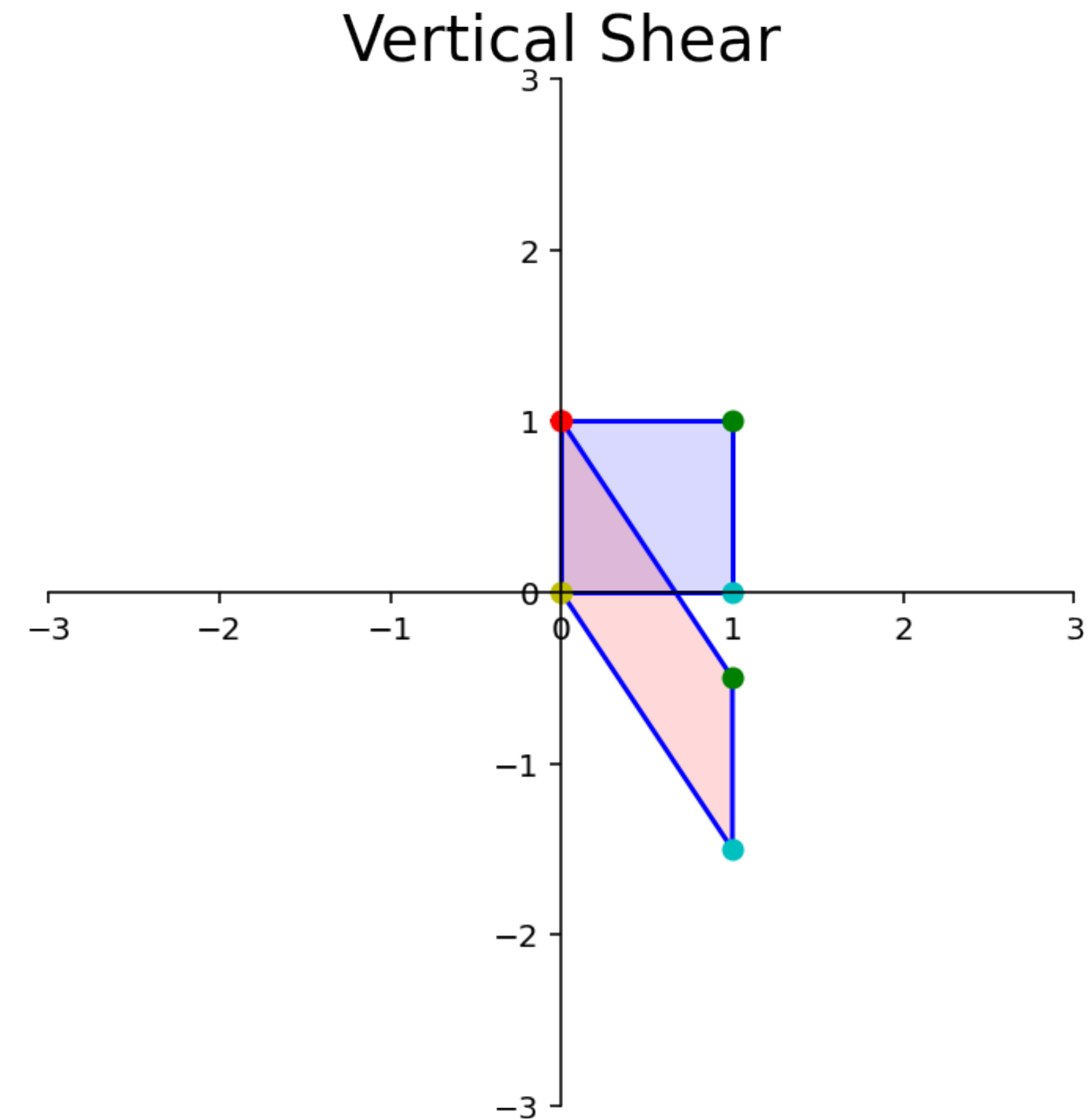
How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture

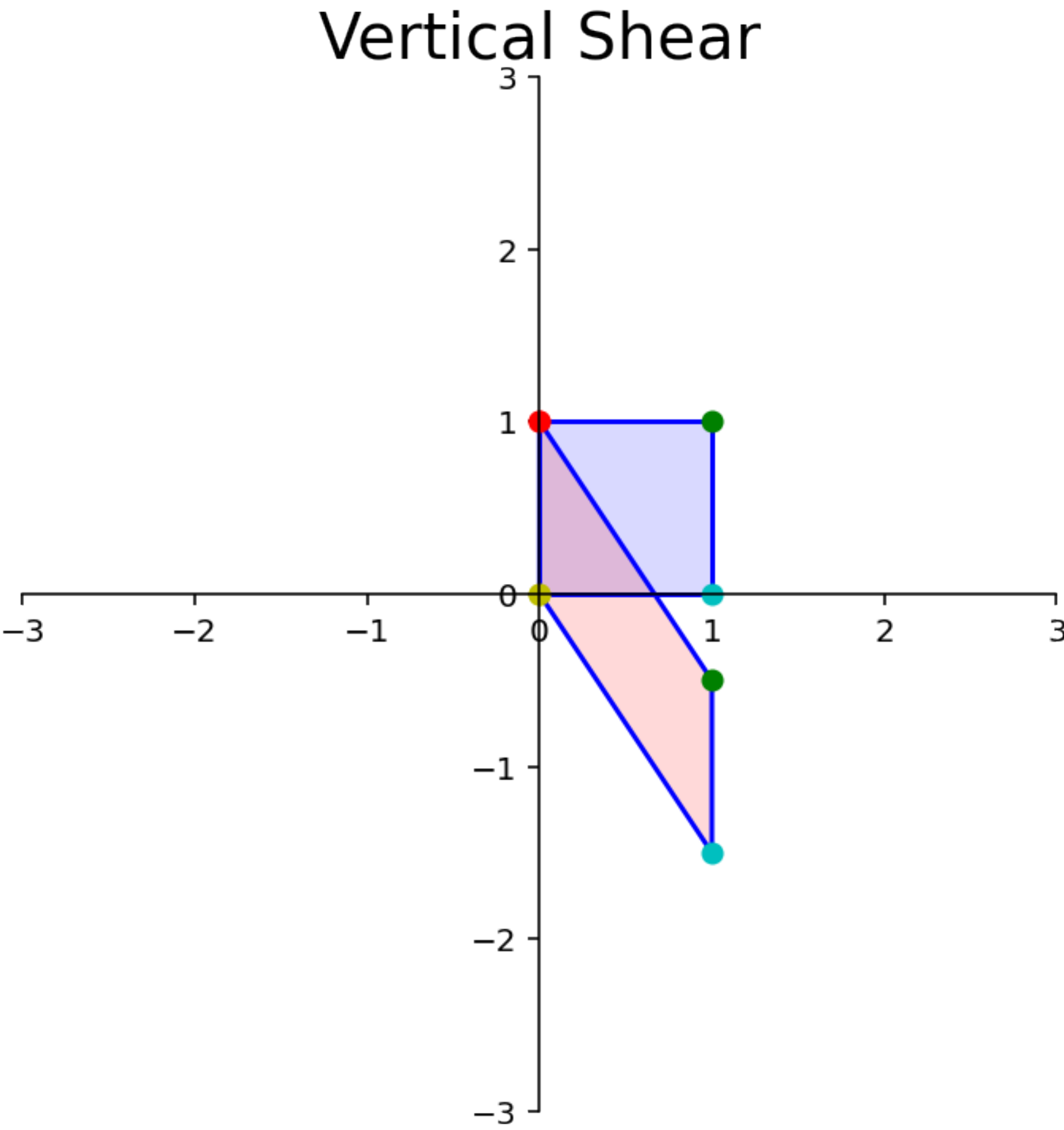
Solution. Find where the standard basis vectors go

Question

Write down the matrix for the following shearing operation using this method



Answer

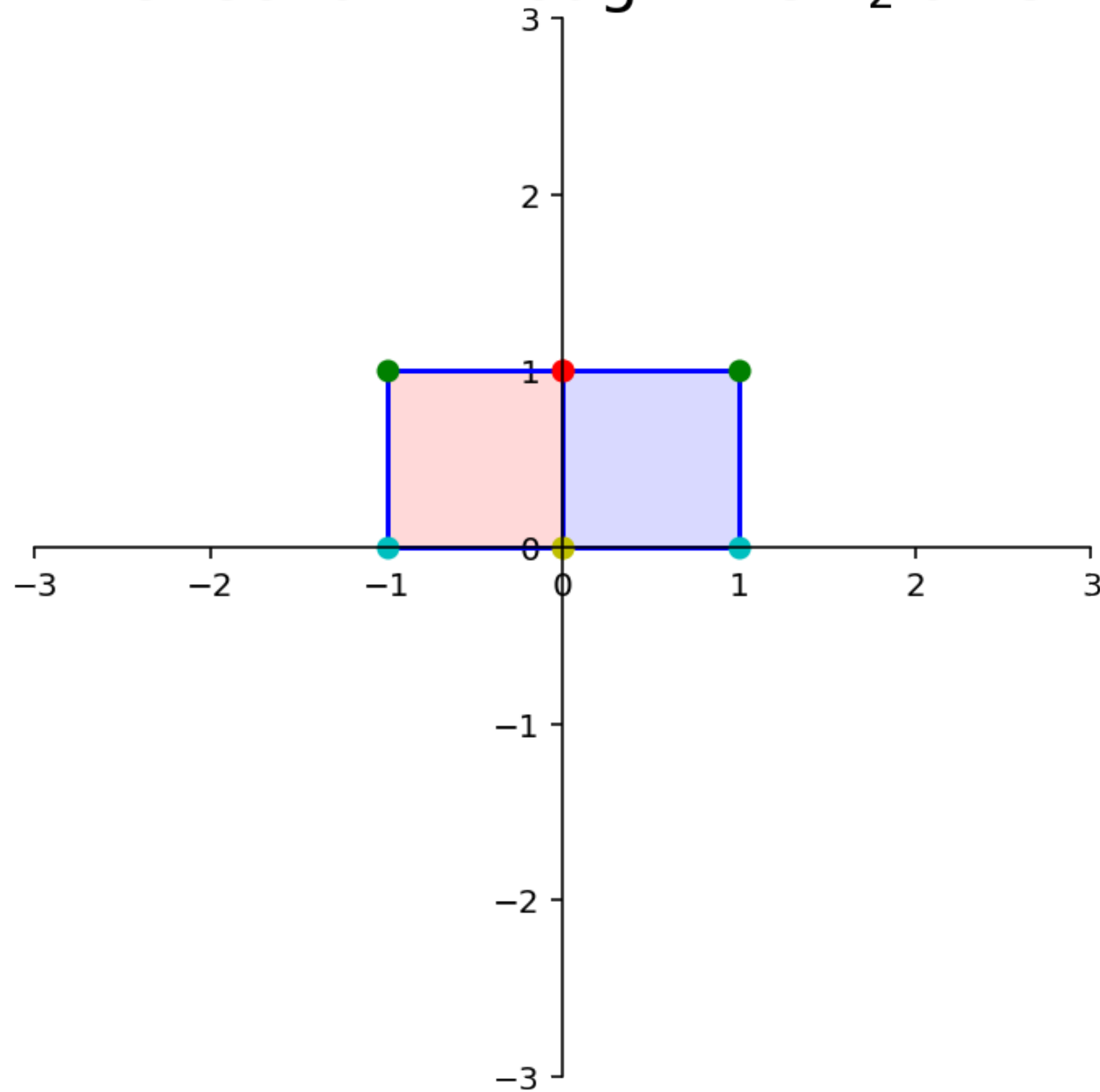


You need to know these matrices,
but you don't need to memorize them

Remember: What does this matrix do
to the unit square? Then build the
matrix from there

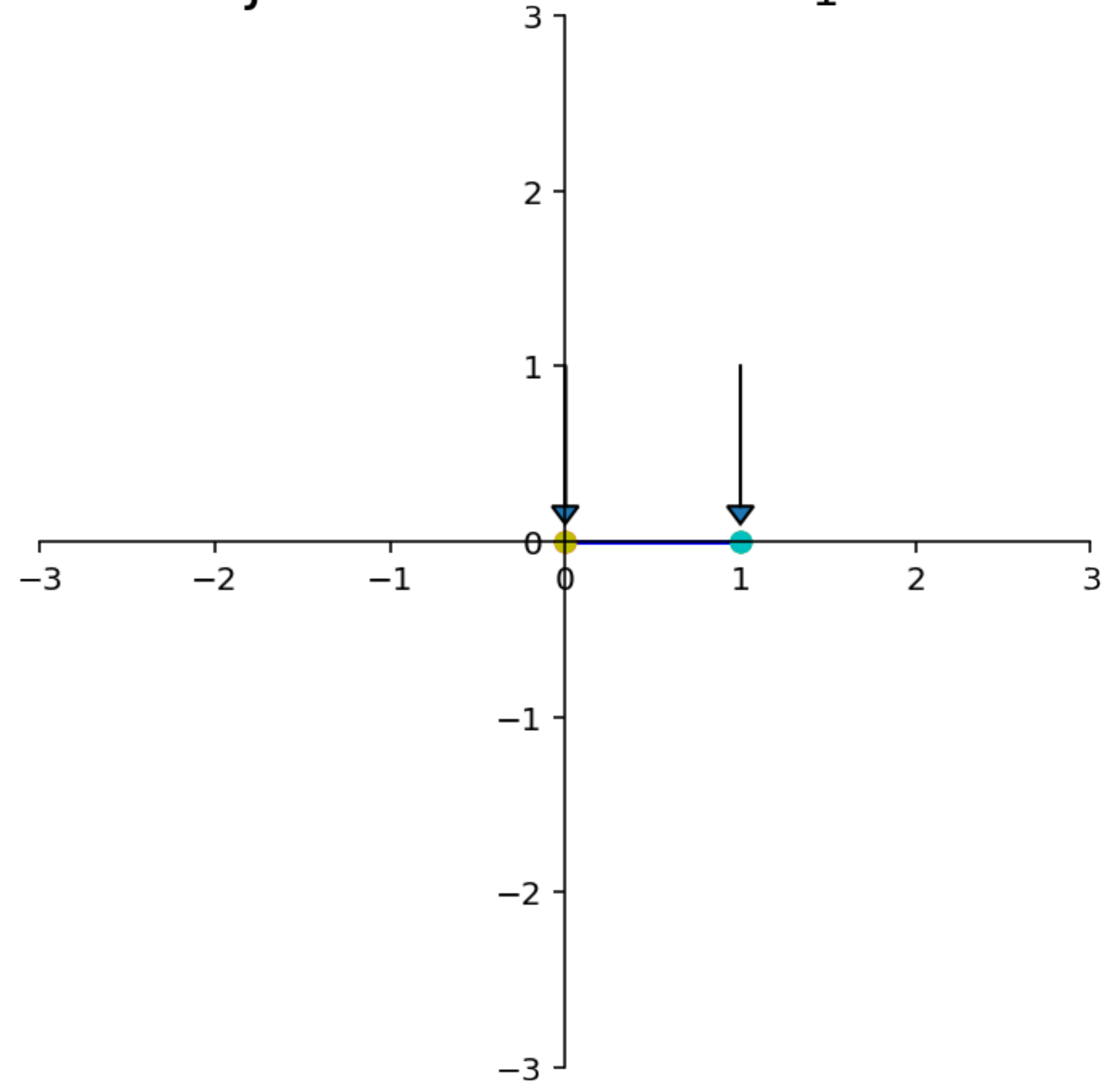
Reflection through the x_2 -axis

Reflection through the x_2 axis

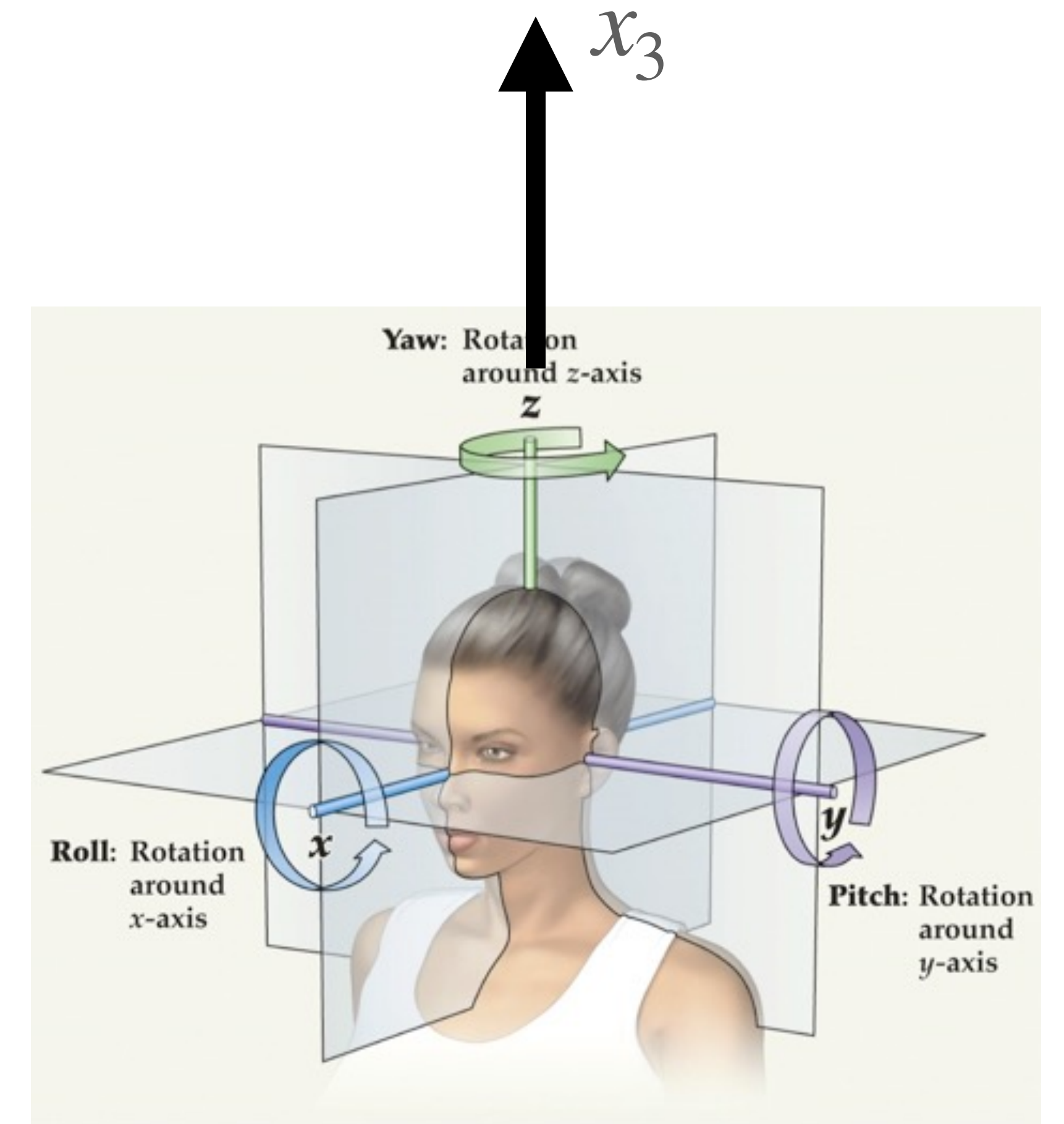
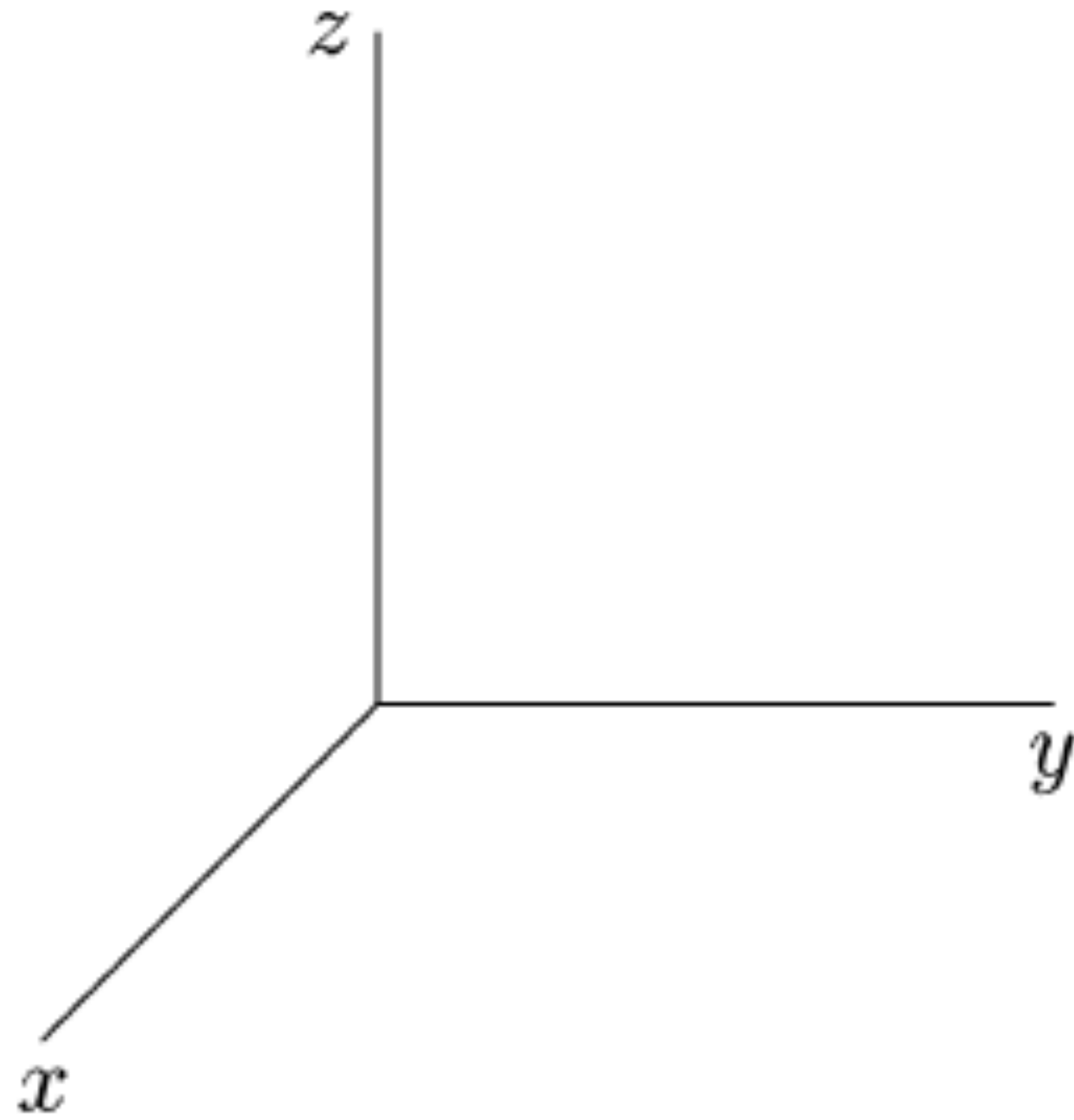


Projections

Projection onto the x_1 axis



A 3D Example: Rotation about the x_3 -Axis (z -Axis)



List of Important 2D Linear Transformations

- » dilation, contraction
- » reflections
- » projections
- » horizontal/vertical contractions
- » horizontal/vertical shearing

Look through the notes for a comprehensive
collection of pictures or...

demo

One-to-One and Onto

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A
transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A
transforms into \mathbf{b}

Recall: A New Interpretation of the Matrix Equation

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Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A
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What about other questions?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Wait

$A\mathbf{x} = \mathbf{0}$ has a
unique solution

\equiv

$A\mathbf{x} = \mathbf{b}$ has at most one
solution

why? :

Onto and One-to-One

Onto and One-to-One

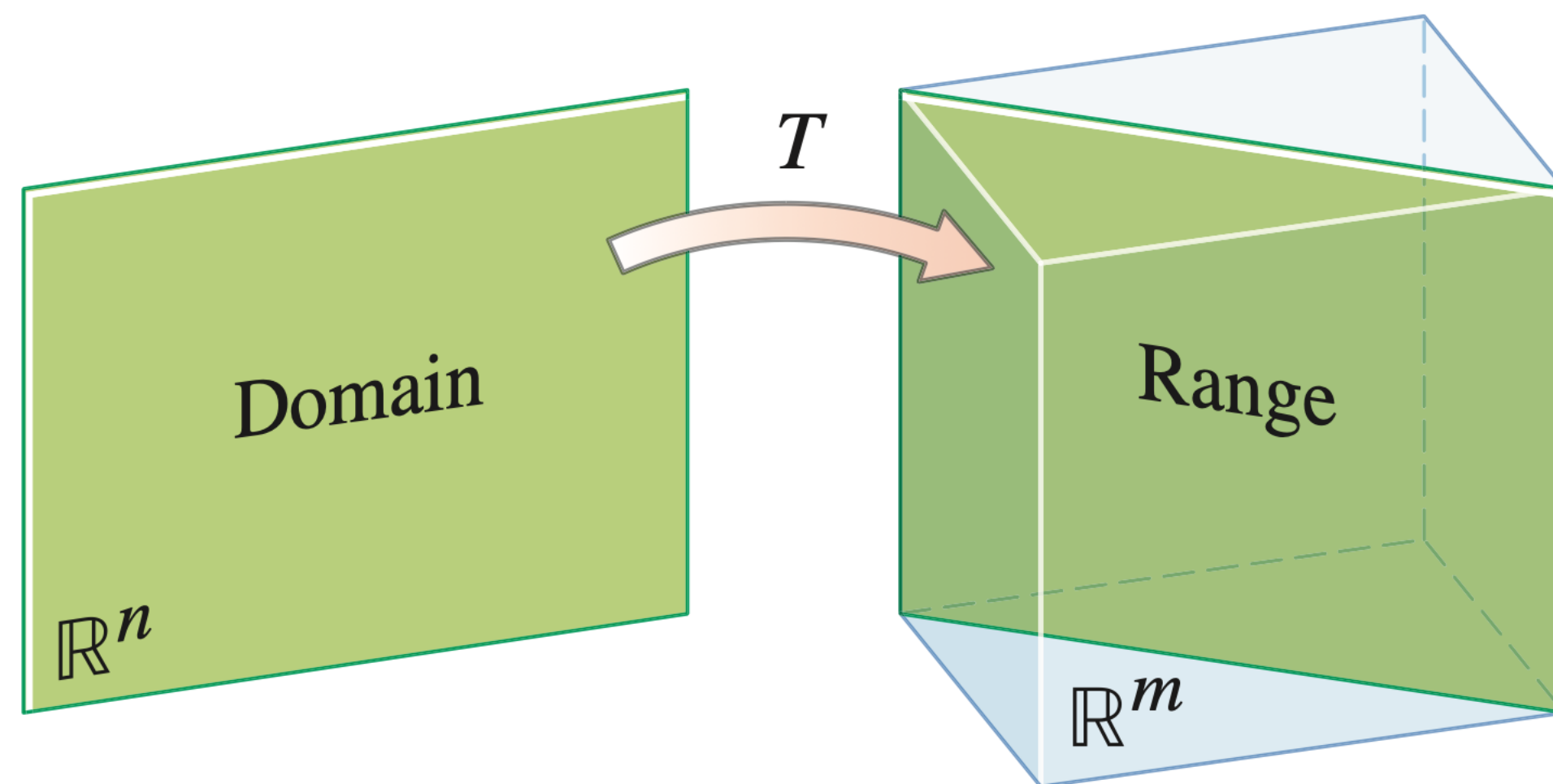
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

Onto and One-to-One

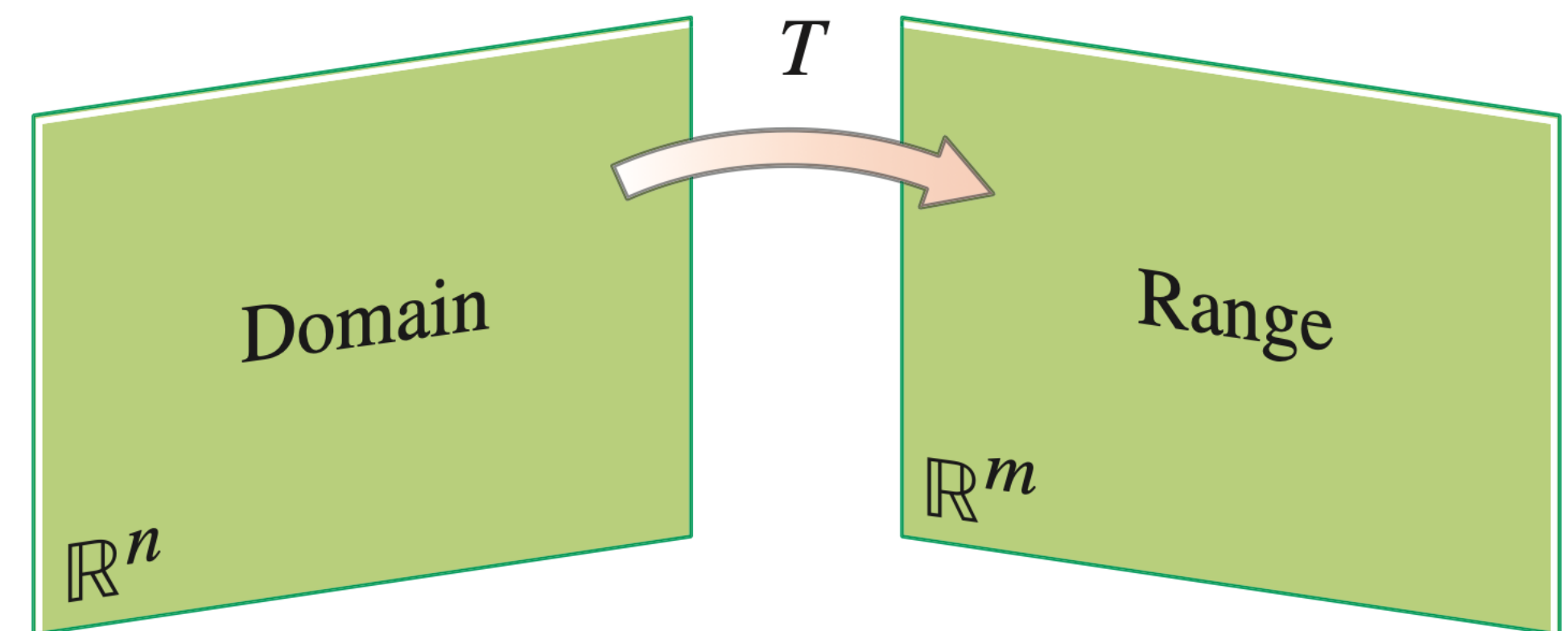
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Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at most one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$)

Onto (Pictorially)

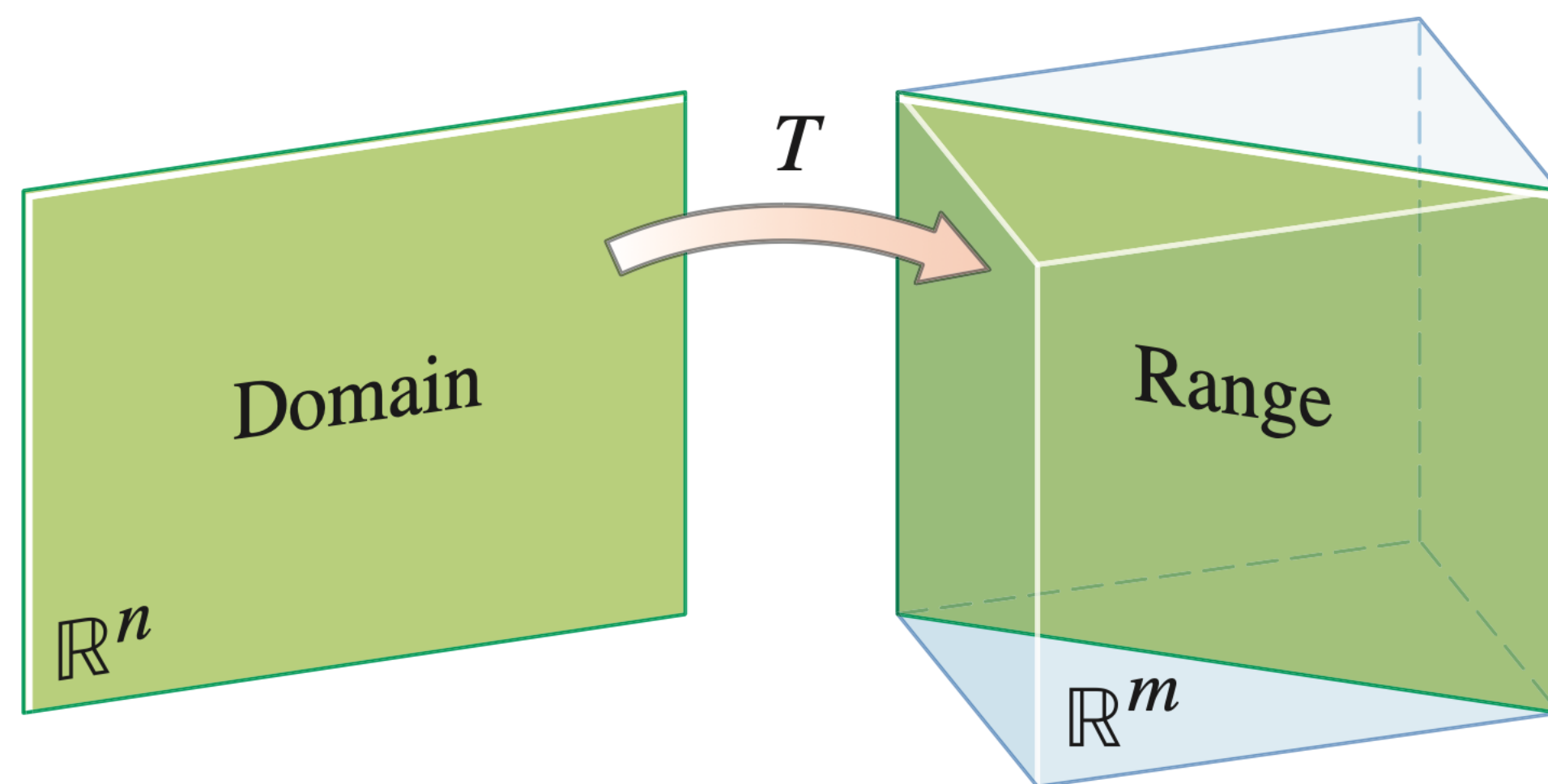


T is *not* onto \mathbb{R}^m

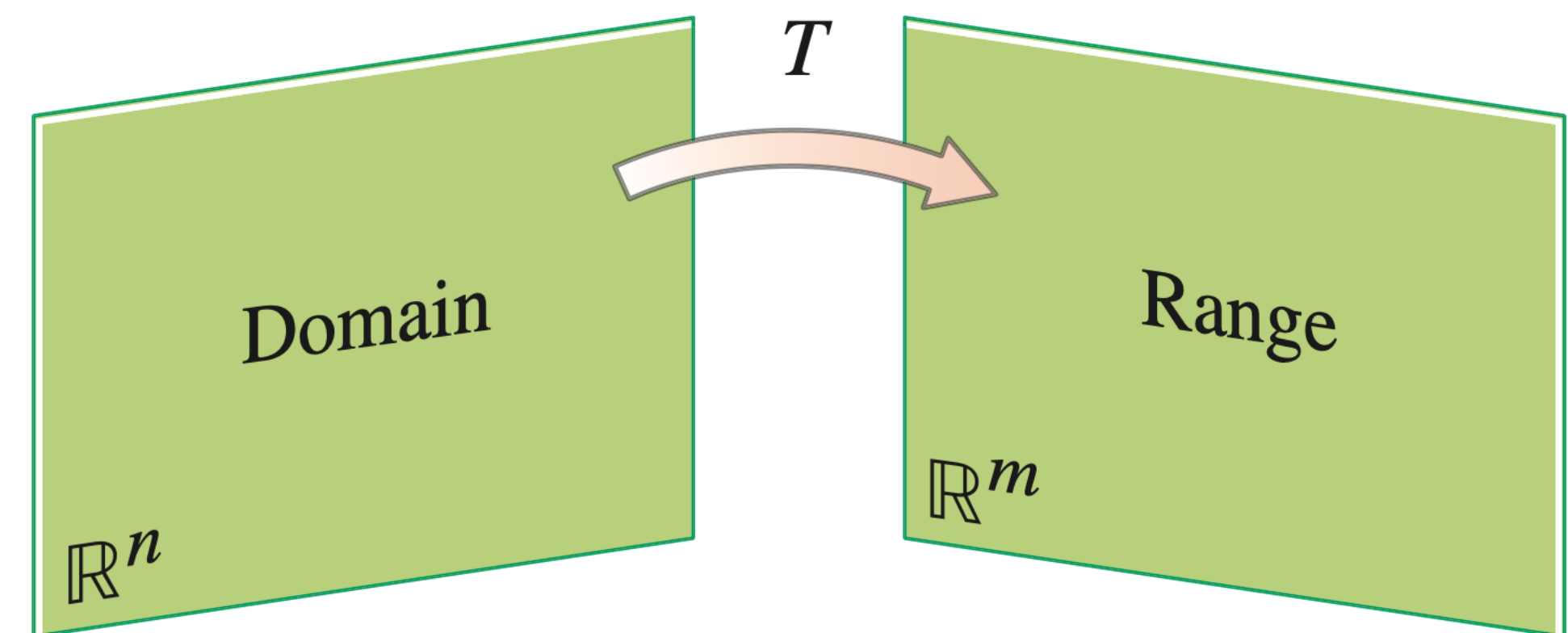


T is onto \mathbb{R}^m

Onto (Pictorially)



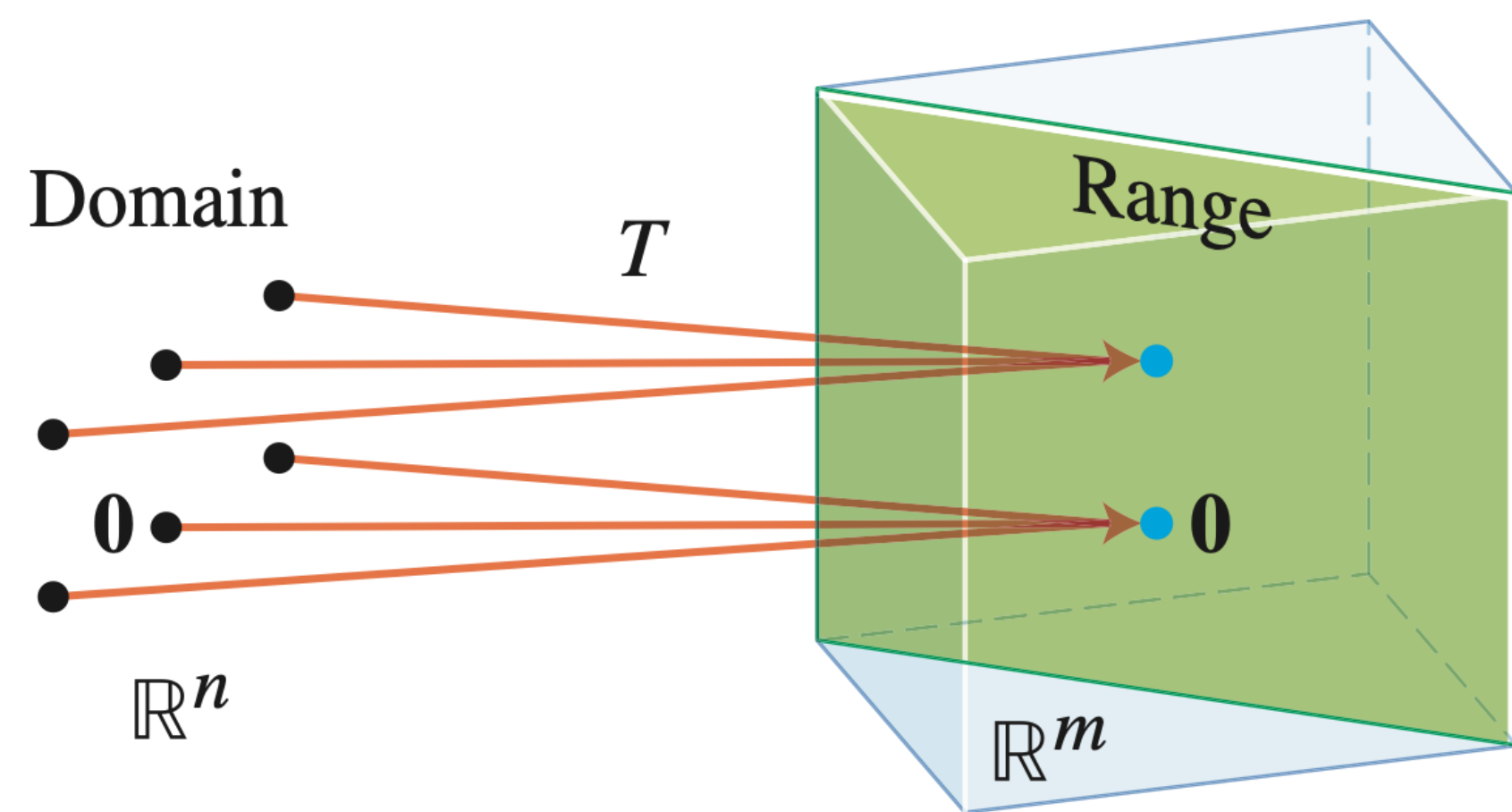
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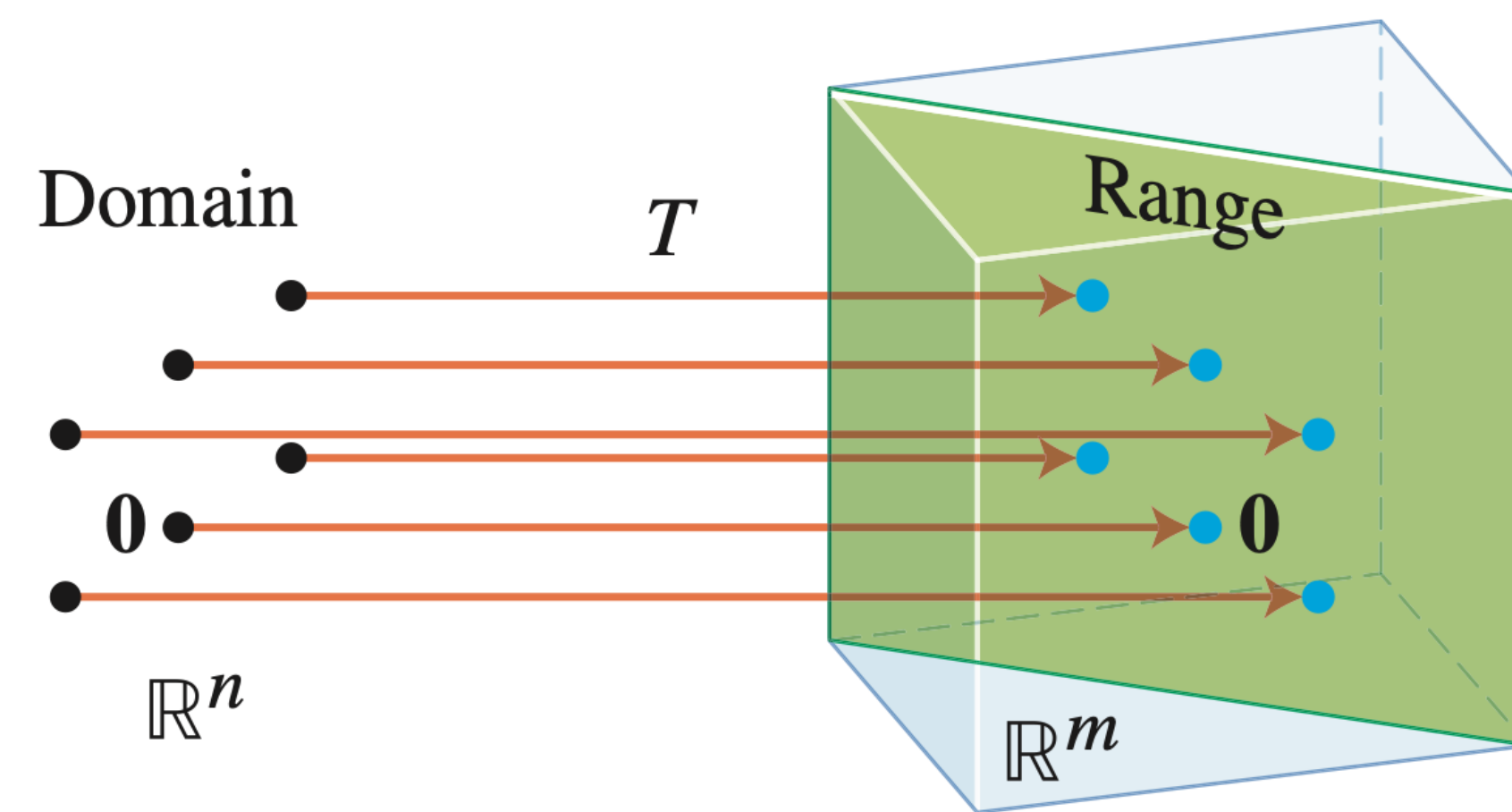
T is onto \mathbb{R}^m

T is onto if its range = its codomain

One-to-One (Pictorially)



T is *not* one-to-one



T is one-to-one

Taking Stock: Onto

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Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

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Taking Stock: Onto

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- » T is onto
- » $A\mathbf{x} = \mathbf{b}$ has a solution for any choice of \mathbf{b}

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- » T is onto
- » $A\mathbf{x} = \mathbf{b}$ has a solution for any choice of \mathbf{b}
- » $\text{range}(T) = \text{codomain}(T)$

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- » $\text{range}(T) = \text{codomain}(T)$
- » the columns of A span \mathbb{R}^m

Taking Stock: Onto

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- » $\text{range}(T) = \text{codomain}(T)$
- » the columns of A span \mathbb{R}^m
- » A has a pivot position in every row

Taking Stock: One-to-One

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Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

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Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

- » T is one-to-one
- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}

Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

- » T is one-to-one
- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
- » $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ implemented by the matrix A

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- » $A\mathbf{x} = \mathbf{b}$ has at most one solution for any \mathbf{b}
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- » The columns of A are linearly independent

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- » The columns of A are linearly independent
- » A has a pivot position in every column

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

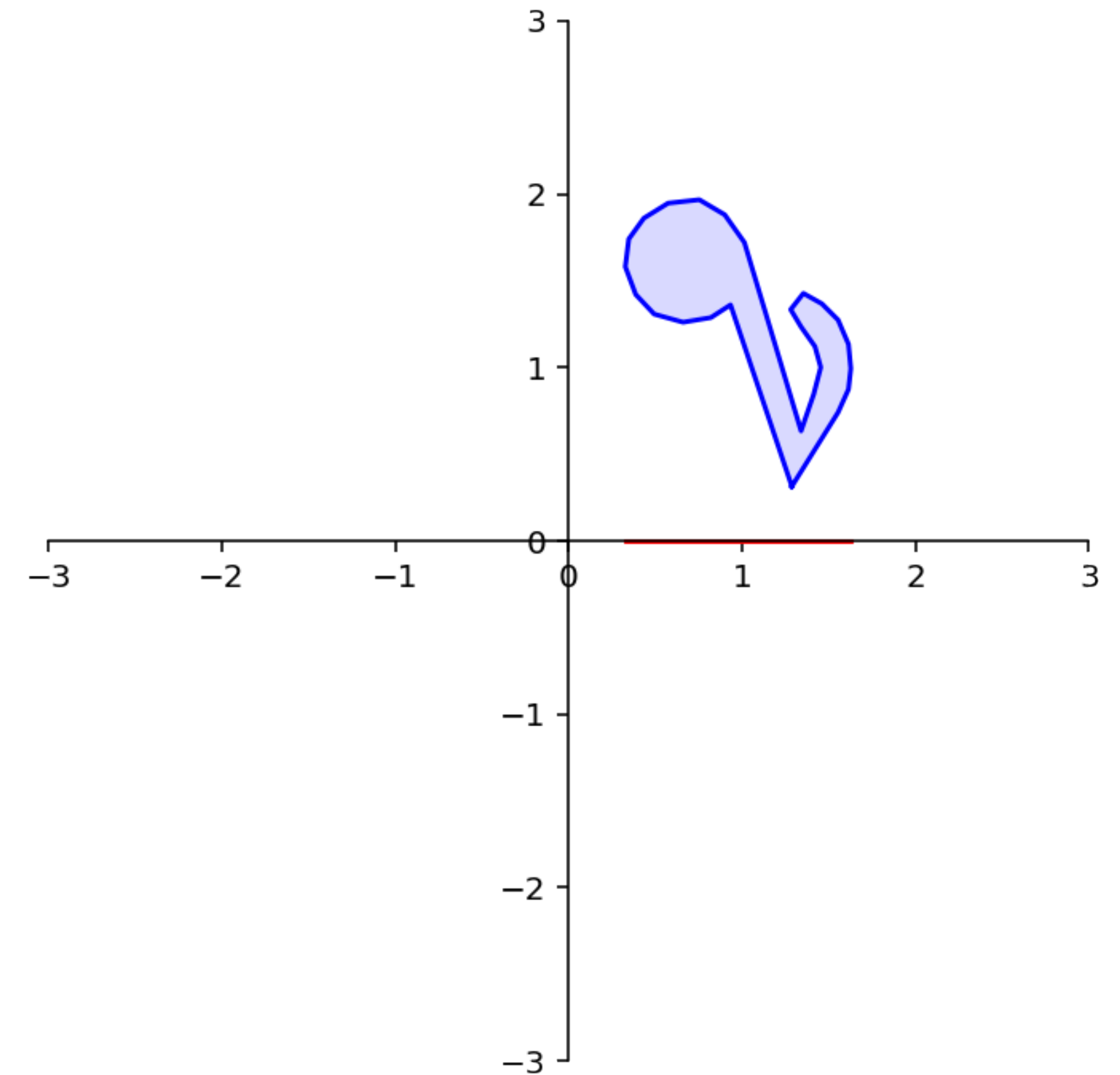
why? :

Example: not 1-1, not onto

Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

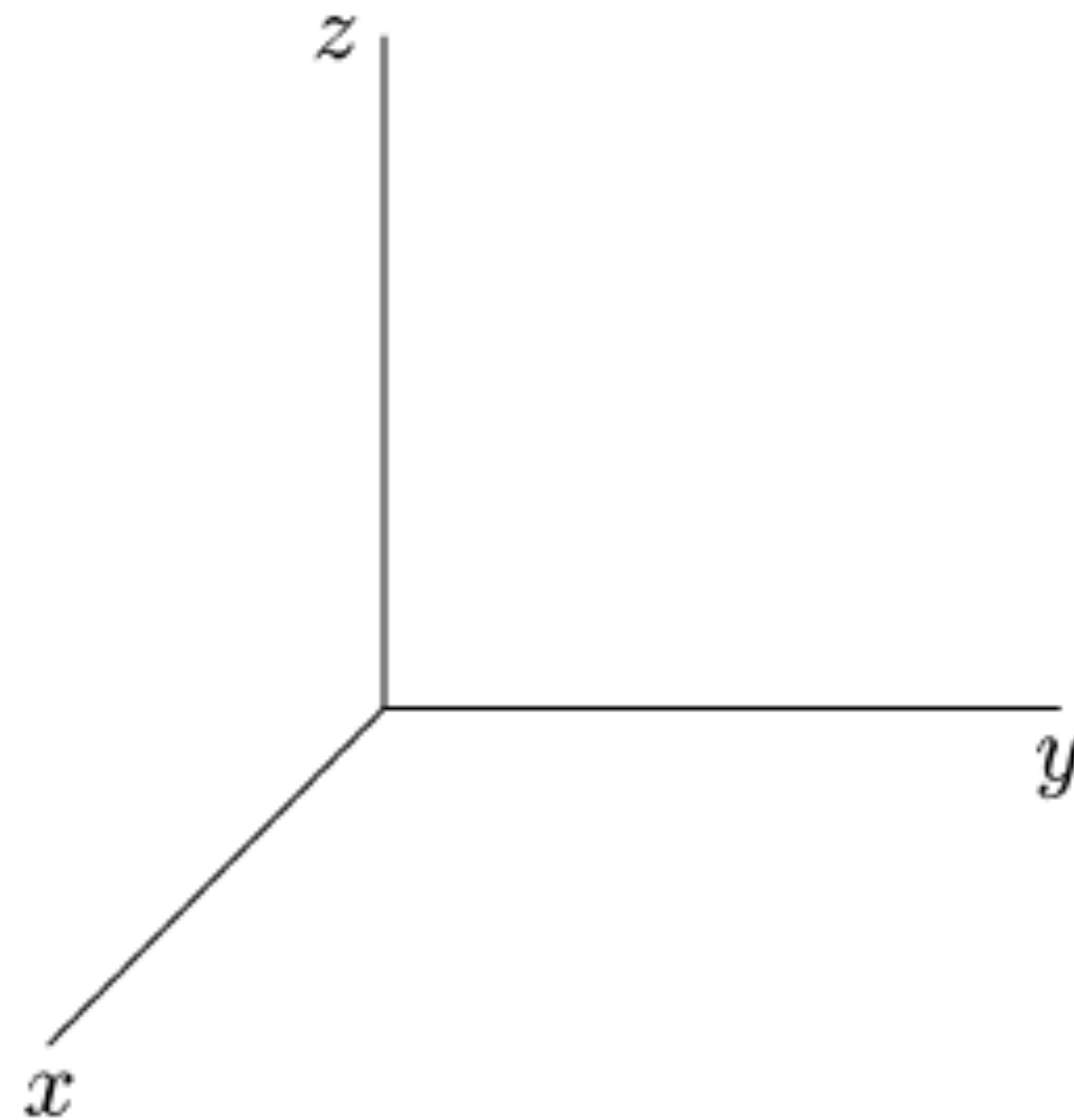


Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why? :



Summary

Matrix transformations and linear transformations are **the same thing**

We can find these matrices by looking at how the transformation behaves on the **standard basis**

We can reason about matrix equations by directly reasoning about the linear transformations