Matrix Inversion & LU Factorization

Geometric Algorithms
Lecture 11

CAS CS 132

Objectives

- Demonstrate how to **invert** a matrix
- Motivate **matrix factorization** in general, and the LU factorization in specific
- Recall elementary row operations and connect them to matrices
- Look at the **LU factorization**, how to find it, and how to use it

Keywords

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

Determinant

Invertible Matrix Theorem

elementary matrices

LU factorization

Matrix Inverses

Basic Algebra

$$2x = 10$$

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Divide on both sides by 2 to get $x = 5$.

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$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

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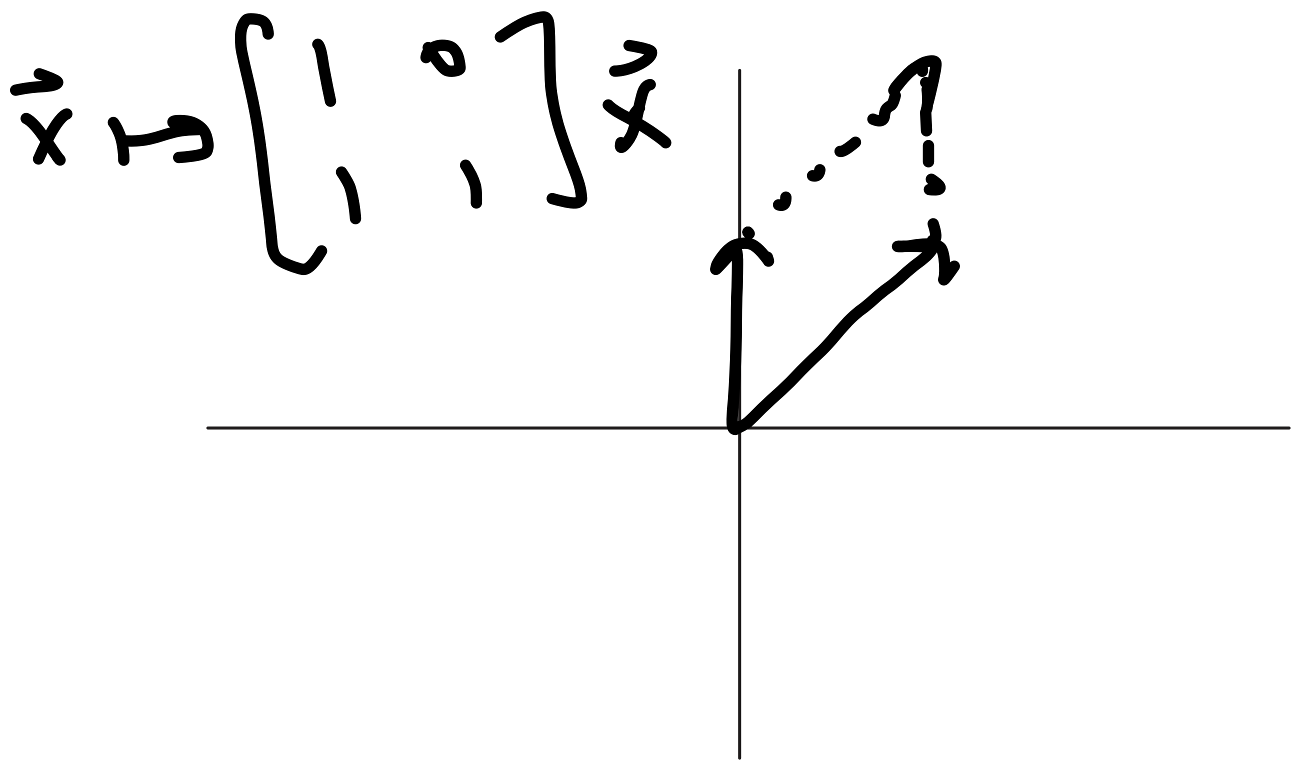
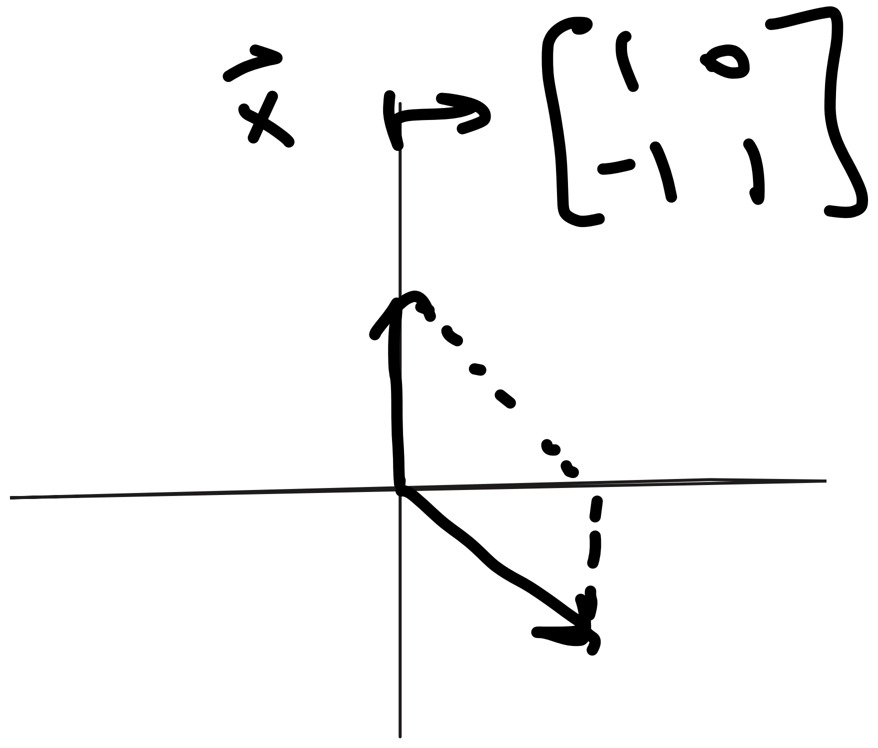
Matrix Inverses





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$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(1) + 0(-1) & 1(0) + 0(1) \\ 1(1) + 1(-1) & 1(0) + 1(1) \end{bmatrix}$$

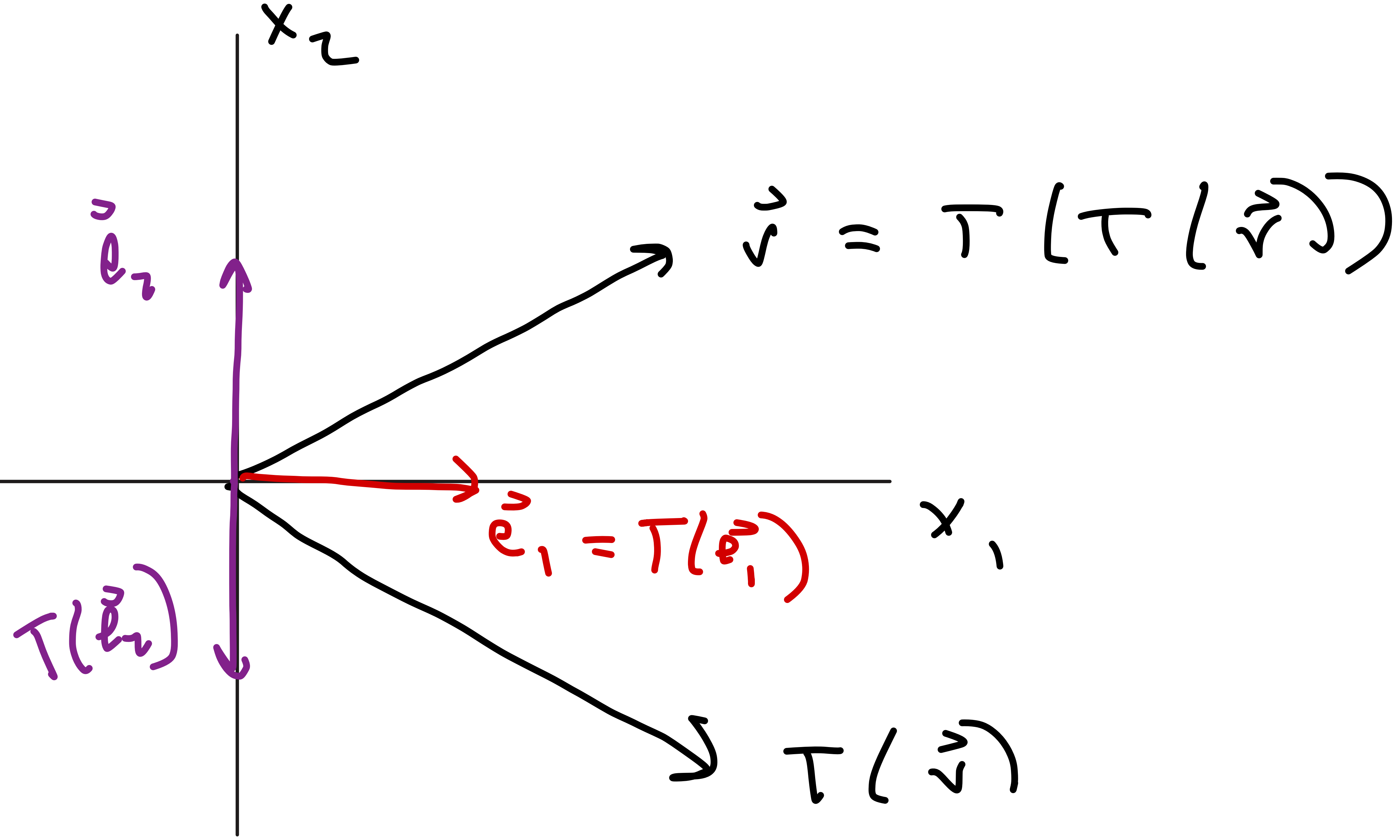
Example: Geometric

$S(T(\vec r))$ $\quad S(\vec x) = A\vec x \quad T(\vec x) = B\vec x \quad S(T(\vec x)) = (AB)\vec x$

Reflection across the x_1–axis in $\mathbb{R}^2$ is it's own inverse.

Verify: $T(\vec r) = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\vec r$

$\begin{bmatrix}1&0\\0&-1\end{bmatrix} \quad T^{-1}(\vec r) = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\vec r$



$$\boxed{\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}}$$

Example: No inverse

Verify:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{b} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow v_3 = 0 \qquad \vec{b} \in \mathbb{R}^3$$

$$A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} =$$

$$\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} =$$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify: $BA = AB = I \qquad AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C$$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a <u>unique</u> solution for any choice of $\mathbf{b}$, then it has

» <u>exactly one</u> solution for any choice of $\mathbf{b}$

$$\vec{x} = A^{-1}\vec{b}$$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» at least one solution for any choice of $\mathbf{b}$

» at most one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

where T is implemented by A

Recall: Onto Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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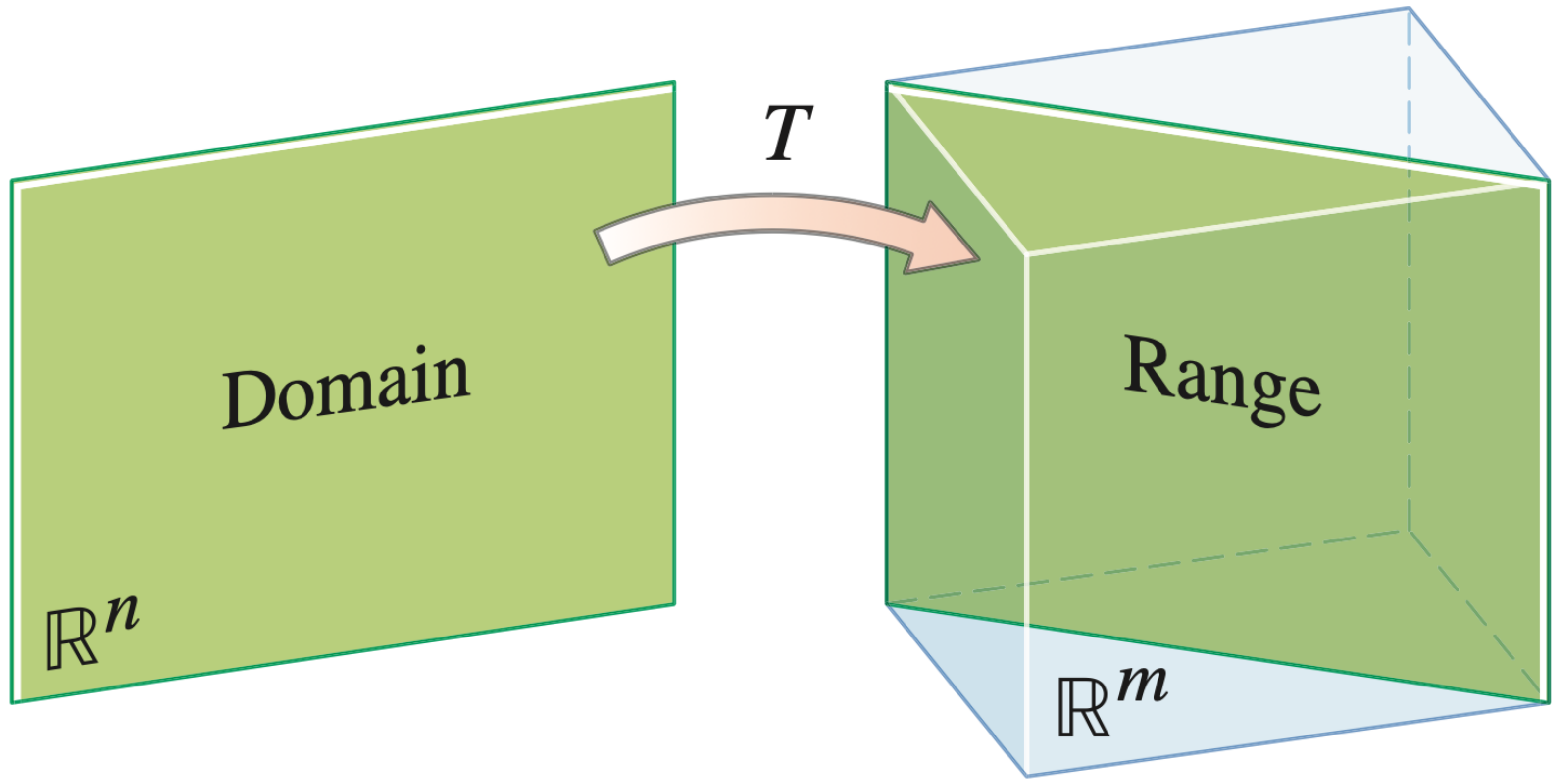

T is *not* onto $\mathbb{R}^m$

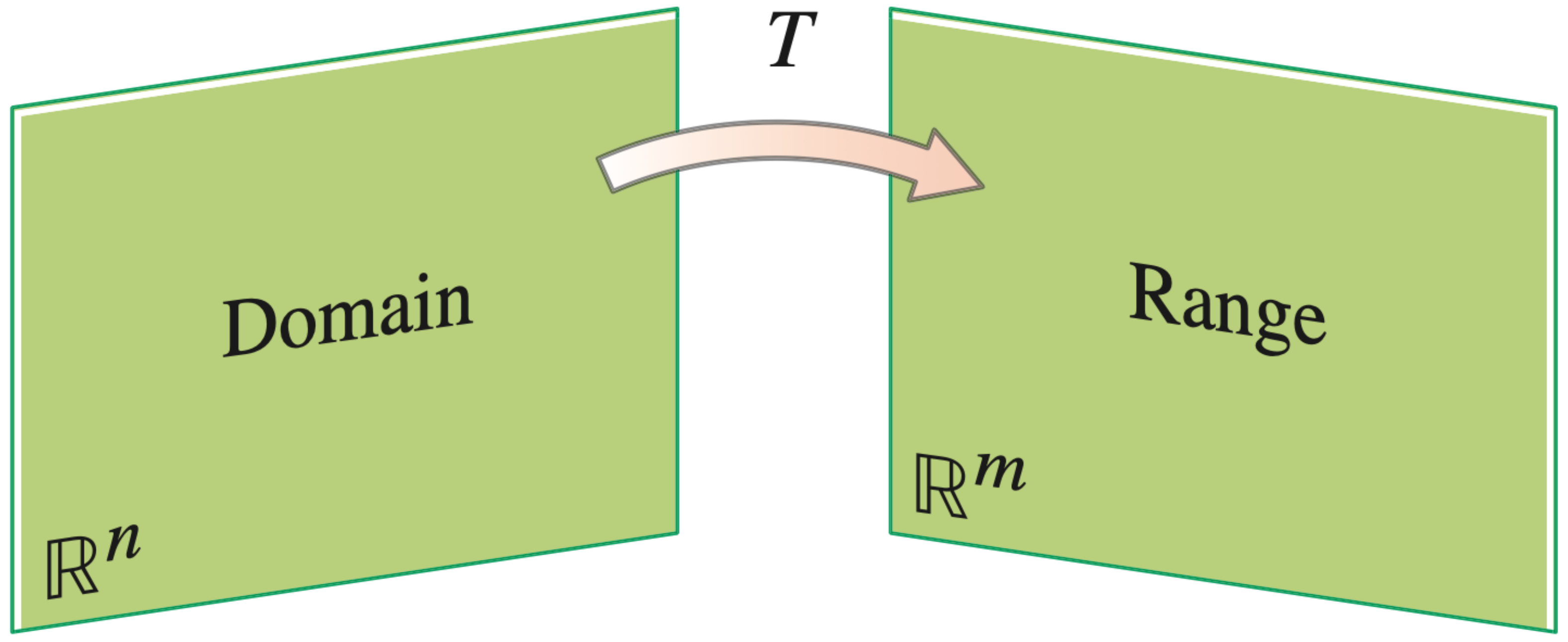

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image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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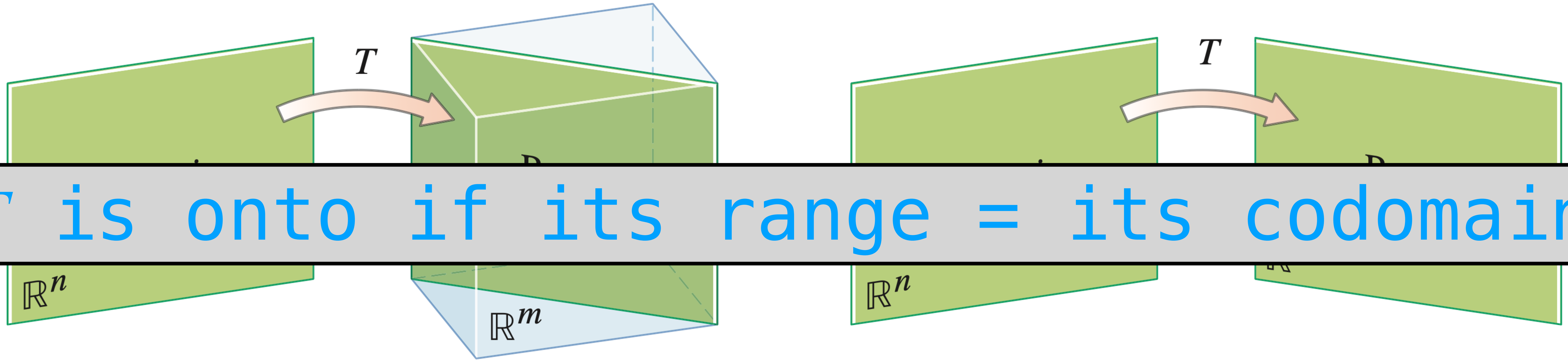

T is onto if its range = its codomain

T is *not* onto $\mathbb{R}^m$

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Recall: One-to-one Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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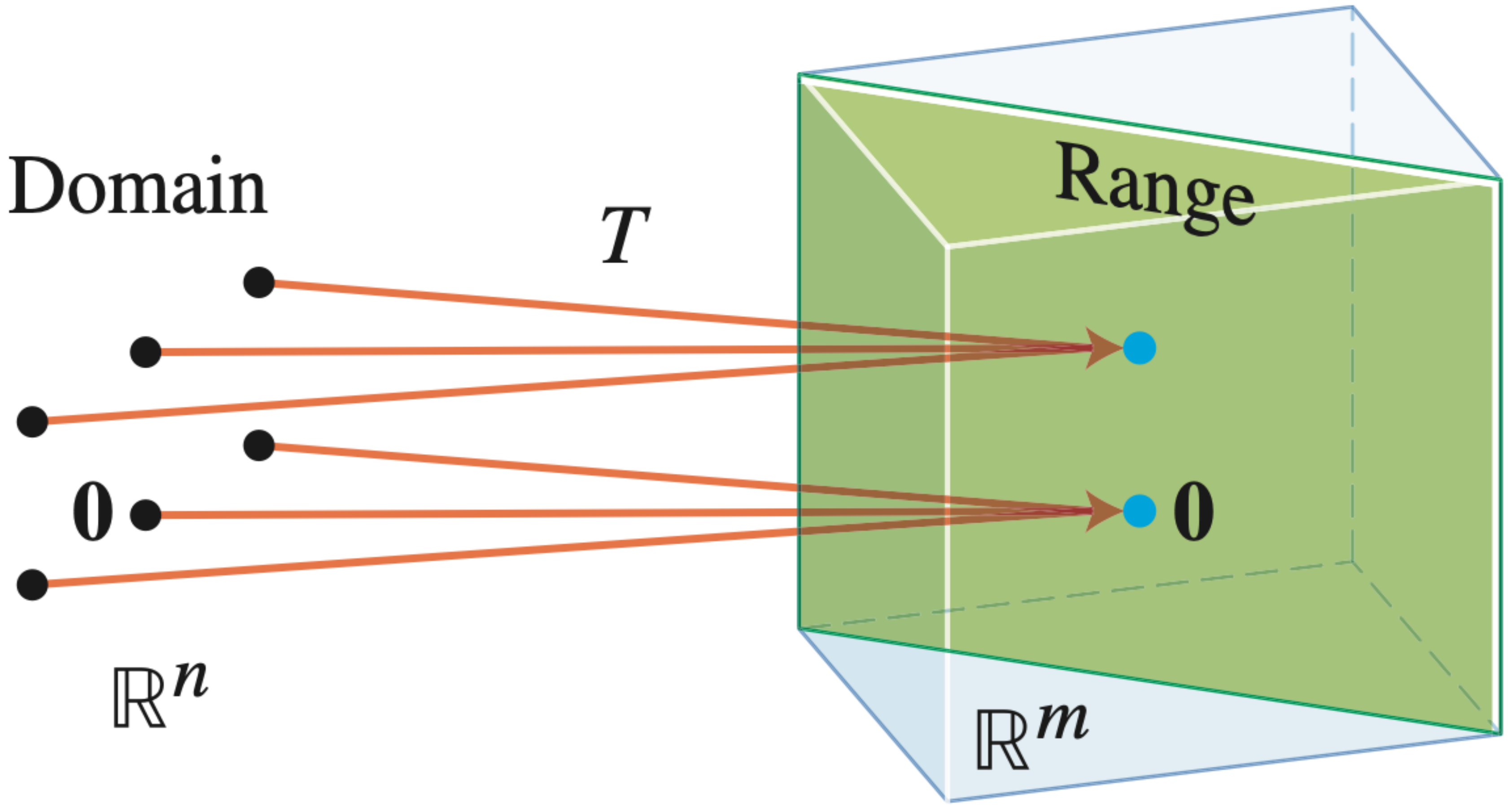
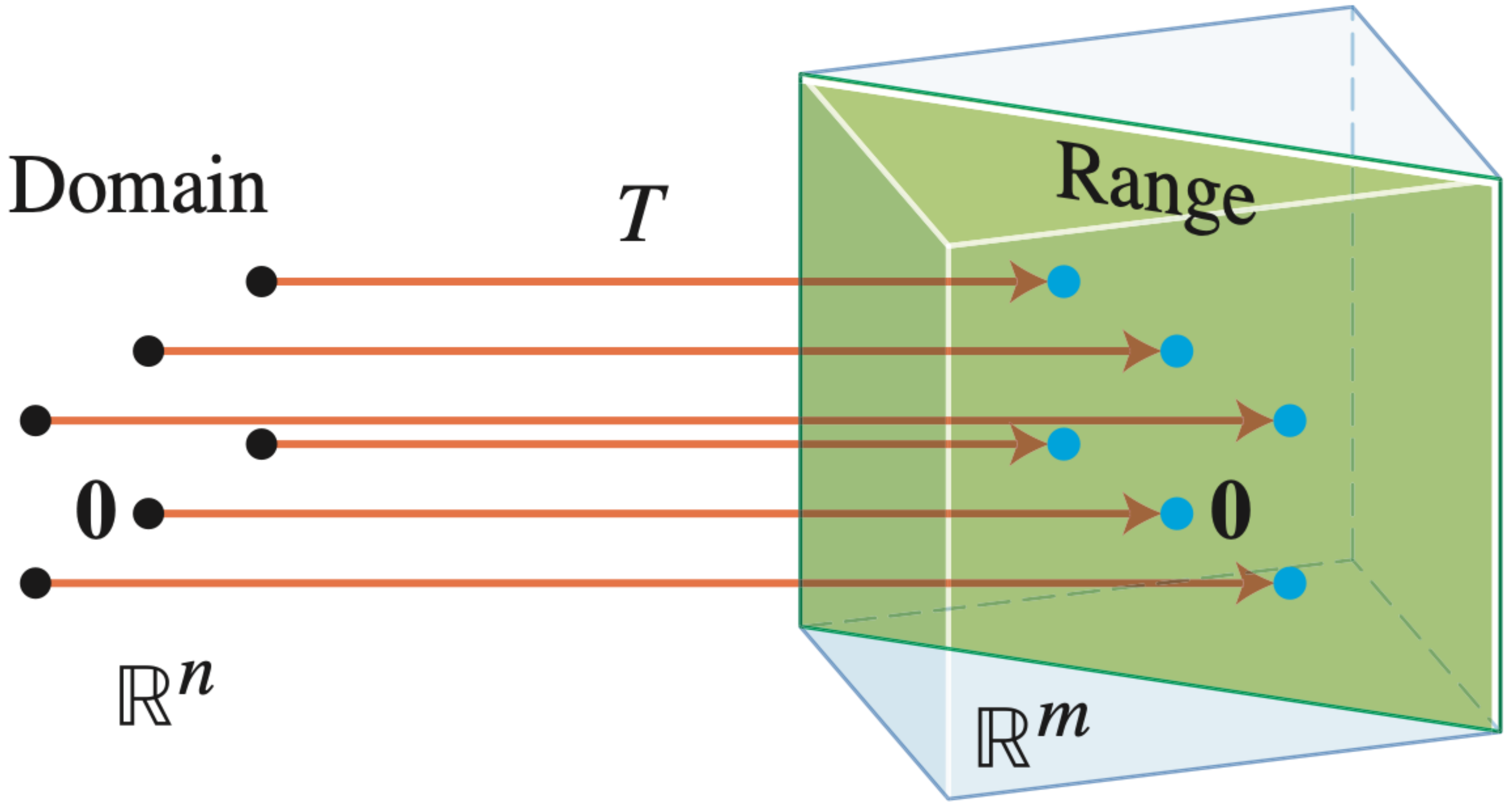


T is *not* one-to-one



T is one-to-one

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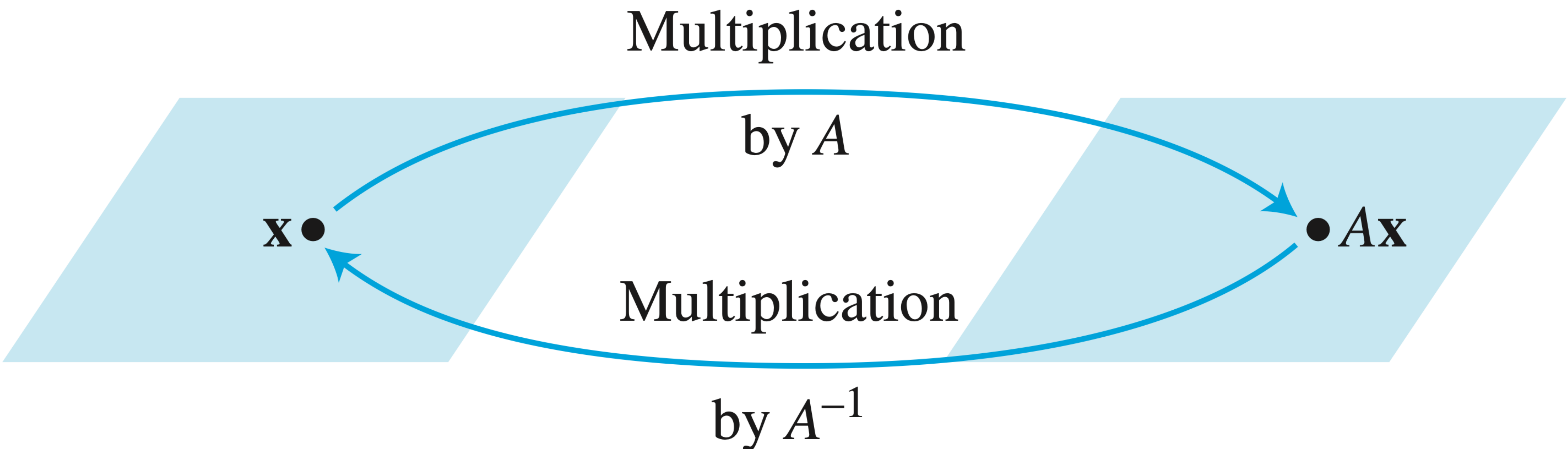
Connection to Transformations

Definition. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$



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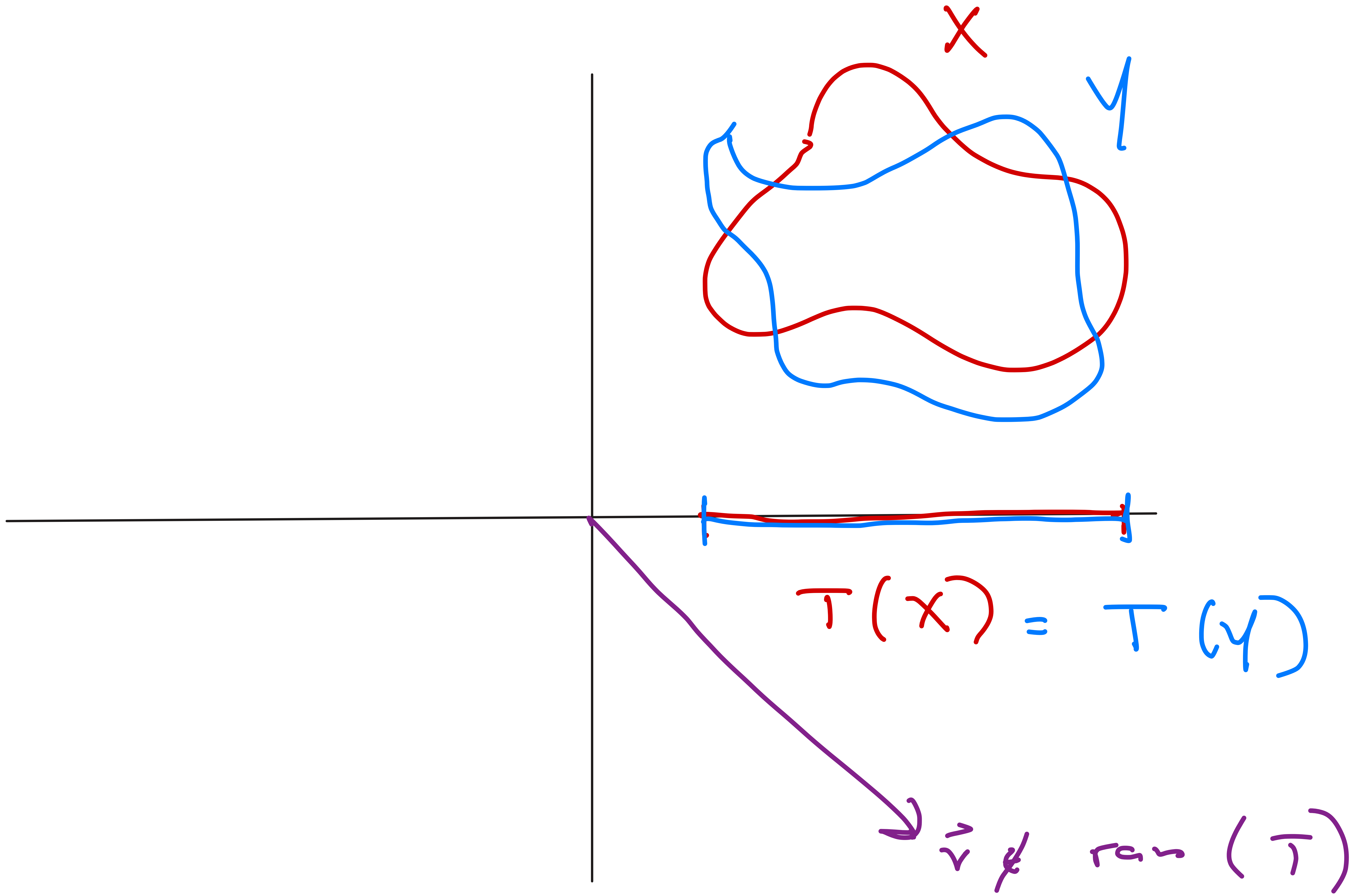
Non-Example. Projection onto the x_1-axis



X
Y
T(X) = T(Y)
$\vec{v} \notin \operatorname{ran}(T)$

Connection to Transformations

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Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$)

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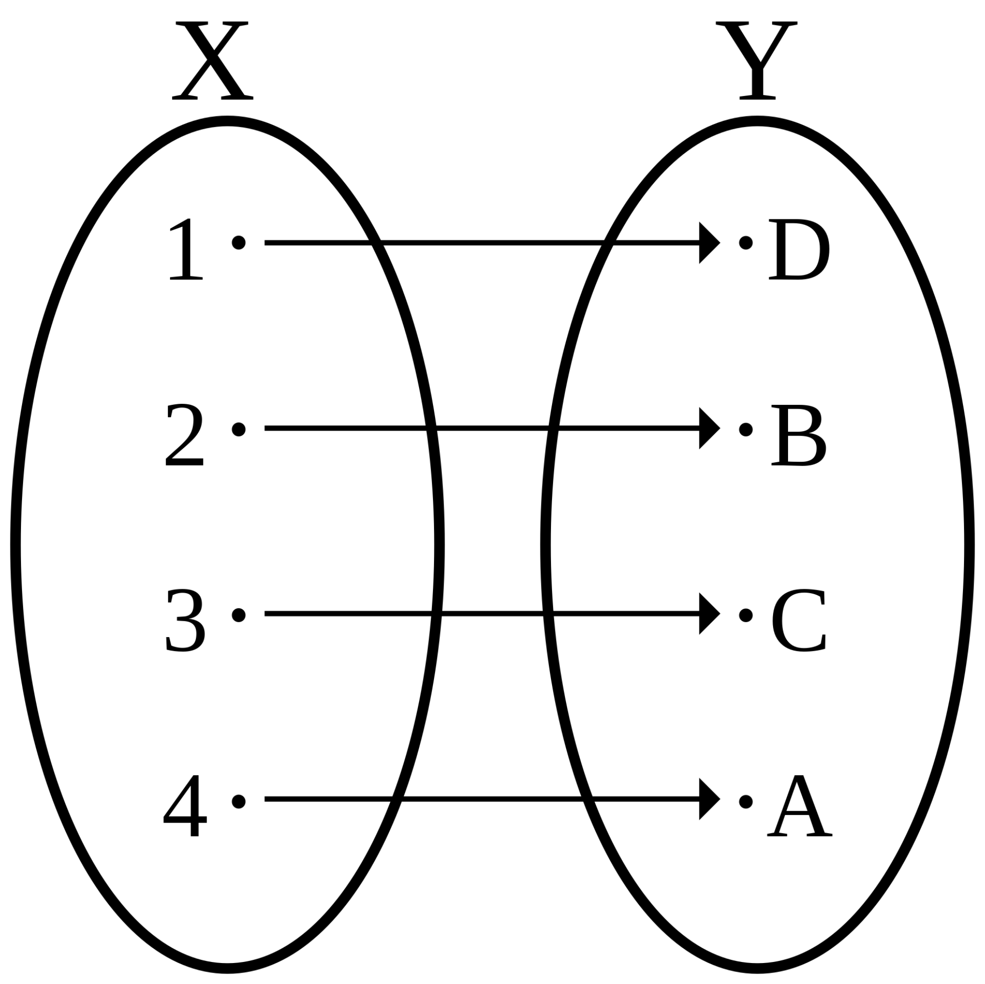
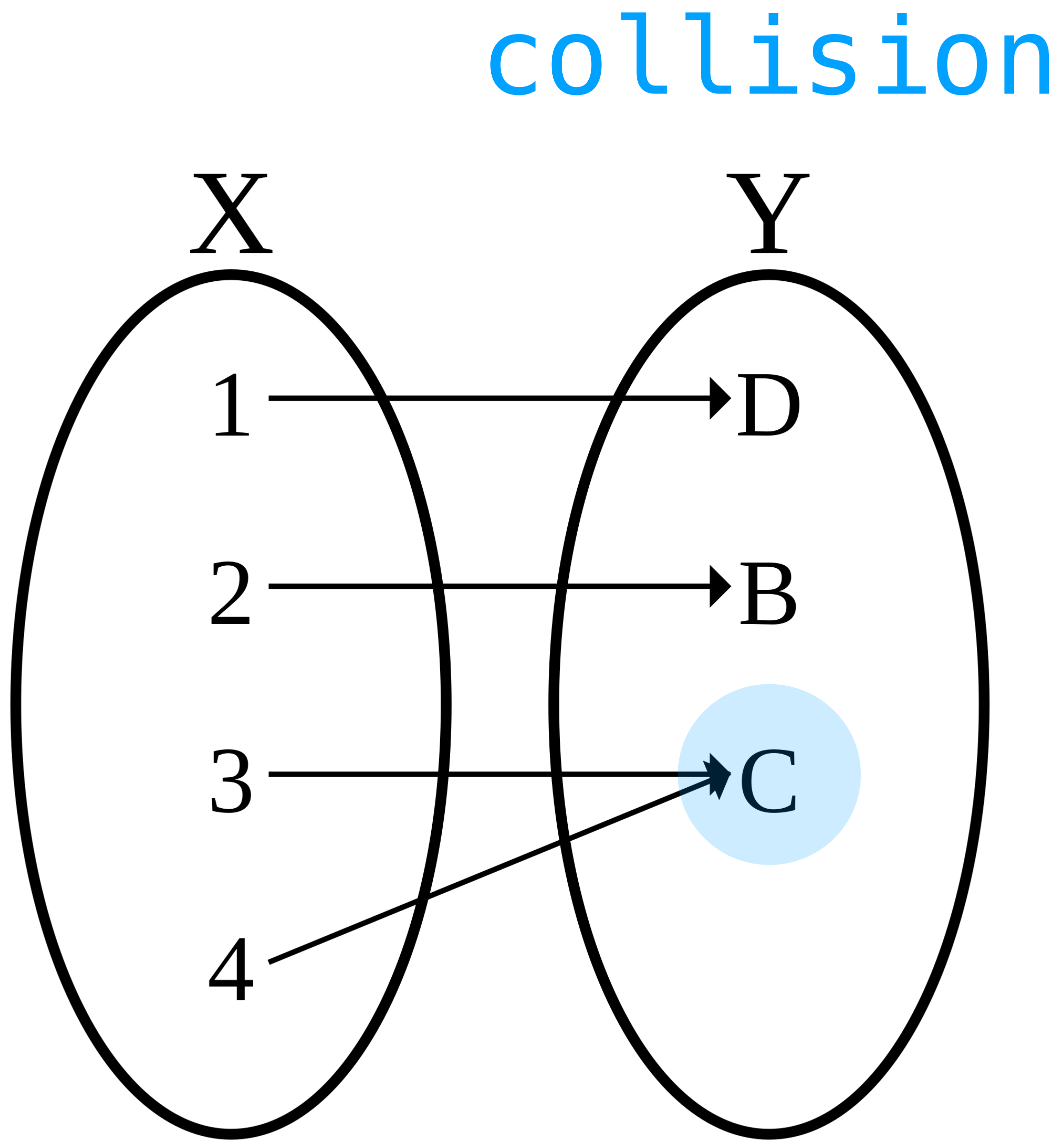
Invertible transformations are 1-1 correspondences

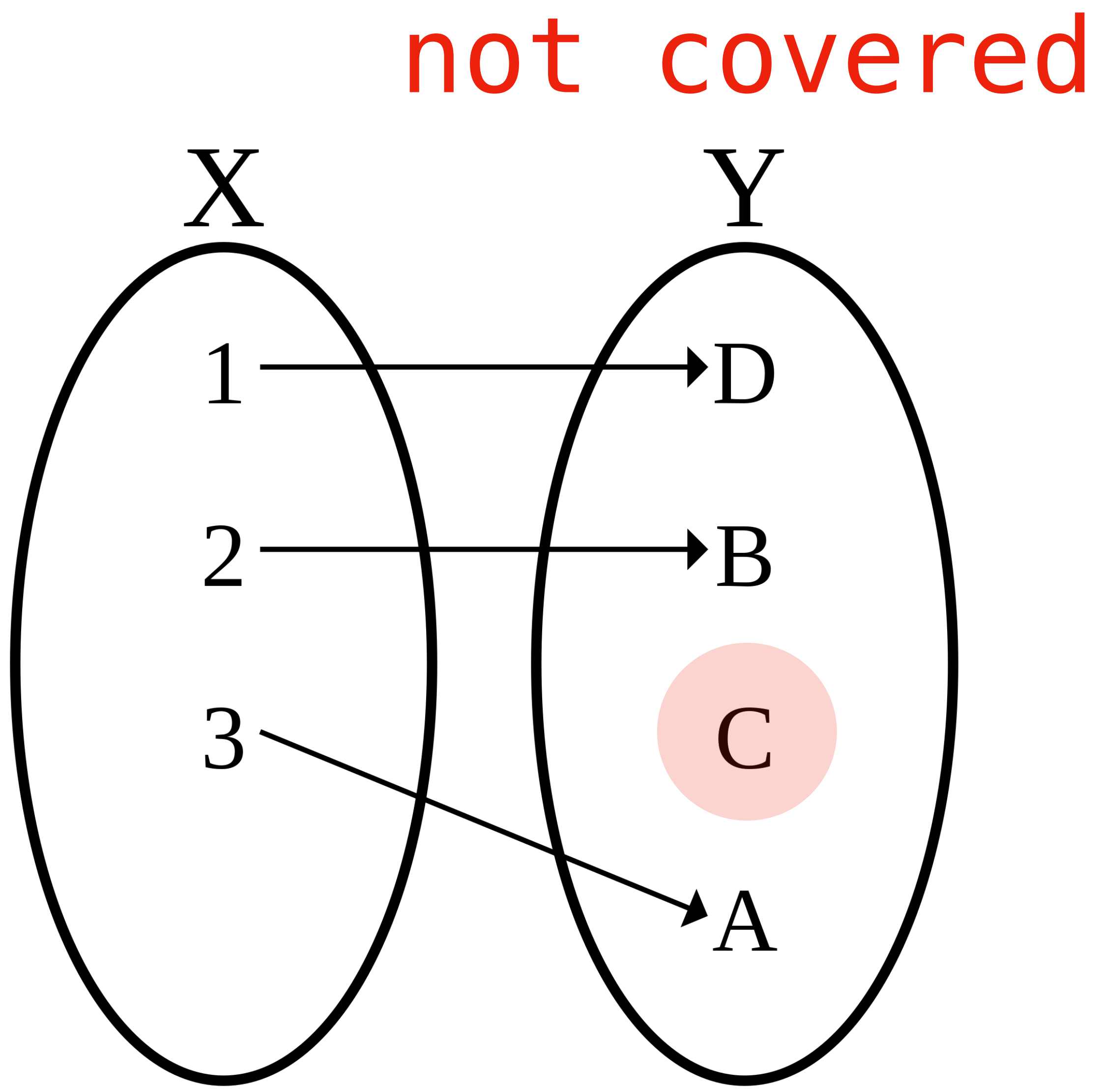
Kinds of Transformations (Pictorially)



1–1 correspondence



onto, not 1–1



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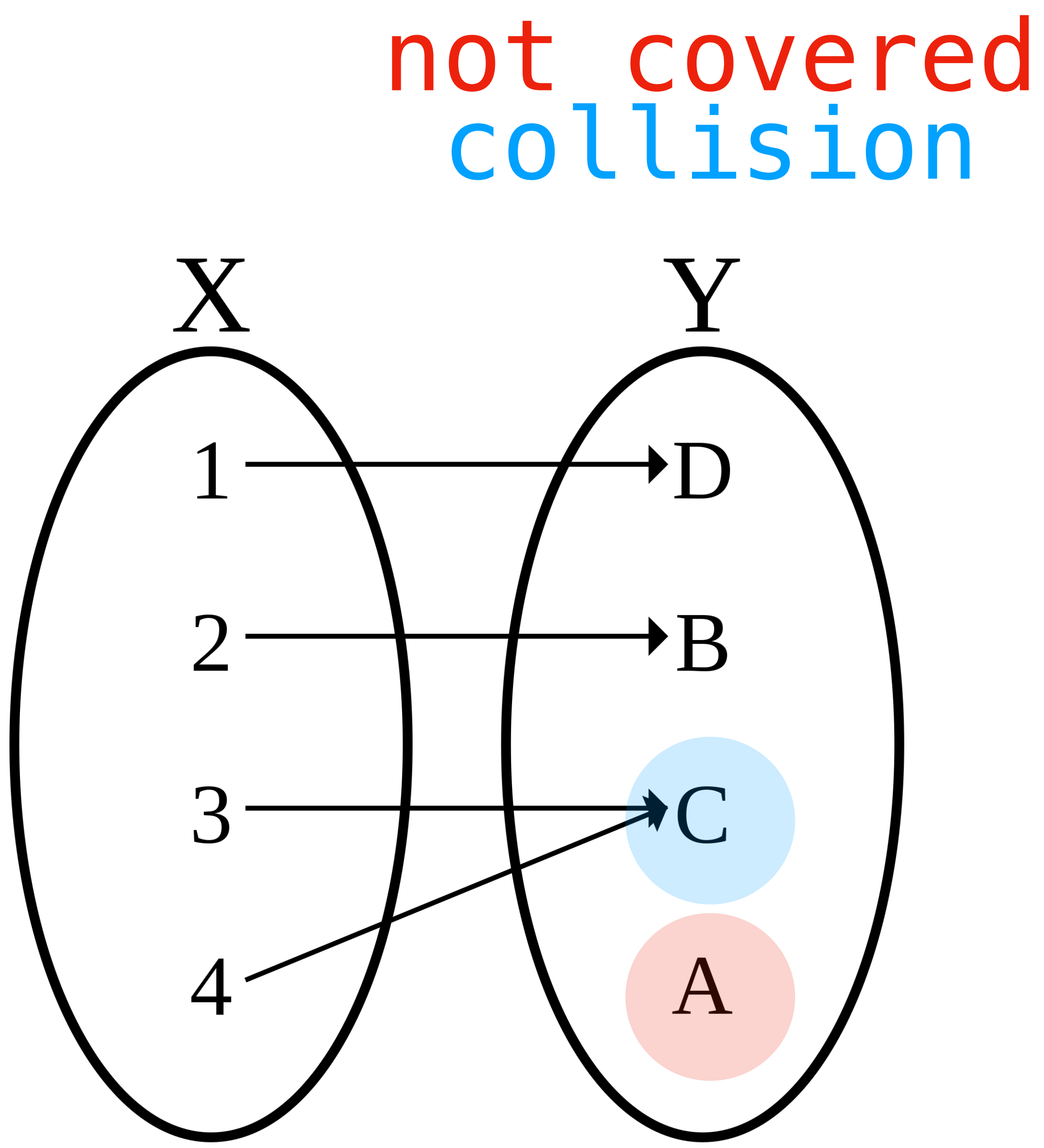

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Computing Matrix Inverses

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it

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Answer 2: the Invertible Matrix Theorem (IMT)

In General

$$A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

$$A\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\|$$

$$\begin{bmatrix}A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3\end{bmatrix}$$

$$//$$

$$\left.\begin{aligned} A\vec{b}_1 &= \vec{e}_1\\ A\vec{b}_2 &= \vec{e}_2\\ A\vec{b}_3 &= \vec{e}_3\end{aligned}\right\} \text{matrix eqs.}$$

In General

$$A\mathbf{b}_1 = \mathbf{e}_1 \qquad A\mathbf{b}_2 = \mathbf{e}_2 \qquad A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns)

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We need to solve 3 matrix equations

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector $\mathbf{e}_i$. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A

This is really the same thing. It's a simultaneous reduction demo

Special Case: 2×2 Matrice Inverses

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

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Is the above matrix invertible?

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Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

$A^{-1}(A\vec{r}) = \vec{r}$

$A\vec{r} = A(A^{-1}(A\vec{r}))$

$A^{-1}A = I$

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$(AB)^T = B^T A^T$

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

$$\begin{tikzcd}
\vec{v} \arrow[r, bend left, "B"] & B\vec{v} \arrow[l, bend left, "B^{-1}"] \arrow[r, bend left, "A"] & A(B\vec{v}) \arrow[l, bend left, "A^{-1}"]
\end{tikzcd}$$

$$B^{-1}A^{-1}AB\vec{v} = \vec{v}$$

$$B\vec{v} = A^{-1}(A(B\vec{v}))$$

$$A\cancel{B}\cancel{B^{-1}}A^{-1} = AIA^{-1} = AA^{-1} = I$$

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen*

Exercise

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at <u>least</u> one solution for every $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has at <u>most</u> one solution for every $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at <u>exactly</u> one solution for every $\mathbf{b}$

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Taking Stock: IMT

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
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These all express the **same thing**

(this is a stronger statement than we just verified)

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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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We only need to check one of these.

Warning. Remember this only applies square matrices.

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

***True** or **False**: If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), the B is also invertible.*

Answer: True

Row reductions don't change the number of pivots.

Question

$A =$

If $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ *is invertible, then is*
$B = [(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3]$ *also invertible? Justify your answer.*

$$A^T = \begin{bmatrix} \text{---} & \vec{a}_1^T & \text{---} \\ \text{---} & \vec{a}_2^T & \text{---} \\ \text{---} & \vec{a}_3^T & \text{---} \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1 + R_2 \\ R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 + 5R_3}} \begin{bmatrix} \text{---} & b_1^T & \text{---} \\ \text{---} & b_2^T & \text{---} \\ \text{---} & b_3^T & \text{---} \end{bmatrix} = B^T$$

$(B^T)^T = B$

Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by row operations

LU Factorization

Matrix Factorization

Matrix Factorization

A **factorization** of a matrix A is an equation which expresses A as a product of one or more matrices, e.g.,

$$A = BC$$

Matrix Factorization

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$$A = BC$$

So far, we've been given two factors and asked to find their product

Factorization is the harder direction

Reasons to Factorize

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Writing A as the product of multiple matrices can

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Writing A as the product of multiple matrices can

» make computing with A faster

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» expose important information about A

Reasons to Factorize

Writing A as the product of multiple matrices can

- » make computing with A faster LU Decomposition
- » make working with A easier
- » expose important information about A

The Problem

Question. For an matrix A, solve the equations

$$A\mathbf{x} = \mathbf{b}_1 \quad , \quad A\mathbf{x} = \mathbf{b}_2 \quad \ldots \quad A\mathbf{x} = \mathbf{b}_{k-1} \quad , \quad A\mathbf{x} = \mathbf{b}_k$$

In other words: we want to solve <u>a bunch</u> of matrix equations over the same matrix

The Problem

Question. For a matrix A, solve (for X) in the equation

$$AX = B$$

where X and B are matrices of appropriate dimension

This is (essentially) the same question

The Problem

Question. Solve $AX = B$

If A is *invertible*, then we have a solution:

Find A^{-1} and then $X = A^{-1}B$

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Question. Solve $AX = B$

If A is *invertible*, then we have a solution:

Find A^{-1} and then $X = A^{-1}B$

What if A^{-1} is not invertible?
Even if it is, can we do it faster?

LU Factorization at a High Level

Given a $m \times n$ matrix A, we are going to factorize A as

echelon form of A

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

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Note. This applies to non-square matrices

What are "L" and "U"?

L stands for "lower" as in *lower triangular*

U stands for "upper" as in *upper triangular*

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}}_{U}$$

The Fundamental Question

$$A = L\boxed{U}$$ echelon form of A

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How do we build L?

The Fundamental Question

$$A = L\boxed{U}$$ echelon form of A

We know how to build U, that's just the forward phase of Gaussian elimination

How do we build L?

The idea. L "implements" the row operations of the forward phase

Elementary Matrices

Recall: Elementary Row Operations

scaling	multiply a row by a number
interchange	switch two rows
replacement	add a scaled equation to another

The First Key Observation

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Elementary row operations are **linear transformations (viewed as transformation on columns)**

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Example: Scale row 2 by 5

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{R_2 \leftarrow 5R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example: Scaling

Restricted to one column, we see this is the above linear transformation

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ 5v_2 \\ v_3 \end{bmatrix}$$

Example: Scaling

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ 5v_2 \\ v_3 \end{bmatrix}$$

Let's build the matrix which implements it:

Another Example: Scaling + Replacement

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ (a_{31} - 2a_{11}) & (a_{32} - 2a_{12}) & (a_{33} - 2a_{13}) \end{bmatrix}$$

$$R_3 \leftarrow (R_3 - 2R_1)$$

Another Example: Scaling + Replacement

$$R_3 \leftarrow (R_3 - 2R_1)$$

Elementary row operations are linear, so they are implemented by matrices

General Elementary Scaling Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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If we want to perform $R_3 \leftarrow kR_3$ then we need the identity matrix but with the entry $A_{33} = k$.

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If we want to perform $R_3 \leftarrow kR_3$ then we need the identity matrix but with the entry $A_{33} = k$.

If we want to perform $R_i \leftarrow kR_i$ then we need the identity matrix but with then entry $A_{ii} = k$.

General Replacement Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix}$$

General Replacement Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix}$$

If we want to perform $R_4 \leftarrow R_4 + kR_1$, then we need the identity matrix but with the entry $A_{41} = k$.

General Replacement Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix}$$

If we want to perform $R_4 \leftarrow R_4 + kR_1$, then we need the identity matrix but with the entry $A_{41} = k$.

If we want to perform $R_i \leftarrow R_i + kR_j$, then we need the identity matrix but with the entry $A_{ij} = k$.

General Swap Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If we want to swap R_2 and R_3, then we need the identity matrix, but with R_2 and R_3 swapped.

Elementary Matrices

Definition. An **elementary matrix** is a matrix obtained by applying a single row operation to the identity matrix I.

Example.

How To: Finding Elementary Matrices

Question. Find the matrix implementing the elementary row operation `op`

Solution. Apply `op` to the identity matrix of the appropriate size

Products of Elementary Matrices

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Taking stock:

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» Remember that Matrix multiplication is transformation composition (i.e., do one then the other)

Products of Elementary Matrices

<u>Taking stock:</u>

» Elementary matrices implement elementary row operations

» Remember that Matrix multiplication is transformation composition (i.e., do one then the other)

So we can implement <u>any</u> sequence of row operations as a product of elementary matrices

How to: Matrices implementing Row Operations

Question. Find the matrix implementing a sequence of row operations op_1, op_2,...

Solution. Apply the row operations in sequence to the identity matrix of the appropriate size

Question

Find the matrix implementing the following sequence of elementary row operations on a $3 \times n$ *matrix.*

$$R_2 \leftarrow 3R_2$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_2 \leftrightarrow R_3$$

Then multiply it with the all-ones 3×3 *matrix.*

Answer

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

Second Key Observation

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Elementary row operations are **invertible** linear transformations

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Elementary row operations are **invertible** linear transformations

This also means the product of elementary matrices is invertible

$$(E_1E_2E_3E_4)^{-1} = E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$$

!! the order reverses !!

Question (Conceptual)

Describe the inverse transformation for each elementary row operation

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Question (Conceptual)

Describe the inverse transformation for each elementary row operation

The inverse of scaling by k is scaling by $1/k$

The inverse of $R_i \leftarrow R_i + kR_j$ is $R_i \leftarrow R_i - kR_j$

The inverse of swapping is swapping again

Recall: Elementary Row Operations

scaling	multiply a row by a number
interchange	switch two rows
replacement	add a scaled equation to another

Recall: Elementary Row Operations

We only need these two for the forward phase

interchange	switch two rows
replacement	add a scaled equation to another

Recall: Elementary Row Operations

We'll assume we only need this

replacement add a scaled equation to another

Reminder: LU Factorization at a High Level

Given a $m \times n$ matrix A, we are going to factorize A as

Echelon form of A

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

Gaussian Elimination and Elementary Matrices

$$A \sim A_1 \sim A_2 \sim \ldots \sim A_k$$

Consider a sequence of elementary row operations from A to an echelon form

Each step can be represent as a **product with an elementary matrix**

Gaussian Elimination and Elementary Matrices

$$A \sim E_1 A \sim E_2 E_1 A \sim \ldots \sim E_k E_{k-1} \ldots E_2 E_1 A$$

Gaussian Elimination and Elementary Matrices

$$A \sim E_1A \sim E_2E_1A \sim \ldots \sim E_kE_{k-1}\ldots E_2E_1A$$

This exactly tells us that if B is the final echelon form we get then

$$B = (E_kE_{k-1}\ldots E_2E_1)A = EA$$

where E implements a <u>sequence</u> of row operations. So:

Gaussian Elimination and Elementary Matrices

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$$B = \overset{\text{Invertible}}{(E_kE_{k-1}\dots E_2E_1)}A = EA$$

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Gaussian Elimination and Elementary Matrices

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This exactly tells us that if B is the final echelon form we get then

$$B = \underbrace{(E_kE_{k-1}\ldots E_2E_1)}^{\text{Invertible}}A = EA$$

where E implements a <u>sequence</u> of row operations. So:

$$A = E^{-1}B = (E_1^{-1}E_2^{-1}\ldots E_{k-1}^{-1}E_k^{-1})B$$

LU Factorization Algorithm

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```
FUNCTION LU_Factorization(A):
```

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    L ← identity matrix
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LU Factorization Algorithm

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FUNCTION LU_Factorization(A):
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        L ← L @ E^{-1}    # note the multiplication on the right
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    FOR each row operation OP in the prev step:
        E ← the matrix implementing OP
        L ← L @ E^-1     # note the multiplication on the right
    RETURN (L, U)
```

we'll see how to do this more efficiently

The forward part of Gaussian elimination <u>is</u> matrix factorization

The "L" Part

$$E = E_k E_{k-1} \cdots E_2 E_1$$

This a product of elementary matrices

So $L = E^{-1} = E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1}$ **!! the order reverses !!**

We won't prove this, but it's worth thinking about: **why is this lower triangular?**

And can we build this in a more efficient way?

demo

How To: LU Factorization by hand

Question. Find a LU Factorization for the matrix A (assuming no swaps)

Solution.

» Start with L as the identity matrix
» Find U by the forward part of GE
» For each operation $R_i \leftarrow R_i + kR_j$, set L_{ij} to $-k$

Analyzing Linear Algebra Algorithms

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We will not use $O(\cdot)$ notation!

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For numerics, we care about number of **FL**oating-oint **OP**erations (FLOPs):

>> addition
>> subtraction
>> multiplication
>> division
>> square root

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>> square root

$2n$ vs. n is very different when $n \sim 10^{20}$

Analyzing LU Factorization

Dominant Terms

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that said, we don't care about *exact* bounds

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A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \to \infty} \frac{f(i)}{g(i)} = 1$$

Dominant Terms

that said, we don't care about *exact* bounds

A function $f(n)$ is ***asymptotically equivalent*** to $g(n)$ if

$$\lim_{i \to \infty} \frac{f(i)}{g(i)} = 1$$

for polynomials, they are equivalent to their dominant term

Dominant Terms

the dominant term of a polynomial is the monomial with the highest degree

$$\lim_{i \to \infty} \frac{3x^3 + 100000x^2}{3x^3} = 1$$

$3x^3$ dominates the function even though the coefficient for x^2 is so large

How To: Solving systems with the LU

Question. Solve the equation $A\mathbf{x} = \mathbf{b}$ given that $A = LU$ is a LU factorization.

Solution. First solve $L\mathbf{x} = \mathbf{b}$ to get a solution $\mathbf{c}$, then solve $U\mathbf{x} = \mathbf{c}$ to get a solution $\mathbf{d}$.

Verify:

How To: Solving systems with the LU

Question. Solve the equation $A\mathbf{x} = \mathbf{b}$ given that $A = LU$ is a LU factorization.

Solution. First solve $L\mathbf{x} = \mathbf{b}$ to get a solution $\mathbf{c}$, then solve $U\mathbf{x} = \mathbf{c}$ to get a solution $\mathbf{d}$.

Why is this better than just solving $A\mathbf{x} = \mathbf{b}$?

FLOPs for Solving General Systems

The following FLOP estimates are based on $n \times n$ matrices

Gaussian Elimination: $\sim \frac{2n^3}{3}$ FLOPS

GE Forward: $\sim \frac{2n^3}{3}$ FLOPS

GE Backward: $\sim 2n^2$ FLOPS

Matrix Inversion: $\sim 2n^3$ FLOPS

Matrix-Vector Multiplication: $\sim 2n^2$ FLOPS

Solving by matrix inversion: $\sim 2n^3$ FLOPS

Solving by Gaussian elimination: $\sim \frac{2n^3}{3}$ FLOPS

FLOPS for solving LU systems

LU Factorization: $\sim \frac{2n^3}{3}$ FLOPS

Solving $L\mathbf{x} = \mathbf{b}$: $\sim 2n^2$ FLOPS (by "forward" elimination)

Solving $U\mathbf{x} = \mathbf{c}$: $\sim 2n^2$ FLOPS (already in echelon form)

Solving by LU Factorization: $\sim \frac{2n^3}{3}$ FLOPS

If you solve several matrix equations for the same matrix, LU factorization is <u>faster</u> than matrix inversion on the first equation, and the same (asymptotically) in later equations (and it works for rectangular matrices).

Other Considerations: Density

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If A doesn't have to many entries (A is **sparse)**, then its likely that L and U won't either.

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If A doesn't have to many entries (A is **sparse)**, then its likely that L and U won't either.

But A^{-1} may have ***many*** entries (A^{-1} is **dense**)

Sparse matrices are faster to compute with and better with respect to storage.

Summary

Matrix inverses allow us to easily solve many matrixes equations over the same A

LU Factorizations allows us to do the same, but more generally more efficiently