Matrix Operations

Geometric Algorithms
Lecture 10

CAS CS 132

Practice Problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_2 \\ 4x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$$

Determine if the above transformation is onto, one-to-one, both, or neither

Answer

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_2 \\ 4x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 - 0 \\ 0 + 0 \\ 3 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

is 1-1

not onto

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$\vec{x} \mapsto \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & -3 \end{bmatrix} \vec{x}$$

3 rows
≤ 2 pivots
NOT ONTO

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim$$

2 cols
$= \cancel{\leq} 2$ pivots
ONE-TO-ONE

Objectives

» Define several important matrix operations

» Motivate and define matrix multiplication and inverses

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

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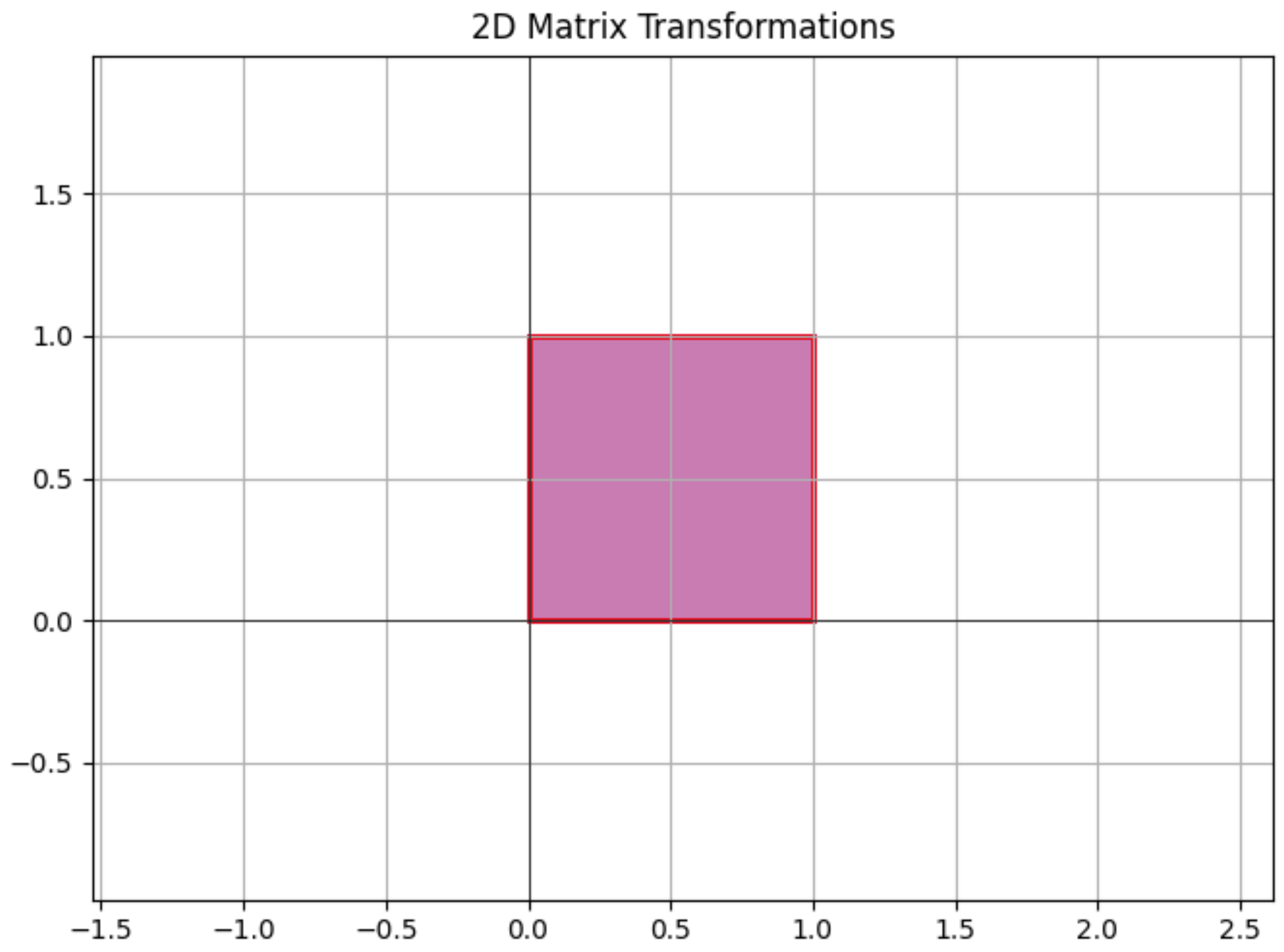
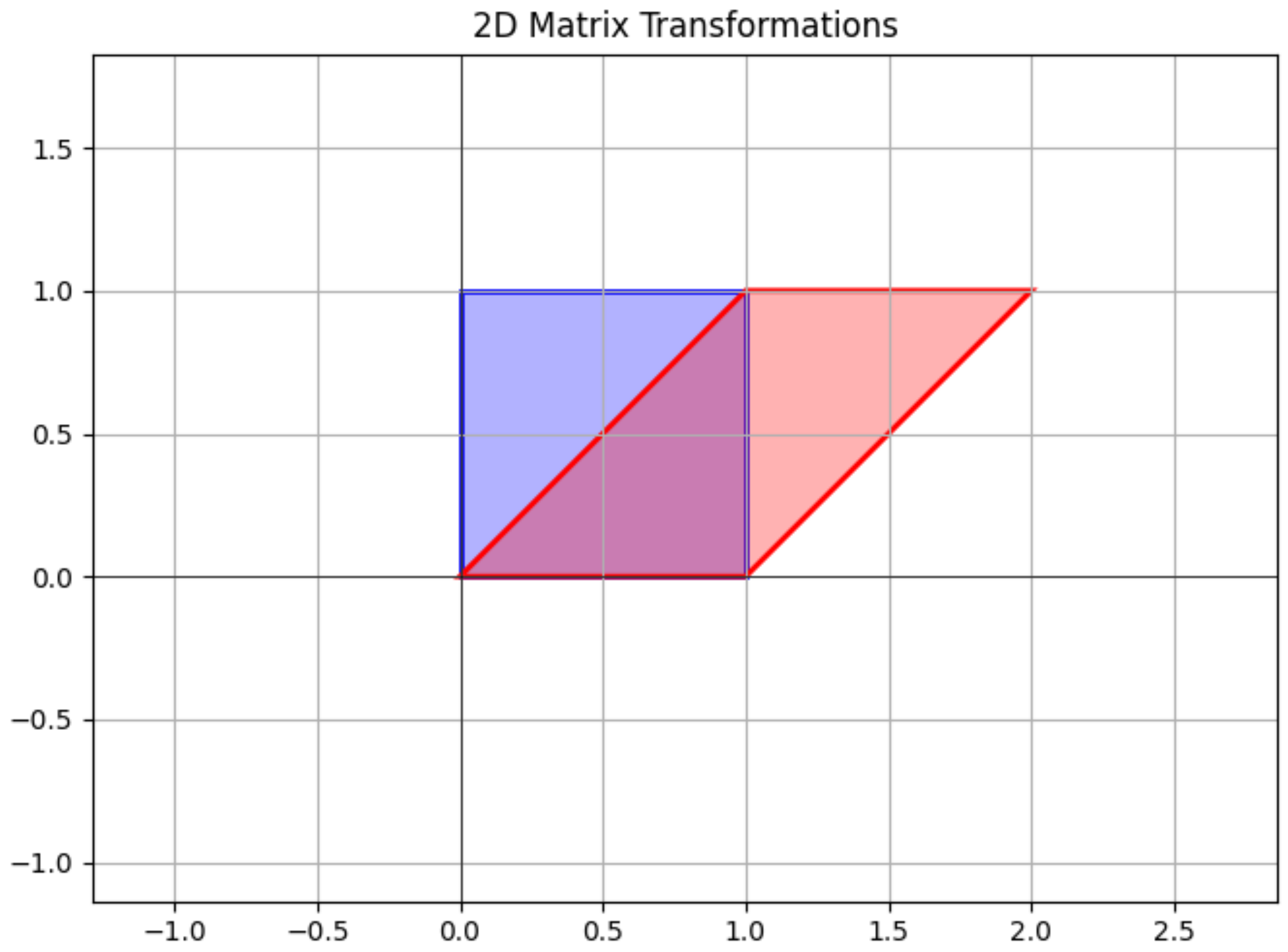
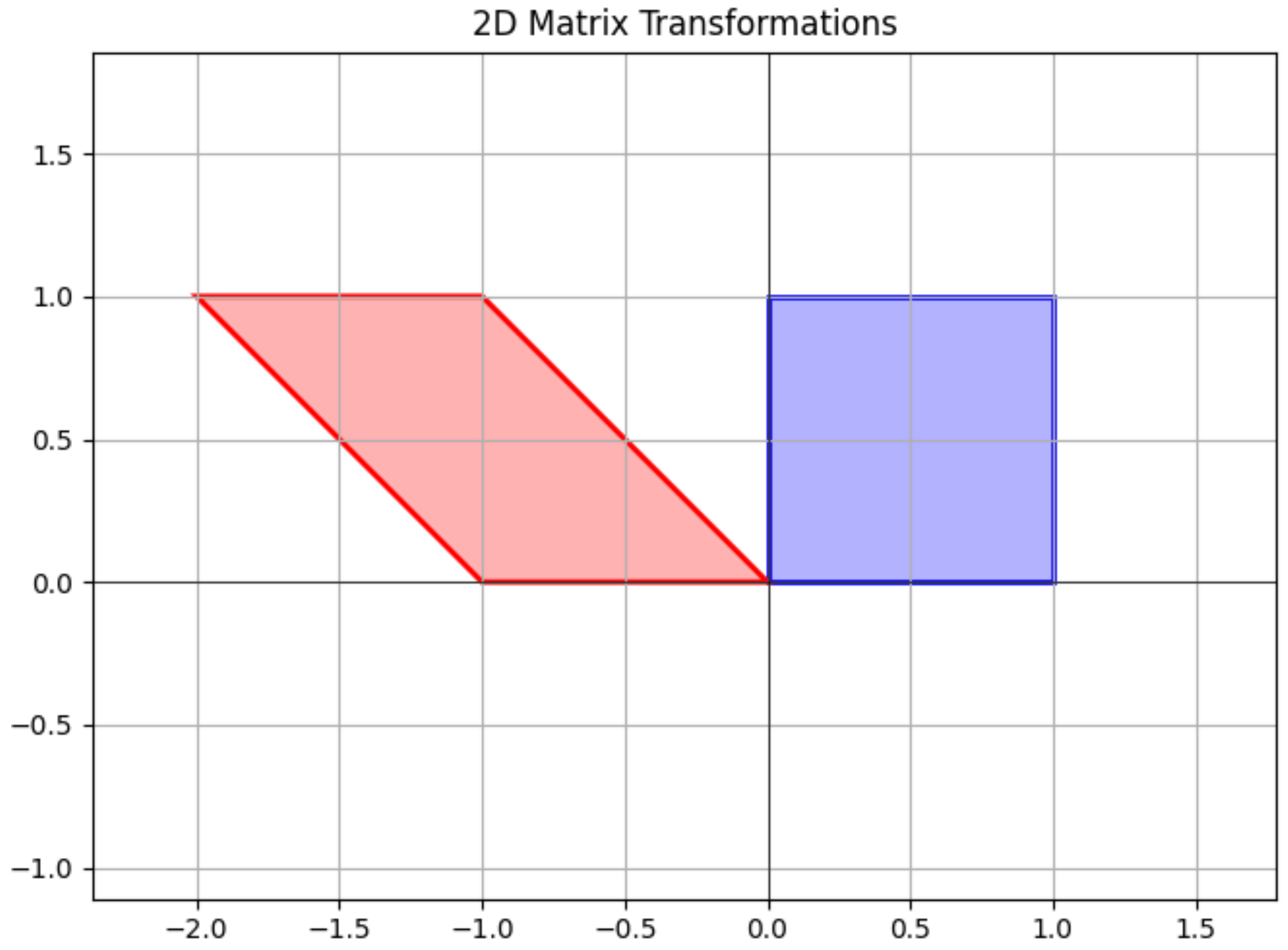
Invertible Matrix Theorem

Composing Linear Transformations

Shearing and Reflecting (Geometrically)







shear

reflect

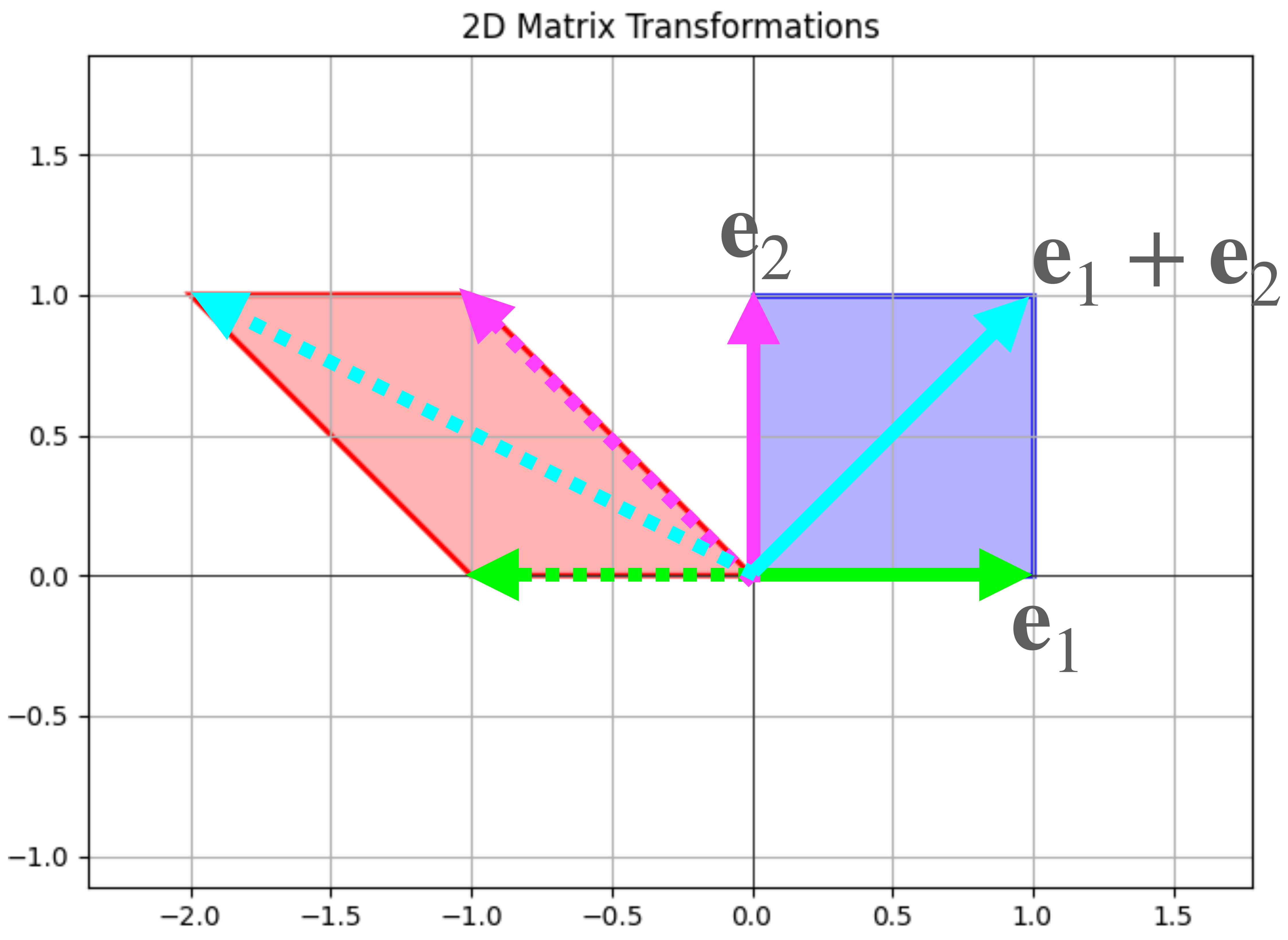
Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Shearing and Reflecting (Algebraically)

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\texttt{reflect}} \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\texttt{shear}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

First multiply by shear matrix, then multiply by reflection matrix

Shearing and Reflecting (Algebraically)

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{reflect}} \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{shear}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

First multiply by shear matrix, then multiply by reflection matrix

This gives us the same transformation

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

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Fact. The composition of two linear transformation is a linear transformation

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Verify: S, T

$$S(T(\vec{u}+\vec{v})) = S(T(\vec{u}) + T(\vec{v})) = S(T(\vec{u})) + S(T(\vec{v})) \checkmark$$

$$S(T(c\vec{v})) = S(cT(\vec{v})) = c\,S(T(\vec{v})) \checkmark$$

The Key Fact

Fact. The composition of two linear transformation is a linear transformation

Verify:

This means the composition of two matrix transformations can be represented as a *single* matrix

The Key Question

Given two linear transformations, how to we compute the matrix which implements their composition?

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Given two linear transformations, how to we compute the matrix which implements their composition?

Matrix Multiplication

Matrix Multiplication

Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$x_1 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$\left[\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

General Composition (2D)

$$A\left(\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = A\left(x_1\vec{b}_1 + x_2\vec{b}_2\right)$$

$$= x_1 A\vec{b}_1 + x_2 A\vec{b}_2$$

$$= \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column

Tracking Dimensions

This only works if the number of columns of the left matrix matches the number of rows of the right matrix

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \end{array} \begin{array}{cc} & k \\ n & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} = \begin{array}{cc} & k \\ m & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

$$(m \times n) \qquad (n \times k) \qquad (m \times k)$$

Important Note

Even if AB is defined, it may be that BA is <u>not</u> defined

Non-Example

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} & \begin{bmatrix}2\\4\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} & \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\end{bmatrix}$$

2×3 2×2 2×3 $\mathbb{R}^2$

Non-Example

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} & \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\end{bmatrix}$$

These are not defined.

Example

$$\underset{2\times 2}{\begin{bmatrix}1&2\\3&4\end{bmatrix}}\;\underset{2\times 3}{\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}} = \underbrace{\begin{bmatrix} \overbrace{\underset{2\times 2}{\begin{bmatrix}1&2\\3&4\end{bmatrix}}\underset{\mathbb{R}^2}{\begin{bmatrix}1\\4\end{bmatrix}}}^{\mathbb{R}^2} & \overbrace{\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}}^{\mathbb{R}^2} & \overbrace{\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}3\\6\end{bmatrix}}^{\mathbb{R}^2} \end{bmatrix}}_{2\times 3}$$

The Key Fact (Restated)

For any matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$ and any vector $\mathbf{v} \in \mathbb{R}^k$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B, the entry in row i and column j of AB is defined above

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1(1) + (0)(0) & -1(1) + 0(1) \\ 0(1) + 1(0) & 0(1) + (1)(1) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

Exercise

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

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What about when the right matrix is a single column?
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Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

Connection with Matrix-Vector Multiplication

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This is just vector multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

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This is just vector multiplication

We can think of $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$ as collection of simultaneous matrix–vector multiplications

Matrix "Interface"

multiplication	what does AB mean when A and B are matrices?
addition	what does $A + B$ mean when A and B are matrices?
scaling	what does cA mean when A is matrix and c is a real number?

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multiplication	what does AB mean when A and B are matrices?
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These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + \left[\mathbf{b}_1 \dots \quad \mathbf{b}_n\right] = \left[(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)\right]$$

Addition is done column-wise (or equivalently, element-wise)

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

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This is exactly the same as vector addition, but for matrices

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise)

e.g. $2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

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This is exactly the same as vector scaling, but for matrices

Algebraic Properties (Addition and Scaling)

$2+3 = 3+2$

$A + B = B + A$

$(A + B) + C = A + (B + C)$

$A + 0 = A$

$r(A + B) = rA + rB$

$(r + s)A = rA + sA$

$r(sA) = (rs)A$

In these properties A, B, and C are matrices of the same size and r and s are scalars ($\mathbb{R}$)

We need to know/memorize these

Algebraic Properties (Addition and Scaling)

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = \cancel{BC + CA}\ BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

In these properties A, B, and C are matrices of the appropriate size so that everything is defined, and r is a scalar

We need to know/memorize these

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$
$T_2 : \mathbb{R}^2 \to \mathbb{R}^3$

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1

(also find a pair where they <u>are</u> the same)

One Answer: Rotation and Reflection

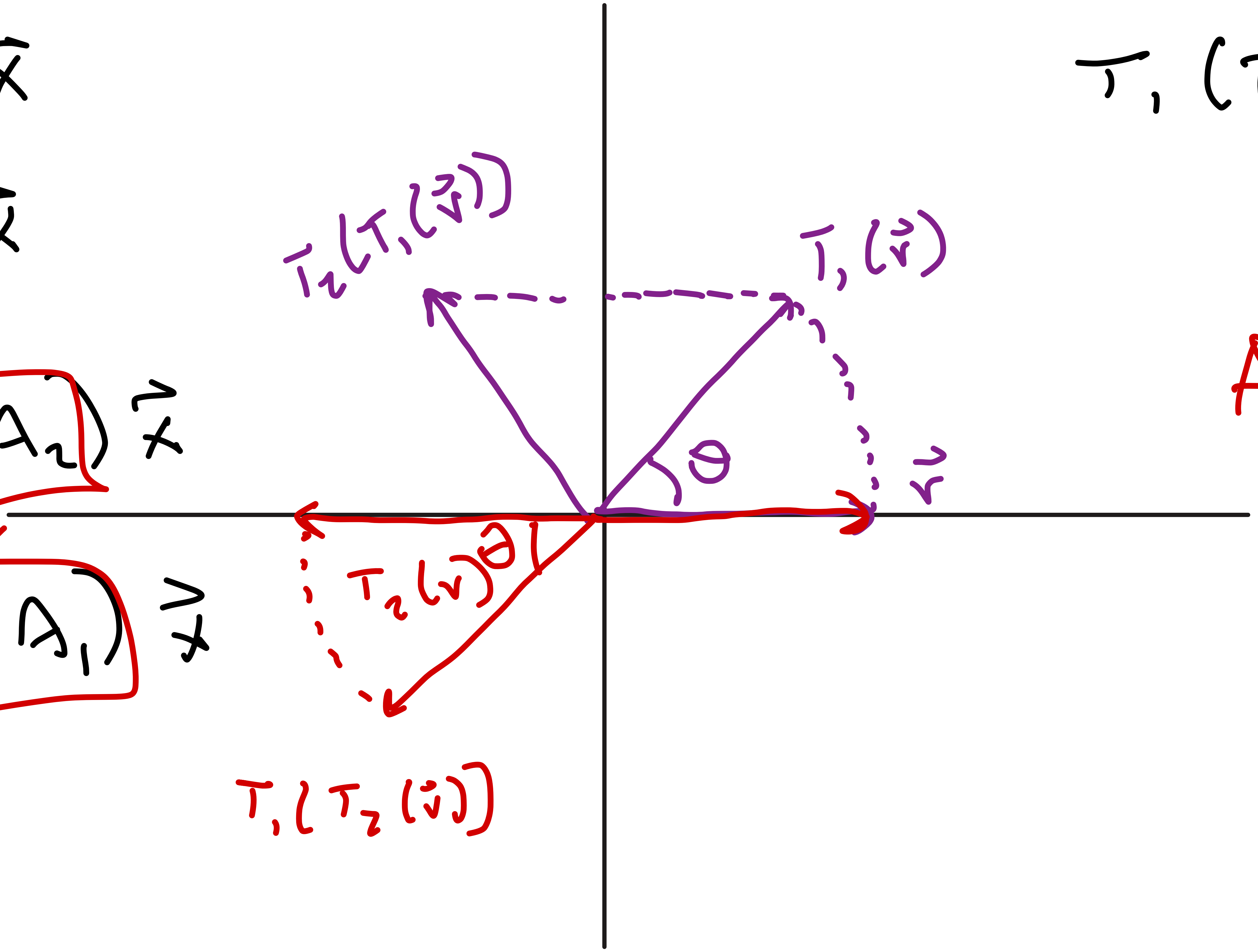
$$T_1(\vec{x}) = A_1\vec{x}$$

$$T_2(\vec{x}) = A_2\vec{x}$$

$$T_1(T_2(\vec{x})) = \boxed{(A_1 A_2)}\,\vec{x}$$

$$\neq$$

$$T_2(T_1(\vec{x})) = \boxed{(A_2 A_1)}\,\vec{x}$$



$$T_1(T_2(\vec{v})) \neq T_2(T_1(\vec{v}))$$

$$A_1 A_2 \neq A_2 A_1$$

More Matrix Operations

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)

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Transpose

Definition. For a $m \times n$ matrix A, the **transpose of** A, written A^T, is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$

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$$(A^T)^T = A$$

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$(cA)^T = cA^T$ (where c is a scalar)

$\boxed{(AB)^T = B^T A^T}$ Important: the order reverses!

not necessarily $= A^T B^T$

Challenge Problem

Demonstrate that $(AB)^T = B^T A^T$ in general.

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

$$\stackrel{?}{=} \cdots$$

Transposes and Inner Products

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For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

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$$\underset{1 \times n}{\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}} = \underset{1 \times 1}{?}$$

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$$\underset{\boxed{1} \times \boxed{n}}{\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}} \underset{\boxed{n} \times \boxed{1}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}} = \underset{\boxed{1} \times \boxed{1}}{?}$$

Transposes and Inner Products

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix Powers

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$n \times n$ $n \times n$

If A is an $n \times n$ matrix, then the product AA is defined

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What should A^0 be? (we want $A^0 A^k = A^{0+k} = A^k$)

$10^0 = 1$, so it stands to reason that $A^0 = I$

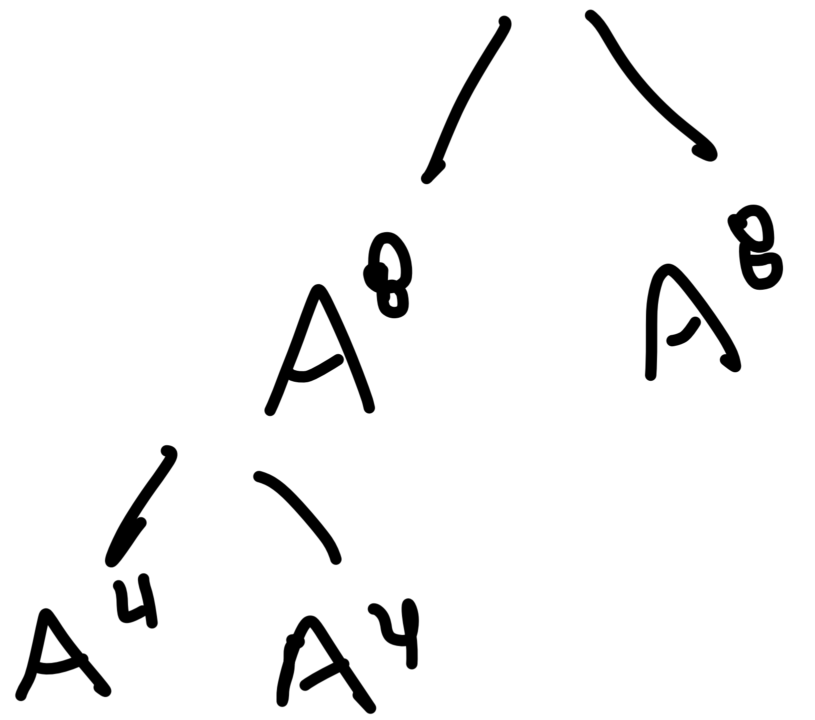
Matrix Powers (Computationally)

We can use **numpy.linalg.matrix_power**

This can be *much* faster than doing a sequence of matrix multiplications, e.g., in the case of

$$A^{16}$$

$$(((A^2)^2)^2)^2$$



Why?:

Final Warnings about Matrix Multiplication

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2. If $AB = AC$ then it is not necessary that $B = C$.

3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Exercise:

Find two nonzero 2×2 *matrices* A *and* B *such that* $AB = 0$

Challenge. *Choose* A *and* B *such that they have* *<u>all</u>* *nonzero entries*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

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`transpose` A^T

So Far: Matrix Operations

`transpose`	A^T
`scaling`	cA

So Far: Matrix Operations

transpose	A^T	
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addition (subtraction)	$A + B$	$A + (-1)B = A - B$

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multiplication (powers)	AB	A^k

What's missing?

Matrix Inverses

Recall: The Identity Matrix

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The identity matrix implements the "do nothing" transformation. For any $\mathbf{v}$,

$$I\mathbf{v} = \mathbf{v}$$

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$$IA = AI = A$$

These may be different sizes

Recall: The Identity Matrix

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}$$

2×2 $\quad$ 2×4 $\quad$ 2×4 $\quad$ 4×4 $\quad$ 2×4

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

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Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Algebra

$$2x = 10$$

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How do we solve this equation?

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How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

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$$2^{-1}(2x) = 2^{-1}(10)$$

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Do all matrices have inverses?

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No. If they did, then every linear system would have a solution

When does a matrix have an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

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» whose columns can have full span and be linearly independent

» that can have inverses

Matrix Inverses

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Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

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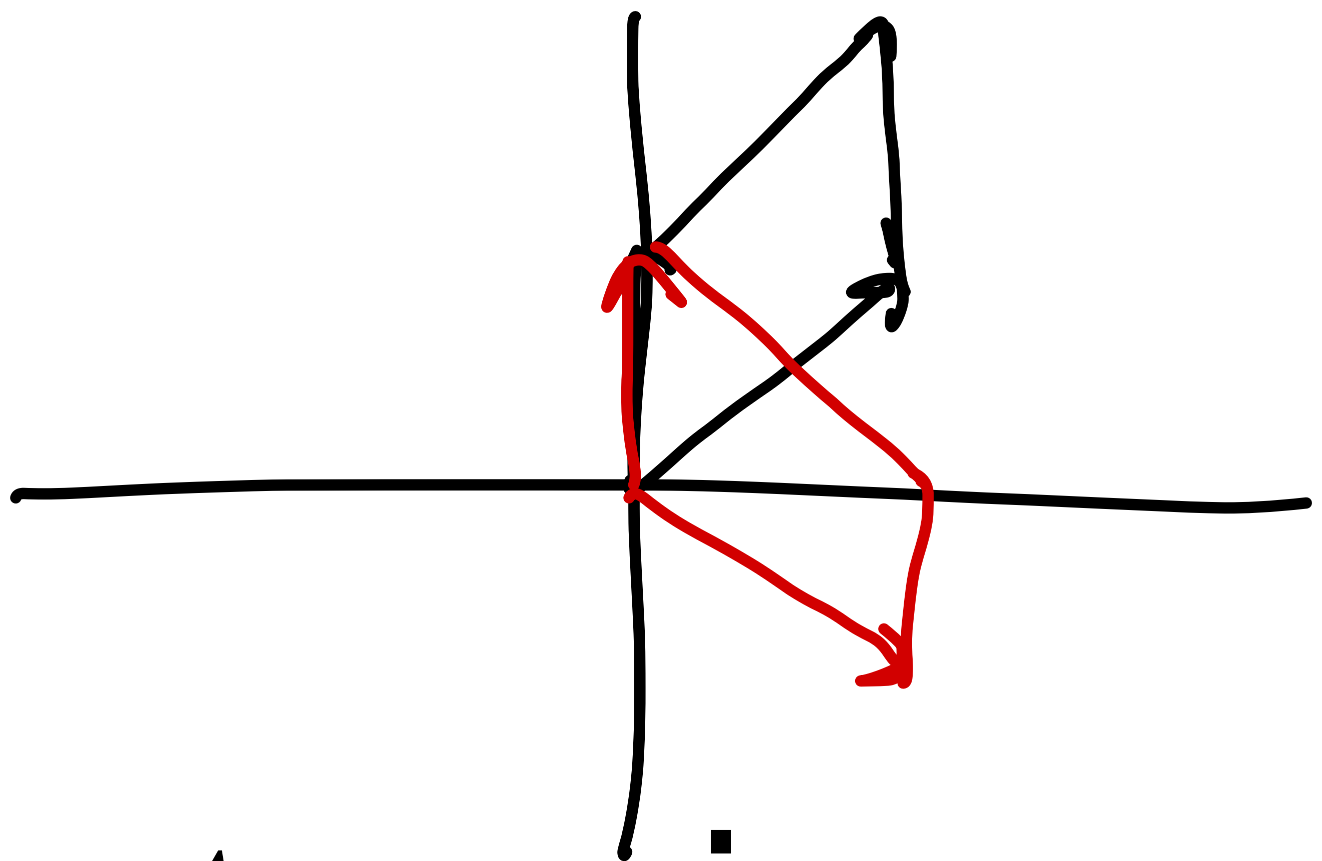
A is **invertible** if it has an inverse. Otherwise it is **singular.**

Matrix Inverses



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$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}$

Example: Geometric

Reflection across the x_1–axis in $\mathbb{R}^2$ is it's own inverse.

Verify:

Example: No inverse

Verify:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of $\mathbf{b}$.

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a <u>unique</u> solution for any choice of $\mathbf{b}$, then it has

» <u>exactly one</u> solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» at least one solution for any choice of $\mathbf{b}$

» at most one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

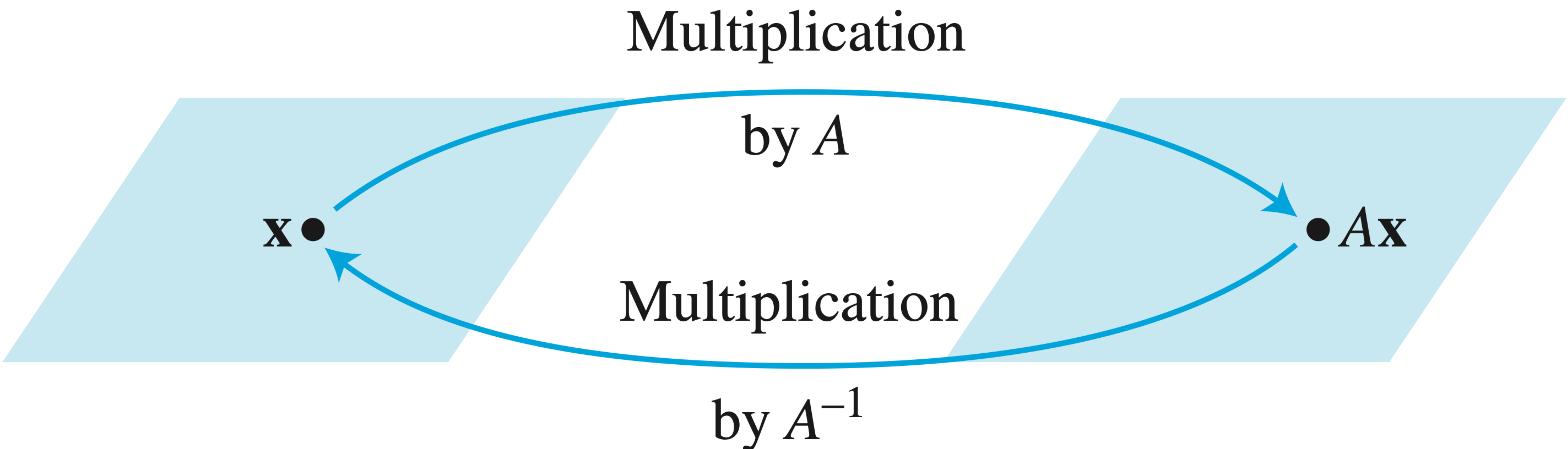
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$



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Non-Example. Projection onto the x_1-axis

Connection to Transformations

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$)

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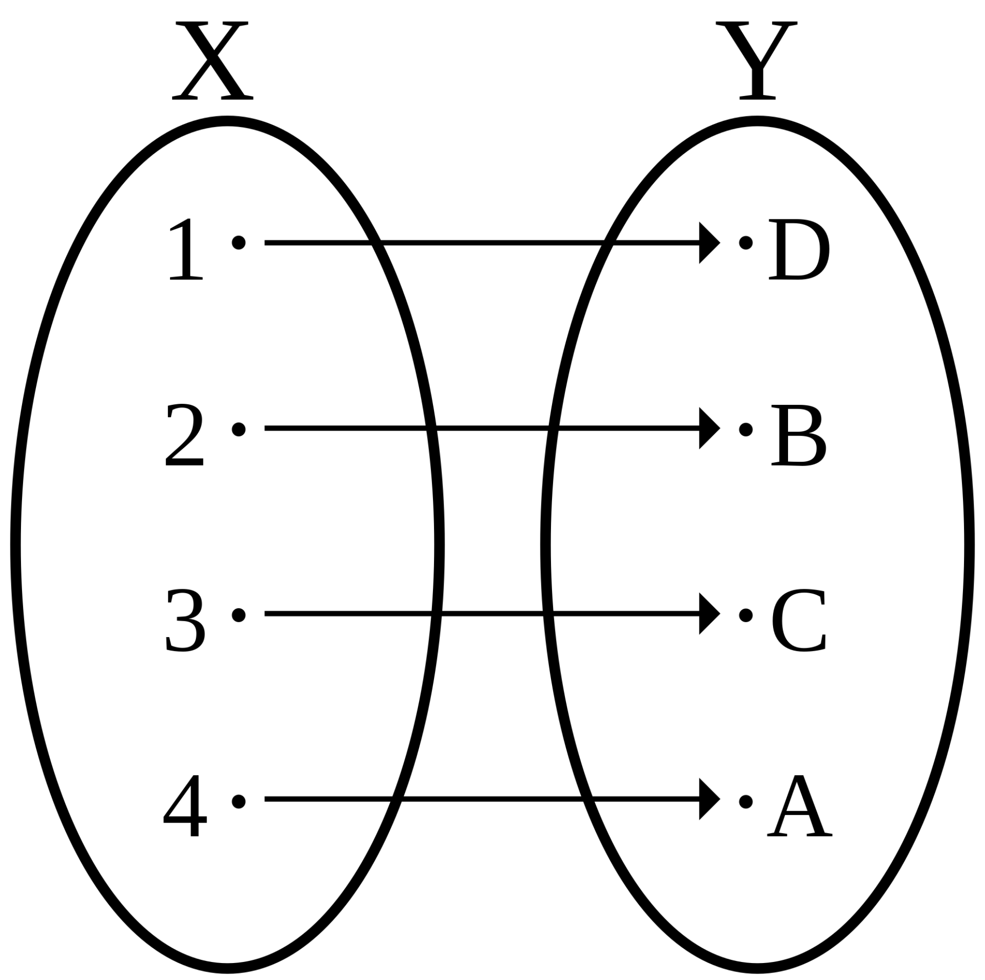
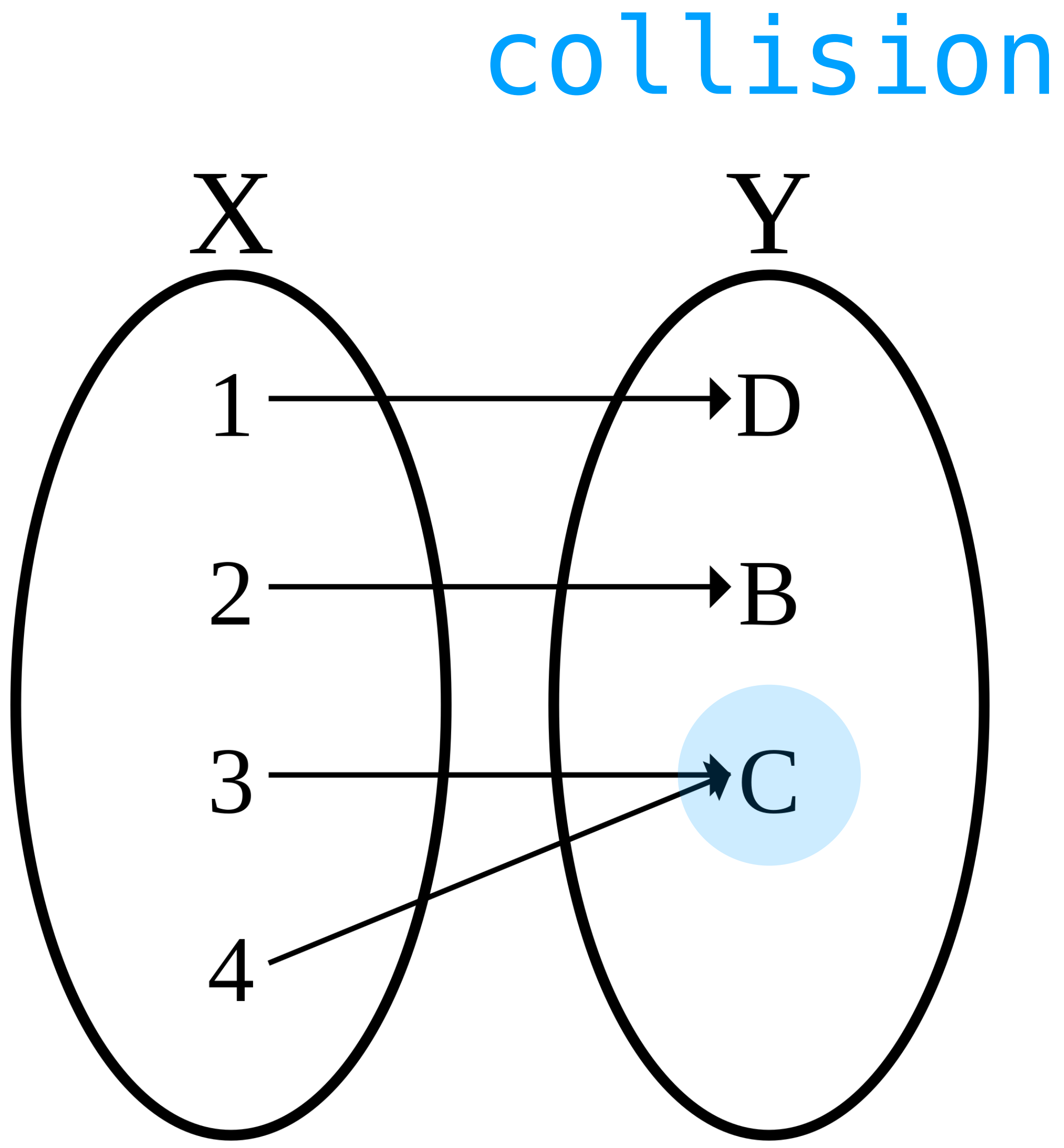
Invertible transformations are 1-1 correspondences

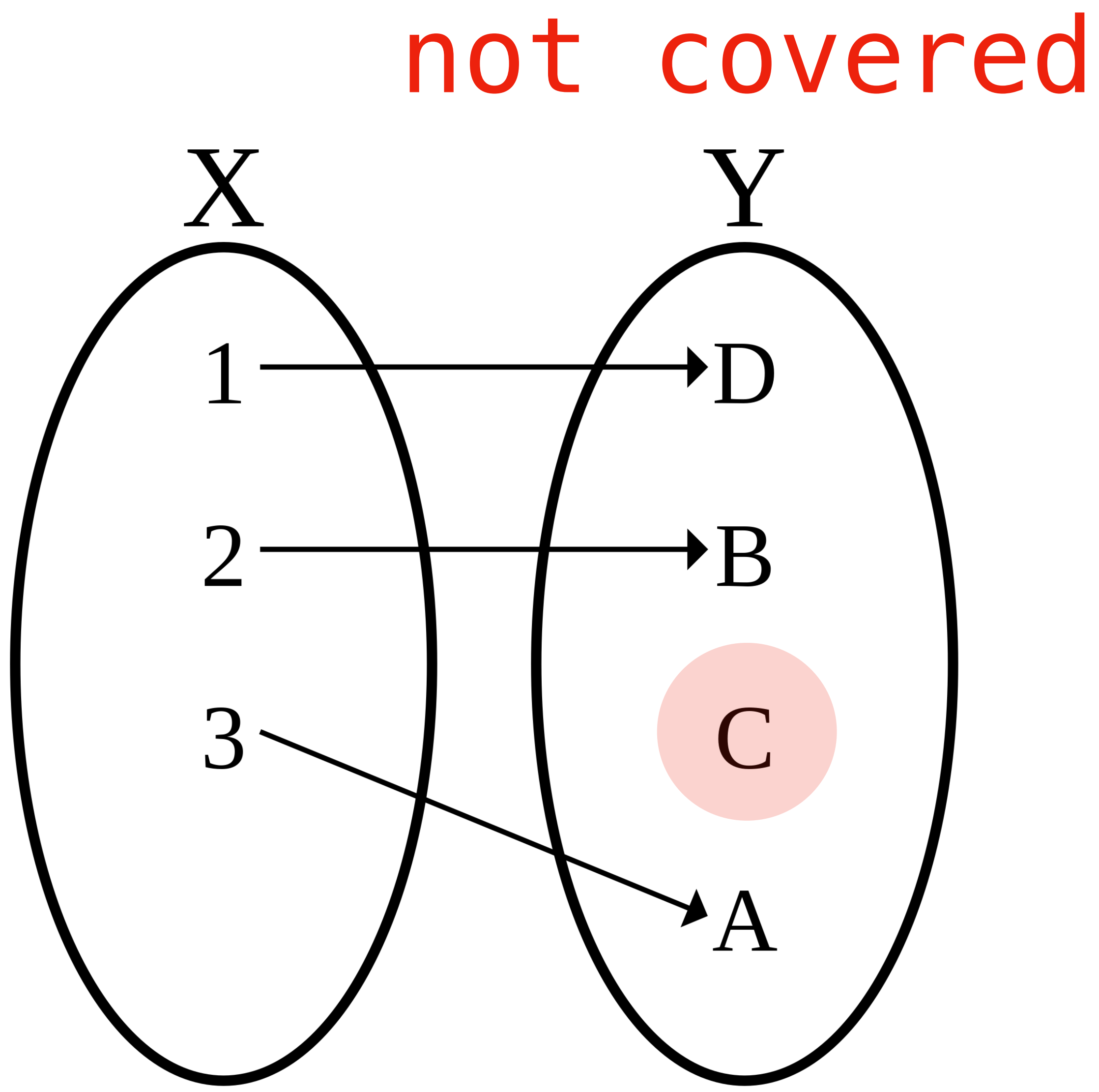
Kinds of Transformations (Pictorially)



1–1 correspondence



onto, not 1–1



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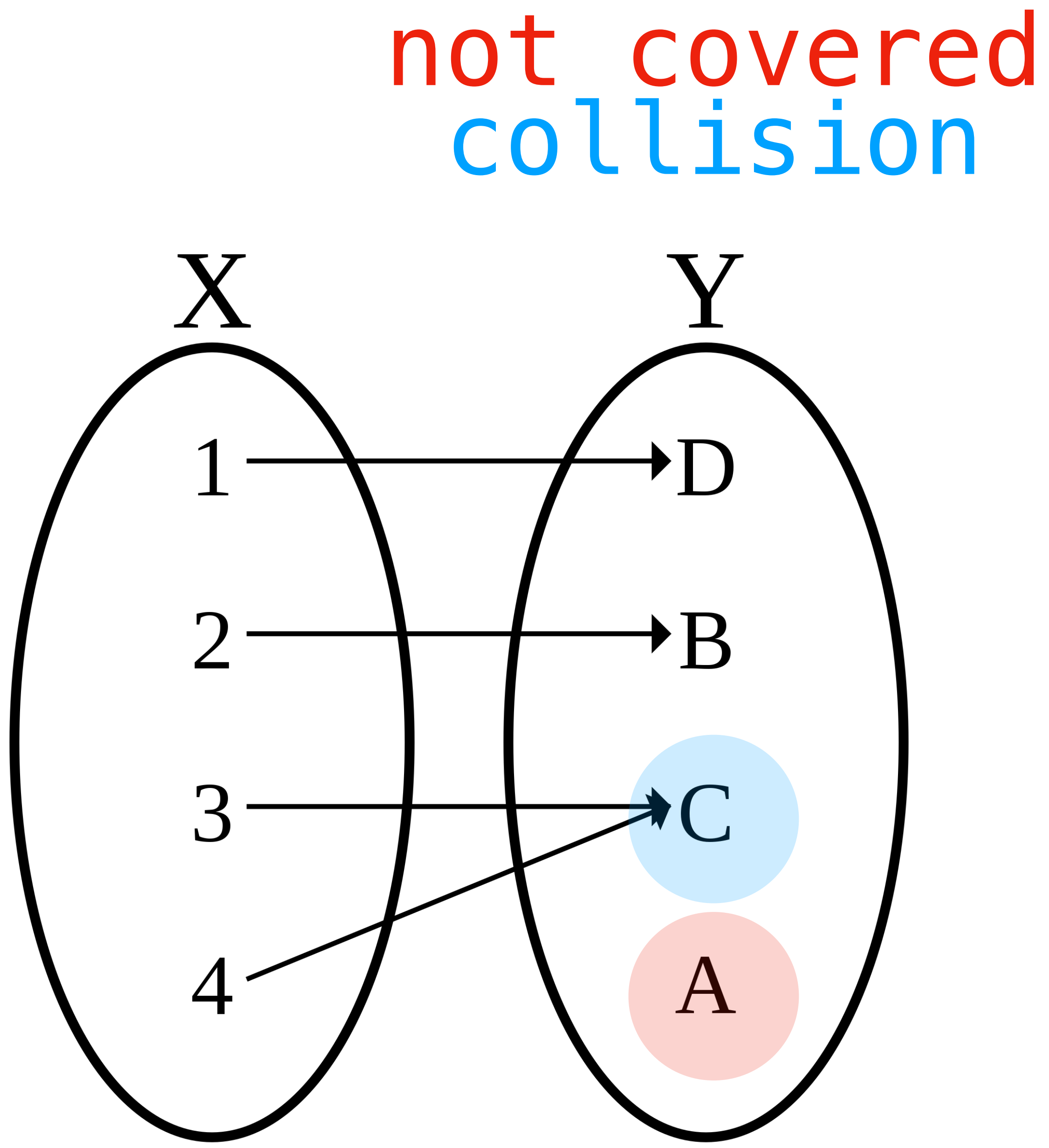

not 1–1, not onto

Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

In General

$$A\mathbf{b}_1 = \mathbf{e}_1 \qquad A\mathbf{b}_2 = \mathbf{e}_2 \qquad A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns)

In General

$$\mathbf{Ab}_1 = \mathbf{e}_1 \qquad \mathbf{Ab}_2 = \mathbf{e}_2 \qquad \mathbf{Ab}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns)

We need to solve 3 matrix equations.

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector $\mathbf{e}_i$. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A

This is really the same thing. It's a simultaneous reduction demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The **determinant** of a 2×2 matrix is the value $ad - bc$

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C, demonstrate that $B^{-1} = C$

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both)

This works because inverses are unique

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen*

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at <ins>least</ins> one solution for every $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has at <ins>most</ins> one solution for every $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at <ins>exactly</ins> one solution for every $\mathbf{b}$

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Verify:

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Verify:

Recall: Onto Transformations

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Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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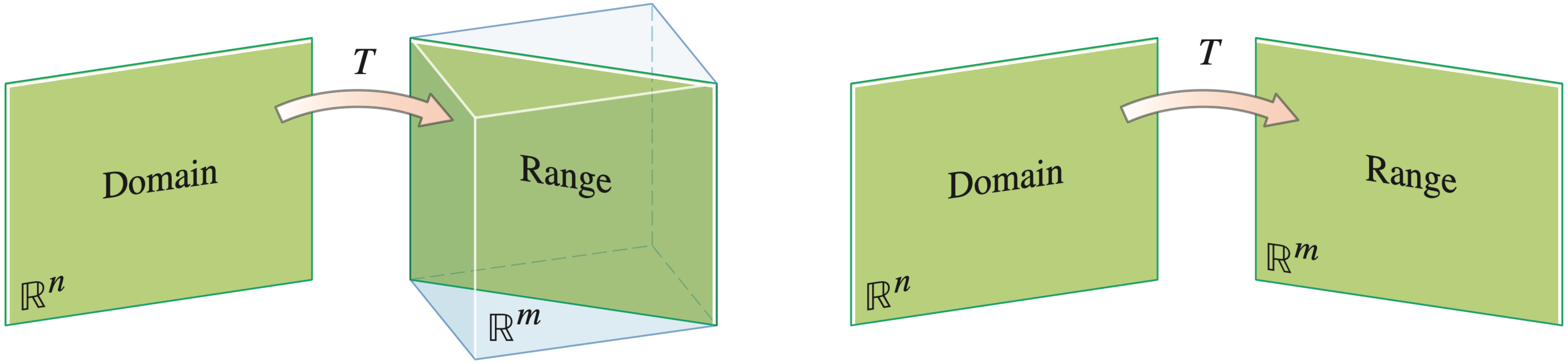

T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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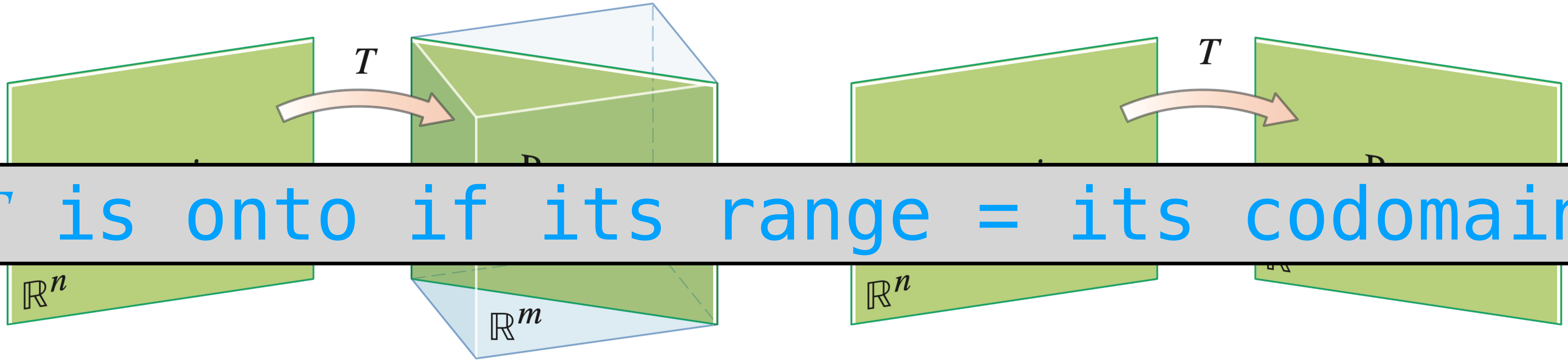

T is onto if its range = its codomain

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Recall: One-to-one Transformations

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Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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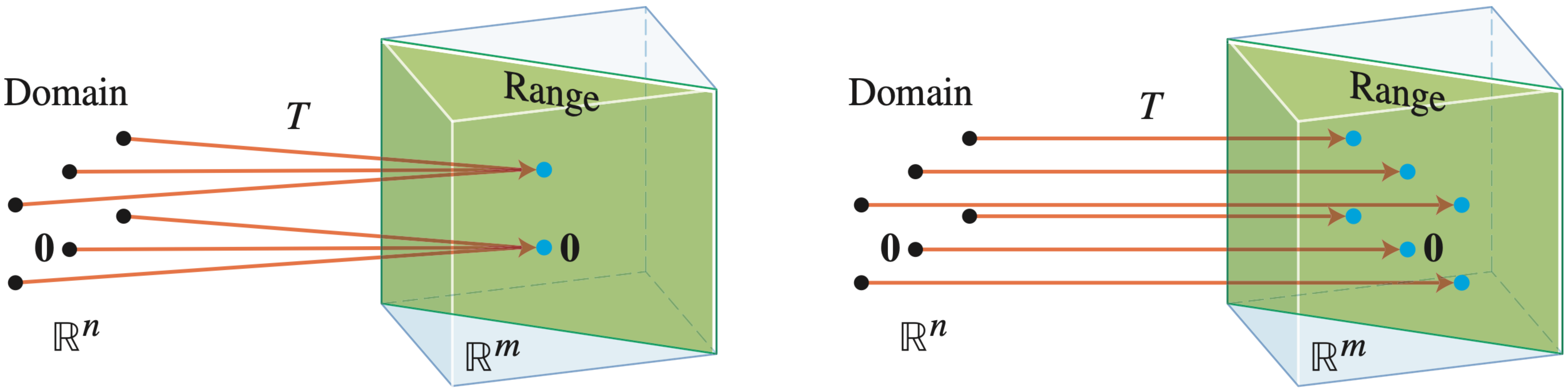

T is *not* one-to-one

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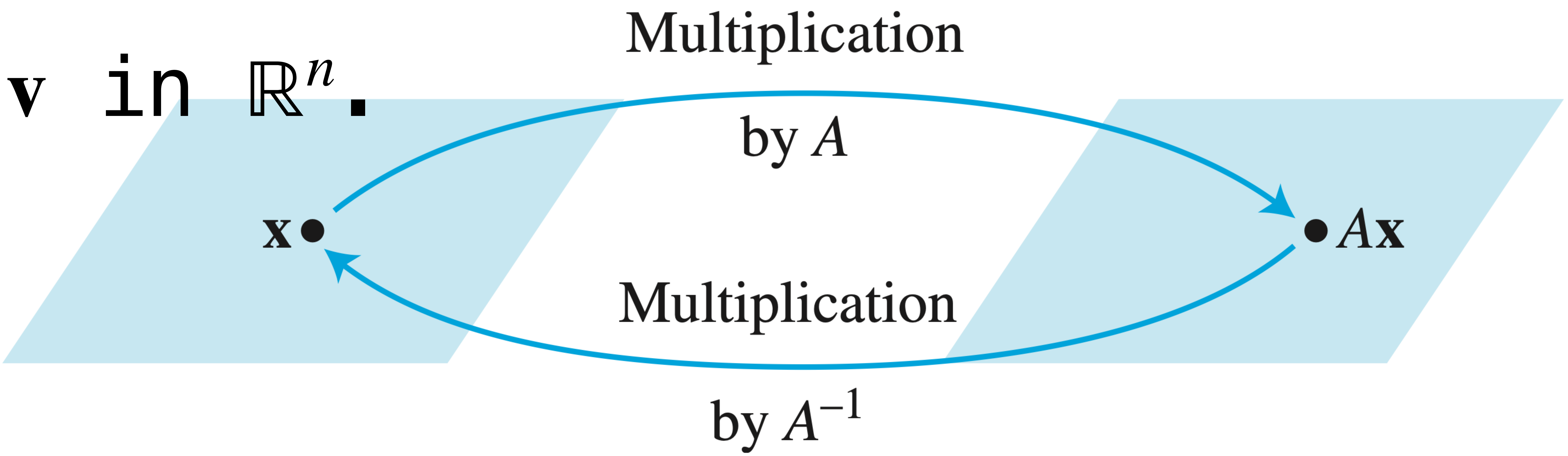
Recall: Invertible Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$S(T(\mathbf{v})) = \mathbf{v}$ and $T(S(\mathbf{v})) = \mathbf{v}$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Definition. A transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
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These all express the **same thing**

(this is a stronger statement than we just verified)

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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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We only need to check one of these.

Warning. Remember this only applies square matrices.

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

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A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

***True** or **False:** If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), the B is also invertible.*

Answer: True

```
Row reductions don't change the number of
pivots.
```

Question

If $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ *is invertible, then is* $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]$ *also invertible? Justify your answer.*

Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by <u>row operations</u>

Summary

The algebra of matrices can help us simplify matrix expressions

The invertible matrix theorem connects all the perspectives we've taken so far