Diagonalization

Geometric Algorithms

Lecture 20

CAS CS 132

Objectives

1. Finish our discussion on the characteristic polynomial
2. Motivate diagonalization via linear dynamical systems and changes of coordinate systems
3. Describe how to diagonalize a matrix

Keywords

multiplicity

similar matrices

diagonalizable matrices

change of basis

eigenbasis

Recap: Characteristic Polynomial

Recall: Determinants and Invertibility

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$\equiv$ λ is an eigenvalue of A

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polynomial in λ

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How To: Finding Eigenvalues

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Question. Determine the eigenvalues of A.

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Solution. Find the *roots* of the characteristic polynomial of A, which is

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viewed as a *polynomial* in λ.

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We'll also use

numpy.linalg.eig(A)

Example

$$A = \begin{bmatrix} 1 & -1 \\ 7 & -3 \end{bmatrix}$$

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes <u>pre-factored</u>:

An Observation: Multiplicity

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Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is <u>at most</u> the multiplicity of λ in $\det(A - \lambda I)$ (and <u>at least</u> 1)

The multiplicity is an upper bound on "how large" the eigenspace is

Example

Let A *be a* 5×5 *matrix with characteristic polynomial* $(x-1)^3(x-3)(x+5)$

- *What is* $\text{rank}(A)$*?*
- *What is the minimum possible rank of* $A - I$*?*

Practice Problem

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

Determine the eigenvalues and an eigenbasis for the above matrix

Answer

$$(\lambda-a)(\lambda-b) = \lambda^2-(a+b)\lambda+ab$$

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 5-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (5-\lambda)(2-\lambda)-4 = \lambda^2-7\lambda+6 = (\lambda-6)(\lambda-1)$$

$$A-6I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 6$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A-I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/4 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1$

$\vec{v}_1 = \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$

eigen basis

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}\right\}$$

Motivating Diagonalization via Linear Dynamical Systems

(briefly)

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We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$

<u>The Question.</u> When can we describe any vector in $\mathbb{R}^n$ as a unique linear combination of eigenvectors of A?

Recall: Linear Dynamical Systems

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$$

$$\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$$

$$\mathbf{v}_4 = A\mathbf{v}_3 = A^4\mathbf{v}_0$$

$$\vdots$$

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$

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$$\vdots$$

multiplying by A changes the state.

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$

demo

Eigenbases and Closed-Form solutions

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Given $\mathbf{v}_k = A\mathbf{v}_{k-1} = A^k\mathbf{v}_0$, if

$$\mathbf{v}_0 = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$$

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eigenvectors of A

$$\mathbf{v}_0 = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$$

then

$$A^k\mathbf{v}_0 = \alpha_1\lambda_1^k\mathbf{b}_1 + \alpha_2\lambda_2^k\mathbf{b}_2 + \alpha_3\lambda_3^k\mathbf{b}_3$$

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closed-form solution

~~Verify:~~

Application: Eigenbases and Limiting Behavior

Theorem. If A has an eigenbasis with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k \mathbf{u}$ for some vector $\mathbf{u}$.

In the long term, the system grows <u>exponentially in λ_1</u>.

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Given a basis $\mathscr{B}$ for $\mathbb{R}^n$, we only need to know how $A \in \mathbb{R}^n$ behaves on $\mathscr{B}$.

Sometimes, A behaves simply on $\mathscr{B}$, as in the case of <u>eigenbases</u>.

What we're really doing is <u>changing our coordinate system</u> to expose a behavior of A.

~~Recap~~: Change of Basis

Recall: Bases define Coordinate Systems

$\vec{x} = \vec{e}_1 + 6\vec{e}_2$

$= \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

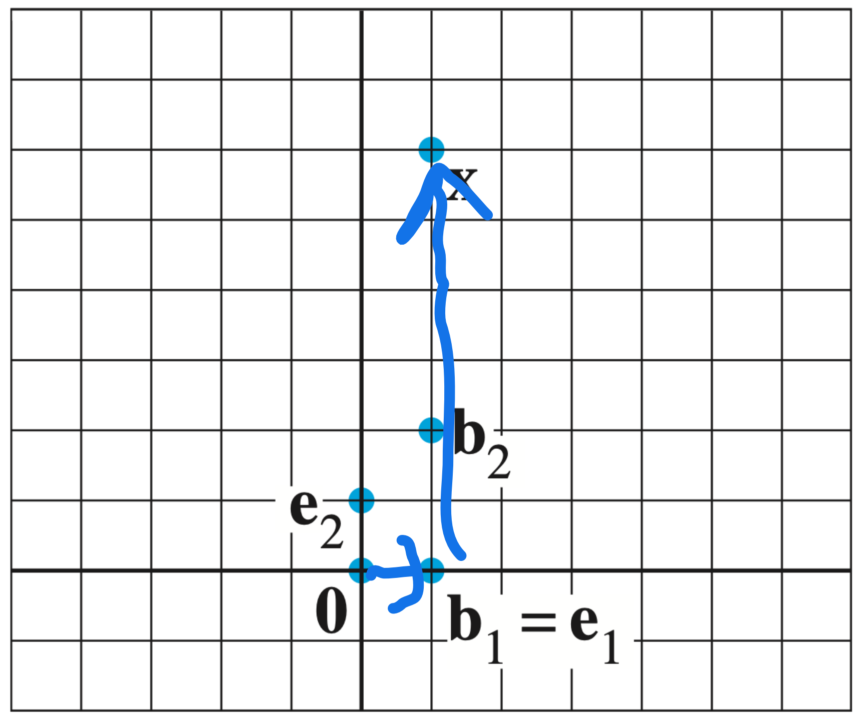


FIGURE 1 Standard graph paper.

$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

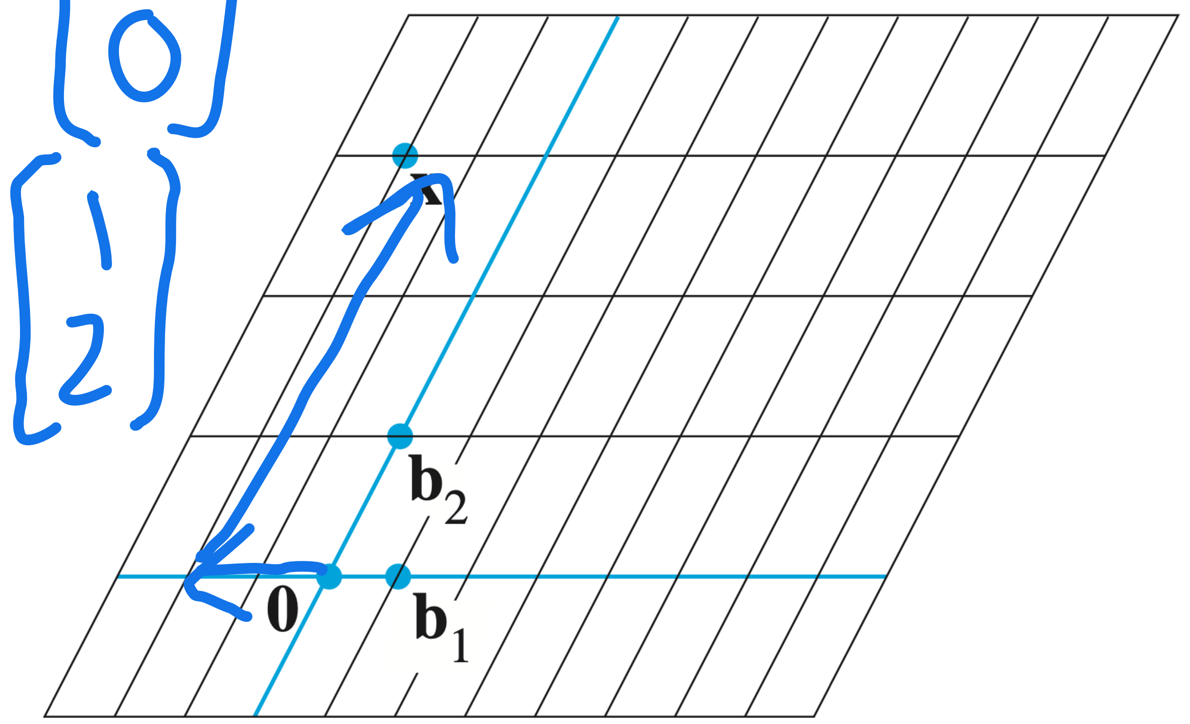


FIGURE 2 $\mathcal{B}$-graph paper.

$[x]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 6 \end{array}\right]$

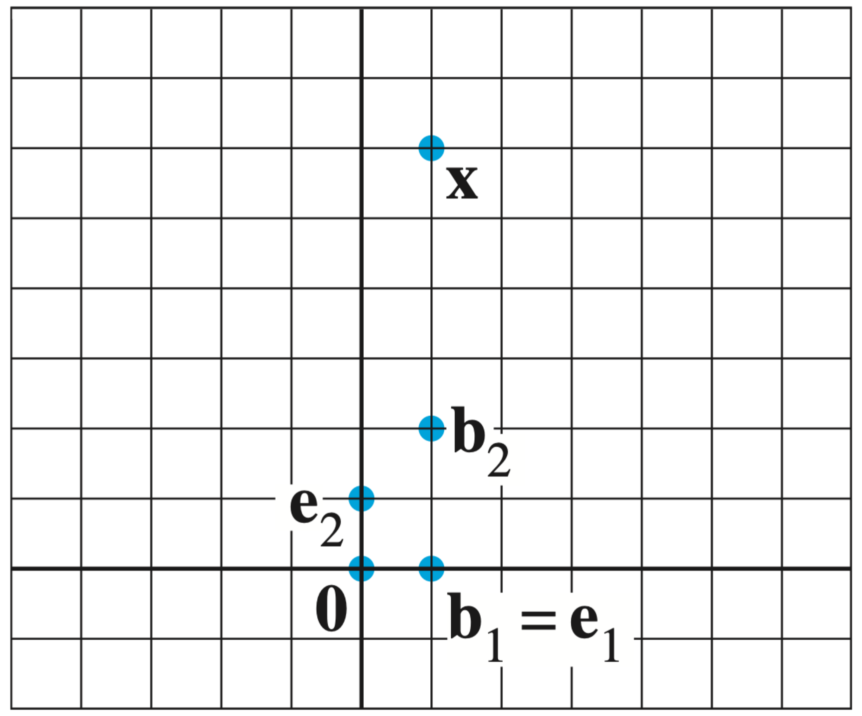
$\vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2$

$[x]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}^{-1} \begin{bmatrix} \vec{x} \end{bmatrix}$

$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \vec{x} \end{bmatrix}$

Linear Algebra and its Applications (Lay, Lay, McDonald)

Recall: Bases define Coordinate Systems



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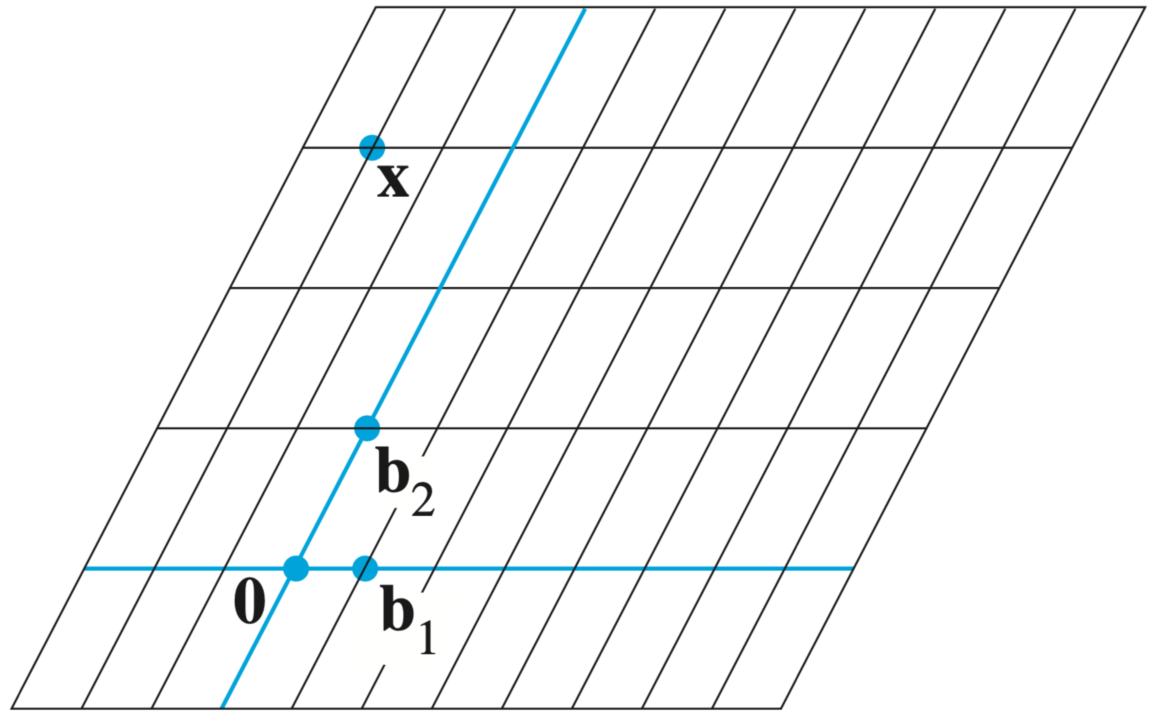

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Given a basis $\mathscr{B}$ of $\mathbb{R}^n$, there is **exactly one way** to write every vector as a linear combination of vectors in $\mathscr{B}$

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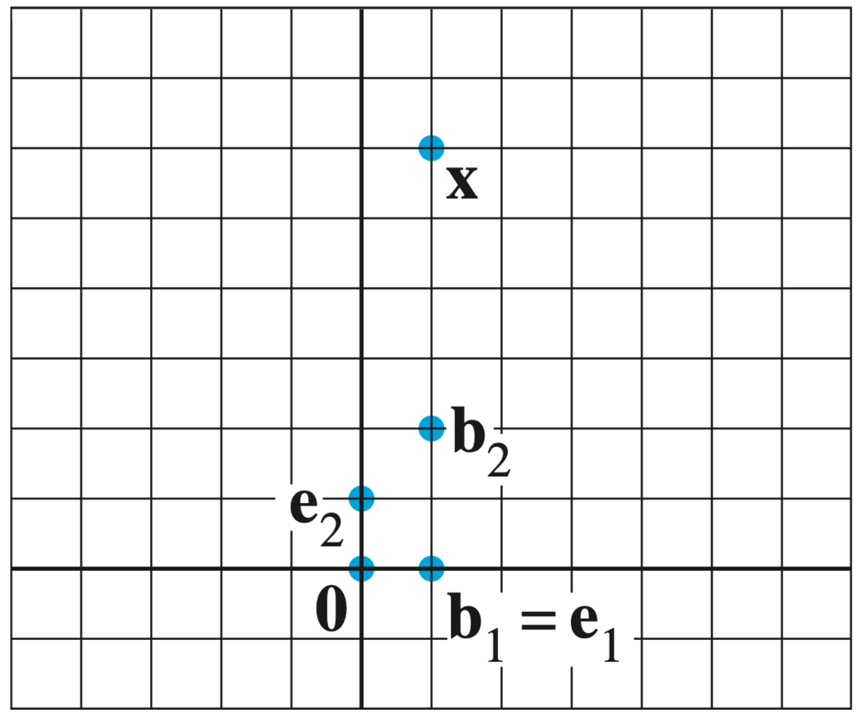

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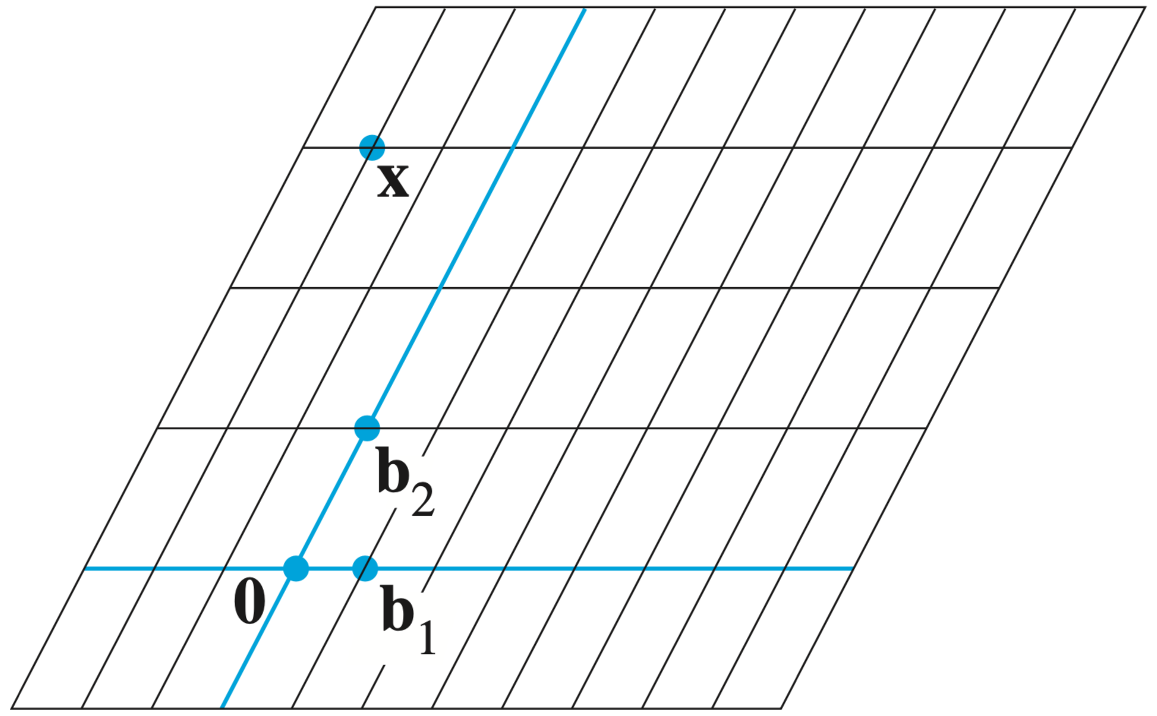

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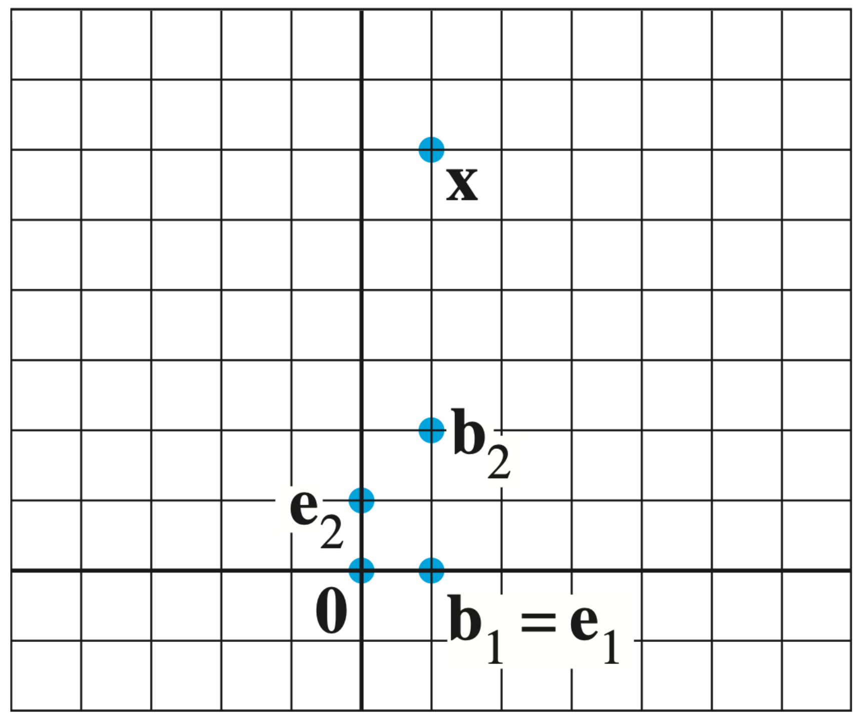

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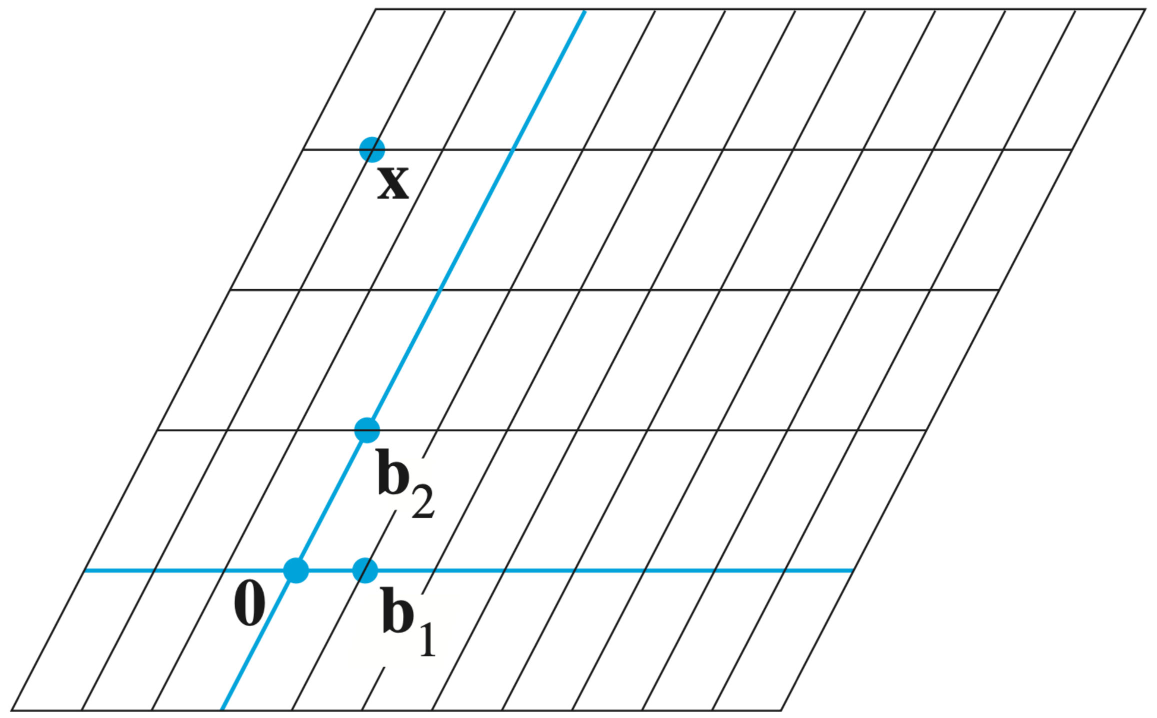

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$\mathscr{B}$ defines a "different grid for our graph paper"

Linear Algebra and its Applications (Lay, Lay, McDonald)

Recall: Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a $\mathbb{R}^n$ and let $\mathscr{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be a basis of $\mathbb{R}^n$ where

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_n\mathbf{b}_n$$

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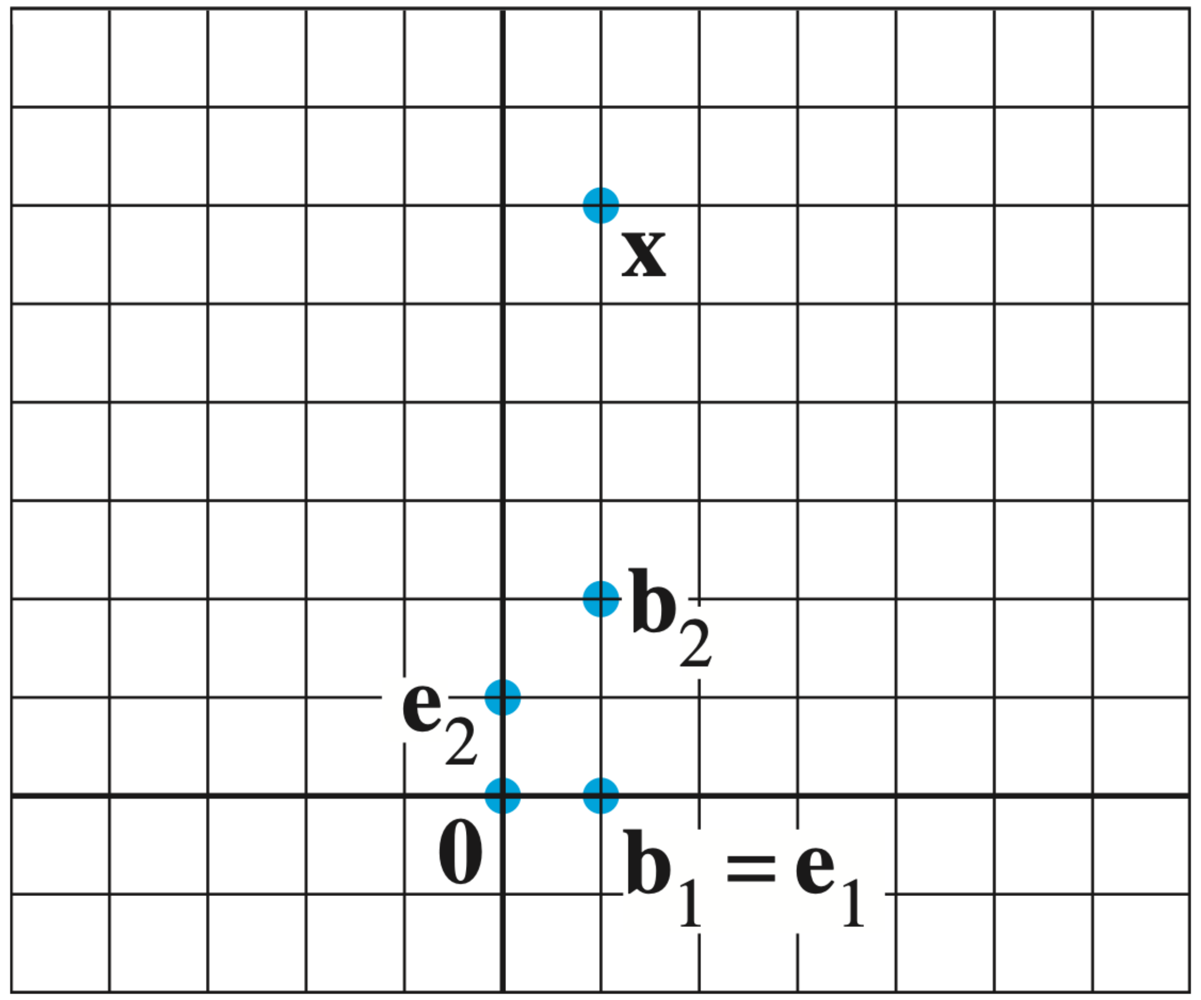
$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Recall: Coordinate Vectors



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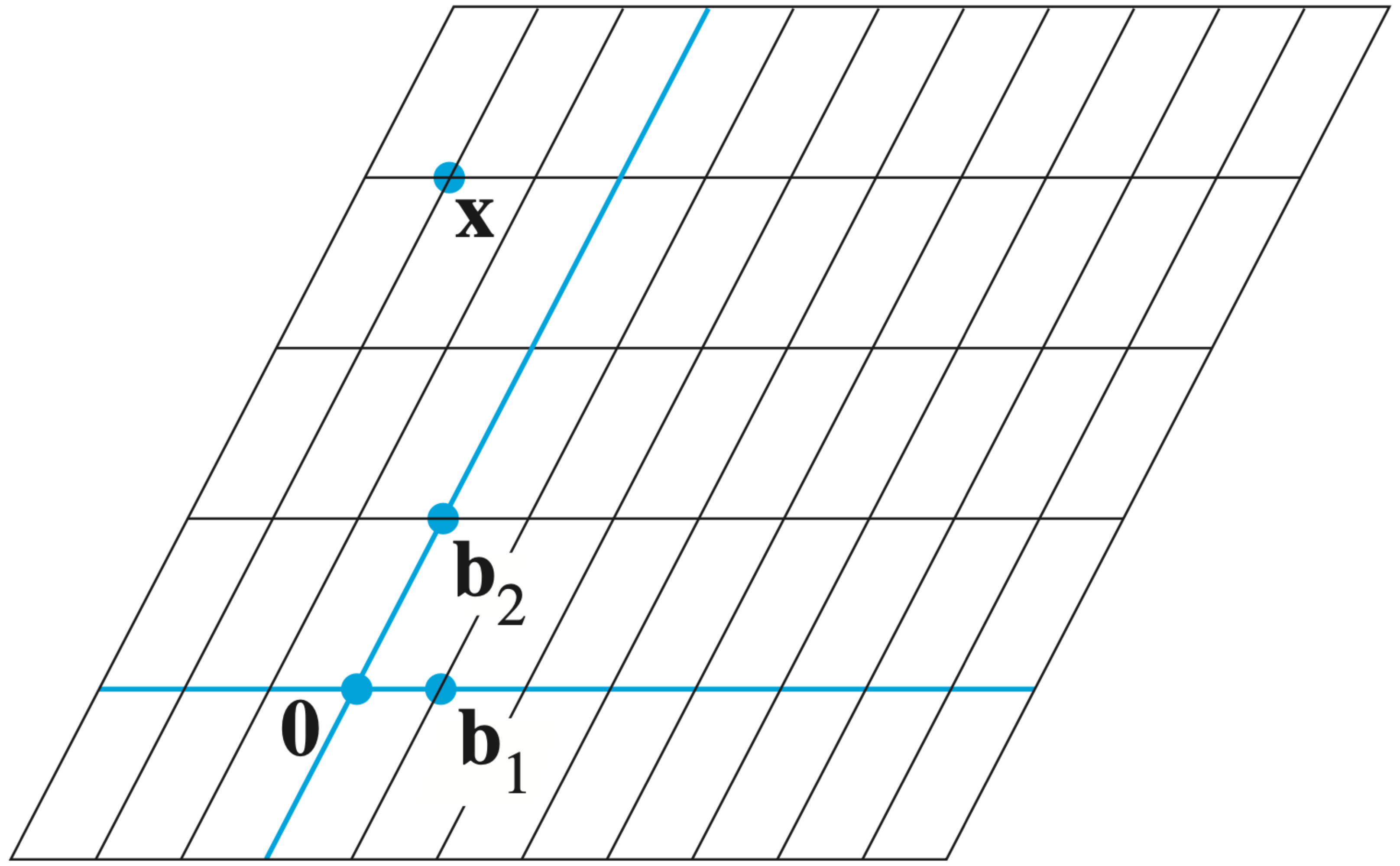


FIGURE 2 $\mathcal{B}$-graph paper.

Question (Conceptual)

We know that if a $n \times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ \dots\ \ \mathbf{b}_n]$ is invertible, then the columns of B form a basis $\mathscr{B}$ of $\mathbb{R}^n$

What is the matrix that implements the transformation

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_n\mathbf{b}_n$*?*

Change of Basis Matrix

Theorem. If $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ form a basis of $\mathbb{R}^n$, then

COB matrix from std basis to $\mathscr{B}$

$$[\mathbf{x}]_{\mathscr{B}} = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]^{-1}\mathbf{x}$$

Matrix inverses perform changes of bases.

$$[x]_{\mathscr{B}} = [C]_{\mathscr{B}}^{\mathscr{C}}[x]_{\mathscr{C}}$$

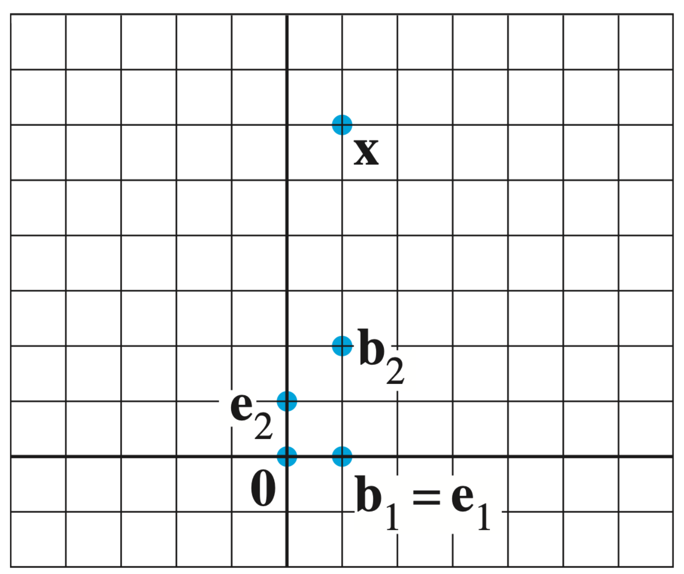
How To: Change of Basis

Question. Given a basis $\mathscr{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of $\mathbb{R}^n$, find the matrix which implements $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$.

Solution. Construct the matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]^{-1}$.

Example

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



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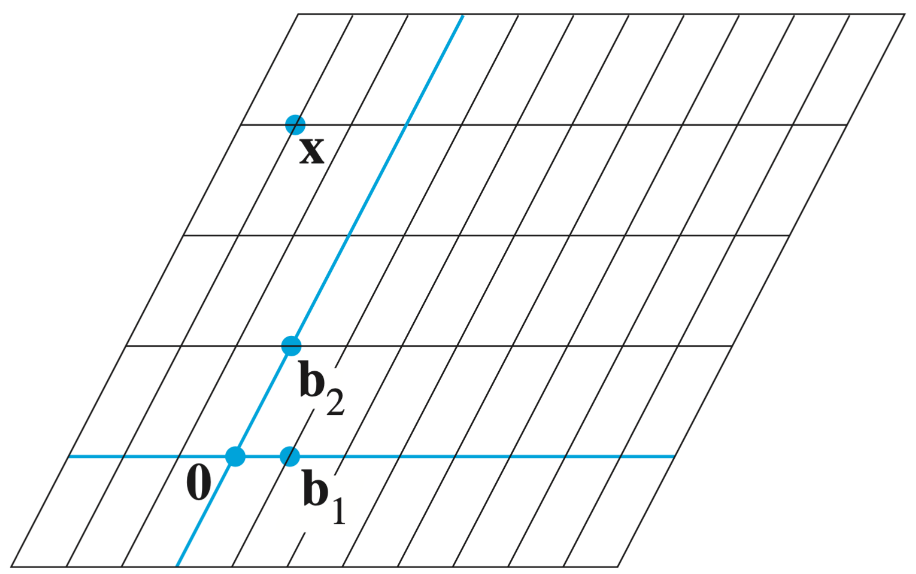

FIGURE 2 $\mathcal{B}$-graph paper.

Write the change-of-bases matrix for the basis $\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right)$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \end{array}\right]$$

Diagonalization

Diagonal Matrices

ex. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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Definition. A $n \times n$ matrix A is **diagonal** if

$$i \neq j \quad \cancel{\text{if and only if}} \quad A_{ij} = 0$$

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Only the diagonal entries can be nonzero

unequal

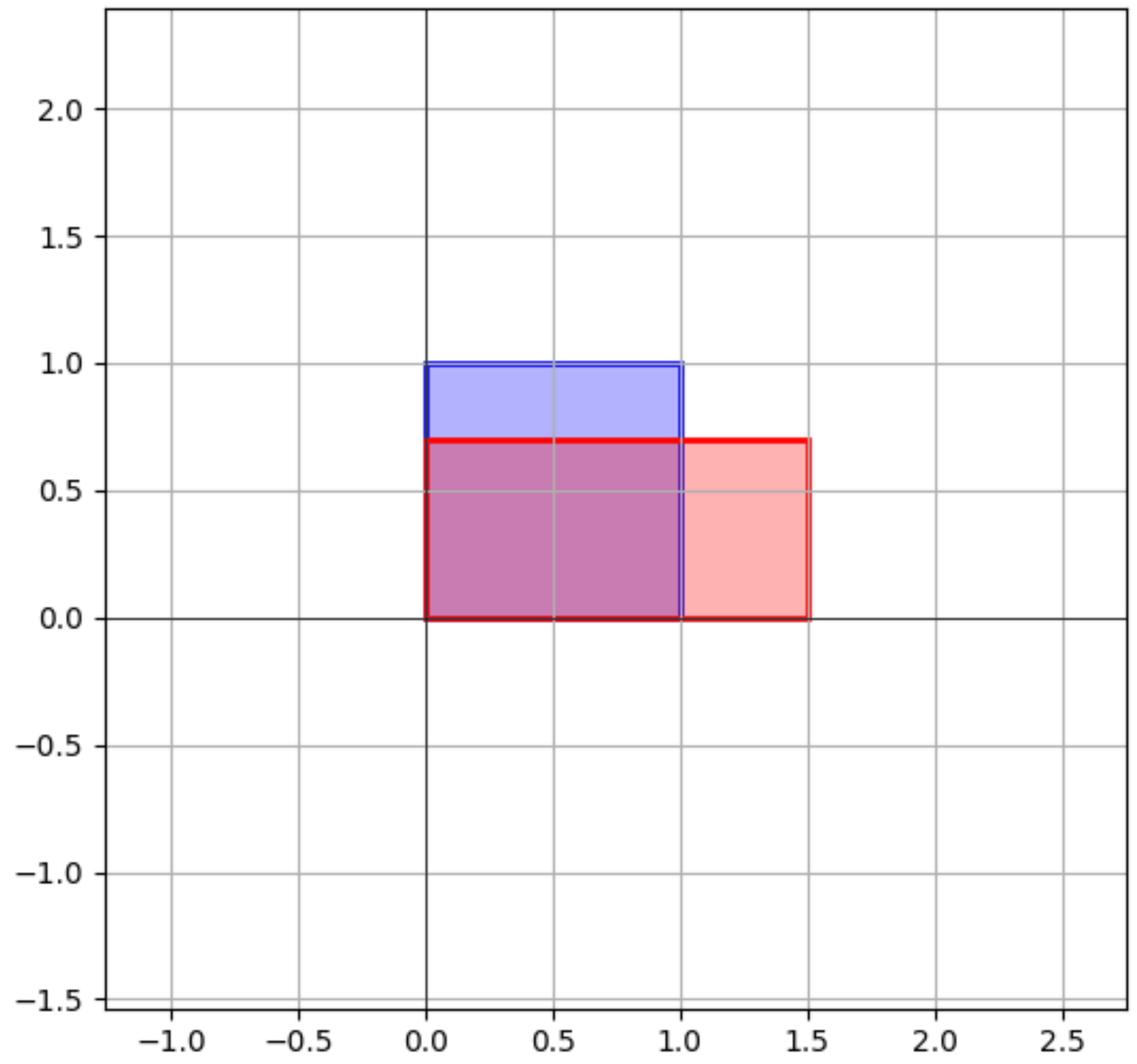
Diagonal matrices are scaling matrices

Recall: Unequal Scaling

The scaling matrix *affects each component of a vector in a simple way*

The diagonal entries <u>scale</u> each corresponding entry

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

High level question: unequal
When do matrices "behave" like scaling matrices "up to" change of basis?

Scaling and Eigenvectors

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The idea. Matrices behave like scaling matrices on eigenvectors.

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathscr{B}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

Scaling and Eigenvectors

The idea. Matrices behave like scaling matrices on eigenvectors.

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The fundamental question:
Can we expose this behavior in terms of a *matrix factorization*?

Recall: Matrix Factorization

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A **factorization** of a matrix A is an equation which expresses A as a product of one or more matrices, e.g.,

$$A = PBP^{-1}$$

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<u>Factorizations can</u>:

- » make working with A easier
- » expose important information about A

Similar Matrices

$$A\vec{x} = PBP^{-1}\vec{x}$$

$$P(B(P^{-1}\vec{x}))$$

$$A = PBP^{-1}$$

Definition. A matrix A is **similar** to a matrix B if there is some invertible matrix P such that $A = PBP^{-1}$

A and B are the same up to a change of basis

$$B = P^{-1}AP = (P^{-1})A(P^{-1})^{-1}$$

Similar Matrices and Eigenvalues

λ

Theorem. Similar matrices have the <u>same eigenvalues</u>.

Verify:

$$A\vec{v} = \lambda\vec{v} \qquad\qquad B\vec{w} = \lambda\vec{w}$$

$$\underset{PBP^{-1}}{\overset{\parallel}{}}\vec{v} = \lambda\vec{v}$$

$$B\underbrace{(P^{-1}\vec{v})}_{\vec{w}} = P^{-1}\lambda\vec{v} = \lambda\underbrace{(P^{-1}\vec{v})}_{\vec{w}}$$

<u>Note</u>: eigenvector has changed

Diagonalizable Matrices

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There is an invertible matrix P and <u>diagonal</u> matrix D such that $A = PDP^{-1}$

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Definition. A matrix A is **diagonalizable** if it is similar to a diagonal matrix

There is an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

Diagonalizable matrices are the same as scaling matrices up to a change of basis

Important: Not all Matrices are Diagonalizable

This is very different from the LU factorization

We will need to figure out which matrices are diagonalizable

<u>Question</u>. Is the zero matrix diagonalizable?

$$D = P\boxed{?}P^{-1}$$
$$= (I)D(I)^{-1}$$

$$0 = (I)0(I)^{-1}$$

Application: Matrix Powers

only take the power of B

Theorem. If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$

It may be easier to take the power of B (as in the case of diagonal matrices)

Verify: $A^2 = (PB\underbrace{P^{-1})(P}_{Id}BP^{-1}) = PB(I)BP^{-1} = PB^2P^{-1}$

How To: Matrix Powers

Question. Given A is diagonalizable, determine A^k

Solution. Find it's diagonalization PDP^{-1} and then compute PD^kP^{-1}

Remember that

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

But how do we find the diagonalization...

Diagonalization and Eigenvectors

Suppose we have a diagonalization

$$A = PDP^{-1}$$

What do we know about it?

Columns of P are eigenvectors

(space)

$A\vec{p}_1 = \square\,\vec{p}_1$

$P\vec{e}_1 = \vec{p}_1$

$P^{-1}\vec{p}_1 = \vec{e}_1$

$$A = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}$$

Verify: $\begin{bmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3\end{bmatrix}\begin{bmatrix}\lambda_1 & & 0\\ 0 & \lambda_2 & \\ & & \lambda_3\end{bmatrix}\begin{bmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3\end{bmatrix}^{-1}\vec{p}_1$

$= \begin{bmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\cancel{\vec{e}_1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3\end{bmatrix}\lambda_1\vec{e}_1 = \lambda_1\vec{p}_1$

$\Rightarrow A\vec{p}_1 = \lambda_1\vec{p}_1$

Columns of P are eigenvectors

$$A = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]^{-1}$$

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In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

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In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

And the entries of D are the **eigenvalues** associated to each eigenvector

Columns of P are eigenvectors

$$A = \overset{\text{eigenbasis}}{[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]} \overset{\text{eigenvalues}}{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}} [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}$$

In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A

And the entries of D are the **eigenvalues** associated to each eigenvector

A diagonalization exposes a lot of information about A

The Diagonalization Theorem

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Theorem. A matrix is diagonalizable if and only if it has an eigenbasis

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(we just did the hard part, if a matrix is diagonalizable then it has an **eigenbasis**)

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(we just did the hard part, if a matrix is diagonalizable then it has an **eigenbasis**)

We can use the same recipe to go in the other direction, given an eigenbasis, we can **build a diagonalization**

Diagonalizing a Matrix

High Level

$$A = PDP^{-1}$$

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The columns of P form an <u>eigenbasis</u> for A

The diagonal of D are the eigenvalues for each column of P

The matrix P^{-1} is a change of basis to this eigenbasis of A

Step 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

`Find all the eigenvalues of` A

`Find the roots of` $\det(A - \lambda I)$

`e.g.`

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

Step 2: Eigenvectors

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

`Find` **`bases`** `of the corresponding eigenspaces`

$1 \leq \dim(\mathrm{Nul}(A+2I)) \leq 2$

`e.g.`

$$\mathrm{Nul}(A - I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Nul}(A + 2I) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 3: Construct P

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

If there are n eigenvectors from the previous step they form an **eigenbasis**

Build the matrix with these vectors as the columns

e.g.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 5: Construct D

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Build the matrix with eigenvalues as diagonal entries

Note the order. It should be the same as the order of columns of P

e.g.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Step 6: Invert P

Find the inverse of P (we know how to do this)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Putting it Together

$$\underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}$$

How to: Diagonalizing a Matrix

Question. Find a diagonalization of $A \in \mathbb{R}^n$, or determine that A is not diagonalizable

Solution.

1. Find the eigenvalues of A, and bases for their eigenspaces. If these eigenvectors don't form a basis of $\mathbb{R}^n$, then A is **not diagonalizable**
2. Otherwise, build a matrix P whose columns are the eigenvectors of A
3. Then build a diagonal matrix D whose entries are the eigenvalues of A *in the same order*
4. Invert P
5. The diagonalization of A is PDP^{-1}

We know how to do every step, its a matter of putting it all together

Example of Failure: Shearing

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The shearing matrix has a single eigenvalue with an eigenspace of dimension 1

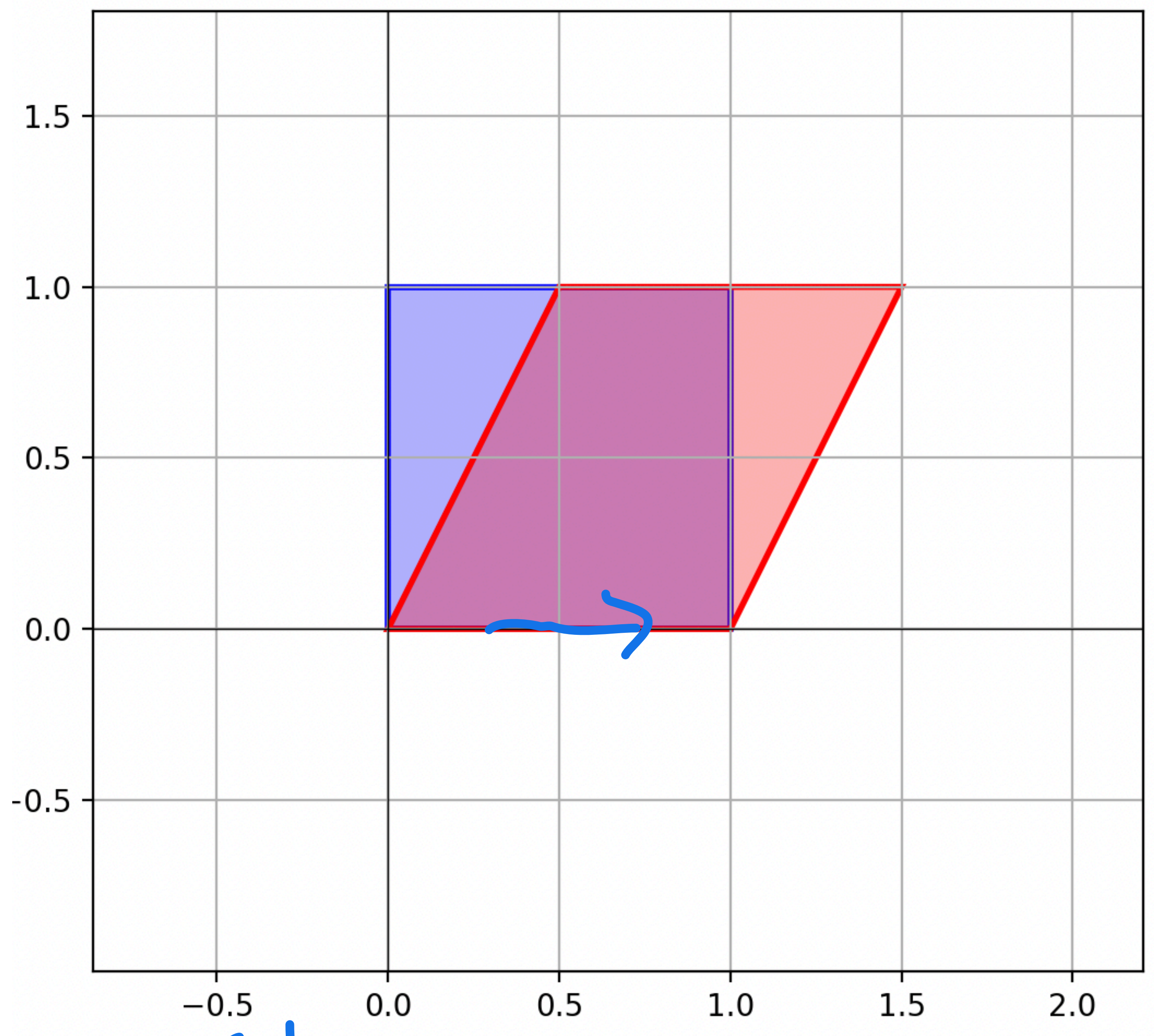
We can't build an eigenbasis of $\mathbb{R}^2$ for A

In other words, A is not diagonalizable

$$\det(A - \lambda I) = (\lambda - 1)^2$$

$$A - I = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Important case: Distinct Eigenvalues

ex. $$\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Theorem. If an $n \times n$ matrix has has n distinct eigenvalues, then it is diagonalizable

This is because eigenvectors with distinct eigenvalues are *linearly independent*

Example

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \qquad \lambda_1 = 1, \quad \lambda_2 = 2 \qquad P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

Find a diagonalization of the above matrix

$$A - I = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

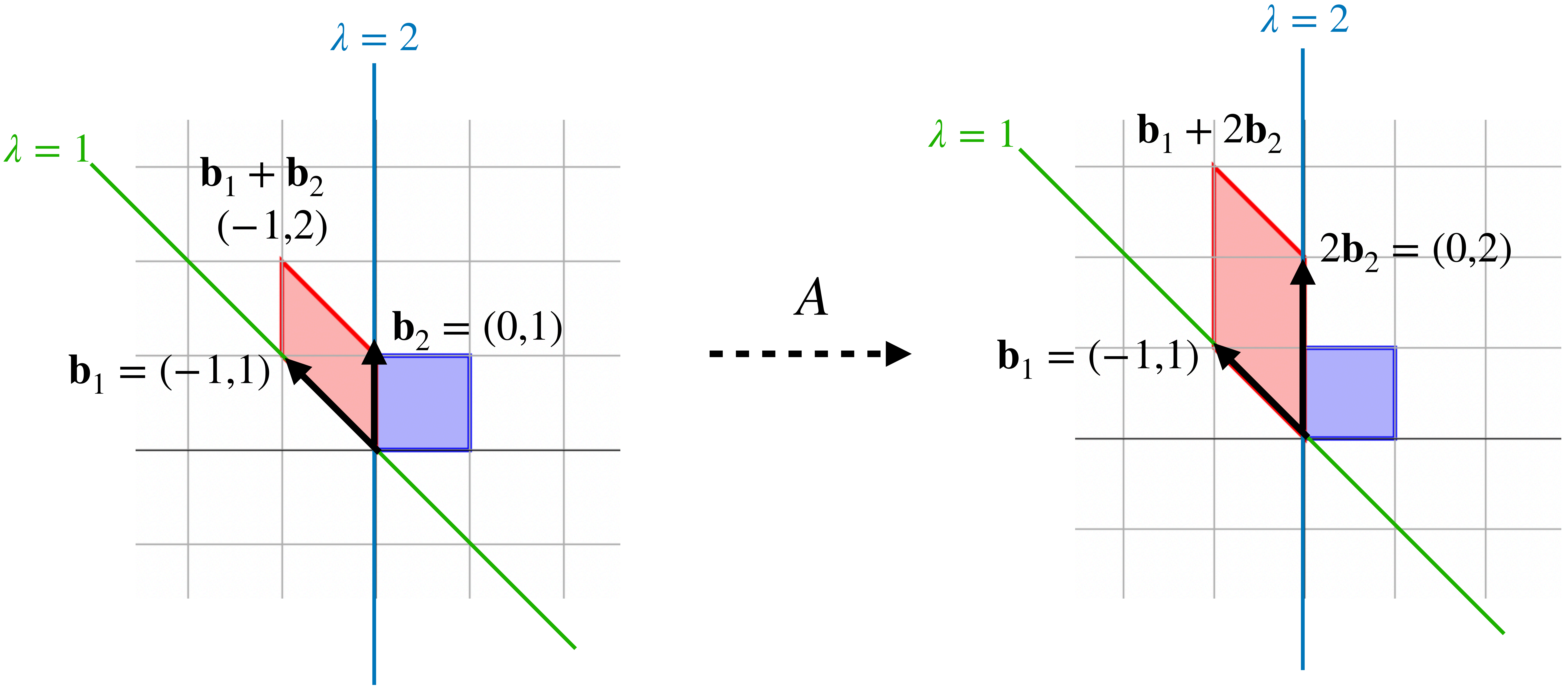
$$A - 2I = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

The Picture

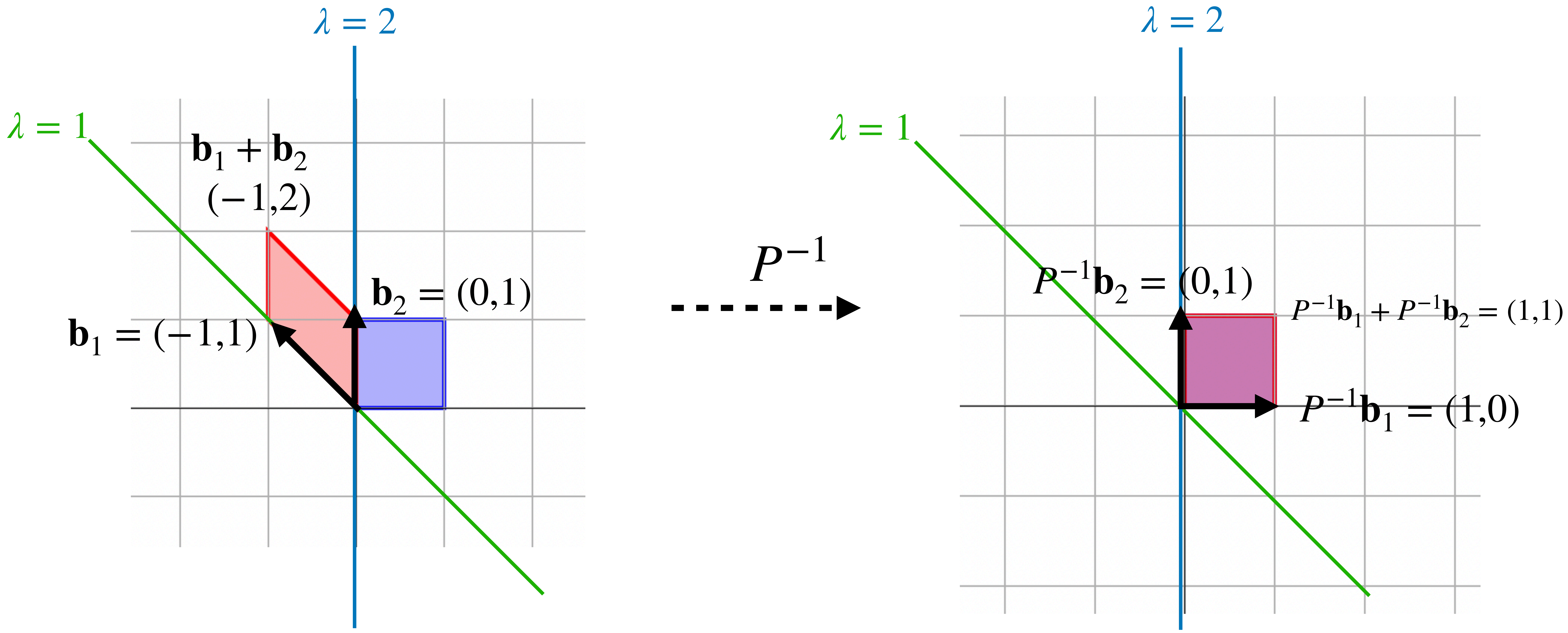
Example (Geometric)

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$



Example (Geometric)

$$P^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

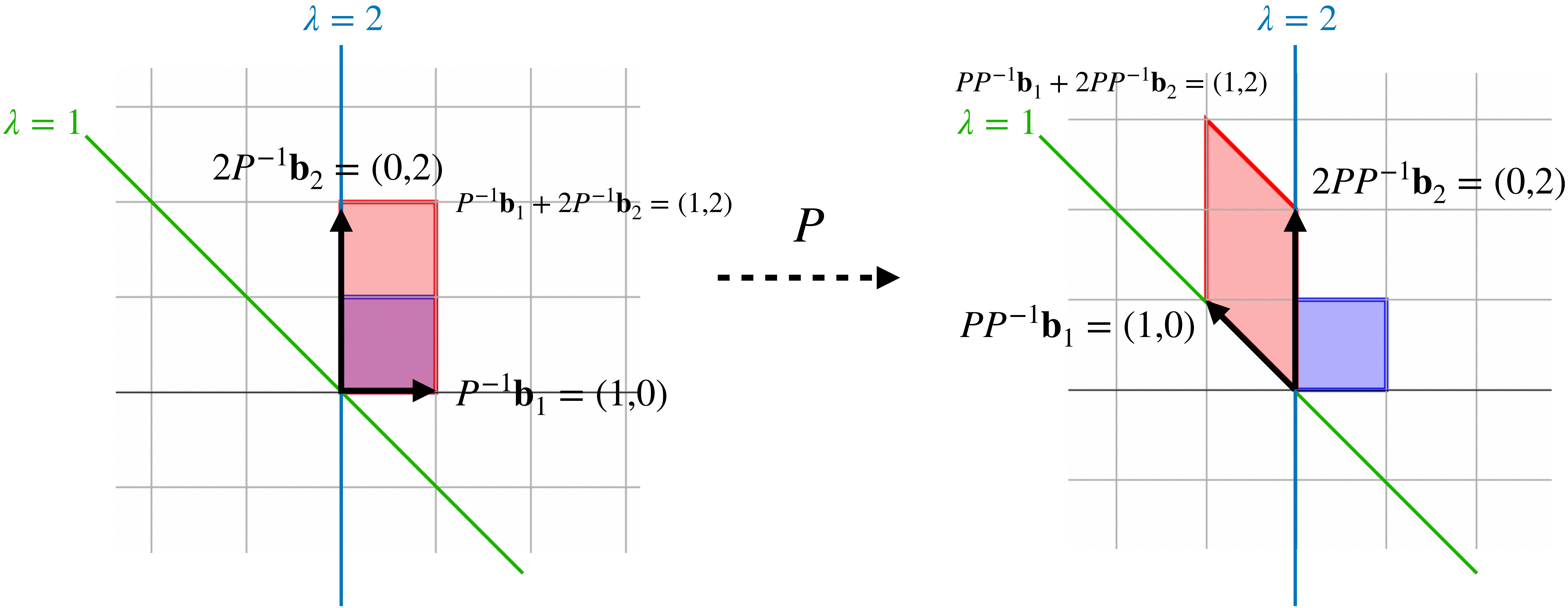


Example (Geometric)

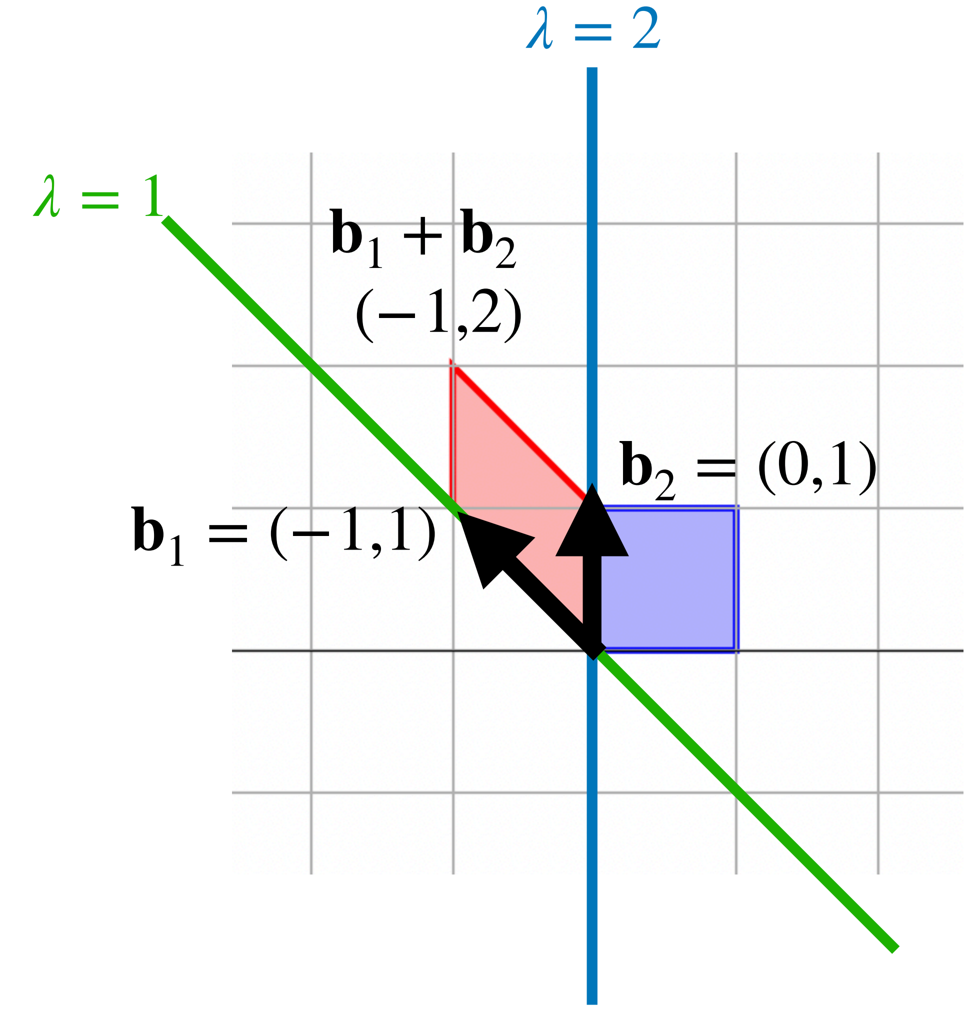
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example (Geometric)

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$



Example (Geometric)



$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$