# Eigenvalues and Eigenvectors

## Geometric Algorithms
## Lecture 18

CAS CS 132

# Practice Problem

*Suppose* $A$ *is a* $234 \times 300$ *matrix. What is the smallest possible value for* $\dim(\text{Nul}(A))$*? What is the largest possible value?*

*What is the smallest possible value for* $\text{rank}(A)$*? What is the largest possible value?*

# Answer

$$A \text{ is } \overset{m \,\times\, n}{234 \times 300}$$

$$\underbrace{\dim(\mathrm{Col}\,A)}_{\substack{\text{"rank"} \\ \leq 234}} + \underbrace{\dim(\mathrm{Nul}\,A)}_{\text{"nullity"}} = n$$

$$300$$

$$\boxed{\begin{array}{c} 66 \leq \dim(\mathrm{Nul}\,A) \leq 300 \\ 0 \leq \dim(\mathrm{Col}\,A) \leq 234 \end{array}}$$

if $\dim(\mathrm{Nul}\,A) = 300$
& $\dim(\mathrm{Col}\,A) = 0$

$$\begin{bmatrix} & & \\ & \underline{0} & \\ & & \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_{300} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

# Objectives

1. <u>Motivate</u> and introduce the fundamental notion of eigenvalues and eigenvectors

2. Determine how to <u>verify</u> eigenvalues and eigenvectors

3. Look at the <u>subspace</u> generated by eigenvectors

4. Apply the study of eigenvectors to <u>dynamical linear systems</u>

# Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

# Motivation

demo

# How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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All matrices do some combination of these things

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In 2D and 3D we've seen:

- rotations
- projections
- shearing
- reflection
- scaling/stretching
- ... **Today's focus**

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We don't need a whole matrix to do scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

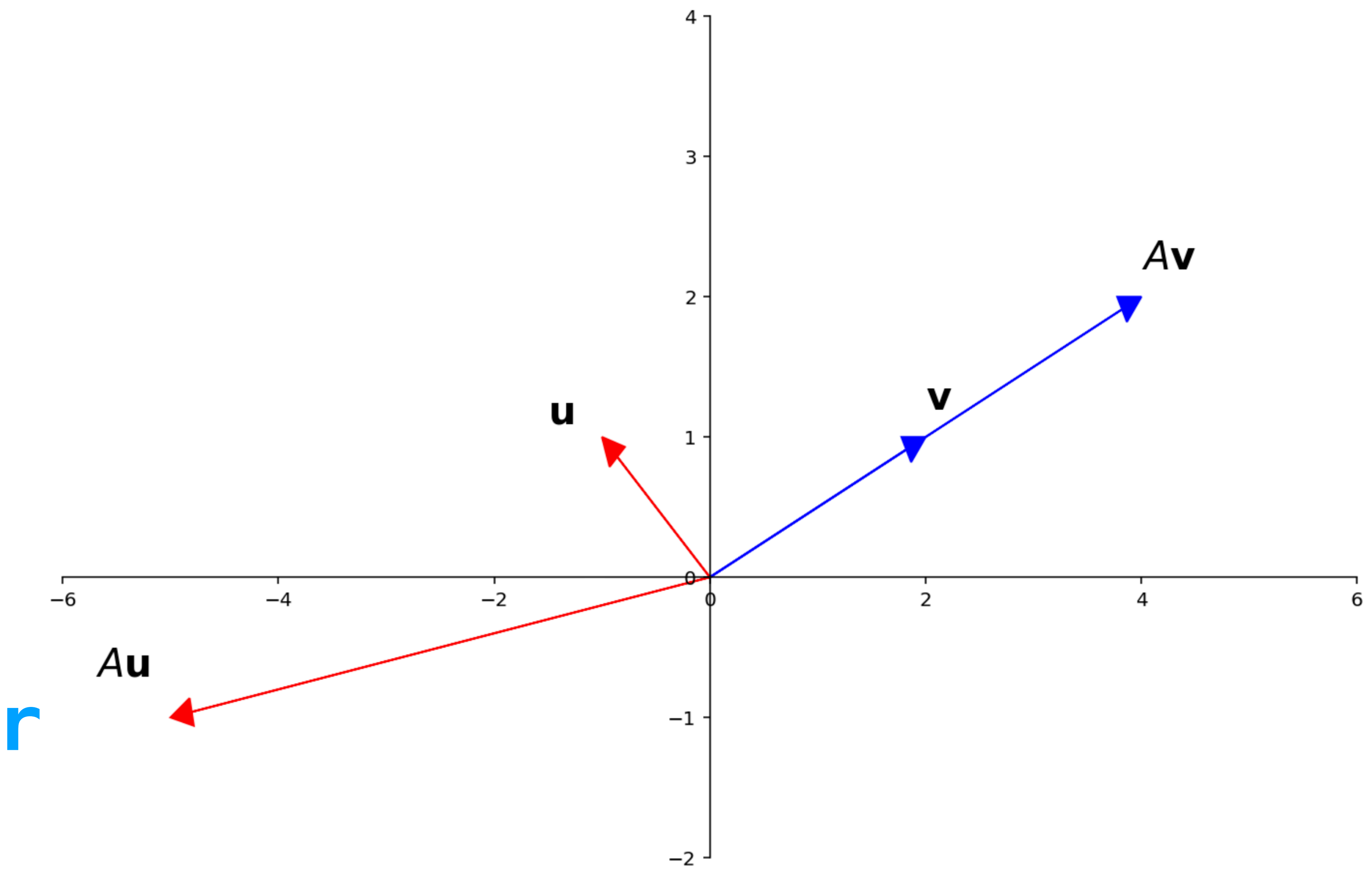
So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what $A$ does to $\mathbf{v}$.

# Eigenvectors (Informal)

eigenvalue

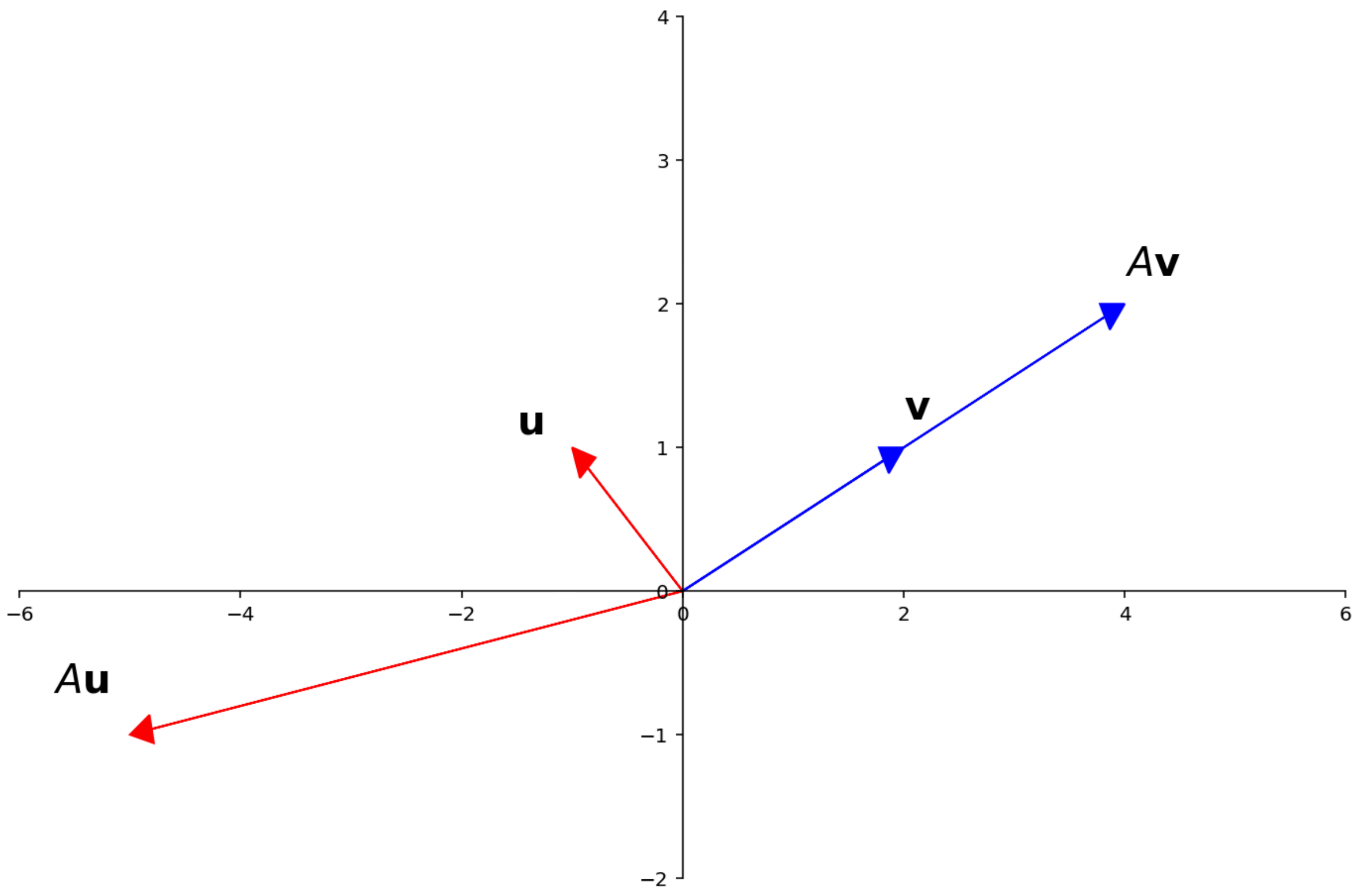
$$A\mathbf{v} = \lambda\mathbf{v}$$

eigenvector



# Eigenvectors (Informal)

$$A\mathbf{v} = \overset{\text{eigenvalue}}{\lambda}\underbrace{\mathbf{v}}_{\text{eigenvector}}$$




Eigenvectors of $A$ are stretched by $A$ without changing their direction.

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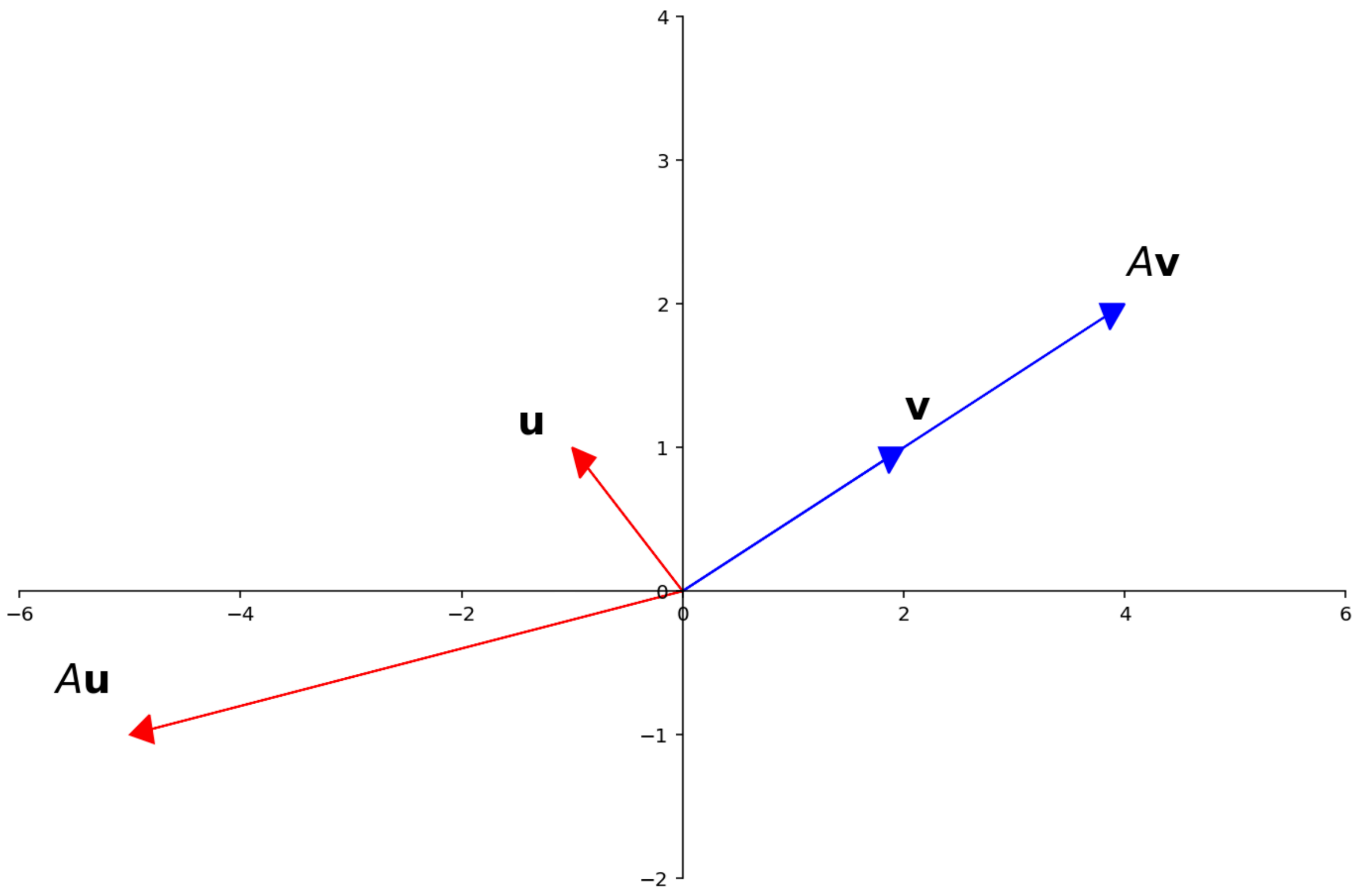



Eigenvectors of $A$ are stretched by $A$ without changing their direction.

The amount they are stretched is called the **eigenvalue.**

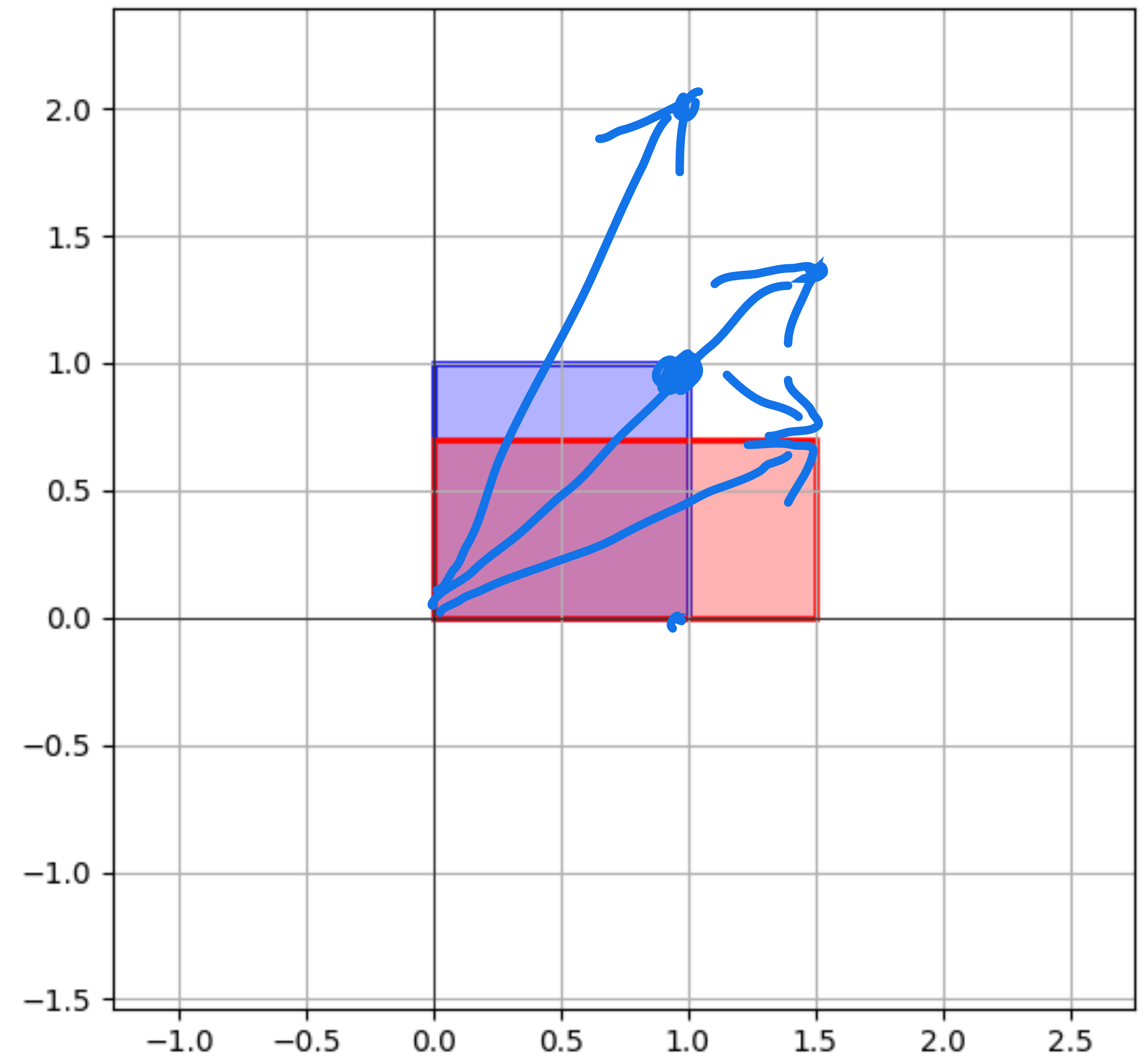
# Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

*It transforms each entry individually and then combines them.*

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = \overset{\lambda_1}{(1.5)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix} = (0.7) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

# Eigenbases (Informal)

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of* $A$. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

# Eigenbases (Informal)

$\lambda_1 \quad \lambda_2 \quad \lambda_3$

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$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

It's "easy to describe" how $A$ transforms $\mathbf{v}$.

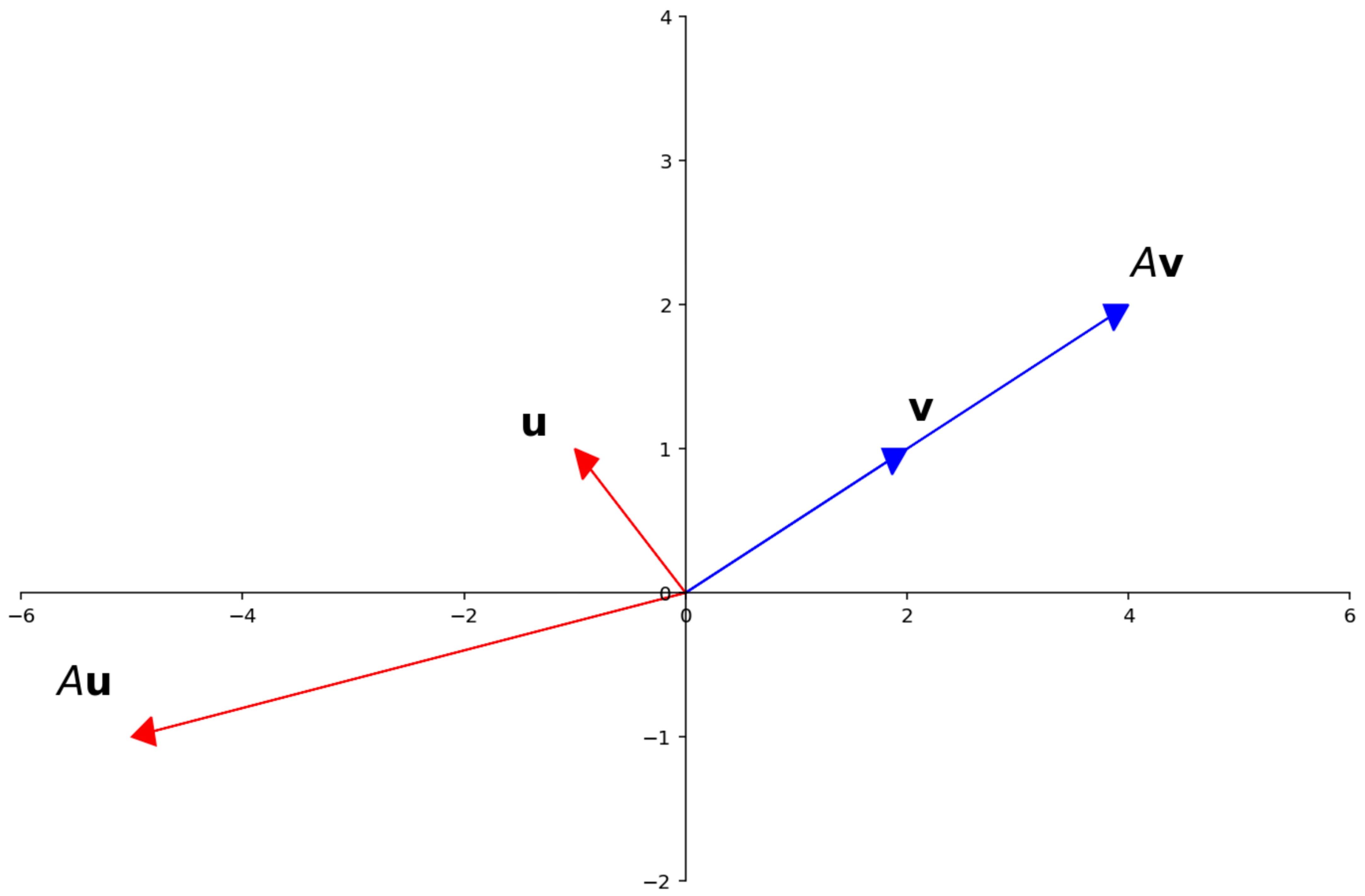
It *transforms each "component" individually and then combines them.*

Verify: $A\vec{v} = A(2\vec{b}_1 - \vec{b}_2 - 5\vec{b}_3) = 2A\vec{b}_1 - A\vec{b}_2 - 5A\vec{b}_3$

$= 2\lambda_1\vec{b}_1 - \lambda_2\vec{b}_2 - 5\lambda_3\vec{b}_3$

# Eigenvalues and Eigenvectors

# Formal Definition



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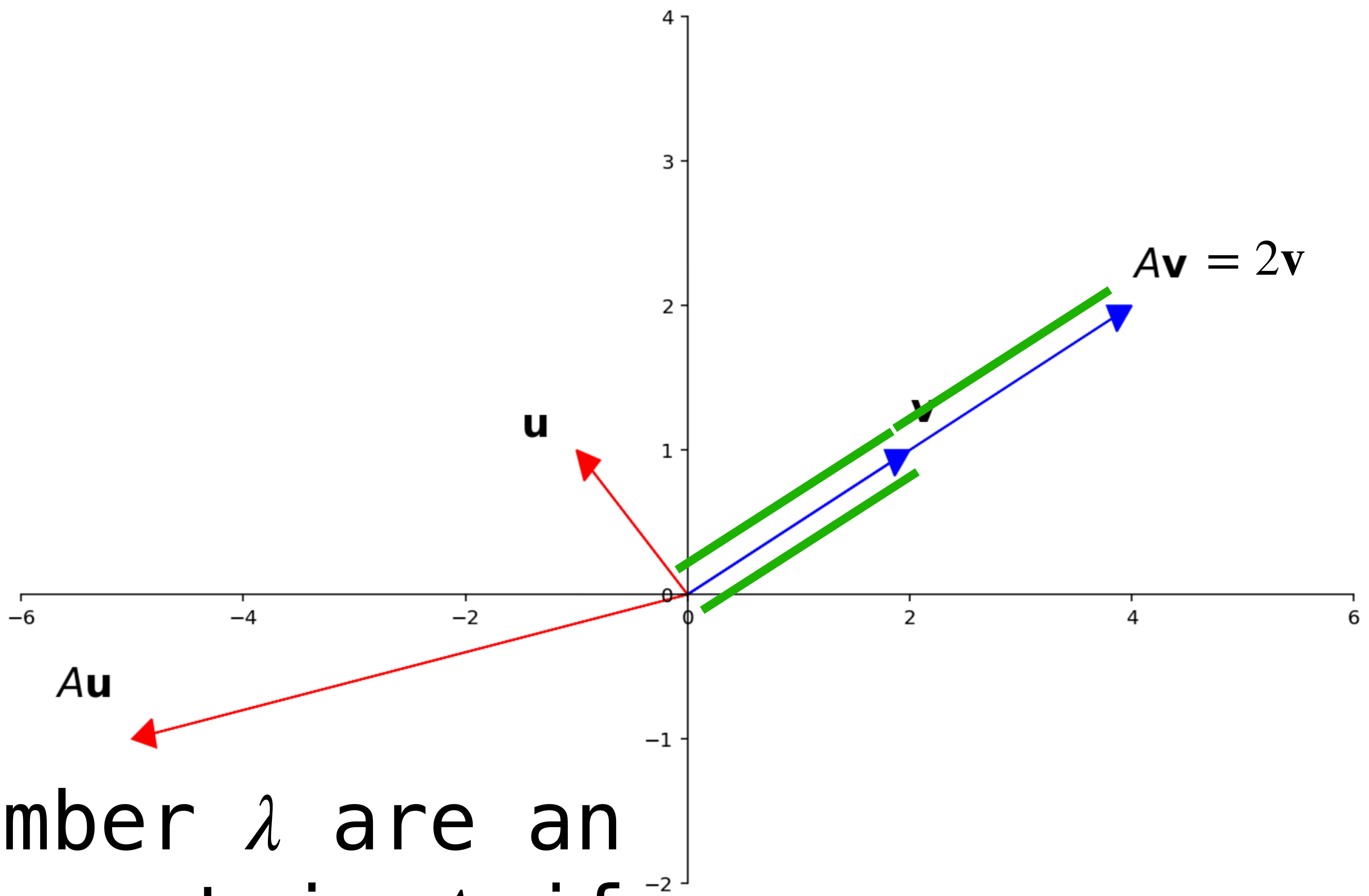


A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number $\lambda$ are an **eigenvector and eigenvalue** for a $n \times n$ matrix $A$ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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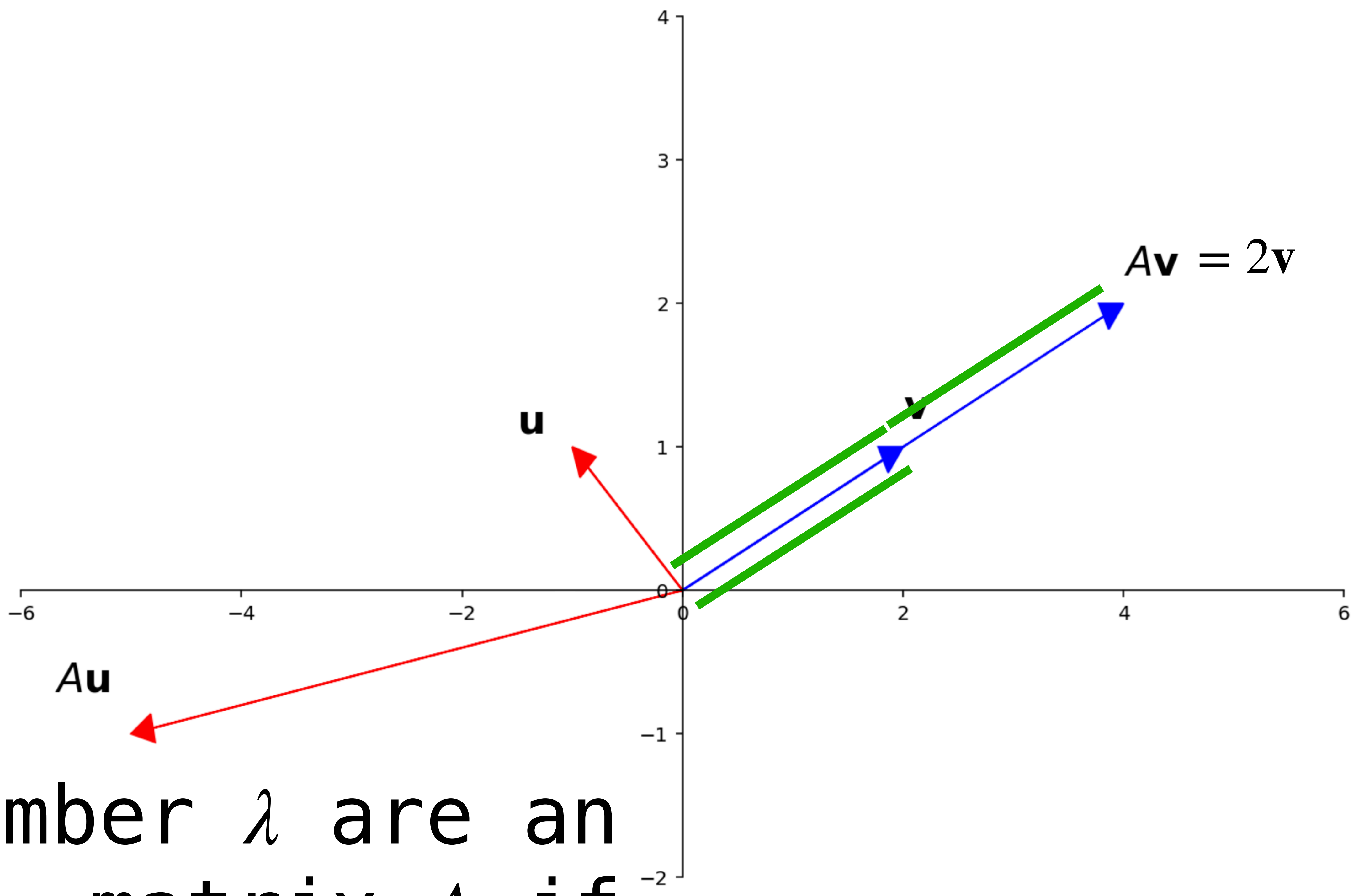


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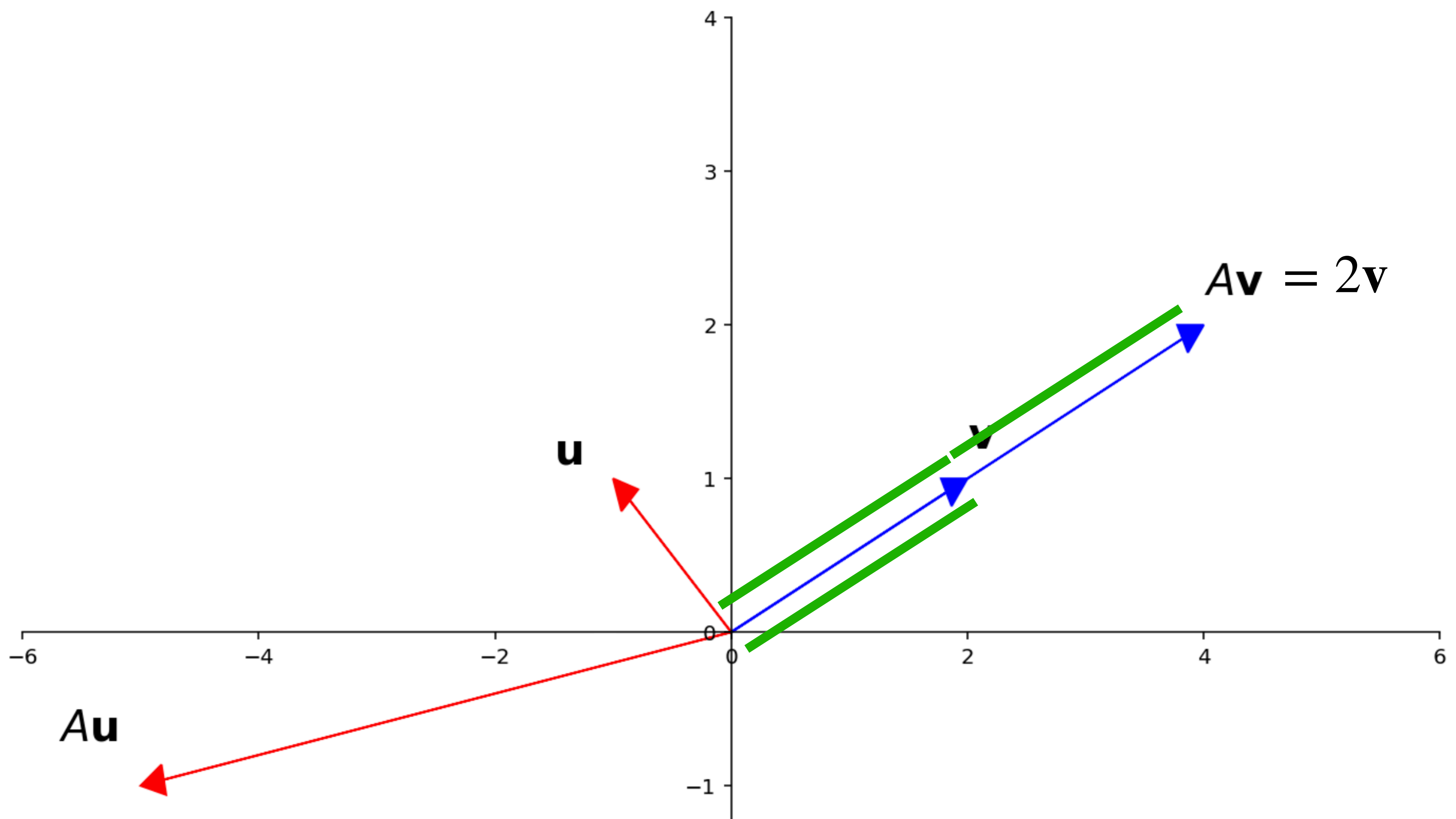
We will say that $\mathbf{v}$ is an eigenvector <u>of/for</u> the eigenvalue $\lambda$, and that $\lambda$ is the eigenvalue <u>of/corresponding to</u> $\mathbf{v}$.

# Formal Definition

$A\vec{0} = \vec{0} = (0)\vec{0}$

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*Note.* Eigenvectors <u>must</u> be nonzero, <u>but</u> it is possible for $0$ to be an eigenvalue.

# What if 0 is an eigenvalue?

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If $A$ has the eigenvalue $0$ with the eigenvector $\mathbf{v}$, then

there is some $\vec{v} \neq 0$ such that

$$\mathbf{A}\mathbf{v} = \mathbf{0}\mathbf{v} = \mathbf{0}$$

what is the set of vectors $\vec{v}$ that satisfy

same as Null(A)

# What if 0 is an eigenvalue?

If $A$ has the eigenvalue $0$ with the eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \mathbf{0}\mathbf{v} = \mathbf{0}$$

In other words,

- » $\mathbf{v} \in \mathrm{Nul}(A)$
- » $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

# Extending the IMT (Again)

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**Theorem.** A $n \times n$ matrix is invertible if and only if it <u>does not</u> have $0$ as an eigenvalue.

$\Updownarrow$ $\mathrm{Nul}(A) = \{0\}$

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**Theorem.** A $n \times n$ matrix is invertible if and only if it <u>does not</u> have $0$ as an eigenvalue.

*To reiterate.* An eigenvalue $0$ is equivalent to

- » $A\mathbf{x} = \mathbf{0}$ has ~~no~~ nontrivial solutions
- » the columns of $A$ are linearly dependent
- » $\text{Col}(A) \neq \mathbb{R}^n$
- » ...

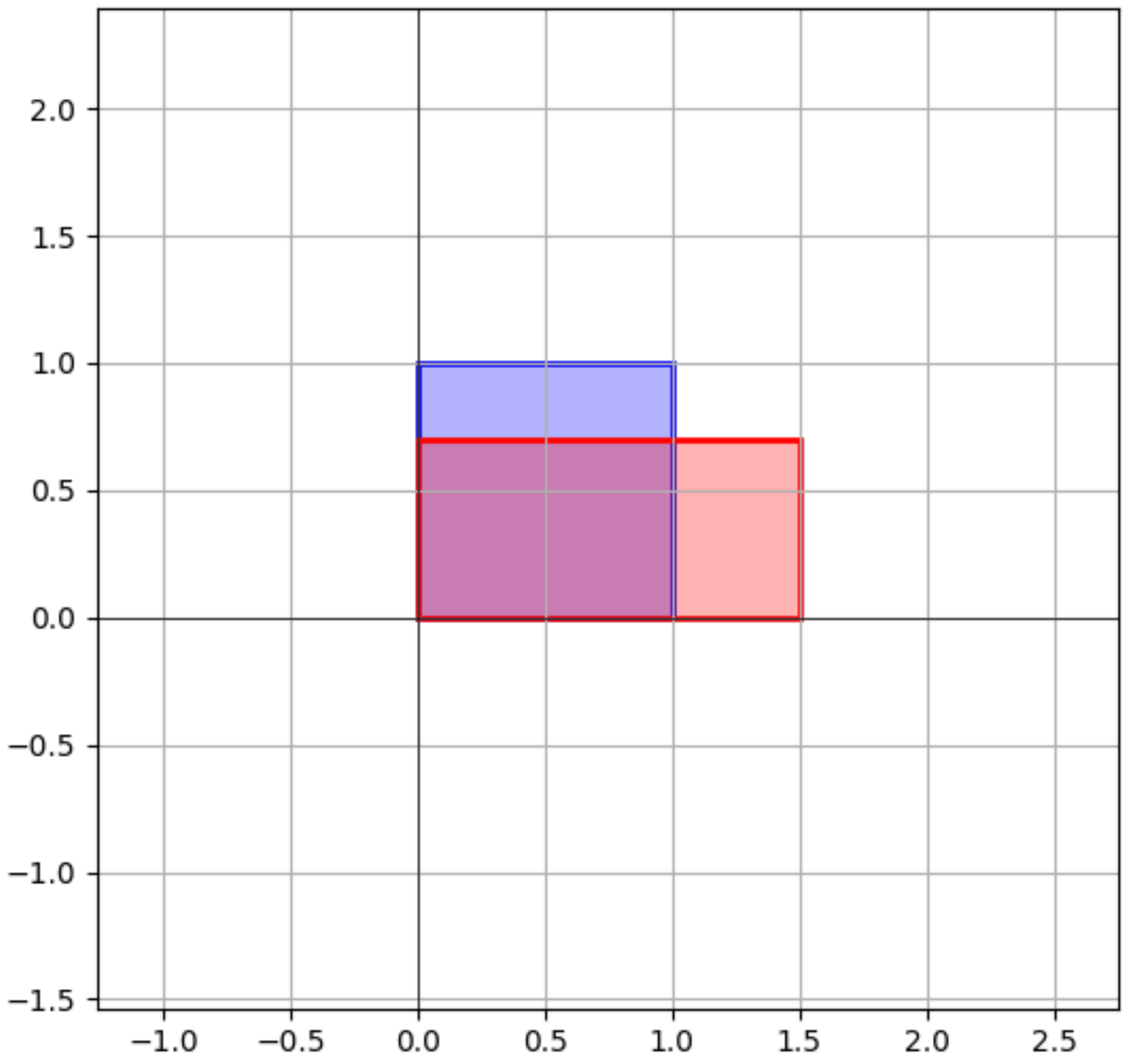
$\dim(\text{Nul } A) > 0$

$$\begin{pmatrix} * & * & & \\ & & * & * \\ & & & \\ & & & \end{pmatrix}$$

some free variables in RREF

# Example: Unequal Scaling

Let's determine it's eigenvalues and eigenvectors:



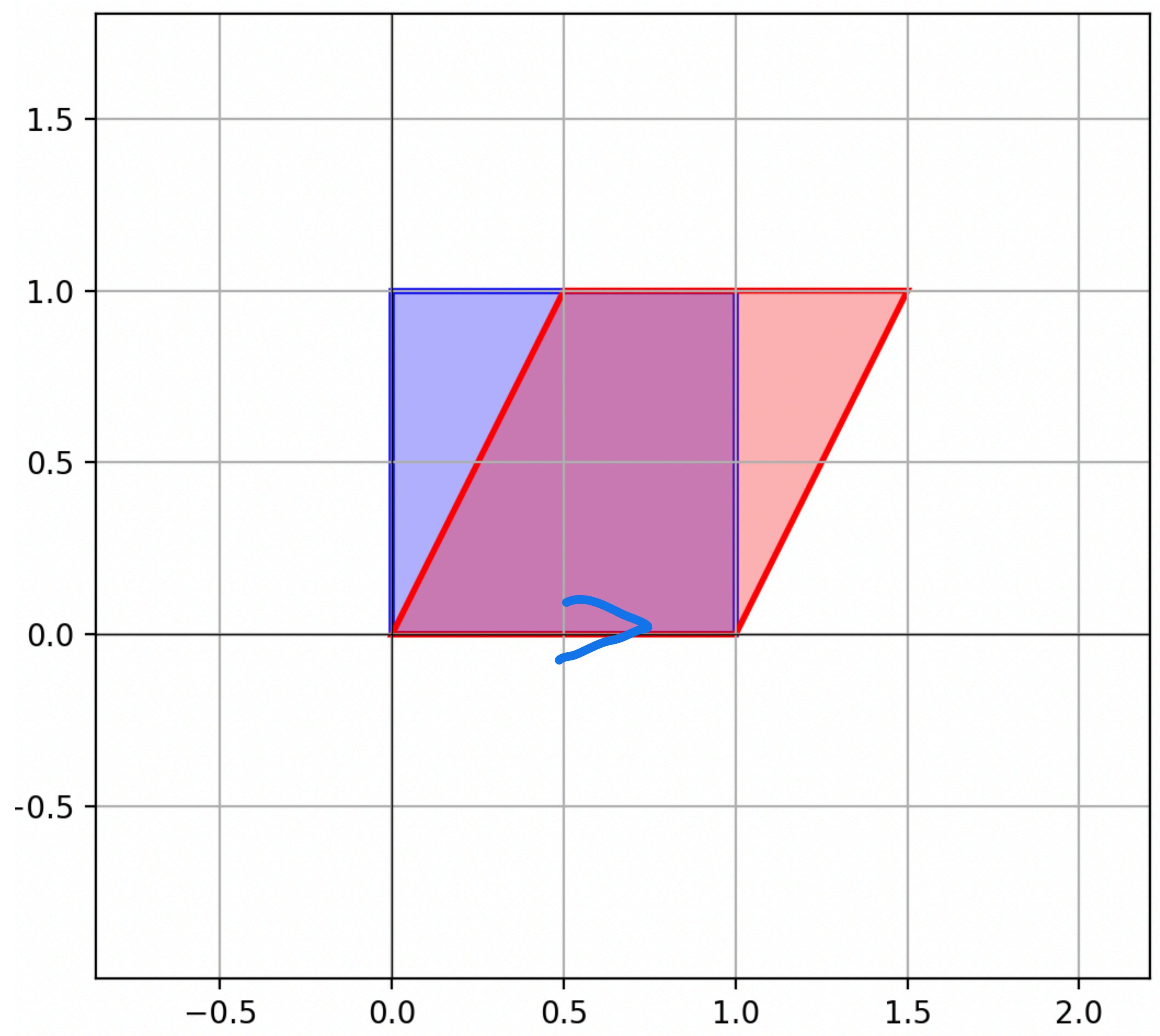

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

# Example: Shearing

Let's determine it's eigenvalues and eigenvectors:

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{(1)}_{\lambda} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

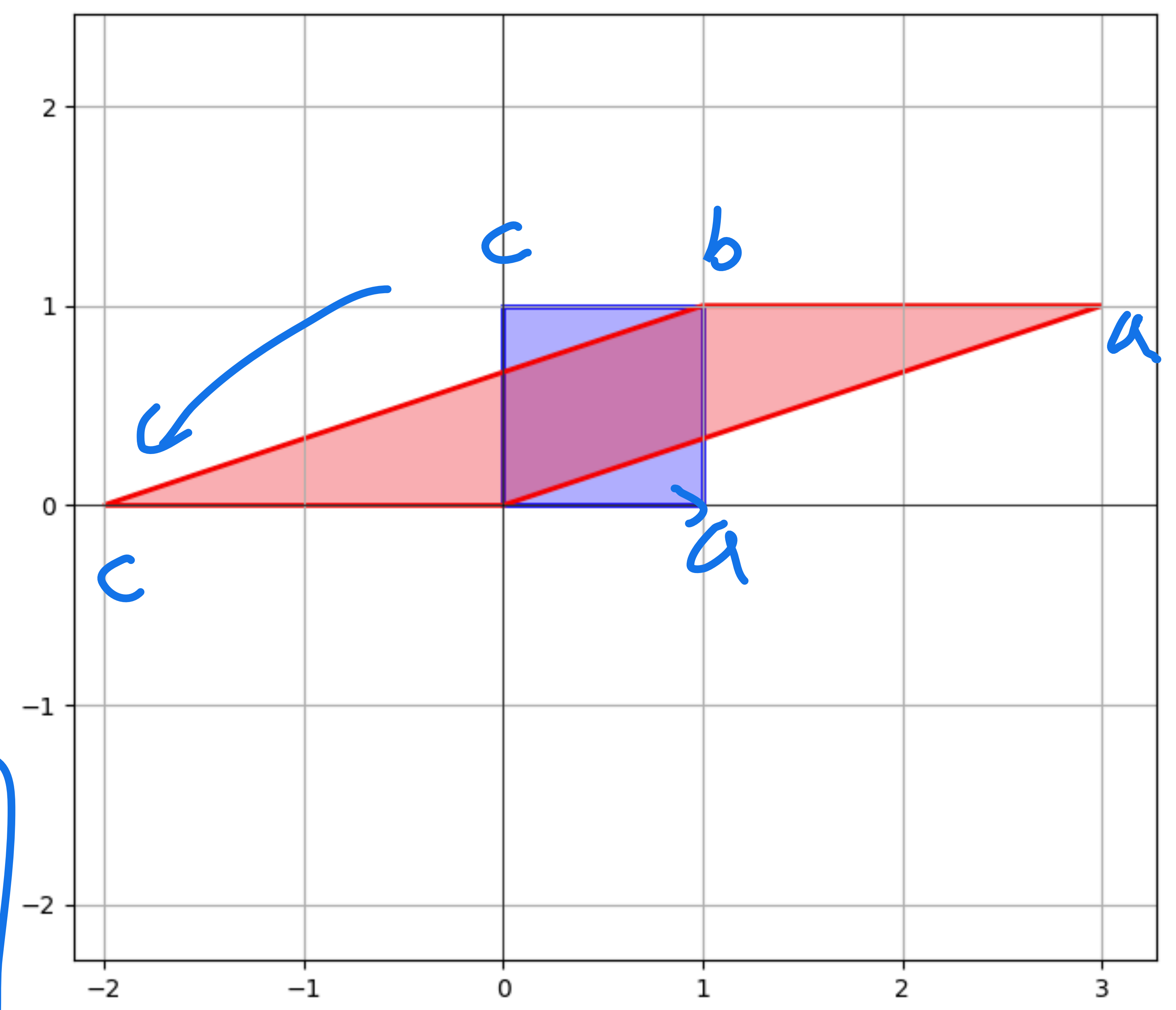
$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$




$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

# Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$




$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \overset{\lambda_u}{(1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underset{\lambda_v}{2} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

# How do we verify eigenvalues and eigenvectors?

# Verifying Eigenvectors

# Verifying Eigenvectors

***Question.*** *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

# Verifying Eigenvectors

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Ask

**Solution.** Easy. Work out the matrix-vector multiplication.

# Verifying Eigenvectors

$\vec{v}_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = \underbrace{(-4)}_{\lambda_1} \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

$\Rightarrow \vec{v}_2$ not an eigenvector

# Verifying Eigenvalues

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This is harder...
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**Question.** Show that $7$ is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

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What vector do we check???

Before we go over how to do this...

# Verifying Eigenvalues (Warm Up)

**Question.** Verify that $1$ is an eigenvalue of

$$\begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

*Hint. Recall our discussion of Markov Chains.*

Solution: A is regular, stochastic $\Rightarrow$ there is a steady state

$$A\vec{v} = \vec{v}$$

# Steady-States and Eigenvectors

Steady-state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue $1$.

How did we find steady-state vectors?:

$$A\vec{v} = \vec{v}$$

$$A\vec{v} - \vec{v} = 0$$

$$(A - I)\vec{v} = 0$$

# Steady-States and Eigenvectors

$\mathbf{v}$ is a steady-state vector* $\equiv$ $\mathbf{v} \in \mathrm{Nul}(A - I)$

*It must also be a probability vector

# Verifying Eigenvalues

This is harder...

**Question.** Show that $\lambda$ is an eigenvalue of $A$.

**Solution:**

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

is there $\vec{v} \neq 0$ in $\mathrm{Nul}(A - \lambda I)$?

# Verifying Eigenvalues

$\mathbf{v}$ `is an eigenvector for` $\lambda \quad \equiv \quad \mathbf{v} \in \mathsf{Nul}(A - \lambda I)$

# Verifying Eigenvalues

$(A-7I)\vec{x}=0$

This is harder... $\overset{?}{=}$ $\vec{x}\in \text{Nul}(A-7I)$

$A$

**Question.** Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

**Solution:** $A-7I=\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\sim\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1=x_2$
$x_2$ free

$\vec{x}=x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7 is an eigenvalue w/ eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ask

# Problem

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\lambda$ $\shortparallel$      $A$ $\shortparallel$

`Verify that` 2 `is an eigenvalue of` $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2-dim'l space of eigenvectors for $\lambda = 2$

$$\vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 = (1/2)x_2 - 3x_3$

$x_2$ free

$x_3$ free

# Answer

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

How many eigenvectors can a matrix have?

# Linear Independence of Eigenvectors

**Theorem.*** If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

*So an $n \times n$ matrix can have at most $n$ eigenvalues.*

Why?: if A has >n eigenvalues

$\Rightarrow$ >n lin. dep. eigenvectors

This is not possible in $\mathbb{R}^n$

*We won't prove this.

# Eigenspace

**Fact.** The set of eigenvectors for a eigenvalue $\lambda$ of $A \in \mathbb{R}^{n \times n}$ form a subspace of $\mathbb{R}^n$.

Verify: closure under scaling ✓

closure under addn: if $\vec{v}, \vec{w}$ eigenvectors

$$A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$$
$$= \lambda\vec{v} + \lambda\vec{w} = \lambda(\vec{v}+\vec{w})$$

Alternate args: eigenspace is just a nullspace $\mathrm{Nul}(A - \lambda I)$

# Eigenspace

**Definition.** The set of eigenvectors for a eigenvalue $\lambda$ of $A$ is called the **eigenspace** of $A$ corresponding to $\lambda$.

It is the same as $\operatorname{Nul}(A - \lambda I)$.

# How To: Basis of an Eigenspace

**Question.** Find a basis for the eigenspace of $A$ corresponding to $\lambda$.

**Solution.** Find a basis for $\mathrm{Nul}(A - \lambda I)$.

**We know how to do this.**

# Example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -1 \\ -4 & 0 & 5 \end{bmatrix}$$

$\mathrm{Nul}(A-I)$

*Determine a basis for the eigenspace corresponding to the eigenvalue* $1$ *:* $\lambda = 1$

$$A - I = \begin{bmatrix} -3 & 0 & 3 \\ 1 & 0 & -1 \\ -4 & 0 & 4 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 = x_3$

$x_2$ free

$x_3$ free

# How do we find eigenvalues?

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**We'll cover this next time...**

# Eigenvalues of Triangular Matrices

**Theorem.** The eigenvalues of a upper/lower triangular matrix are its entries along the diagonal.

Verify: $\begin{bmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} = a_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(A - a_{22}I) = \begin{bmatrix} a_{11}-a_{22} & * & * \\ 0 & 0 & * \\ 0 & 0 & a_{33}-a_{22} \end{bmatrix} \Rightarrow (A - a_{22}I)\vec{x} = 0$ has nontrivial soln's

free variable

# Example

$\lambda_1, \lambda_2 = 0 \quad A - 0I = A$

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -8/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_3$ eigen

$x_1 = -2x_2$
$x_2$ free
$x_3 = 0$

$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 =$

$\lambda_3 = 2$

$$A - 2I = \begin{bmatrix} 1 & 6 & -8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

*Determine the eigenvectors and values of the above matrix:*

$$\sim \begin{bmatrix} 1 & 0 & 10 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -10x_3$
$x_2 = 3x_3$
$x_3$ free

$\vec{x} = x_3 \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_3 =$

Verfying

$$A \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -30 + 18 - 8 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -20 \\ 6 \\ 2 \end{bmatrix}$$

$\lambda_1 = 3$ exercise

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

# Linear Dynamical Systems

# Recall: Linear Dynamical Systems

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**Definition.** A **(discrete time) linear dynamical system** is described by a $n \times n$ matrix $A$. It's **evolution function** is the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

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Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after $i$ time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

# Recall: Linear Dynamical Systems

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The possible states of the system are vectors in $\mathbb{R}^n$.

$A$ tells us how our system evolves over time.

Given an **initial state vector** $\mathbf{v}_0$, we can determine the **state vector** of the system after $i$ time steps:

$$\mathbf{v}_i = A\mathbf{v}_{i-1}$$

# Recall: State Vectors

$$\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number $k$ time steps

This is also called a *recurrence relation* or a *linear difference function*

# Recall: State Vectors

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0)$$

$$\boxed{\mathbf{v}_k = A^k\mathbf{v}_0}$$

$$\mathbf{v}_5 = A\mathbf{v}_4 = A(AAAA\mathbf{v}_0)$$

$$\vdots$$

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It's defined in terms of $A$ itself, which doesn't tell us much about how the system behaves

*It's also difficult computationally because matrix multiplication is expensive*

# (Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain $A^k$ or previously defined terms

# (Closed-Form) Solutions

A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is does not contain $A^k$ or previously defined terms

In other word, it does not depend on $A^k$ and is not **recursive**

# Solutions with Eigenvectors as Initial States

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It's easy to give a closed-form solution if the initial state is an eigenvector:

$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

$$v_1 = A v_0 = \lambda v_0$$

$$v_2 = A v_1 = A(\lambda v_0) = \lambda A v_0 = \lambda^2 v_0$$

# Solutions with Eigenvectors as Initial States

It's easy to give a closed-form solution if the initial state is an eigenvector:

**No dependence on $A^k$ or $\mathbf{v}_{k-1}$**

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**<u>The Key Point.</u> This is still true of sums of eigenvectors.**

# Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for $A$ which span all of $\mathbb{R}^2$:

$\lambda_1 > \lambda_2 \geq 0$

$\lambda_1, \lambda_2$ eigenvalues

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2$$

$$A^k\vec{v} = a_1A^k\vec{b}_1 + a_2A^k\vec{b}_2 = a_1\lambda_1^k\vec{b}_1 + a_2\lambda_2^k\vec{b}_2$$

e.g. $2\lambda_1^k\vec{b}_1 + 3\lambda_2^k b_2$

$$\frac{A^k\vec{v}}{\lambda_1^k} = a_1\vec{b}_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{b}_2 \Rightarrow A^k\vec{v} \sim a_1\lambda_1^k\vec{b}_1$$

# Eigenvectors and Growth in the Limit

**Theorem.** For a linear dynamical system $A$ with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{\cancel{k}\,n}$ of $A$ with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k \geq 0$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant $c_1$ (in other words, in the long term, the system grows <u>exponentially in $\lambda_1$</u>).

Verify: argued in $k=2$

$$\vec{v}_0 = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$$

$$A^k\vec{v}_0 = a_1\lambda_1^k\vec{b}_1 + \cdots + a_n\lambda_n^k\vec{b}_n$$

# Eigenbases

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**Definition.** An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix $A$ is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of $A$.

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*We can represent vectors as **unique** linear combinations of eigenvectors.*

***Not all matrices have eigenbases.***

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \lambda = 1 \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

# Eigenbases and Growth in the Limit

**Theorem.** For a linear dynamical system $A$ with initial state $\mathbf{v}_0$, if $A$ has an eigenbasis $\mathbf{b}_1, \ldots, \mathbf{b}_k$, then

$$\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$$

for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of $A$ and $\mathbf{b}_1$ is its eigenvalue.**

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for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of $A$ and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.