Eigenvalues and Eigenvectors

Geometric Algorithms
Lecture 18

CAS CS 132

Practice Problem

Suppose A *is a* 234×300 *matrix. What is the smallest possible value for* $\dim(\mathsf{Nul}(A))$*? What is the largest possible value?*

What is the smallest possible value for $\mathsf{rank}(A)$*? What is the largest possible value?*

Answer

$$A \text{ is } \overset{m}{234} \overset{\times}{\times} \overset{n}{300}$$

$$\underbrace{\dim(\mathrm{Col}\, A)}_{\text{"rank"}} + \underbrace{\dim(\mathrm{Nul}\, A)}_{\text{"nullity"}} = n$$

$$66 \le \dim(\mathrm{Nul}\, A) \le 300$$

$$0 \le \dim(\mathrm{Col}\, A) \le 234$$

if $\dim(\mathrm{Nul}\, A) = 300$

&

$\dim(\mathrm{Col}\, A) = 0$

A is 0 matrix

Objectives

1. Motivate and introduce the fundamental notion of eigenvalues and eigenvectors
2. Determine how to verify eigenvalues and eigenvectors
3. Look at the subspace generated by eigenvectors
4. Apply the study of eigenvectors to dynamical linear systems

Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

Motivation

demo

How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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All matrices do some combination of these things

* square matrices

How can matrices transform vectors?*

`In 2D and 3D we've seen:`

- rotations
- projections
- shearing
- reflection
- scaling/stretching
- ... **Today's focus**

All matrices do some combination of these things

* square matrices

What's special about scaling?

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`We don't need a whole matrix to do scaling`

$$\mathbf{x} \mapsto c\mathbf{x}$$

What's special about scaling?

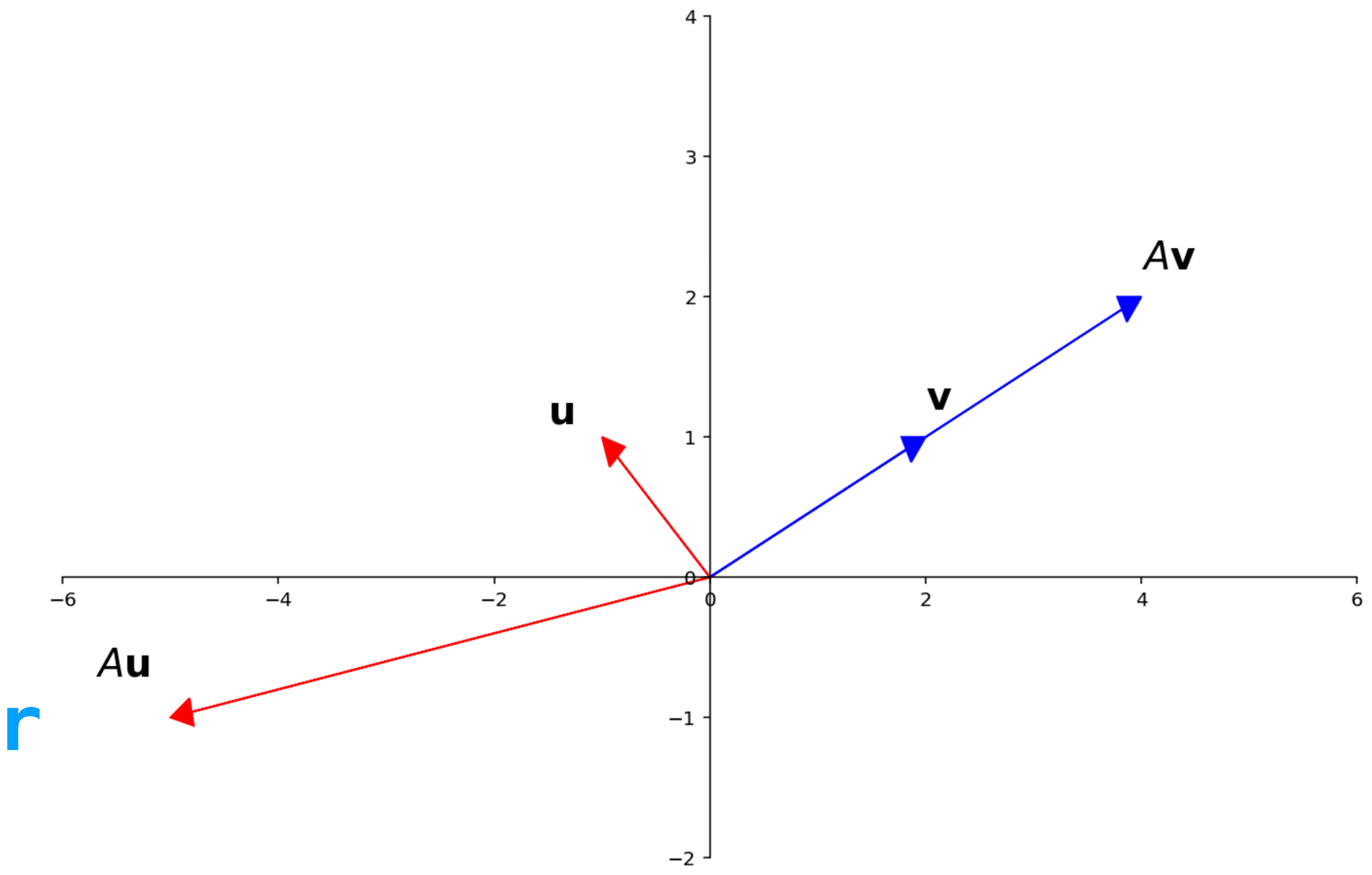
We don't need a whole matrix to do scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what A does to $\mathbf{v}$.

Eigenvectors (Informal)

$$A\mathbf{v} = \overset{\text{eigenvalue}}{\lambda}\underset{\text{eigenvector}}{\mathbf{v}}$$



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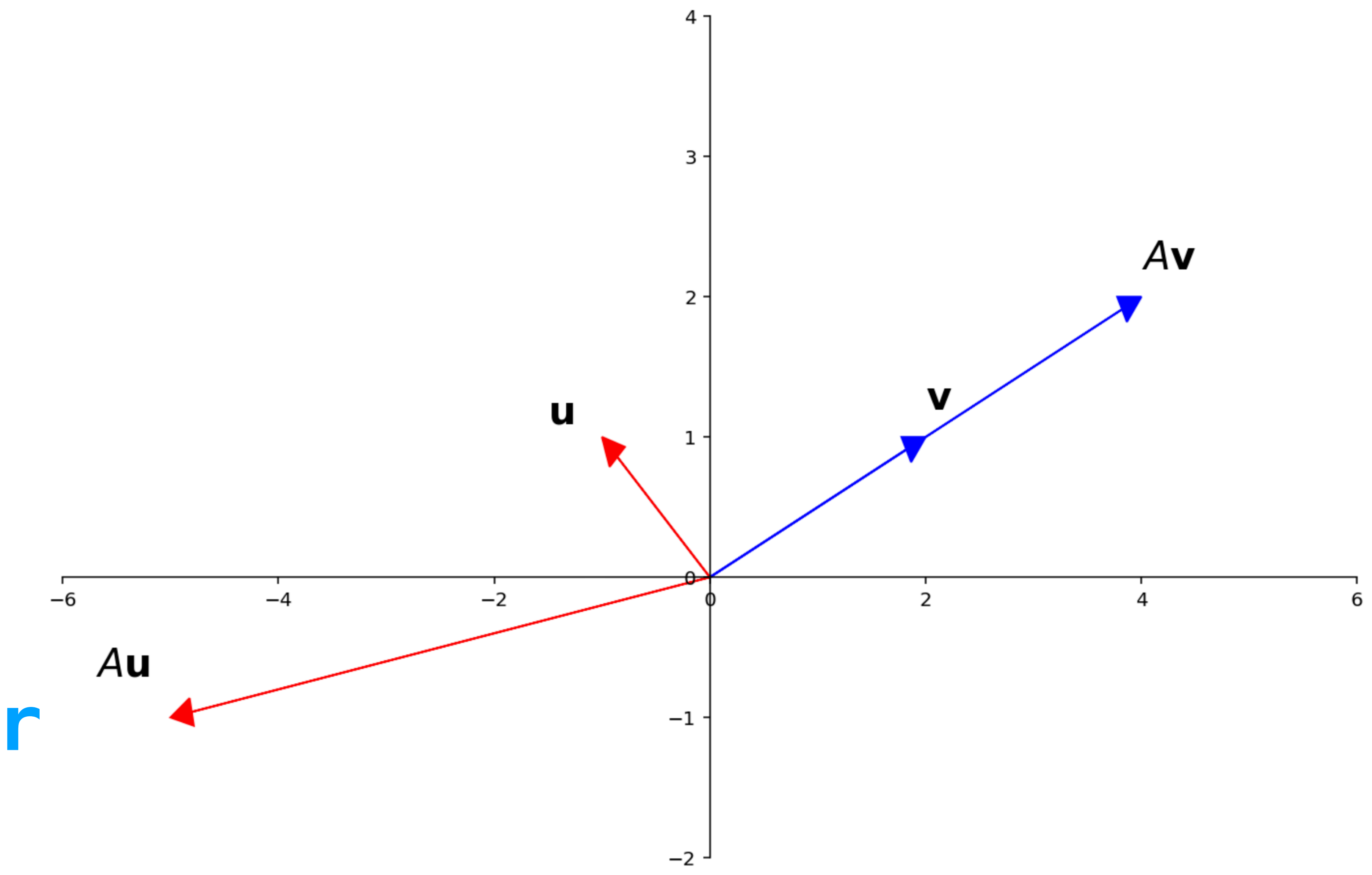

Eigenvectors of A are stretched by A without changing their direction.

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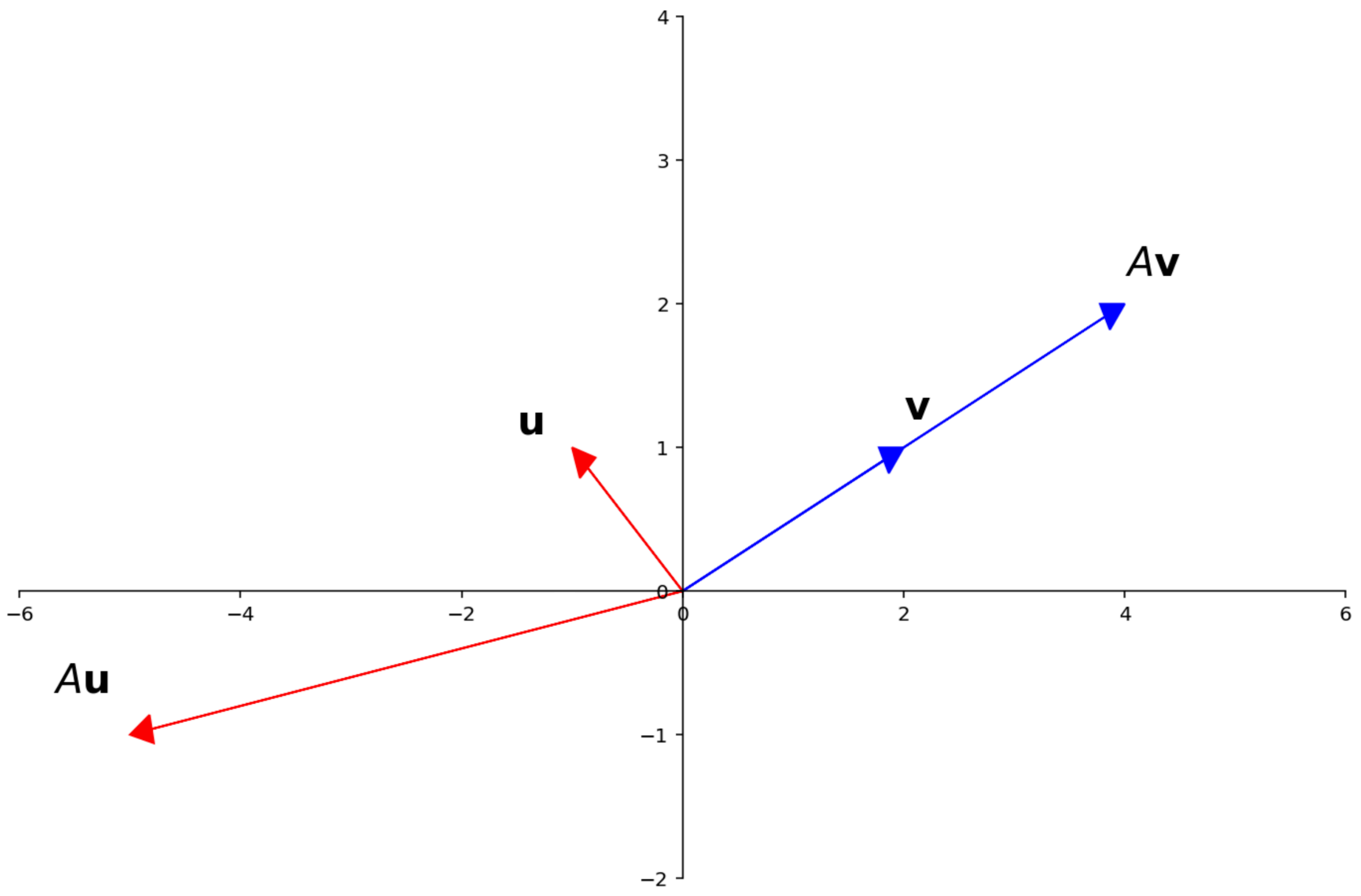


Eigenvectors of A are stretched by A without changing their direction.

The amount they are stretched is called the **eigenvalue.**

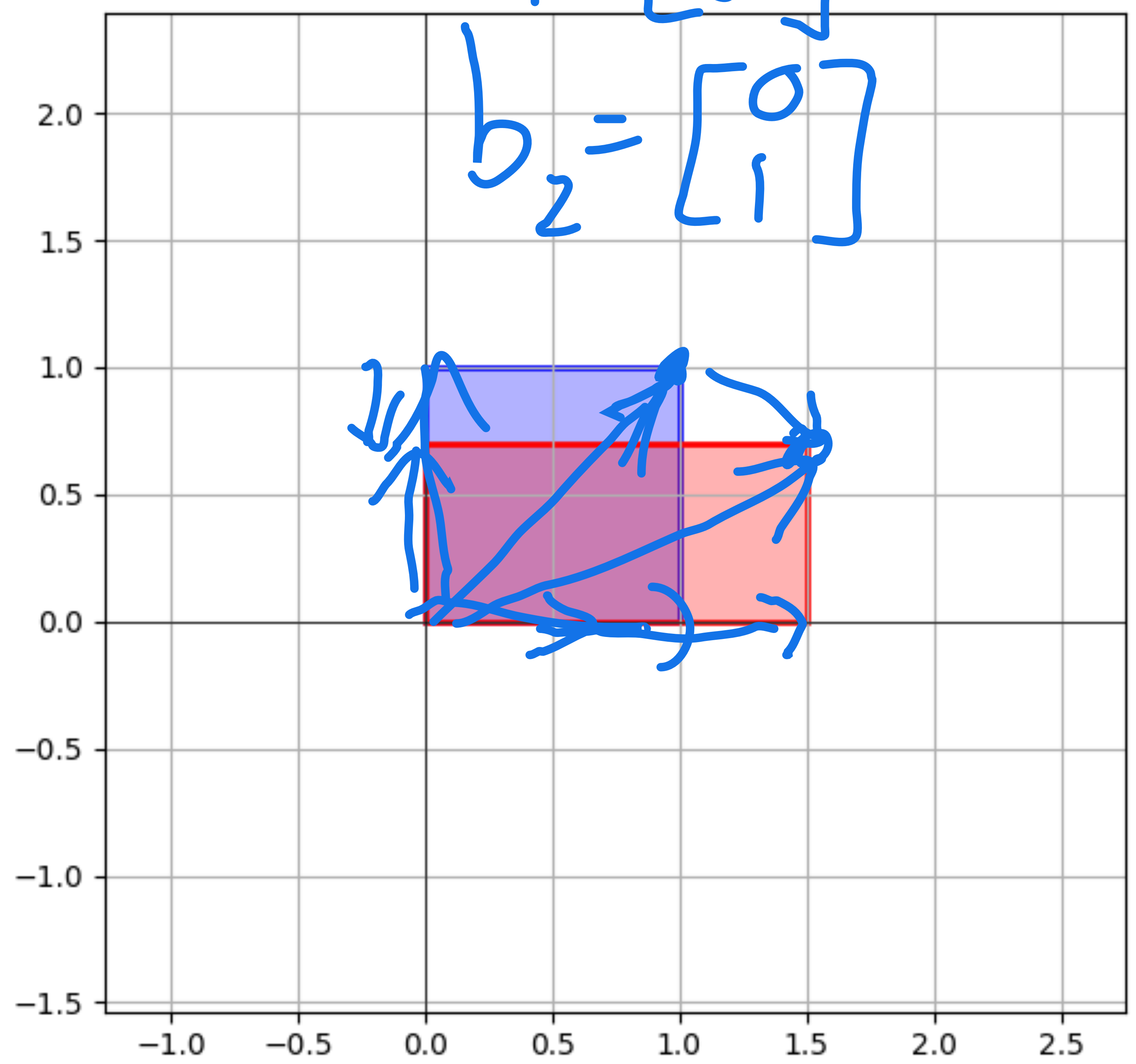
Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

It transforms each entry individually and then combines them.

$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = (1.5) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} " & " \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix} = (0.7) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenbases (Informal)

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of A*. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

Eigenbases (Informal)

$\lambda_1, \lambda_2, \lambda_3$

Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of A*. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

It's "easy to describe" how A transforms $\mathbf{v}$.

It *transforms each "component" individually and then combines them.*

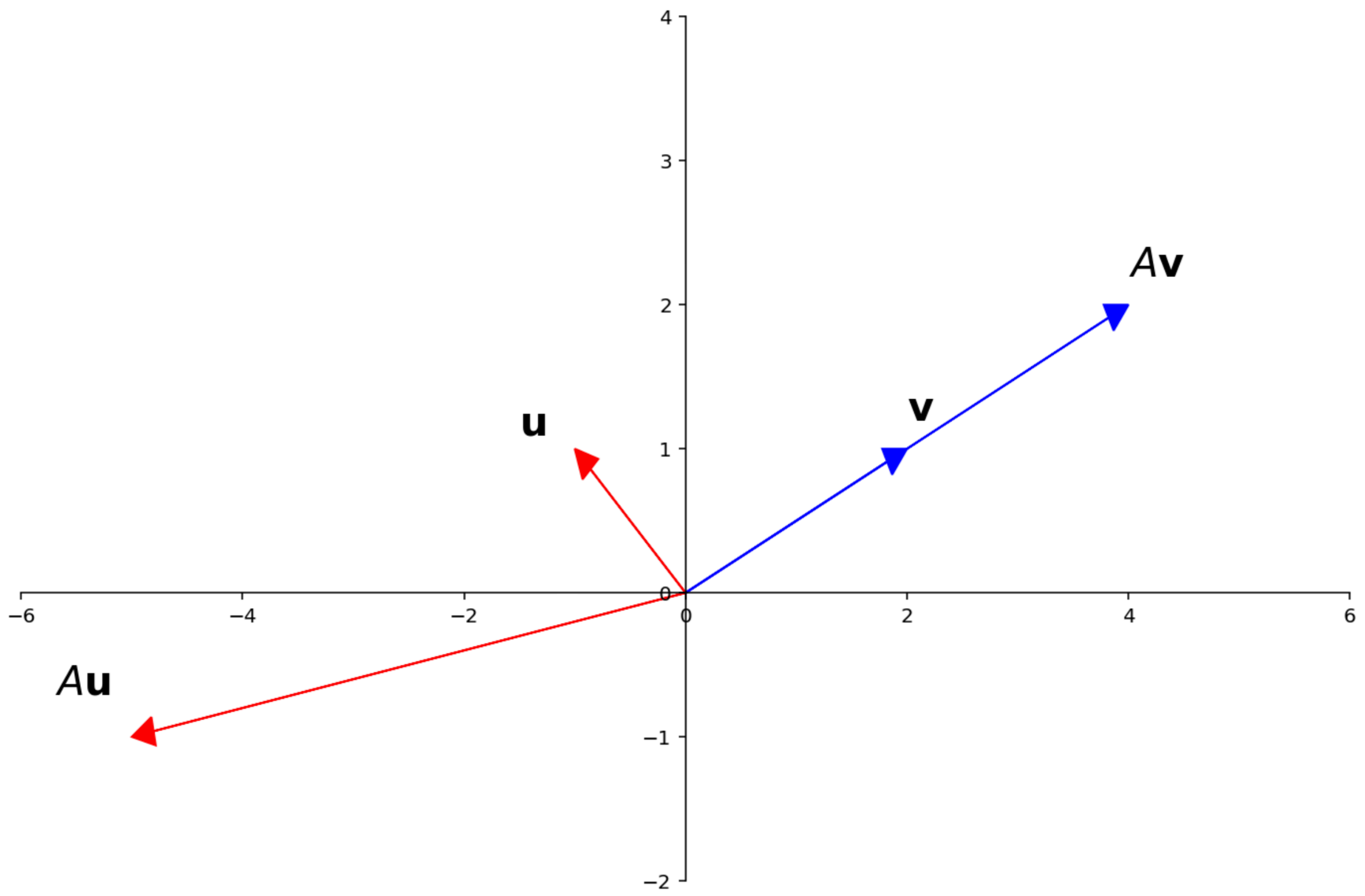
Verify: $A\vec{v} = A(2\vec{b}_1 - \vec{b}_2 - 5\vec{b}_3) = A(2\vec{b}_1) - A\vec{b}_2 - A(5\vec{b}_3)$

$= 2A\vec{b}_1 - A\vec{b}_2 - 5A\vec{b}_3$

$= 2\lambda_1\vec{b}_1 - \lambda_2\vec{b}_2 - 5\lambda_3\vec{b}_3$

Eigenvalues and Eigenvectors

Formal Definition



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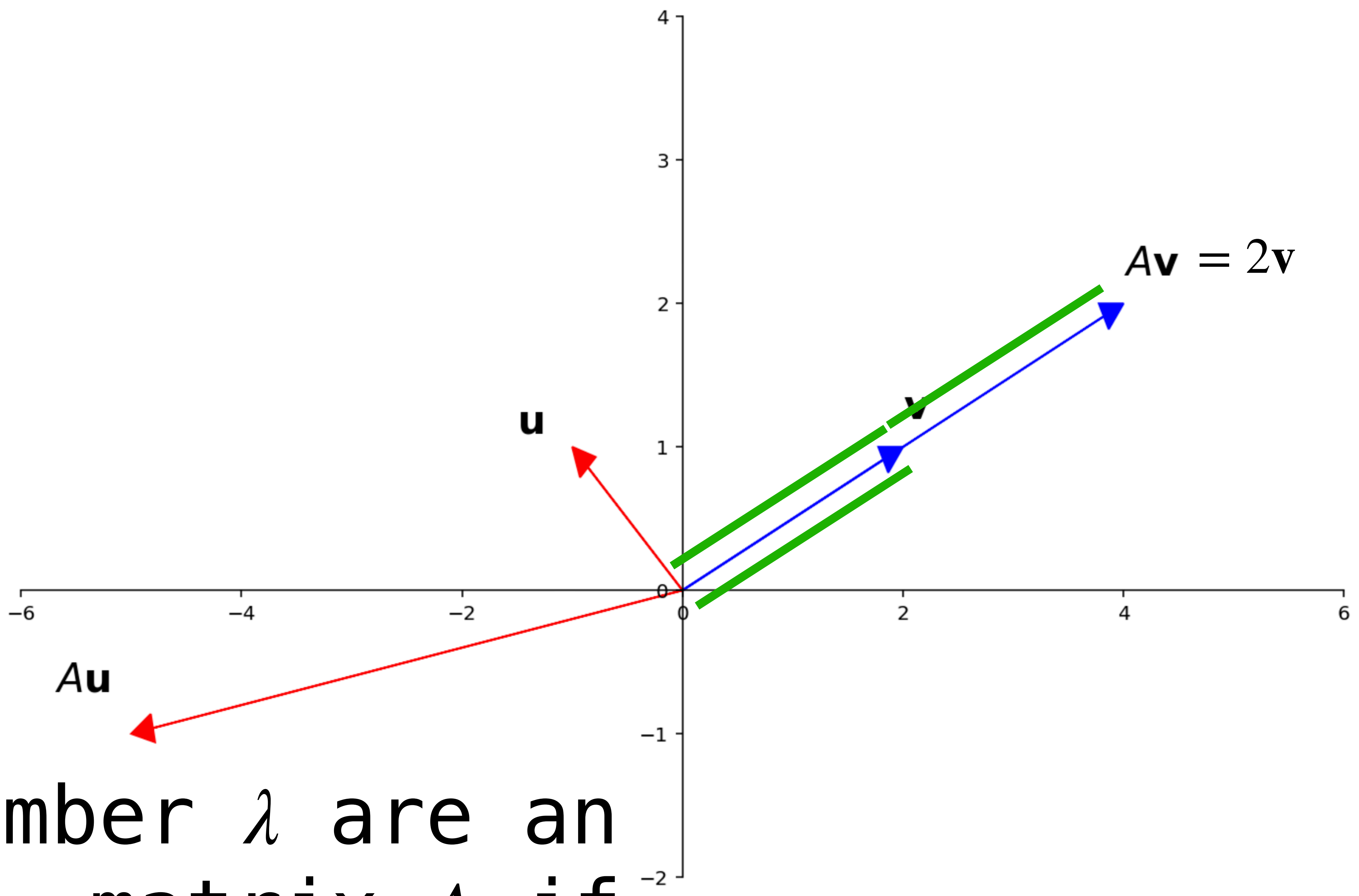
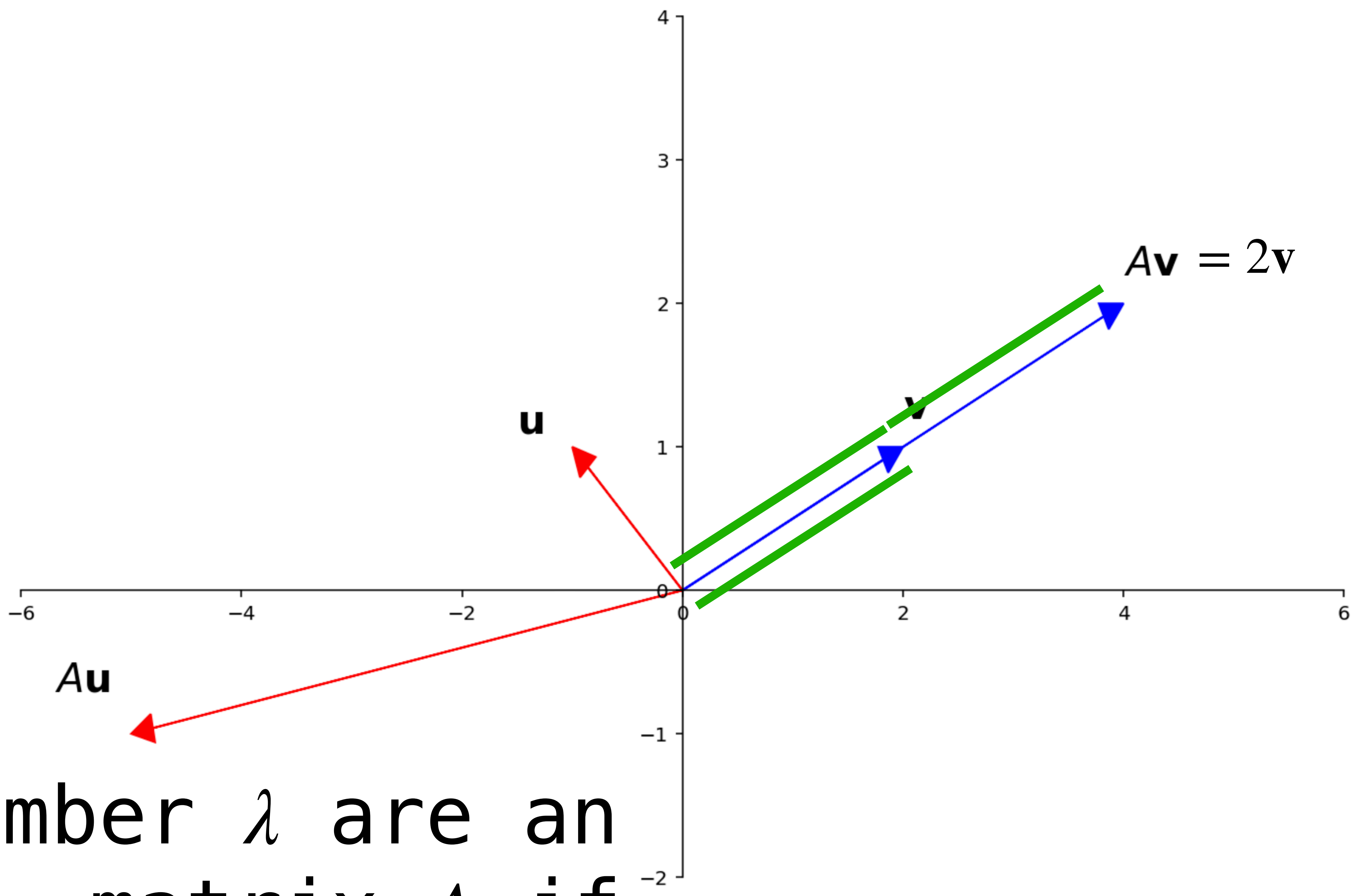


A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

Formal Definition



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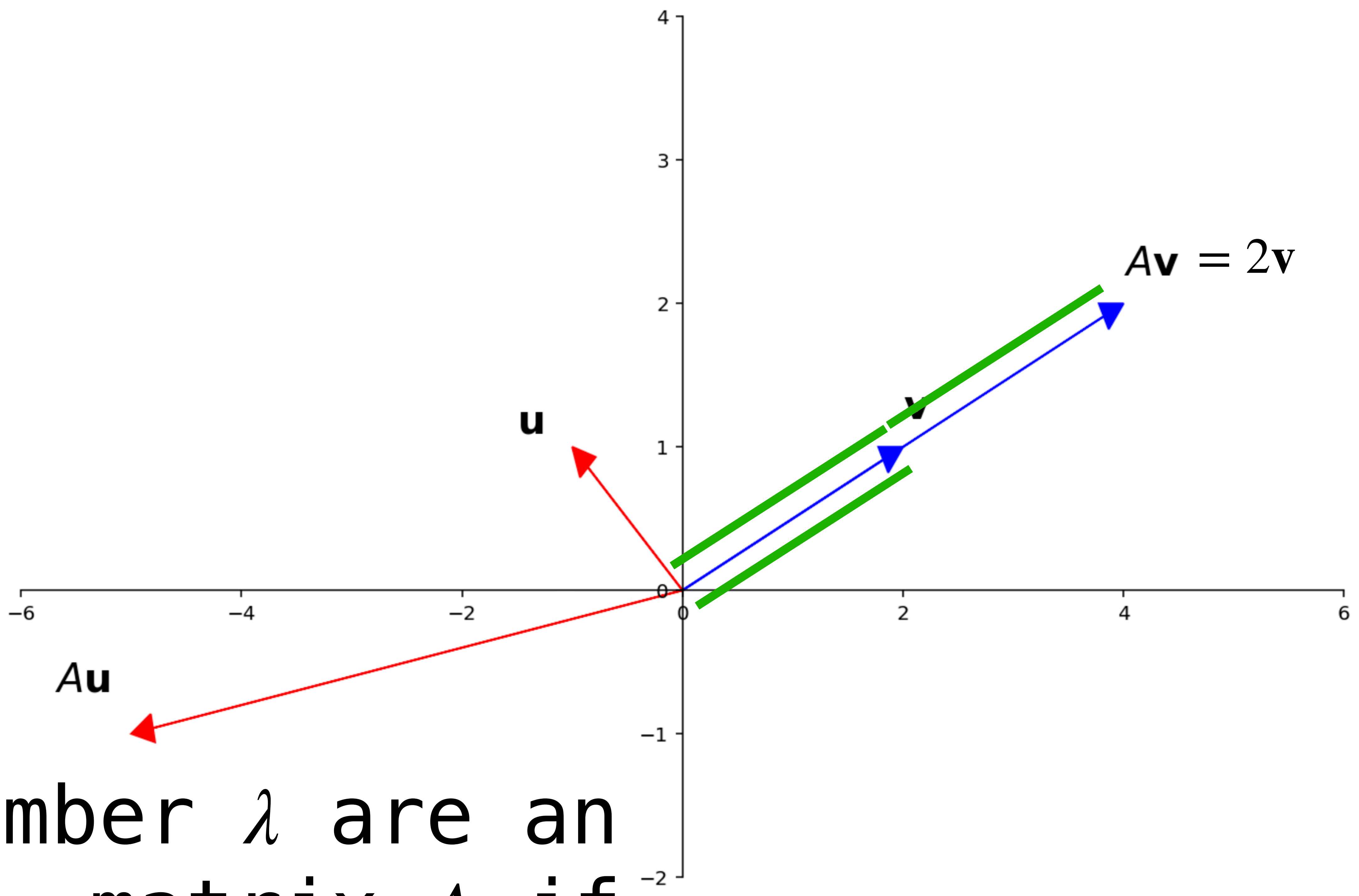
We will say that $\mathbf{v}$ is an eigenvector <u>of/for</u> the eigenvalue λ, and that λ is the eigenvalue <u>of/corresponding to</u> $\mathbf{v}$.

Formal Definition

$A\vec{0} = 0 \cdot \vec{0}$



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

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Note. Eigenvectors <u>must</u> be nonzero, but it is possible for 0 to be an eigenvalue.

What if 0 is an eigenvalue?

What if 0 is an eigenvalue?

If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

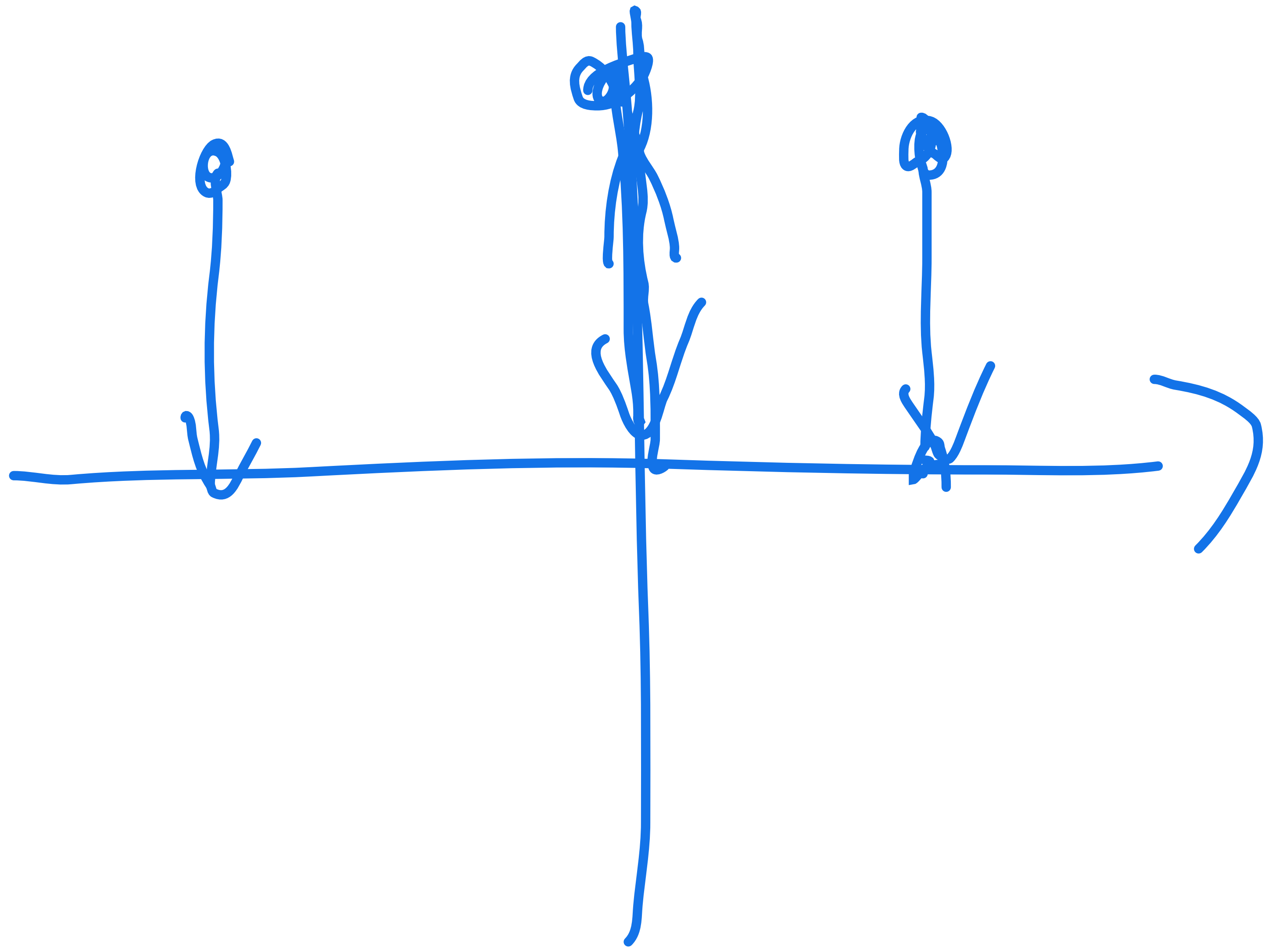
$$\{\mathbf{v} \mid \underline{A\mathbf{v} = 0\mathbf{v} = \mathbf{0}}\} = \mathrm{Null}(A)$$

What if 0 is an eigenvalue?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$$



In other words,

- $\mathbf{v} \in \mathrm{Nul}(A)$
- $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

Extending the IMT (Again)

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Theorem. A $n \times n$ matrix is invertible if and only if it <u>does not</u> have 0 as an eigenvalue.

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To reiterate. An eigenvalue 0 is equivalent to

Extending the IMT (Again)

Theorem. A $n \times n$ matrix is invertible if and only if it <u>does not</u> have 0 as an eigenvalue.

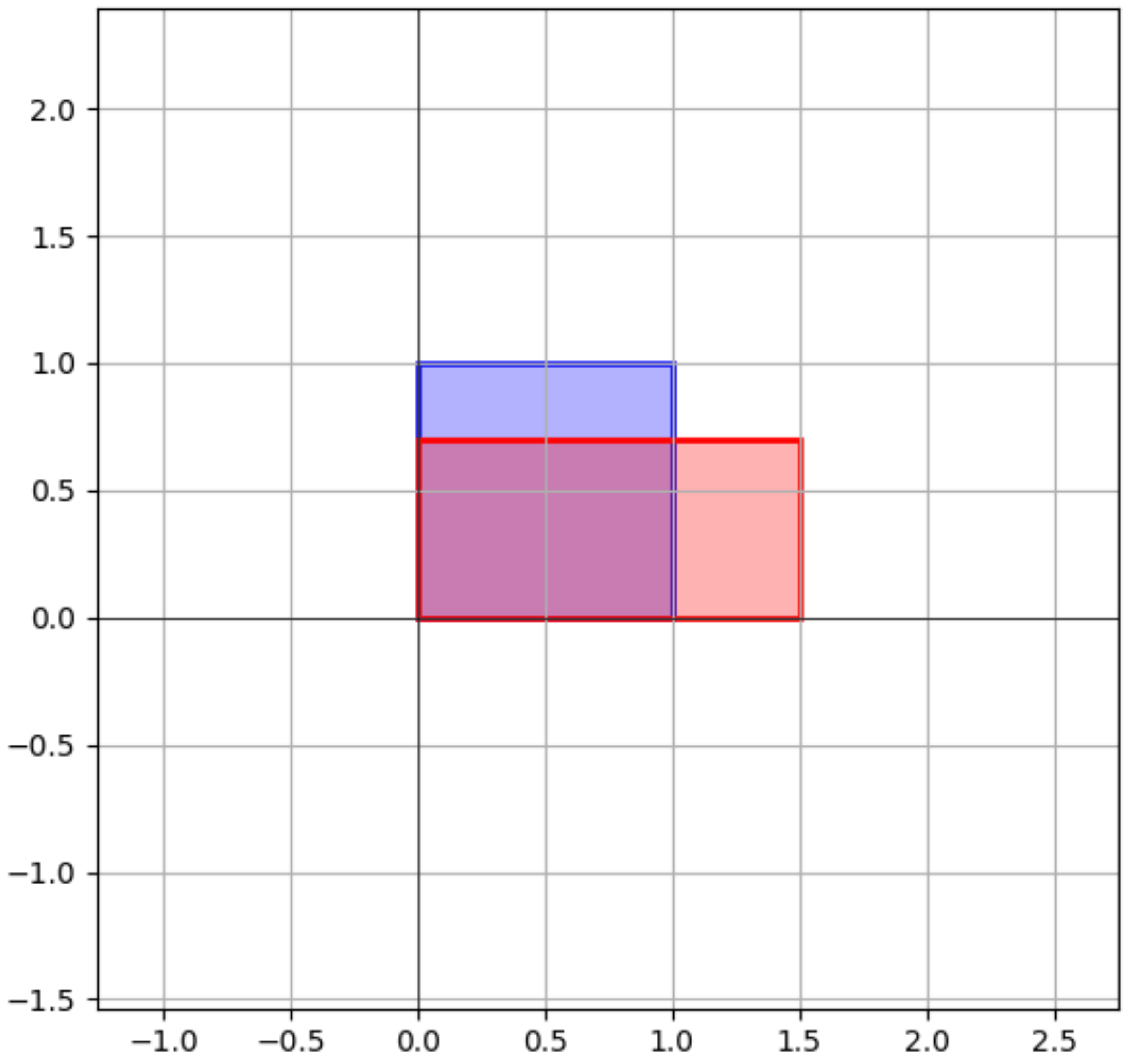
$\text{Nul}\,A = \{0\}$

To reiterate. An eigenvalue 0 is equivalent to

» $A\mathbf{x} = \mathbf{0}$ has ~~no~~ nontrivial solutions
» the columns of A are linearly dependent
» $\text{Col}(A) \neq \mathbb{R}^n$
» ...

Example: Unequal Scaling

Let's determine it's eigenvalues and eigenvectors:



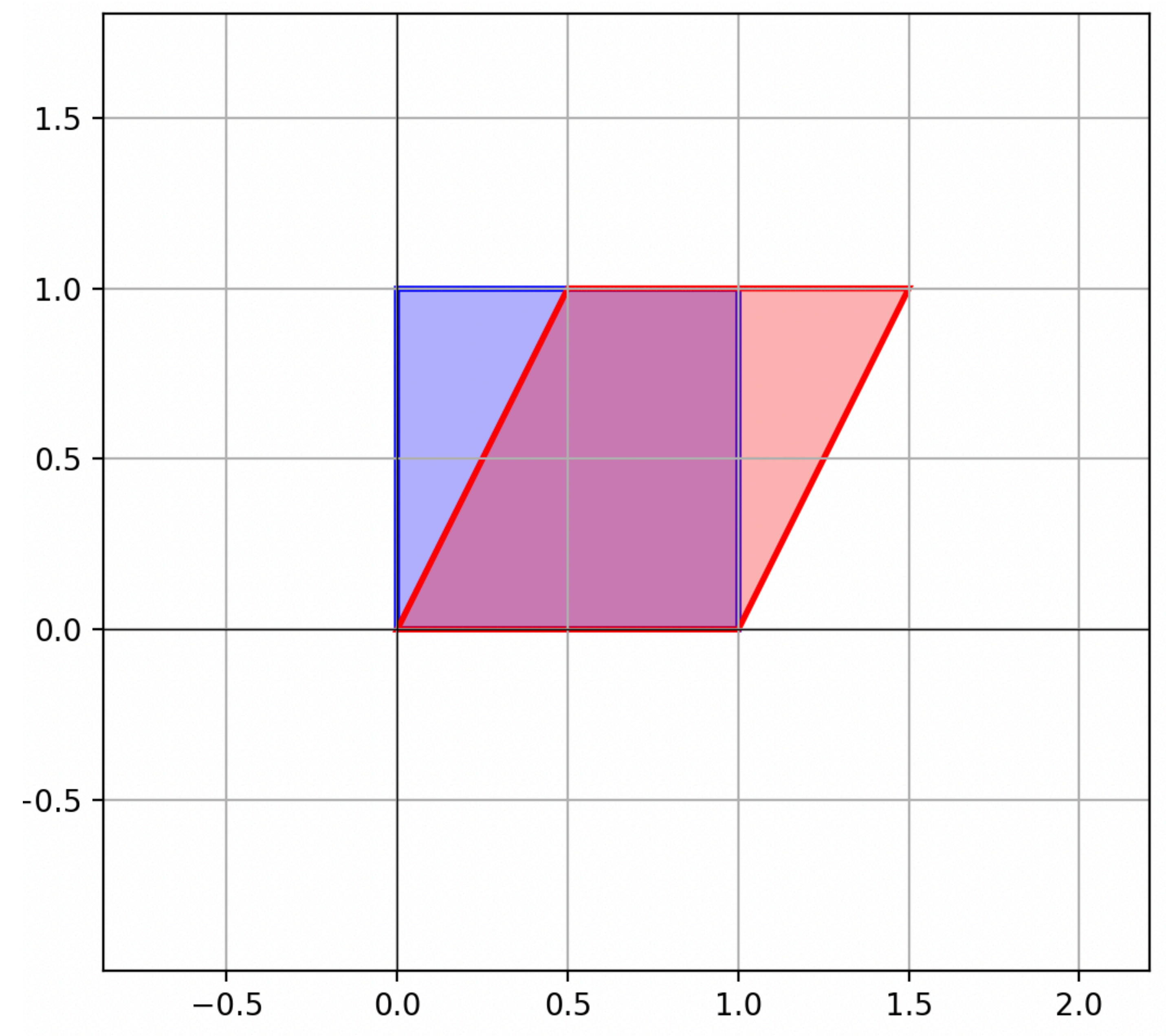
$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

1.5 0.7

Example: Shearing

Let's determine it's eigenvalues and eigenvectors:

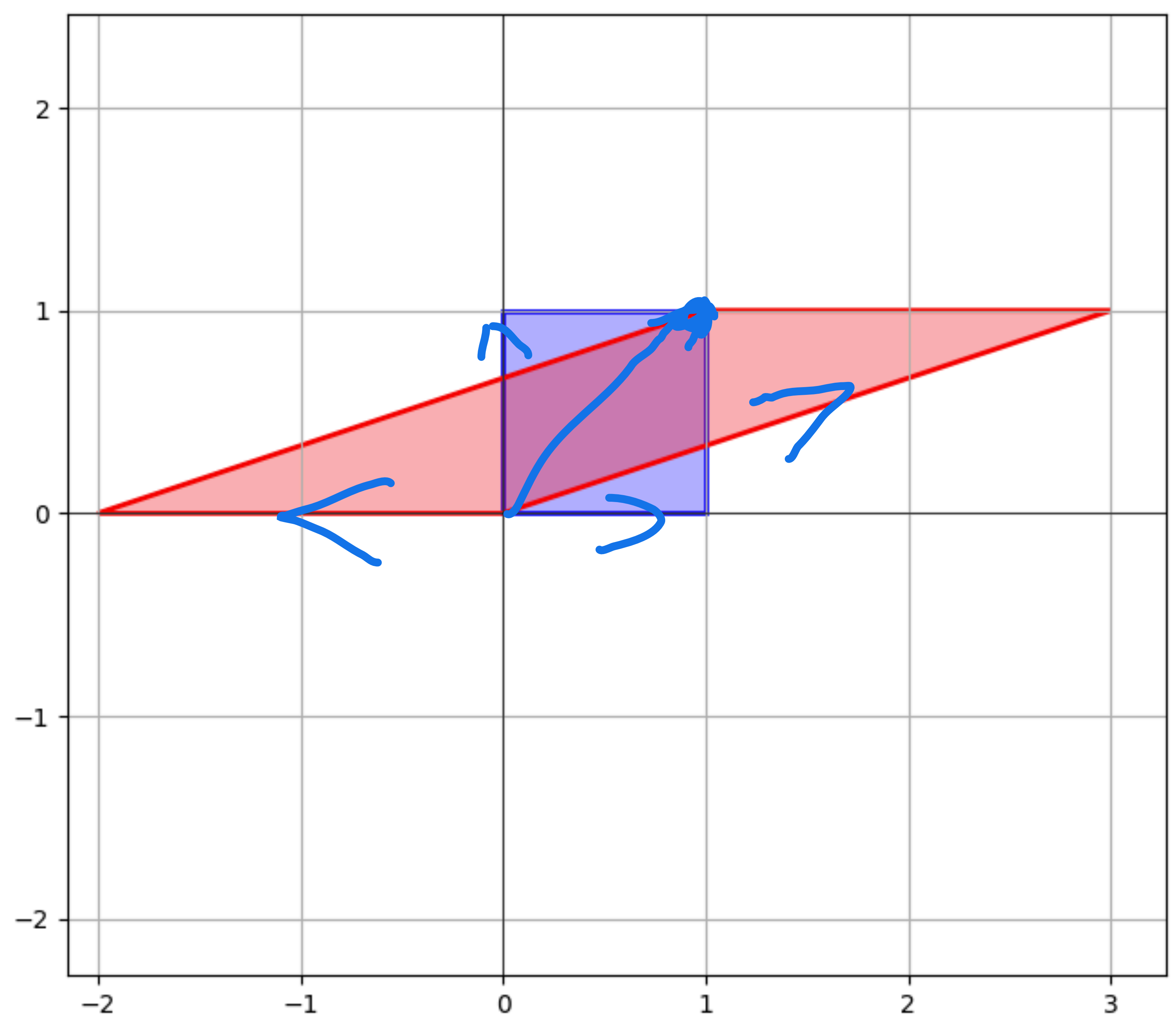
$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{(1)}_{\lambda} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$



$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underset{\lambda_u}{(1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underset{\lambda_v}{2} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

How do we verify eigenvalues and eigenvectors?

Verifying Eigenvectors

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.* ask

Solution. Easy. Work out the matrix-vector multiplication.

Verifying Eigenvectors

$\vec{v}_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = \underset{\lambda_1}{(-4)}\begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \Rightarrow \vec{v}_2 \text{ not an eigenvector}$$

Verifying Eigenvalues

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This is harder...
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Question. Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

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What vector do we check???

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What vector do we check???

Before we go over how to do this...

Verifying Eigenvalues (Warm Up)

Question. Verify that 1 is an eigenvalue of

$$A = \begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

$$(A - I)\vec{x} = 0$$

Hint. Recall our discussion of Markov Chains.

Solution: A is regular $\Rightarrow$ there is a unique steady state

$$A\vec{x} = \vec{x} = (1)\vec{x}$$

$$\Leftrightarrow A\vec{x} - \vec{x} = 0 \Leftrightarrow (A - I)\vec{x} = 0$$

Steady-States and Eigenvectors

Steady-state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue 1.

How did we find steady-state vectors?:

look for (nontrivial) sol'ns to $(A-I)\vec{x}=0$

Steady-States and Eigenvectors

$$\mathbf{v} \text{ is a steady-state vector}^{*} \equiv \mathbf{v} \in \mathrm{Nul}(A - I)$$

$*$It must also be a probability vector

Verifying Eigenvalues

This is harder...

Question. Show that λ is an eigenvalue of A.

(there exists $\vec{v} \neq 0$)

Solution:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

$$\vec{v} \in \mathrm{Nul}(A - \lambda I)$$

Verifying Eigenvalues

$\mathbf{v}$ is an eigenvector for λ $\equiv$ $\mathbf{v} \in \mathrm{Nul}(A - \lambda I)$

(and $\vec{v} \neq 0$)

if just $\{0\}$

λ not an eigenvalue

Verifying Eigenvalues

$(A-7I)\vec{x}=0$

This is harder...

$$A \text{ "=" } \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \quad \lambda \text{ "=" } 7$$

Question. Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Solution: $A-7I=\begin{bmatrix} -6 & 6 \\ 6 & -5 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1=x_2$

x_2 free

Yes!

Nonzero sol'ns

$\vec{x}=x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem

Ask

$(A-2I)\vec{x}=0$

$A\vec{x}=2\vec{x}$

Verify that $\overset{\lambda}{2}$ *is an eigenvalue of* $\overset{A}{\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}}$

$$A-2I=\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_1 - x_2 + 6x_3 = 0$

$2x_1 = x_2 - 6x_3$

$x_1 = \left(\frac{1}{2}\right)x_2 - 3x_3$

x_2 free

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x} = x_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

How many eigenvectors can a matrix have?

Linear Independence of Eigenvectors

Theorem.* If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

So an $n \times n$ matrix can have at most n eigenvalues.

Why?: more than n eigenvalues $\Rightarrow$ more than n lin. ind. eigenvectors

not possible

*We won't prove this.

Eigenspace

Fact. The set of eigenvectors for a eigenvalue λ of $A \in \mathbb{R}^{n\times n}$ form a subspace of $\mathbb{R}^n$.

Verify: $\mathrm{Nul}(A - \lambda I)$

closure under add'n: $\vec{v}, \vec{w}$ eigenvectors $\quad A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \lambda\vec{w}$

$= \lambda(\vec{v}+\vec{w})$

closure under scaling: $\vec{v}$ eigenvector $\quad A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$

Eigenspace

Definition. The set of eigenvectors for a eigenvalue λ of A is called the **eigenspace** of A corresponding to λ.

It is the same as $\mathrm{Nul}(A - \lambda I)$.

How To: Basis of an Eigenspace

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\mathrm{Nul}(A - \lambda I)$.

We know how to do this.

Example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -1 \\ -4 & 0 & 5 \end{bmatrix} \quad \vec{x} \in \mathrm{Nul}(A - I)$$

$$(A - I)\vec{x} = 0 \qquad \lambda = 1$$

Determine a basis for the eigenspace corresponding to the eigenvalue 1*:*

$$A - I = \begin{bmatrix} -3 & 0 & 3 \\ 1 & 0 & -1 \\ -4 & 0 & 4 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = x_3$$

$$x_2 \text{ free} \Leftrightarrow x_2 = x_2$$

$$x_3 \text{ free} \Leftrightarrow x_3 = x_3$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

How do we find eigenvalues?

How do we find eigenvalues?

We'll cover this next time...

Eigenvalues of Triangular Matrices

$(A-\lambda I)\vec{v}=0$ Does this have nontrivial sol'ns?

upper or lower

Theorem. The eigenvalues of a triangular matrix are its entries along the diagonal.

Verify:

$$A=\begin{bmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33} \end{bmatrix} \qquad A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}a_{11}\\0\\0\end{bmatrix}=a_{11}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$A-a_{22}I=\begin{bmatrix} a_{11}-a_{22} & * & * \\ 0 & 0 & * \\ 0 & 0 & a_{33}-a_{22} \end{bmatrix}$$

free variable! nontrivial sol'ns exist

Still have to do $\lambda_3 = 2$

~~(Do just 2)~~

Example

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

λ_1, λ_2, λ_3

$(A - \lambda I)\vec{v} = 0$

$xy = 0$

$A\vec{x} = 0$

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Determine the eigenvectors and values of the above matrix:

$$\begin{bmatrix} 0 & 6 & -8 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

x_1 free

$x_2 = 0$

$x_3 = 0$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{v}_1$

$\lambda_2 = 0 \quad A - 0I = A$

$$\begin{bmatrix} 1 & 2 & -8/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2$

x_2 free

$x_3 = 0$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\vec{v}_2$

$\lambda_3 = 2$

$$A - 2I = \begin{bmatrix} 1 & 6 & -8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_3 \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$$

Linear Dynamical Systems

$$\vec{v}_3 = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$$

Recall: Linear Dynamical Systems

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Definition. A **(discrete time) linear dynamical system** is described by a $n \times n$ matrix A. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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The possible states of the system are vectors in $\mathbb{R}^n$.

A tells us how our system evolves over time.

Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after i time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

Recall: State Vectors

$$
\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}
$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number k time steps

This is also called a *recurrence relation* or a *linear difference function*

Recall: State Vectors

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0)$$

$$\boxed{\mathbf{v}_k = A^k\mathbf{v}_0}$$

$$\mathbf{v}_5 = A\mathbf{v}_4 = A(AAAA\mathbf{v}_0)$$

$$\vdots$$

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It's defined in terms of A itself, which doesn't tell us much about how the system behaves

It's also difficult computationally because matrix multiplication is expensive

(Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

(Closed-Form) Solutions

A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain A^k or previously defined terms

In other word, it does not depend on A^k and is not **recursive**

Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

$$v_1 = A v_0 = \lambda_1 \vec{v}_0$$

$$v_2 = A v_1 = A \lambda_1 \vec{v}_0 = \lambda_1 \lambda_1 \vec{v}_0 = \lambda^2 \vec{v}_0$$

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<u>The Key Point.</u> This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

$A^k\vec{v} \sim a_1\lambda_1^k\vec{b}_1$

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for A which span all of $\mathbb{R}^2$:

λ_1, λ_2

$\lambda_1 > \lambda_2 \geq 0$

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2$$

$$A^k\vec{v} = a_1 A^k\vec{b}_1 + a_2 A^k\vec{b}_2 = \underline{a_1\lambda_1^k\vec{b}_1 + a_2\lambda_2^k\vec{b}_2}$$

$$A^2\vec{v} = A(a_1\lambda_1\vec{b}_1 + a_2\lambda_2\vec{b}_2)$$

$$= a_1\lambda_1 A\vec{b}_1 + a_2\lambda_2 A\vec{b}_2 = a_1\lambda_1^2\vec{b}_1 + a_2\lambda_2^2\vec{b}_2$$

$$\frac{A^k\vec{v}}{\lambda_1^k} = a_1\vec{b}_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{b}_2$$

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{\cancel{k}\,n}$ of A with eigenvalues

$$|\lambda_1| > |\lambda_2| \ldots \geq |\lambda_{\cancel{k}\,n}|$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows exponentially in λ_1).

Verify: argued in case $k=2$

$$A^k \vec{v} = a_1 \lambda_1^k \vec{v}_1 + \cdots + a_n \lambda_n^k \vec{v}_n$$

If eigenbasis exists, this is closed-form

Eigenbases

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Not all matrices have eigenbases.

Example: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\lambda = 1 \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Eigenbases and Growth in the Limit

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for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

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The largest eigenvalue describes the long-term exponential behavior of the system.