# The Characteristic Equation

## Geometric Algorithms
## Lecture 19

CAS CS 132

# Practice Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

*Determine the dimension of the eigenspace of* $A$ *for the eigenvalue 4.*

*(try not to do any row reductions)*

# Answer

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

# Objectives

1. Briefly <u>recap</u> eigenvalues and eigenvectors
2. Get a primer on <u>determinants</u>
3. Determine how to <u>find eigenvalues</u> (not just verify them)

# Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

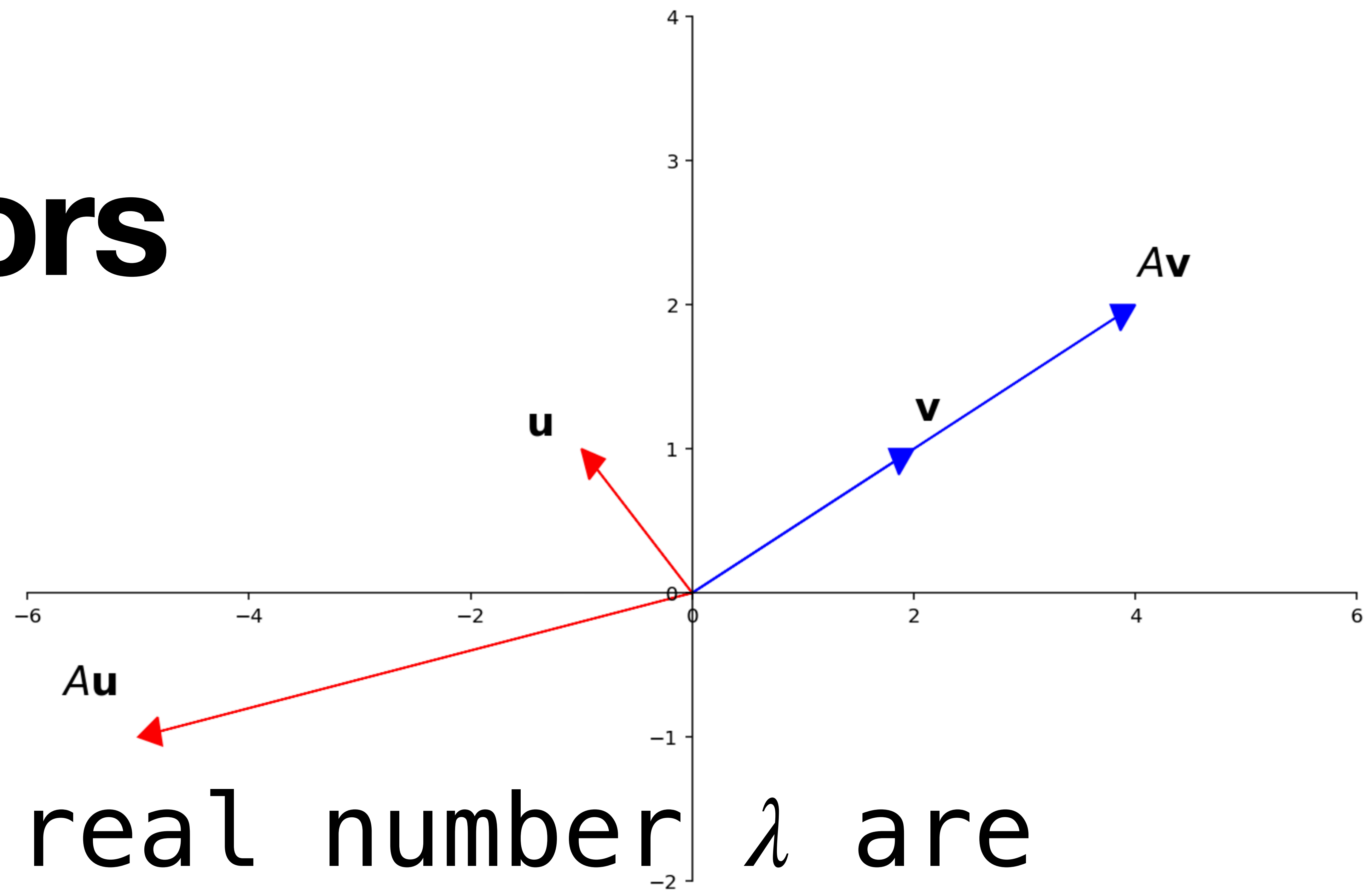
polynomial roots

triangular matrices

multiplicity

# Recap

# Recall: Eigenvalues/vectors




A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number $\lambda$ are an **eigenvector and eigenvalue** for a $n \times n$ matrix $A$ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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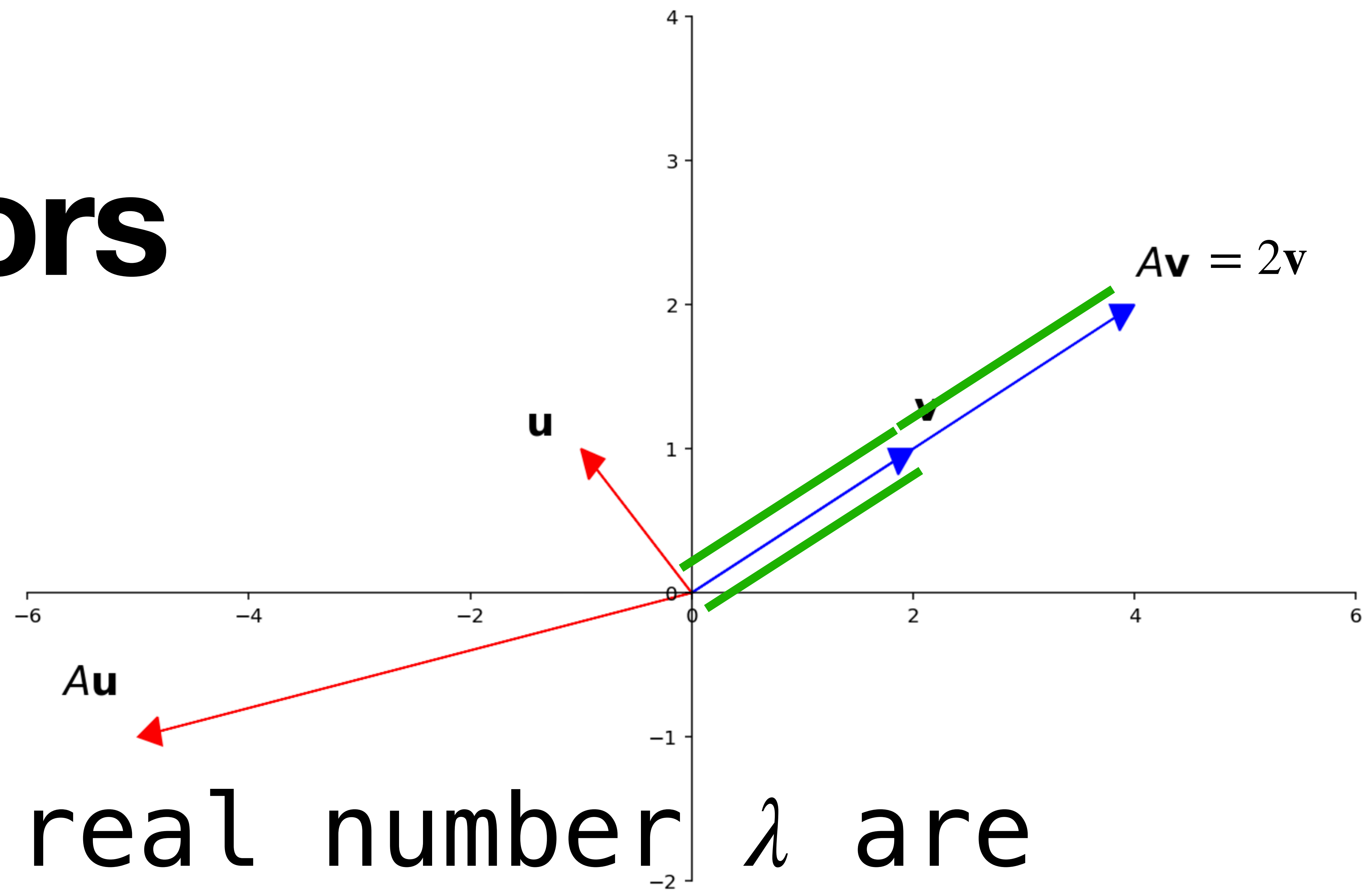


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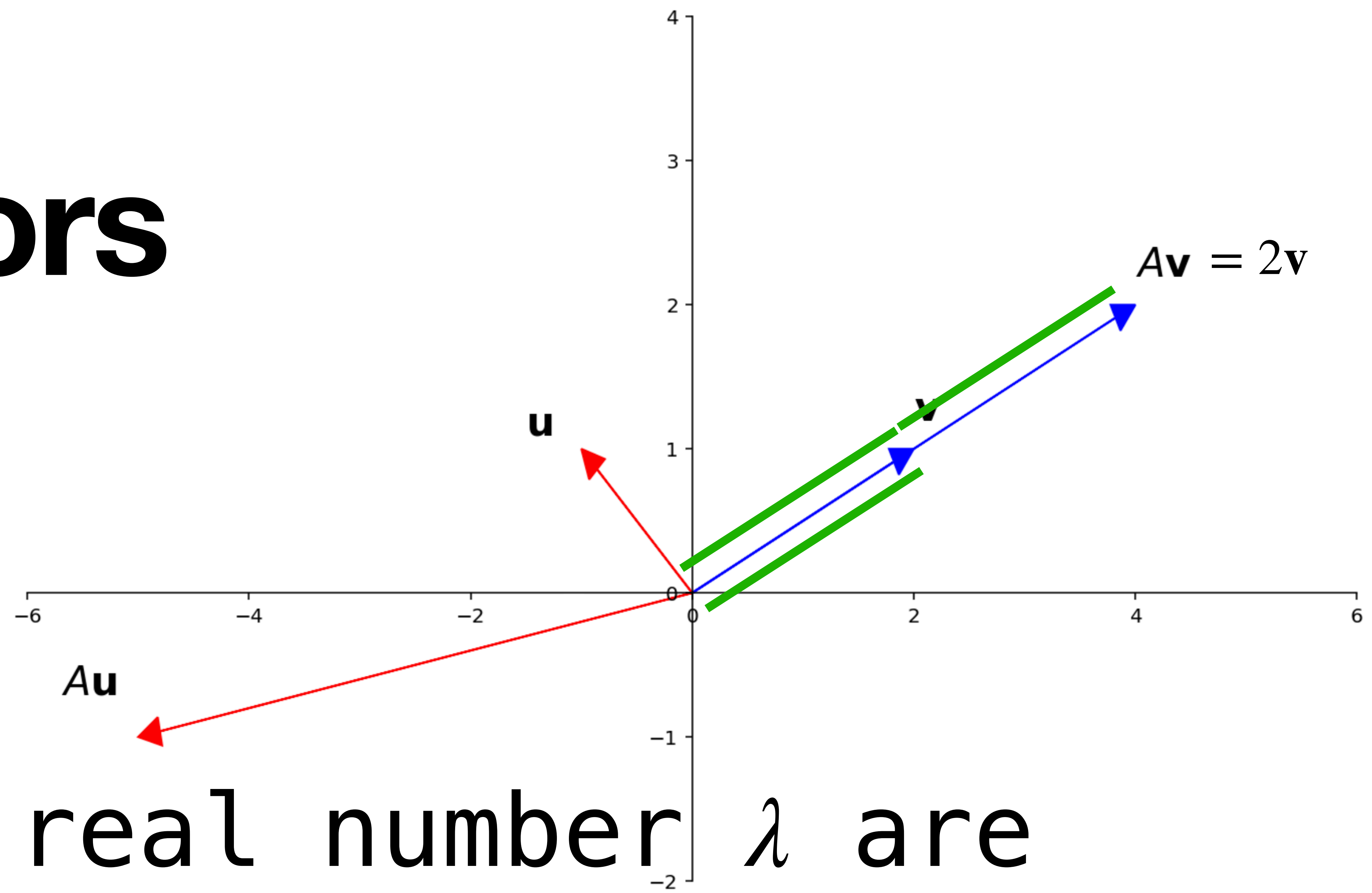


A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number $\lambda$ are an **eigenvector and eigenvalue** for a $n \times n$ matrix $A$ if

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**$\mathbf{v}$ is "just scaled" by $A$, not rotated**

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix}$$

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*If we don't need the vector we can just show that $A - \lambda I$ is **not** invertible (by IMT).*

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(we did this for our recap problem)

How do eigenvectors relate to linear dynamical systems?

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In other word, it does not depend on $A^k$ and is not **recursive**

# Example

$$\mathbf{v}_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_{k-1} \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine a closed for the above linear dynamical system.

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It's easy to give a closed-form solution if the initial state is an eigenvector:

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**<u>The Key Point.</u> This is still true of sums of eigenvectors.**

# Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2$ for $A$ which span all of $\mathbb{R}^2$:

# Eigenvectors and Growth in the Limit

**Theorem.** For a linear dynamical system $A$ with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of $A$ with eigenvalues

$$\lambda_1 > \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant $c_1$ (in other words, in the long term, the system grows <u>exponentially in $\lambda_1$</u>).

Verify:

# Eigenbases

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*We can represent vectors as **unique** linear combinations of eigenvectors.*

***Not all matrices have eigenbases.***

# Eigenbases and Growth in the Limit

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for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of $A$ and $\mathbf{b}_1$ is its eigenvalue.**

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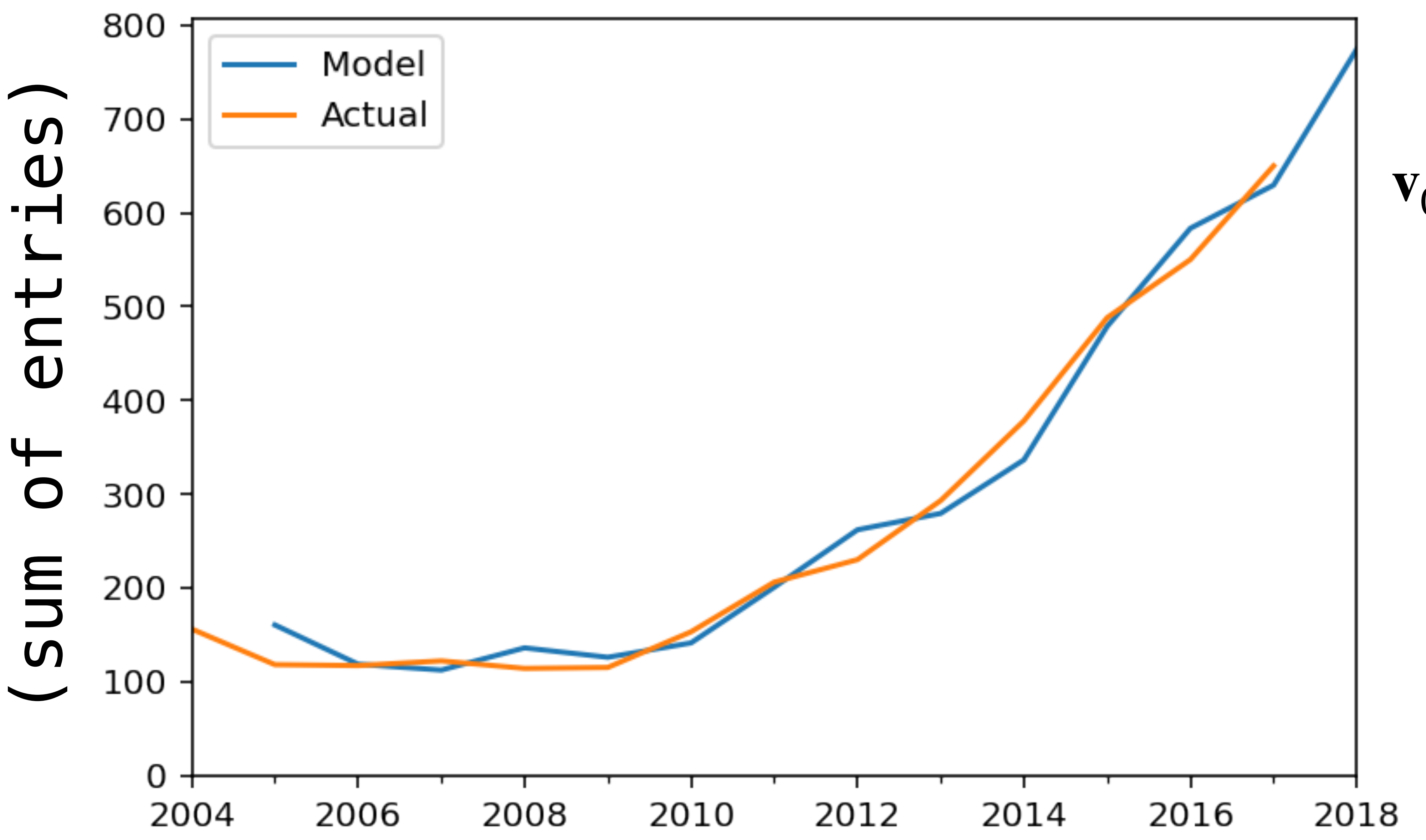
for some constant $c_1$, where where $\lambda_1$ is the **largest eigenvalue of $A$ and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

# Example: CS Major Growth

see the notes for more details




$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \text{\# of year 1 students enrolled in 2024} \\ \text{\# of year 2 students enrolled in 2024} \\ \text{\# of year 3 students enrolled in 2024} \\ \text{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

($A$ is determined by least squares)

**This is clearly exponential. If we want to "extract" the exponent, we need to look at the <u>largest eigenvalue</u>.**

moving on...

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**Solution (Idea).** Can we somehow "solve for $\lambda$" in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

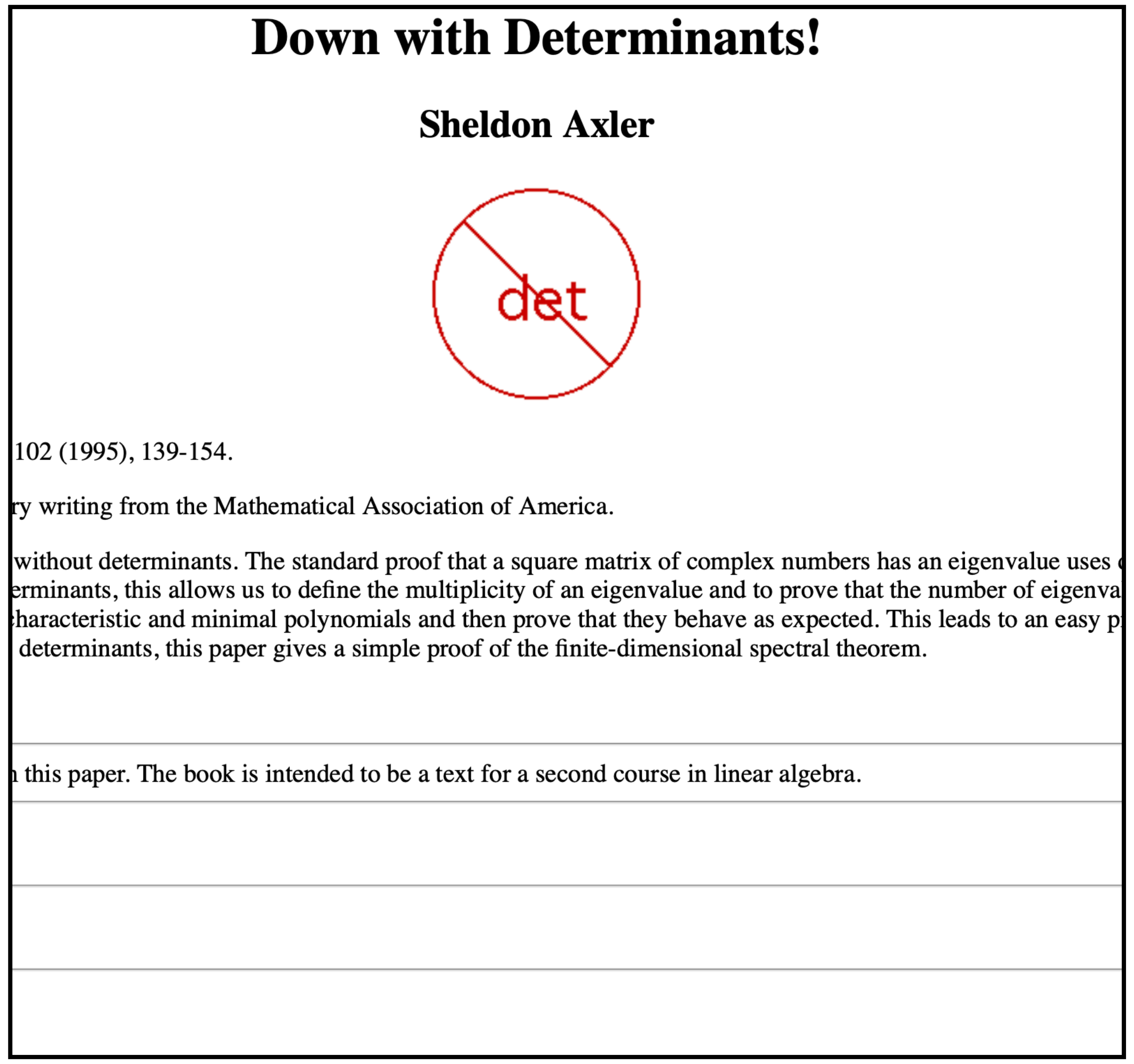
# Determinants

# An Aside: Determinants are Mysterious

Determinants are strangely polarizing

Some people love them, some people hate them

We'll only scratch the surface...



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In broad strokes, it's a big sum of products of entries of $A$.

# A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

We can think of this function as a procedure:

```
FUNCTION det(A):
  total = 0
  FOR all matrix B we can get by swapping a bunch of rows of A:
    s = 1 IF (# of swaps necessary) is even ELSE −1
    total += s ∗ (product of the diagonal entries of B)
  RETURN total
```

# The Determinant of $2 \times 2$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow^{0} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$(-1)^0 ad$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to^1 \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$(-1)^1 cb$$

# The Determinant of $3 \times 3$ matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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# Another Perspective

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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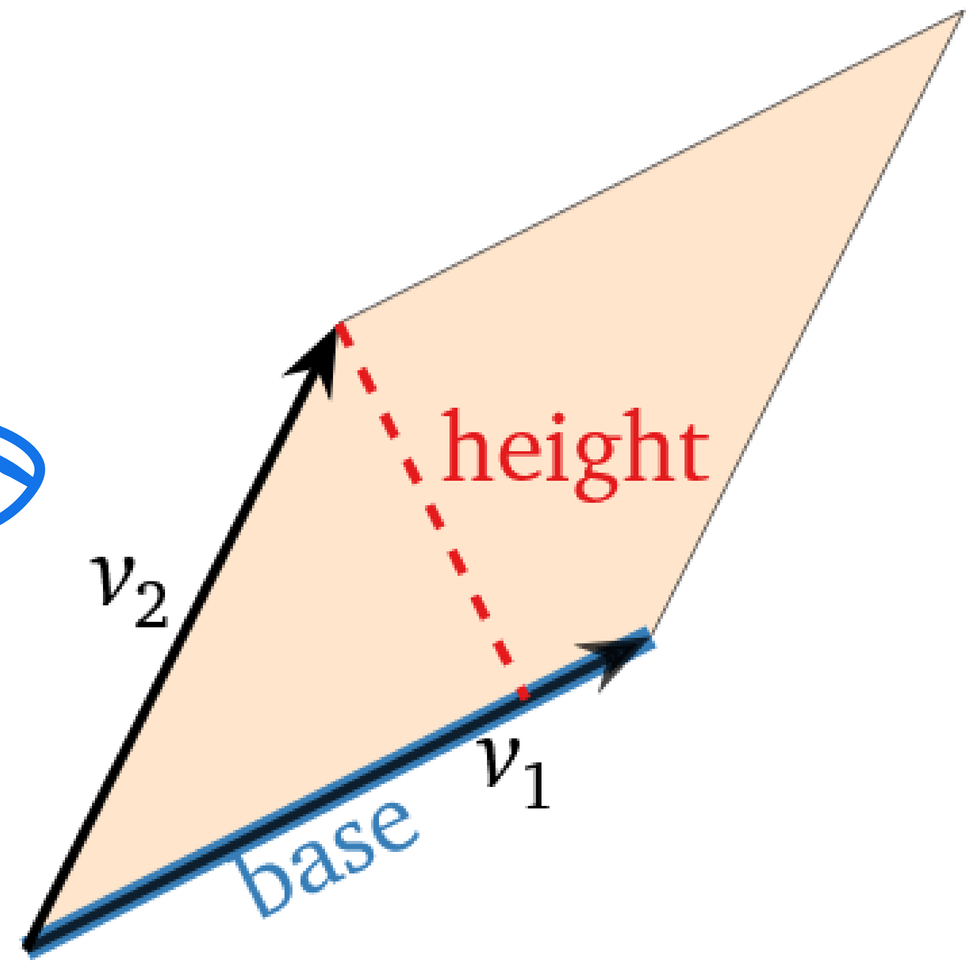
So we can yet again extend the IMT:

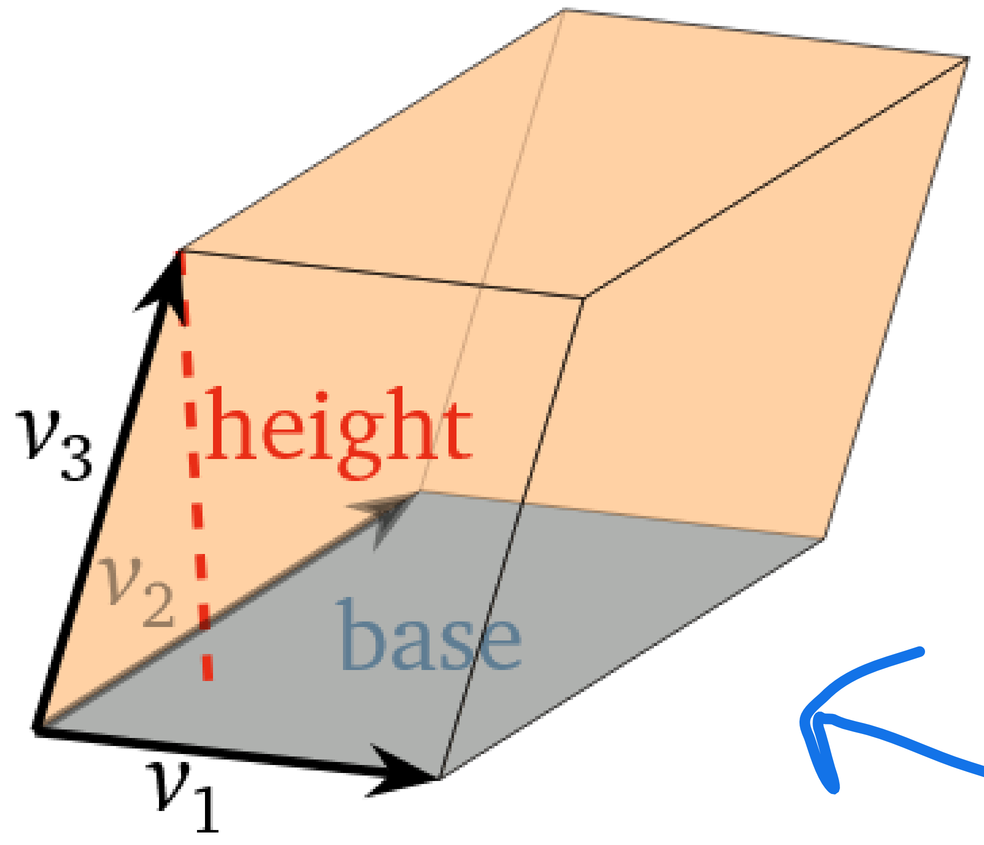
- » $A$ is invertible
- » $\det(A) \neq 0$
- » $0$ is not an eigenvalue

*These must be all true or all false.*

# A Geometric Interpretation: Volume

$$\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\right| = \text{vol}(P)$$




$$\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\right| = \text{vol}(P)$$

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product of diagonal entries

0 if $A$ is not invertible

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# Example

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

# Example (Again)

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

The definition holds no matter which sequence of row operations you use.

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3. Determine the product of entries along the diagonal of $U$, call this $P$.

4. The determinant of $A$ is $\dfrac{(-1)^s P}{c}$.

# Question

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

*Find the determinant of the above matrix.*

# Answer

# The Shorter Version

Beyond small matrices, we'll just use a computer

**With NumPy:**

*numpy.linalg.det(A)*

# Properties of Determinants

# Properties of Determinants (1)

$$\det(AB) = \det(A)\det(B)$$

It follows that $AB$ is invertible if and only if $A$ and $B$ are invertible

(we won't verify this)

# Example Question

*Use the fact that* $\det(AB) = \det(A)\det(B)$ *to give an expression for* $\det(A^{-1})$ *in terms of* $\det(A)$*.*

*Hint. What is* $\det(I)$*?*

# Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that $A^T$ is invertible if and only if $A$ is invertible.

(we also won't verify this)

# Example Question

*If* $A^{-1} = A^T$*, then what are the possible values of* $\det(A)$*?*

# Properties of Determinants (3)

**Theorem.** If $A$ is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:

# Answer

# Characteristic Equation

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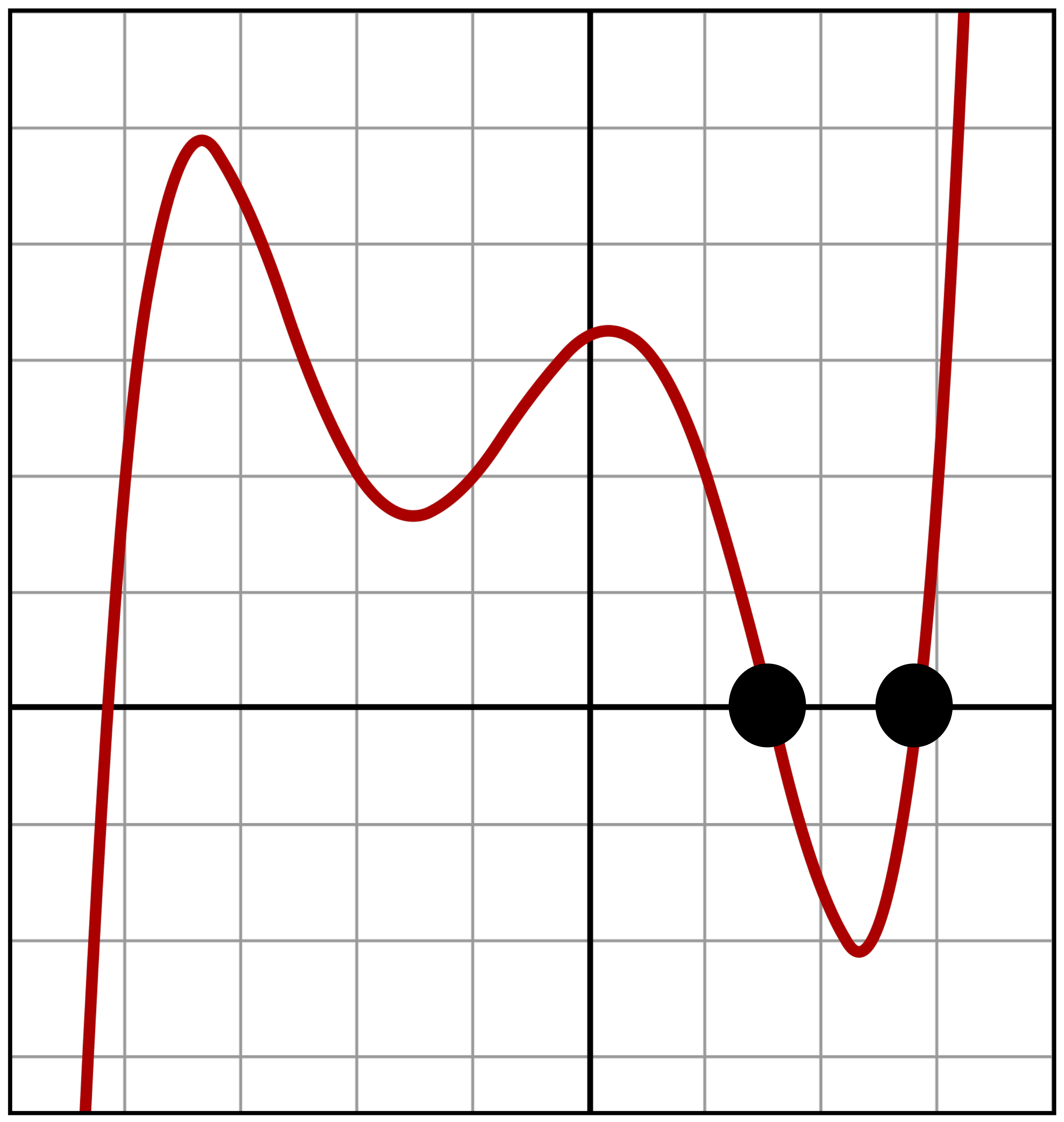
**But a matrix may not have numbers as entries.**

We might think of the matrix $A - \lambda I$ has having *polynomials* as entries.

Then $\det(A - \lambda I)$ is a **polynomial.**

# Reminder: Polynomial Roots



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

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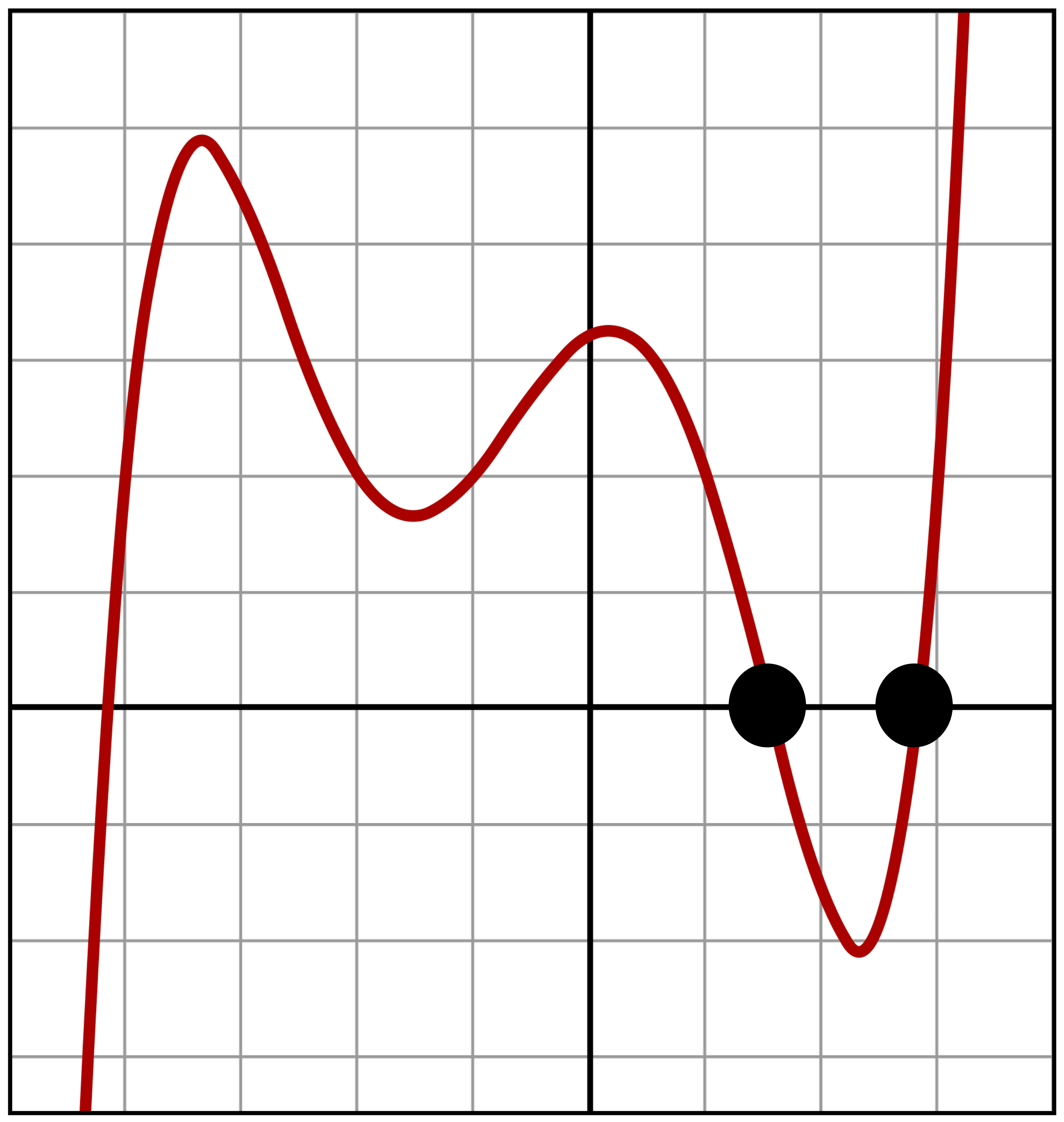

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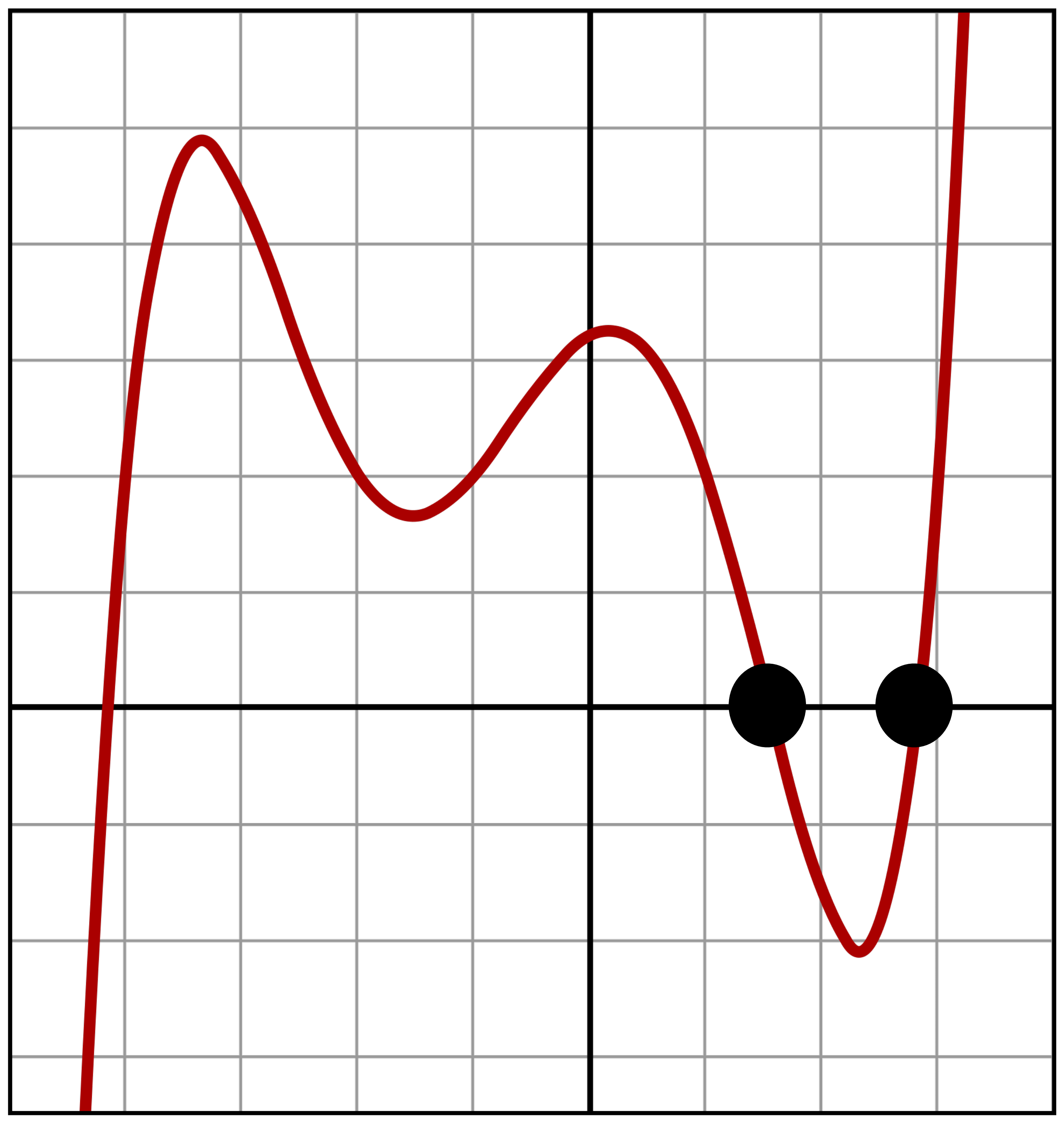

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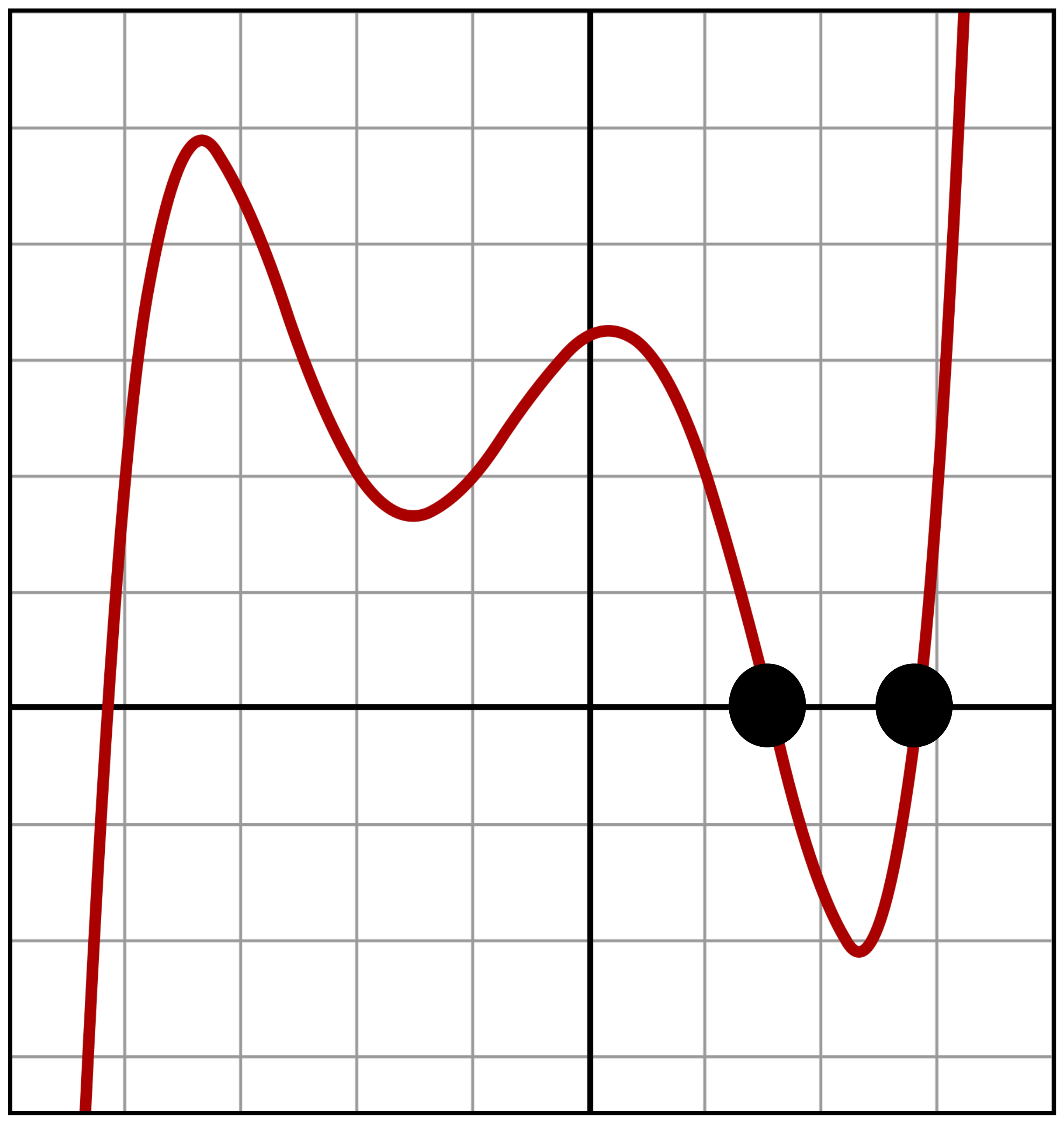
If $r$ is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



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**Definition.** The **characteristic polynomial** of a matrix $A$ is $\det(A - \lambda I)$ viewed as a polynomial in the *variable* $\lambda$.

**This is a polynomial with the eigenvalues of $A$ as roots.**

So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

# "Deriving" the characteristic polynomial

Q: When is $\lambda$ an eigenvalue for $A$?

A: When $(A-\lambda I)\vec{v}=0$ has nontrivial solutions.

$$\Updownarrow \quad (A-\lambda I \text{ not invertible})$$

$$\det(A-\lambda I)=0$$

Hence, the characteristic polynomial

# Example: $2 \times 2$ Matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

# Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes <u>pre-factored</u>:

# How To: Finding Eigenvalues

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**Question.** Find all eigenvalues of the matrix $A$.

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**Solution.** Find the roots of the characteristic polynomial of $A$.

# An Observation: Multiplicity

$$\boxed{\lambda^1}(\lambda - 1)^{\boxed{2}}(\lambda - 4)^{\boxed{1}}$$ multiplicities

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

**Is the multiplicity meaningful in this context?**

# Multiplicity and Dimension

**Theorem.** The dimension of the eigenspace of $A$ for the eigenvalue $\lambda$ is <u>at most</u> the multiplicity of $\lambda$ in $\det(A - \lambda I)$.

**The multiplicity is an upper bound on "how large" the eigenspace is.**

# Example

Let $A$ be a $5 \times 5$ matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$.

» What is $\operatorname{rank}(A)$?

» What is the minimum possible rank of $A - I$?